DIFFERENTIAL CALCULUS *for* JEE Main & Advanced

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DIFFERENTIAL CALCULUS *for* JEE Main & Advanced

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PREFACE

This book is meant for students who aspire to join the Indian Institute of Technologies (IITs) and various other engineering institutes through the JEE Main and Advanced examinations. The content has been devised to cover the syllabi of JEE and other engineering entrance examinations on the topic *Differential Calculus*. The book will serve as a text book as well as practice problem book for these competitive examinations.

As a tutor with more than eighteen years of teaching this topic in the coaching institutes of Kota, I have realised the need for a comprehensive textbook in this subject.

I am grateful to McGraw-Hill Education for providing me an opportunity to translate my years of teaching experience into a comprehensive textbook on this subject.

This book will help to develop a deep understanding of Differential Calculus through graphs and problem solving. The detailed table of contents will enable teachers and students to easily access their topics of interest.

Each chapter is divided into several segments. Each segment contains theory with illustrative examples. It is followed by Concept Problems and Practice Problems, which will help students assess the basic concepts. At the end of the theory portion, a collection of Target Problems have been given to develop mastery over the chapter.

The problems for JEE Advanced have been clearly indicated in each chapter.

The collection of objective type questions will help in a thorough revision of the chapter. The Review Exercises contain problems of a moderate level while the Target Exercises will assess the students' ability to solve tougher problems. For teachers, this book could be quite helpful as it provides numerous problems graded by difficulty level which can be given to students as assignments.

I am thankful to all teachers who have motivated me and have given their valuable recommendations. I thank my family for their whole-hearted support in writing this book. I specially thank Mr. Devendra Kumar and Mr. S. Suman for their co-operation in bringing this book.

Suggestions for improvement are always welcomed and shall be gratefully acknowledged.

Vinay Kumar

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1

LIMITS

1.1 Introduction

Let us introduce the notion of limit of a function which plays an important role in mathematical analysis and the study of calculus. The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry.

We use limits to describe the way a function f varies. Some functions vary continuously i.e. small changes in x produce only small changes in f(x). Other functions can have values that jump or vary erratically.

Sometimes we face with problems whose solutions involve the use of limits. According to a formula of geometry, the area of a circle of radius r is πr^2. How is such a formula derived? The usual way is to inscribe regular polygons in the circle, find the areas of these polygons, and then determine the “limiting value” of these areas as the number of sides of the polygons increase without bound. Thus, even such seemingly simple formula as that for the area of a circle depends on the concept of limit for its derivation.

We also use limits to define tangent to graphs of functions. This geometrical application leads to the important concept of derivative of a function, which quantifies the way a function's value changes.

The need for evaluating the limit of a function arises in science and engineering when we come across situations where a function (denoting a physical quantity) is not defined at x = a, however the value of the function as x takes values very very close to ‘a’ symbolize a useful physical quantity, for example instantaneous velocity, acceleration etc.

Neighbourhood of a Point

Any open interval containing a point a as its midpoint is called a neighbourhood of a.

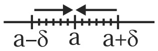
We denote neighbourhoods by N(a), $N_1(a)$, $N_2(a)$, etc.

Since a neighbourhood N(a) is an open interval symmetric about a, it consists of all real x satisfying $a - \delta < x < a + \delta$ for some $\delta > 0$. The positive number δ is called the radius of the neighbourhood.

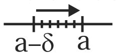
We designate N(a) by N(a, δ) if we wish to specify its radius. The inequalities $a - \delta < x < a + \delta$ are equivalent to $-\delta < x - a < \delta$, and to $|x - a| < \delta$. Thus, N(a, δ) consists of all points x whose distance from a is less than δ.

Meaning of $x \to a$

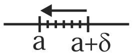
The symbol $x \to a$ is called as 'x tends to a' or 'x approaches a'. It implies that x takes values closer and closer to 'a' but not 'a'.



Sometimes we need to consider values of x approaching 'a' from only one side of 'a'.

If x approaches 'a' from the left of 'a' then we use the symbol: $x \to a^-$ or $x \to a - 0$. 

Similarly, if x approaches 'a' from the right of 'a' then we use the symbol : $x \to a^+$ or $x \to a + 0$.



Limit of a Function

Let us consider a function y = f(x) of a continuous variable x. Suppose that the independent variable x approaches a number 'a'. This means that x is made to assume values which become arbitrarily close to 'a' but are not equal to 'a'. To describe such a situation we say that x tends to 'a' or x approaches 'a' and write $x \to a$. If there is a number ℓ such that as x approaches 'a', either from the right or from the left, f(x) approaches ℓ, then ℓ is called the limit of f(x) as x approaches 'a'.

Informal Definition of Limit

Let f(x) be defined on an open interval about 'a', except possibly at 'a' itself. If f(x) gets arbitrarily close to ℓ for all x sufficiently close to 'a', we say that f approaches the limit ℓ as x approaches 'a', and we write

$$\lim_{x\to a} f(x) = \ell.$$

(the abbreviation "lim" means "limit")

It is evident that if a function has a limit for $x \to a$, this limit is unique, since the values of the function corresponding to the values of x approaching 'a' must become arbitrarily close to a constant and hence cannot be simultaneously close to more than one constant number.

An alternative notation for $\lim_{x\to a} f(x) = \ell$ is $f(x) \to \ell$ as $x \to a$ which is usually read as "f(x) approaches ℓ as x approaches a".

Study Tip

1. A number ℓ is said to be a limiting value only if it is finite and real, otherwise we say that the limit does not exist or dne (for brevity).
2. Note that 'a' need not be in the domain of f.

   Even if 'a' happens to be in the domain of f, the value f(a) plays no role in determining whether $\lim_{x\to a} f(x) = \ell$. It is only the behaviour of f(x) for x near a that concerns us.
3. We also note that the three statements,

   $\lim_{x\to a} f(x) = \ell$, $\lim_{x\to a} (f(x) - \ell) = 0$, $\lim_{x\to a} |f(x) - \ell| = 0$, are all equivalent.

Let us consider the function $f(x) = \dfrac{x}{x}$.

This function is equal to 1 for all values of x except x = 0. It is not equal to 1 when x = 0; it is in fact not defined at x = 0. For when we say that f(x) is defined for x = 0 we mean that we can calculate its value for x = 0 by putting x = 0 in the formula which defines f(x). In this case we cannot. When we put x = 0 in f(x) we get 0/0, which is meaningless.

Thus $f(x) = \dfrac{x}{x}$ is a function which differs from y = 1 solely in that it is not defined for x = 0.

We have $\lim_{x\to 0} \dfrac{x}{x} = 1$, since x/x is equal to 1 so long as x differs from zero, however small the difference may be.

On the other hand there is of course nothing to prevent the limit of f(x) as x tends to zero from being equal to f(0), the value of f(x) for x = 0.

Thus if f(x) = x, then f(0) = 0 and $\lim_{x\to 0} f(x) = 0$.

Note: That the statement $\lim_{x\to 0} f(x) = \ell$ is a statement about the values of f(x) when x has any value distinct from but differing by little from zero. It is not a statement about the value of f(x) when x = 0. When we make the statements we assert that, when x is nearly equal to zero, f(x) is nearly equal to ℓ. We assert nothing about what happens when x actually equal to 0.

Caution

A wrong statement about limits :

The number ℓ is the limit of f(x) as x approaches a if f(x) gets closer to ℓ as x approaches a.

Consider f(x) = x + [x] +1 at x = 0.

As x approaches 0, the function f(x) gets closer to 1/2, from both sides but 1/2 is not the limit because f(x) does not get arbitrarily close to 1/2. For example the function cannot attain 0.4995 or 0.5002 by using x sufficiently close to 0.

Suppose we are asked to sketch the graph of the function f given by $f(x) = \dfrac{x^3 - 1}{x - 1}$, $x \neq 1$.

For all values other than x = 1, we can use standard curve-sketching techniques. However, at x = 1, it is not clear what to do. To get an idea of the behaviour of the graph of f near x = 1, we can use two sets of x-values – one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in table.

x approaches 1 from the left → ← x approaches 1 from the right

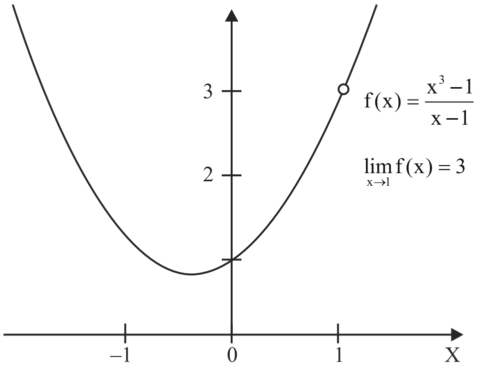
x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
f(x)	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

f(x) approaches 3. → ← f(x) approaches 3.

When we plot these points, it appears that the graph of f is a parabola that has a hole at point (1, 3), as shown in the figure. Although x cannot equal 1, we can move sufficiently close to 1, and as a result f(x) moves arbitrarily close to 3.

Using limit notation, we write $\lim_{x\to 1} f(x) = 3$.

It is read as "the limit of f(x) as x approaches 1 is 3."



Example 1: Evaluate $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x-2}$

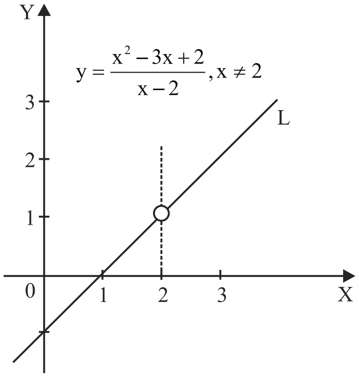
Solution: Consider the function f defined by

$$f(x)=\frac{x^2-3x+2}{x-2},\ x\neq 2$$

The domain of f is the set of all real numbers except 2, which has been excluded because substitution of x = 2 in the expression for f(x) yields the undefined term $\dfrac{0}{0}$.

On the other hand, $x^2-3x+2=(x-1)(x-2)$ and $f(x)=\dfrac{(x-1)(x-2)}{x-2}=x-1$, provided $x\neq 2$ ....(1)

The graph of the function y = x – 1 is a straight line L; so the graph of f(x) is the line L with a hole at the point (2, 1).



Although the function f is not defined at x = 2, we know its behaviour from values of x near 2. The graph makes it clear that if x is close to 2, then f(x) is close to 1. In fact, the values of f(x) can be brought arbitrarily close to 1 by taking x sufficiently close to 2.

We express this fact by writing $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x-2}=1$, which means that the limit of $\dfrac{x^2-3x+2}{x-2}$ is 1 as x approaches 2.

Example 2: Let $f(x)=\dfrac{x^3-1}{x^2-1}$. How does f(x) behave when x is near 1 but is not 1 itself ?

Solution: There are two influences acting on the fraction $(x^3-1)/(x^2-1)$ when x is near 1. On the one hand, the numerator x^3-1 approaches 0; thus there is an influence pushing the fraction towards 0. On the other hand the denominator x^2-1 also approaches 0; division by a small number tends to make a fraction large. How do these two opposing influences balance out?

We rewrite the quotient $(x^3-1)/(x^2-1)$ as follows: When $x\neq 1$, we have

$$\frac{x^3-1}{x^2-1}=\frac{(x^2+x+1)(x-1)}{(x+1)(x-1)}=\frac{x^2+x+1}{x+1}.$$

So the behaviour of $(x^3-1)/(x^2-1)$ for x near 1, but not equal to 1, is the same as the behaviour of $\dfrac{x^2+x+1}{x+1}$.

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{x^2+x+1}{x+1}.$$

Now, as x approaches 1, x^2+x+1 approaches 3 and x + 1 approaches 2. Thus, $\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x+1}=\dfrac{3}{2}$, from which it follows that $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{3}{2}$.

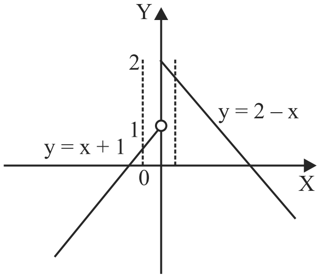
We can also express this as follows:

As $x\to 1$, $\dfrac{x^3-1}{x^2-1}\to\dfrac{3}{2}$.

Example 3: Find the limit of the function

$$f(x)=\begin{cases}x+1 & ,\ x<0\\ 2-x & ,\ x\geq 0\end{cases}\quad \text{as } x\to 0.$$

Solution: Using the graph of y = f(x), we see that the function approaches 1 as x approaches 0 from the left of 0. Also the function approaches 2 as x approaches 0 from the right of 0.



Since the function does not approach the same level from both sides of x = 0, $\lim\limits_{x\to 0}f(x)$ does not exist.

Formal Definition of Limit

The conceptual problems in trying to give an exact meaning to the expression $\lim\limits_{x\to a}f(x)=b$ revolve around phrases such as "arbitrarily close," "sufficiently near," and "arbitrarily small." After all, there is a no such thing in any absolute sense as a small positive real number. The number 0.000001 is small in most contexts, but in comparison with 0.000000000001 it is huge. However, we can assert that one number is smaller than another. Moreover, the actual closeness of one number x to another number a is just the distance between them : it is $|x-a|$. One way to say that a function f takes on values arbitrarily close to a number ℓ is to state that, for any small positive real number ε, there are numbers x such that $|f(x) - \ell| < \varepsilon$. We are stating that no matter what positive number ε is selected, 10^{-7}, or 10^{-17}, or 10^{-127}, there are numbers x so that the distance between f(x) and ℓ is smaller than ε.

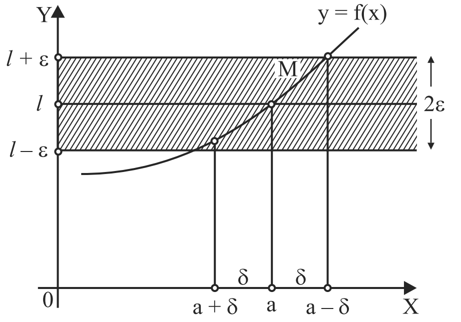
$\varepsilon - \delta$ definition

Let f be a function defined on an open interval containing a except possibly at a and let ℓ be a real number. The statement $\lim\limits_{x \to a} f(x) = \ell$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - \ell| < \varepsilon$.

The inequality $0 < |x - a|$ that appears in the definition is just a fancy way of saying "x is not a." The inequality $|x - a| < \delta$ asserts that x is within a distance δ of a. The two inequalities may be combined as the single statement $0 < |x - a| < \delta$, which describes the open interval $(a - \delta, a + \delta)$ from which a is deleted.

Here we wish to say that f(x) is arbitrarily close to ℓ whenever x is sufficiently close, but not equal to a. What does "sufficiently close" mean? " The answer is this : If an arbitrary $\varepsilon > 0$ is chosen with which to measure the distance between f(x) and ℓ, then it must be the case that there is a number $\delta > 0$ such that whenever x is in the domain of f and within a distance δ of a, but not equal to a then the distance between f(x) and ℓ is less than ε.

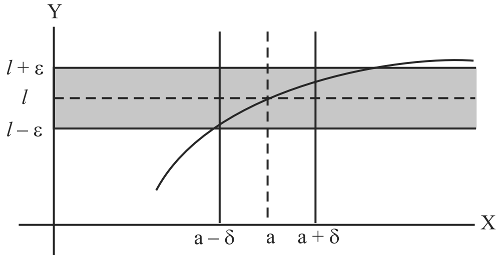
This is illustrated on the graph of the function y = f(x) as shown in the figure. Since from the inequality $|x - a| < \delta$ there follows the inequality $|f(x) - l| < \varepsilon$, this means that for all points x that are not more distant from the point a than δ, the points M of the graph of the function y = f(x) lie within a band of width 2ε bounded by the lines $y = \ell - \varepsilon$ and $y = \ell + \varepsilon$.



First $\varepsilon > 0$ is chosen arbitrarily. There must then exist a number $\delta > 0$ such that whenever x lies in the interval $(a - \delta, a + \delta)$, and $x \neq a$ then the point (x, f(x)) lies in the shaded rectangle.

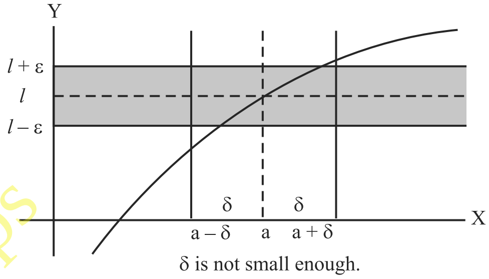
In other words, ε is the challenge. The response is δ. Usually, the smaller ε is, the smaller δ will have to be. The geometric significance of the definition is shown in the Figure 1. The narrow horizontal band of width 2ε is again the challenge. The response is a sufficiently narrow vertical band, of width 2δ, such that the part of the graph within that vertical band (except perhaps at x = a) also lies in the challenging horizontal band of width 2ε.

In Figure 2 the vertical band shown is not narrow enough to meet the challenge of the horizontal band shown. But the vertical band shown in Figure 3 is sufficiently narrow.



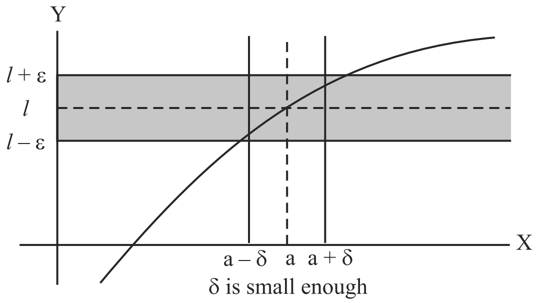
The number δ is a response

Figure 1



δ is not small enough.

Figure 2



δ is small enough

Figure 3

Example 4: Prove that $\lim\limits_{x \to 2} (3x + 1) = 7$.

Solution: Let an arbitrary $\varepsilon > 0$ be given. For the inequality $|(3x + 1) - 7| < \varepsilon$ to be fulfilled it is necessary to have the following inequalities fulfilled :

$$|3x - 6| < \varepsilon, \quad |x - 2| < \frac{\varepsilon}{3}, \quad -\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}$$

Thus, given any ε, for all values of x satisfying the inequality $|x - 2| < \frac{\varepsilon}{3} = \delta$, the value of the function 3x + 1 will differ from 7 by less than ε. And this means that 7 is the limit of the function as $x \to 2$.

Example 5: Examine the limit of the function

$y = \frac{x^2 - 1}{x - 1}$ as x tends to 1.

Solution: The function is defined for every value of x other than 1 and $y = \frac{x^2 - 1}{x - 1} = x + 1$, when $x \neq 1$.

Firstly consider the behaviour of the values of y for values of x greater than 1. Clearly, y is greater than 2 when x is greater than 1.

If, x, while remaining greater than 1 takes up values whose difference from 1 constantly diminishes, then y, while remaining greater than 2, takes up values whose difference from 2 constantly diminishes also.

In fact, difference between y and 2 can be made as small as we like by taking x sufficiently near 1.

For instance, consider the number .001. Then

$|y - 2| = y - 2 = x + 1 - 2 < .001 \Leftrightarrow x < 1.001.$

Thus, for every value of x which is greater than 1 and less than 1.001, the absolute value of the difference between y and 2 is less than the number .001 which we had arbitrarily selected.

Instead of the particular number .001, we now consider any small positive number ε. Then

$$y - 2 = x - 1 < \varepsilon \Leftrightarrow x < 1 + \varepsilon.$$

Thus, there exists an interval $(1, 1 + \varepsilon)$, such that the value of y, for any value of x in this interval, differs from 2 numerically, by a number which is smaller than the positive number ε, selected arbitrarily.

Thus, the limit of y as x approaches 1 from the right is 2 and we have $\lim_{x \to 1^+} y = 2$.

We now consider the behaviour of the values of y for values of x less than 1. When x is less than 1, y is less than 2.

If, x, while remaining less than 1, takes up values whose difference from 1 constantly diminishes, then y, while remaining less than 2, takes up values whose difference from 2 constantly diminishes also.

Let now, ε be any arbitrarily assigned positive number, however small. We then have,

$$|y - 2| = 2 - y = 2 - (x + 1) = 1 - x < \varepsilon$$

so that for every value of x less than 1 but > 1 – ε, the absolute value of the difference between y and 2 is less than the number ε.

Thus, the limit of y, as x approaches 1, from the left is 2 and we write $\lim_{x \to 1^-} y = 2$.

Combining the conclusions arrived at in the last two cases, we see that corresponding to any arbitrarily assigned positive number ε, there exists an interval $(1 - \varepsilon, 1 + \varepsilon)$ around 1, such that for every value of x in this interval, other than 1 where the function is not defined, y differs from 2 numerically by a number which is less than ε, i.e., we have $|y - 2| < \varepsilon$ for any x, other than 1, such that $|x - 1| < \varepsilon$.

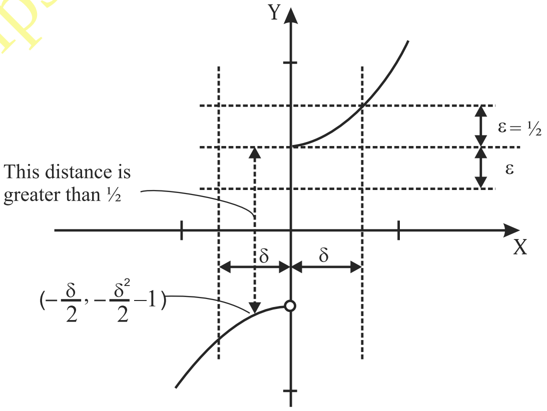
Example 6: Consider the function

$$g(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ -(x^2 + 1), & x < 0 \end{cases}$$. Prove that $\lim_{x \to 0} g(x) \neq 1$.

Solution To do this, we must establish the negation of the limit condition. There is an $\varepsilon > 0$ such that, for any $\delta > 0$, there is a number x in the domain of a g such that $0 < |x| < \delta$ and $|g(x) - 1| \geq \varepsilon$. There are many possible choices for ε. To be specific, take $\varepsilon = \frac{1}{2}$. We must now show that for every positive number δ, there is a nonzero number x in the open interval $(-\delta, \delta)$, such that the distance between g(x) and 1 is greater than or equal to $\frac{1}{2}$. See figure. Take $x = -\frac{\delta}{2}$. This number is nonzero, lies in $(-\delta, \delta)$, and furthermore

$$g(x) = g\left(-\frac{\delta}{2}\right) = -\frac{\delta^2}{4} - 1 < -1.$$

Hence $|g(x) - 1| > 2 \geq \frac{1}{2}$.


Y
ε = ½
ε
This distance is greater than ½
X
δ
δ
$(-\frac{\delta}{2}, -\frac{\delta^2}{2} - 1)$

Notice in the definition that the hypothesis $0 < |x - a|$ simply means $x \neq a$. This condition releases the point of approach a from the responsibility of having any image at all; and even if there is an image f(a) the condition $|f(a) - \ell| < \in$ is allowed to fail. Briefly, the idea of limit discounts what happens precisely at a, but is vitally concerned with images of the "neighbours" of a.

There is no practical need of applying ∈–δ definition everywhere since it involves lengthy and complicated calculations. We shall later on derive some simple rules for finding limits.

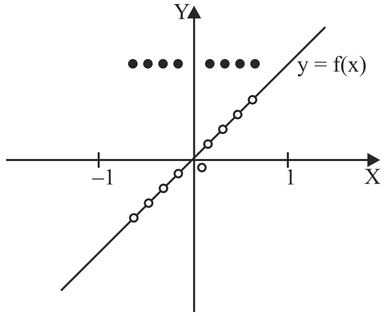
Example 7: Let the function f be defined as follows:

$f\left(\frac{1}{n}\right) = 1$ for every nonzero integer n,

f(x) = x for every other real number x. Prove that $\lim_{x \to 0} f(x)$ does not exist.

Solution: The graph of f is the line y = x, with the exception that when x = 1/n, n a nonzero integer, y = 1. It is shown in the figure.

Since f(0) = 0 and f(x) is close to 0 when x is close to 0, x ≠ 1/n, we might suspect that $\lim\limits_{x\to 0} f(x) = 0$.

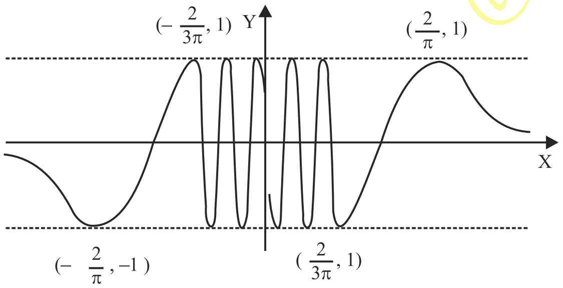


However, this is not correct. For if we take the neighbourhood N = (–1/2, 1/2) of 0, there is no deleted neighbourhood D = (-δ, 0) ∪ (0, δ) of 0 such that f(x) is in N for every x in D.

We can always find a positive integer n such that 1/n < δ and for this number f(1/n) is not in N since f(1/n) = 1. In a similar way, we can show that $\lim\limits_{x\to 0} f(x)$ is not equal to any other number. Therefore, this limit does not exist.

Example 8: Prove that $\lim\limits_{x\to 0}$ (sin 1/x) does not exist.

Solution: The graph of y = sin 1/x is drawn below.



The function oscillates between –1 and 1 more and more rapidly as x approaches closer and closer to zero from either side. If we take any interval containing 0, however small it may be, then for an infinite number of points of this interval the function assumes the values 1 and –1. There can, therefore, exist no number which differs from sin 1/x by a number less than an arbitrarily assigned positive number for values of x near 0. Hence $\lim\limits_{x\to 0}$ (sin 1/x) does not exist.

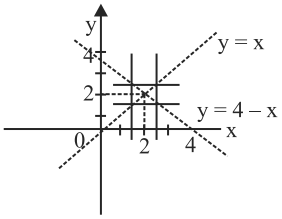
Example 9: $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ as x→ 0.

Solution: The function oscillates between the graphs of y = –x² and y = x². As x approaches closer and closer to zero from either side, the two functions approach 0, and hence f(x) → 0 as x→ 0.

Example 10: Find the limit of the function

$$f(x) = \begin{cases} x, & x \in Q \\ 4 - x, & x \notin Q \end{cases}$$ as x→ 2 and x→ 4.

Solution: We draw the rough sketch of y = f(x).



If x approaches 2 by taking rational values, then the function f(x) approaches 2 using y = x. And if x approaches 2 by taking irrational values, then the function f(x) also approaches 2 using y = 4 – x. Hence, the limit of f(x) as x→ 2 is 2.

$$\lim_{x\to 2} f(x) = 2$$

However, if x approaches 4 by taking rational values, then the function f(x) approaches 4 using y = x. And if x approaches 4 by taking irrational values, then the function f(x) approaches 0 using y = 4 – x. Since the function does not approach the same level, the limit $\lim\limits_{x\to 4} f(x)$ does not exist.

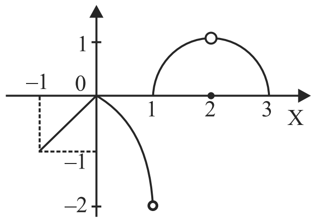
There are many such interesting functions that have unusual limit behaviour. An often cited one is the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

This function has no limit at any real number.

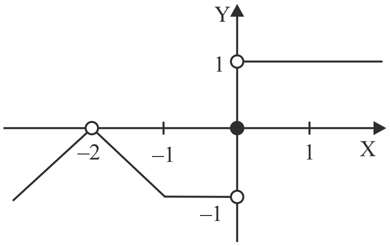
Practice Problems A

1. For the function graphed here, find the following limits:



(i) $\lim\limits_{x\to 2} f(x)$ (ii) $\lim\limits_{x\to 1} f(x)$

2. For the function graphed here, find the following limits:



(i) $\lim\limits_{x\to -2} f(x)$ (ii) $\lim\limits_{x\to 0} f(x)$ (iii) $\lim\limits_{x\to -1} f(x)$

3. If $\lim\limits_{x\to 1} f(x) = 5$ must f be defined at x = 1 ?

If it is, must f(1) = 5 ? Can we conclude anything about the values of f at x = 1 ? Explain.

4. If f(1) = 5, must $\lim\limits_{x\to 1} f(x)$ exist ? If it does, then must $\lim\limits_{x\to 1} f(x) = 5$? Can we conclude anything about $\lim\limits_{x\to 1} f(x)$?

5. Let $f(x) = \begin{cases} x & \text{if x is rational} \\ -x & \text{if x is not rational} \end{cases}$

(a) Does $\lim_{x\to 1} f(x)$ exist ?
(b) Does $\lim_{x\to \sqrt{2}} f(x)$ exist ?
(c) Does $\lim_{x\to 0} f(x)$ exist ?
(d) For which numbers a does $\lim_{x\to a} f(x)$ exist?

6. Let $f(x) = \begin{cases} x^2 & \text{if x is an irrational number} \\ 1 & \text{if x is an rational number} \end{cases}$.

Prove that f(x) has a limit at the points x = 1 and x = –1 and does not have a limit at other points.

7. Find $\lim\limits_{x\to 0} \cot^{-1} x^2$.

8. Find $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\cos x}$.

9. Find $\lim\limits_{x\to 0} \cos \pi/x$.

1.2 Concept of Infinity

Suppose that n assumes successively the values 1, 2, 3,.... Then as n gets larger and larger and there is no limit to the extent of its increase. However, large a number we may think of, a time will come when n has become larger than this number.

It is convenient to have a short phrase to express this unending growth of n and we shall say that n tends to infinity, or $n \to \infty$, When we say that n 'tends to ∞' we mean simply that n is supposed to assume a series of values which increase beyond all limit.

There is no real number 'infinity'. This implies that the equation $n = \infty$ is meaningless. A number n cannot be equal to ∞, because 'equal to ∞' means nothing. So far in fact the symbol ∞ means nothing at all except in the one phrase 'tends to ∞', the meaning of which we have explained above. Later on we shall learn how to attach a meaning to other phrases involving the symbol ∞, but we bear in mind

(i) that ∞, by itself means nothing, although phrases containing it sometimes mean something,

(ii) that in every case in which a phrase containing the symbol ∞ means something it will do so simply because we have previously attached a meaning to this particular phrase by means of a special definition.

Limit of a Sequence

Definition A number ℓ is said to be the limit of the function y = f(n) of the integral argument n or the limit of the sequence $y_1, y_2, \ldots, y_n$, if for all sufficiently large integral values of n the corresponding values y_n of the function become arbitrarily close to the number ℓ.

Let us consider a function of an integral argument. Usually such an argument is denoted by the letter n and the values of the function by some other letter supplied with a subscript indicating the value of the integral argument. For instance, if y = f(n) is a function of the integral argument n we write $y_n = f(n)$. Given such a function, we say that the values

$$y_1 = f(1),\ y_2 = f(2),\ \ldots,\ y_n = f(n),\ \ldots,$$

assumed by the function form a sequence.

If there is a sequence $y_1, y_2, y_3, \ldots$, this assigns, to every natural number n, a value $y_n = f(n)$. For instance, the terms of the geometric progression $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ .... are the subsequent values of the function $f(n) = \frac{1}{2^n}$.

We now discuss the question 'what properties has f(n) for sufficiently large values of n ?' i.e. 'how does f(n) behaves as n tends to ∞ ?'

It may occur that, as n increases, the values $y_n = f(n)$ become arbitrarily close to a number ℓ. Then we say that the number ℓ is the limit of the function f(n) of the integral argument n or that the sequence $y_1, y_2, \ldots, y_n, \ldots$ has the limit ℓ, as $n \to \infty$, and write

$$\lim_{n\to\infty} f(n) = \ell \text{ or } \lim_{n\to\infty} y_n = \ell$$

Consider the function 1/n for large values of n. Instead of saying '1/n is small for large values of n' we say '1/n tends to 0 as n tends to ∞ . Similarly we say that '1– (1/n) tends to 1 as n tends to ∞'.

We shall say that 'the limit of 1/n as n tends to ∞ is 0', a statement which we express as $\lim\limits_{n\to\infty}\frac{1}{n}=0$.

We shall also sometimes write '$1/n \to 0$ as $n \to \infty$' which may be read '1/n tends to 0 as n tends to ∞'.

In the same way we shall write

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1 \text{ or, } \quad 1-(1/n)\to 1 \text{ as } n\to\infty.$$

Now let us consider $f(n) = n^2$. Then 'n^2 is large when n is large'. And it is natural in this case to say that 'n^2 tends to ∞ as n tends to ∞', and we write $n^2 \to \infty$.

Finally consider the function $f(n) = -n^2$. In this case f(n) is large, but negative, when n is large and we say that '$-n^2$ tends to $-\infty$ as n tends to ∞' and write $-n^2 \to -\infty$.

Definition The function f(n) is said to tend to the limit ℓ as n tends to ∞ , if, however small be the positive number ε, f(n) differs from ℓ by less than ε for sufficiently large values of n; that is to say if, however small be the positive number ε, we can determine a number N(ε) corresponding to ε, such that f(n) differs from ℓ by less than ε for all values of n greater than or equal to N(ε).

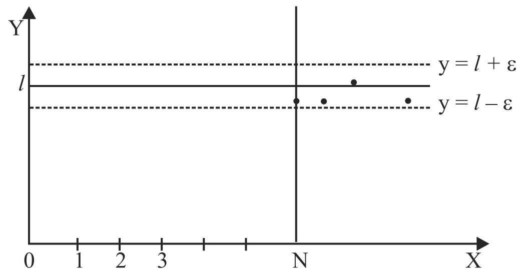
The definition may be stated more shortly as follows : if, given any positive number, ε, however small, we can find N(ε) so that $|f(n)-\ell| < \varepsilon$ when $n \ge N(\varepsilon)$, then we say that f(n) tends to the limit ℓ as n tends to ∞ , and write

$$\lim_{n\to\infty} f(n) = \ell$$

The definition of a limit may be illustrated geometrically as follows. The graph of f(n) consists of a number of points corresponding to the values n = 1, 2, 3....

Draw the line $y = \ell$, and the parallel lines $y = \ell - \varepsilon$, $y = \ell + \varepsilon$ at distance ε from it. Assume that for each positive ε , a number N can be found, depending on ε , such that the part of the graph to the right of n = N lies within the band. Then we say that 'as x approaches ∞, f(n) approaches ℓ' and write

$$\lim_{n\to\infty} f(n) = \ell$$



$\lim\limits_{n\to\infty} f(n) = l$

Note: The limit ℓ may itself be one of the actual values of f(n). Thus if f(n) = 0 for all values of n, it is obvious that $\lim\limits_{n\to\infty} f(n) = 0$.

On the other hand the limit itself need not (and in general will not) be the value of the function for any value of n. This is sufficiently obvious in the case f(n) = 1/n. The limit is zero; but the function is never equal to zero for any value of n.

A limit is not a value of the function. It is something quite distinct from these values.

For the function f(n) = 1 the limit is equal to all the values of f(n).

For f(n) = 1+ (1/n), it is not equal to any value of f(n). For $f(n) = (\sin\frac{1}{2}n\pi)/n$, (whose limits as n tends to ∞ is easily seen to be 0, since $\sin\frac{1}{2}n\pi$ is never numerically greater than 1), the limit is equal to the value which f(n) assumes for all even values of n, but the values assumed for odd values of n are all different from the limit and from one another.

Definition The function f(n) is said to tend to ∞ (positive infinity) with n if, when any number M, however large, is assigned, we can determine N so that f(n) > M when n ≥ N; that is to say if, however large M may be, f(n) > M for sufficient large values of n.

Another form of statement is, if we can make f(n) as large as we please by sufficiently increasing n.

A function may be always numerically very large when n is very large without tending either to ∞ or to –∞. A sufficient illustration of this is given by $f(n) = (-1)^n n$.

Example 1: Find $\lim\limits_{n\to\infty} \sin n\theta\pi$, if θ is rational.

Solution: Let $f(n) = \sin n\theta\pi$ and θ = p/q, where p and q are positive integers. Let n = aq + b, here a is the quotient and b the remainder when n is divided by q.

Then $\sin(np\pi/q) = (-)^{ap}\sin(bp\pi/q)$.

Suppose, for example, p even. Then, as n increases from 0 to q – 1, f(n) takes the values

$$0,\ \sin\frac{p\pi}{q},\ \sin\frac{2p\pi}{q}\ ....,\ \sin\frac{(q-1)p\pi}{q}$$

When n increases from q to 2q–1 these values are repeated and so also as n goes from 2q to 3q – 1, 3q to 4q–1, and so on. Thus the values of f(n) form a cyclic repetition of a finite series of different values. It is evident that when this is the case f(n) cannot tend to a limit, nor to ∞ , nor to –∞, as n tends to infinity. The limit does not exist.

Oscillating Functions

When f(n) does not tend to a limit, nor to ∞ , nor to –∞, as n tends to ∞, we say that f(n) oscillates as n tends to ∞.

The simplest example of an oscillatory function is given by $f(n) = (-1)^n$, which is equal to 1 when n is even and to –1 when n is odd. In this case the values recur cyclically.

Now consider $f(n) = (-1)^n + n^{-1}$, the values of which are $-1 + 1, 1 + \frac{1}{2}, -1 + \frac{1}{3}, 1 + \frac{1}{4}, -1 + \frac{1}{5}, \ldots$

When n is large every value is nearly equal to 1 or –1, and obviously f(n) does not tend to a limit or to ∞ or to –∞ and therefore it oscillates, but the values do not recur. It is to be observed that in this case every value of f(n) is numerically less than or equal to $\frac{3}{2}$.

If f(n) oscillates as n tends to ∞, then f(n) will be said to oscillate finitely or infinitely according as is or is not possible to assign a number K such that all the values of f(n) are numerically less than K,

i.e. $| f(n) | < K$ for all values of n.

For example, consider $f(n) = n\{1 + (-1)^n\}$.

In this case the values of f(n) are

$$0, 4, 0, 8, 0, 12, 0, 16, \ldots$$

The odd terms are all zero and the even terms tend to ∞. Thus f(n) oscillates infinitely.

But the function $f(n) = \sin n\theta\pi$, θ is rational, oscillates finitely.

Now consider the sequence

$$\sin\frac{\pi}{2}, \sin\frac{2\pi}{2}, \sin\frac{3\pi}{2}, \ldots, \sin\frac{n\pi}{2}, \ldots$$

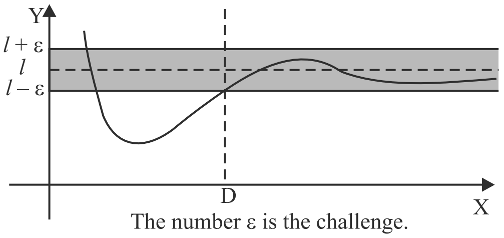
It does not possess a limit since, for n = 1, 2, 3, 4, $y_n = \sin\frac{n\pi}{2}$ consecutively takes on the values 1, 0, –1, 0 which then are repeated indefinitely in the same order as n increases. Therefore the number to which y_n tends, as $n \to \infty$, does not exist.

Limit of a Function as $x \to \infty$

Definition Let f(x) be defined for all x beyond some number c. If for each positive number ε, there is a number D such that for all x > D it is true that

$$| f(x) - \ell | < \varepsilon \text{, then we write } \lim_{x\to\infty} f(x) = \ell.$$

The positive number ε is the challenge, and D is a response. The smaller ε is, the larger D usually must be chosen. The geometric meaning of the precise definition of $\lim_{x\to\infty} f(x) = \ell$ is shown in the figure.



The number ε is the challenge.
The number D is a response.

Draw two lines parallel to the x axis, one of height $\ell + \varepsilon$ and one of height $\ell - \varepsilon$. They are the two edges of an endless band of width 2ε. Assume that for each positive ε, a number D can be found, depending on ε, such that the part of the graph to the right of x = D lies within the band. Then we say that "as x approaches ∞, f(x) approaches ℓ" and write as

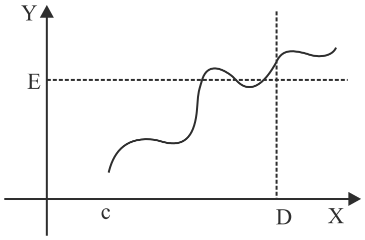
$$\lim_{x\to\infty} f(x) = \ell$$

We have a similar definition for $\lim_{x\to-\infty} f(x) = \ell$.

Note that the definition of the limit of a sequence can be regarded as a special case of the definition of the limit of a function, as its argument becomes infinitely large and assumes only integral values.

Definition We assume that f(x) is defined for all x greater than some number c. If for each number E there is a number D such that for all x > D it is true that f(x) > E, then we write $\lim_{x\to\infty} f(x) = \infty$.

For each possible choice of a horizontal line, say, at height E, if we are far enough to the right of f, we stay above that line. That is , there is a number D such that if x > D, then f(x) > E, as illustrated in the figure.



D is a reply (so is any larger number)

We have similar definitions for $\lim_{x\to\infty} f(x) = -\infty$,

$$\lim_{x\to-\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty.$$

For example, if a > 1, the function $y = a^x$ becomes infinite for $x \to \infty$ i.e. $\lim_{x\to\infty} a^x = \infty$.

If 0 < a < 1 we have $\lim_{x\to-\infty} a^x = \infty$.

Further, $\lim_{x\to\infty} \log_a x = \infty$, if a > 1

$$\lim_{x\to\infty} \log_a x = -\infty \text{, if } 0 < a < 1$$

Note: We say that a limit, such as $\lim_{x\to\infty} f(x)$, exists when the limit is a real number, but not when the limit is ∞ or –∞.

Bounded Function

The function f(x) in a given interval of x is said to have

(i) a **lower bound** if there exists a finite number A, such that $f(x) \ge A$, for all values of x in the given interval.

(ii) an **upper bound** if there exists a finite number B such that $f(x) \le B$, for all values of x in the given interval.

A function is said to be **bounded** in an interval if it has a lower bound as well as an upper bound.

Thus, a function is bounded if its values are contained between two numbers A and B i.e. $A \le f(x) \le B$.

Then, the graph of the function $y = f(x)$ lies between the lines $y = A$ and $y = B$.

Alternatively, a function f(x) is called bounded in a given interval of x if these exists a number $M > 0$ such that $|f(x)| \le M$ for all values of x in the given interval. If there is no such number M, the function is called unbounded.

The graph of a bounded function (considered within the given interval) is completely contained between the straight lines $y = -M$ and $y = M$.

For instance, the functions $y = \sin x$ and $y = \cos x$ are bounded for all x while the function $y = a^x$ when considered on the whole x–axis, is unbounded.

It should be stressed that when speaking of the boundedness of a function it is necessary to indicate the interval on which the function is considered.

For example, the function $y = \frac{1}{x}$ is bounded in the interval $(1, \infty)$ and unbounded in the interval (0, 1).

The function $y = \tan x$ is bounded in the interval $\left(0, \frac{\pi}{4}\right)$ and unbounded in the interval $\left(0, \frac{\pi}{2}\right)$.

Definition The function f(x) is called bounded as $x \to a$ if there exists a neighbourhood of the point a, in which the given function is bounded.

Definition The function $y = f(x)$ is called bounded as $x \to \infty$ if there exists a number $N > 0$ such that for all values of x satisfying the inequality $x > N$, the function f(x) is bounded.

From the definition of a bounded function f(x) it follows that if $\lim\limits_{x \to a} f(x) = \infty$ or $\lim\limits_{x \to \infty} f(x) = \infty$ that is, if f(x) is an **infinitely large function**, it is unbounded. The converse is not true: **an unbounded function may not be infinitely large**.

For example, the function $y = x \sin x$ as $x \to \infty$ is unbounded because, for any $M > 0$, values of x can be found such that $|x \sin x| > M$. But the function $y = x \sin x$ is not infinitely large because it becomes zero when $x = 0, \pi, 2\pi, \ldots$.

Note:

1. If a function f(x) increases as $x \to \infty$ and remains bounded then it possesses a limit.
2. The product of a bounded function by an infinitesimally small quantity is an infinitesimally small quantity.

Let f(x) be an infinitesimally small quantity and g(x) a bounded function in a neighbourhood of the point a to which x tends i.e. $|g(x)| \le M$. Then we have $\lim\limits_{x \to a} f(x)g(x) = 0$.

Infinite Limits

Suppose that when $x \to a$ the "limiting behaviour" of a function $y = f(x)$ is such that its absolute value increases unlimitedly then we say that the function f(x) approaches an infinitely large quantity as $x \to a$.

Definition A function $y = f(x)$ is said to approach infinity as $x \to a$ if for all the values of x lying sufficiently close to a the corresponding values of the function f(x) become greater than any given arbitrarily large positive number.

If a function f(x) approaches infinity as $x \to a$ we write $\lim\limits_{x \to a} f(x) = \infty$.

A function $y = f(x)$ which approaches infinity as $x \to a$ does not have a limit in the ordinary sense. To extend the ordinary terminology and to characterize the behaviour of the function whose absolute value $|f(x)|$ increases indefinitely we say that the function $y = f(x)$ tends to infinity or has an infinite limit.

Suppose that a function $y = f(x)$ tending to infinity as $x \to a$ only assumed positive (negative) values in a neighbourhood of the point c, then we say that the function f(x) approaches plus (minus) infinity as $x \to a$. In these cases we write, respectively,

$$\lim_{x \to a} f(x) = \infty \text{ and } \lim_{x \to a} f(x) = -\infty$$

Consider the following examples:

(i) The function $y = \frac{x+1}{x-1}$ approaches infinity as $x \to 1$, it tends to $-\infty$ if x remains less than 1 and to ∞ if x remains greater than 1.

(ii) The function $y = \frac{1}{x^2}$ has the limit ∞ as $x \to 0$.

(iii) $\lim\limits_{x \to 1} \frac{-1}{(x-1)^2} = -\infty$.

Note: In what follows, when speaking about a limit, we shall mean that the limit is finite unless it is stipulated that it is infinite.

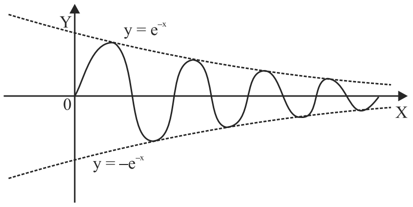
For example, since $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, we say that $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$ exists. However, although $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$, we do not say that $\lim\limits_{x \to 0} \frac{1}{x^2}$ exists.

Consider some more examples :

$\lim\limits_{x \to \infty} \cot^{-1} x = \pi$, $\lim\limits_{x \to -\infty} x^2 = \infty$, $\lim\limits_{x \to \infty} \sin x$ dne.

Example 2: Show that $\lim\limits_{x \to \infty} e^{-x} \sin x = 0$.

Solution: $-e^{-x} \le e^{-x} \sin x \le e^{-x}$



We can see from the graph that $\lim\limits_{x\to\infty} e^{-x} \sin x = 0$.

Example 3: Evaluate $\lim\limits_{x\to\infty} 2^x \sin^{-1}\sin(x)$.

Solution: $-\dfrac{\pi}{2}.2^x \le 2^x \sin^{-1}\sin(x) \le \dfrac{\pi}{2} 2^x$

The function oscillates between $-\infty$ and ∞.
Hence the limit does not exist.

Example 4: Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^2}{x^2+1}$.

Solution: We have

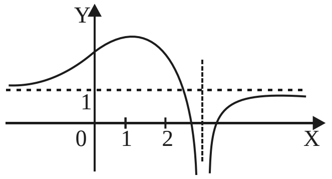
$$\frac{2x^2}{x^2+1} = \frac{2(x^2+1)-2}{x^2+1} = 2 - \frac{2}{x^2+1}$$

and as $x \to \infty$, $\dfrac{2}{x^2+1} \to 0$.

Hence, $\lim\limits_{x\to\infty} \dfrac{2x^2}{x^2+1} = 2$.

Concept Problems A

1. Find the following limits for the function shown in the figure:



(i) $\lim\limits_{x\to 3^-} f(x)$ (ii) $\lim\limits_{x\to 3} f(x)$

(iii) $\lim\limits_{x\to -\infty} f(x)$ (iv) $\lim\limits_{x\to \infty} f(x)$

2. Let $f(x) = \begin{cases} \dfrac{1}{x+2}, & x < -2 \\ x^2 - 5 & -2 < x \le 3 \\ \sqrt{x+13} & x > 3 \end{cases}$

Find (i) $\lim\limits_{x\to -2} f(x)$, (ii) $\lim\limits_{x\to 0} f(x)$, (iii) $\lim\limits_{x\to 3} f(x)$.

3. Find $\lim\limits_{n\to\infty} [\alpha n]$, where α is any positive number and [.] denotes G.I.F.

4. Find $\lim\limits_{n\to\infty} \{n - (-1)^n\} + n\{1 - (-1)^n\}$

5. Find $\lim\limits_{n\to\infty} (\sin n\theta\pi)/\{n(n-(-1)^n)\}$, where θ is any real number.

6. Find $\lim\limits_{n\to\infty} \sin n\theta\pi$, if θ is integral.

7. Find $\lim\limits_{n\to\infty} \dfrac{1}{\cos^2 \frac{1}{2} n\pi + n \sin^2 \frac{1}{2} n\pi}$.

8. Find $\lim\limits_{n\to\infty} \dfrac{1}{n\left(\cos^2 \frac{1}{2} n\pi + n \sin^2 \frac{1}{2} n\pi\right)}$.

1.3 Theorems on Limits

Let a and b be real numbers and let n be a positive integer.

(i) $\lim\limits_{x\to a} b = b$ **(ii)** $\lim\limits_{x\to a} x = a$ **(iii)** $\lim\limits_{x\to a} x^n = a^n$

Let f and g be two functions such that

$\lim\limits_{x\to a} f(x) = \ell$, $\lim\limits_{x\to a} g(x) = m$.

Then

(iv) Constant Multiple rule

$\lim\limits_{x\to a} k(f(x)) = k.\lim\limits_{x\to a} f(x) = k.\ell$,

where k is independent of x.
A constant factor may be taken outside the limit sign.

(v) Sum rule

$\lim\limits_{x\to a} (f+g)(x) = \lim\limits_{x\to a} [f(x) + g(x)]$

$= \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = \ell + m$,

i.e., the limit of the sum of two functions is equal to the sum of their limits.

(vi) Difference rule

$\lim\limits_{x\to a} (f-g)(x) = \lim\limits_{x\to a} [f(x) - g(x)]$

$= \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = \ell - m$,

i.e., the limit of the difference of two functions is equal to the difference of their limits.

The limit of an algebraic sum of two, three or, in general, any definite number of functions is equal to the algebraic sum of the limits of these functions.

(vii) Product rule

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a}[f(x).\, g(x)]$$

$$= \lim_{x\to a} f(x).\ \lim_{x\to a} g(x) = \ell m,$$

i.e., the limit of the product of two functions is equal to the product of their limits.

The limit of a product of two, three or, in general, any definite number of functions is equal to the product of the limits of these functions.

(viii) Quotient rule

$$\lim_{x\to a}(f/g)(x) = \lim_{x\to a}[f(x)/g(x)]$$

$$= \lim_{x\to a} f(x) / \lim_{x\to a} g(x) = \ell / m,\ (m \ne 0)$$

i.e., the limit of the quotient of two functions is equal to the quotient of their limits provided the limit of the divisor is not zero.

(ix) Power rule

For any positive integer n, $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, provided $\lim\limits_{x\to c} f(x) > 0$ when n is even.

Also $\lim\limits_{x\to a}(f(x))^{m/n} = \left(\lim\limits_{x\to a} f(x)\right)^{m/n}$ where m/n is a rational number and the limit on the right exists.

The limit of a power is the power of the limit.

(x) Composition rule

If f is continuous* at x = ℓ, then

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(\ell).$$

Specially, $\lim\limits_{x\to a} \ell n(f(x) = \ell n\left(\lim\limits_{x\to a} f(x)\right) = \ell n\, \ell$

provided $\ell > 0$.

*We shall learn about continuity in the next chapter. A function is f(x) said to be continuous at x = a if $\lim\limits_{x\to a}$ f(x) = f(a).

These results are of fundamental importance but their formal proofs will not be given in this book.

Note:

(i) For any polynomial p(x) = $c_0 + c_1x + ... + c_nx^n$ and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1a + ... = c_na^n = p(a).$$

(ii) Consider the rational function

$f(x) = \dfrac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials. For any real number a,

(a) if q(a) ≠ 0, then $\lim\limits_{x\to a} f(x) = f(a)$.

(b) if q(a) = 0 but p(a) ≠ 0, then $\lim\limits_{x\to a} f(x)$ does not exist.

(c) if q(a) = 0 and p(a) = 0, then $\lim\limits_{x\to a} f(x)$ may or may not exist. This will be dealt later on in detail.

Example 1: Evaluate

(i) $\lim\limits_{x\to 2}(x+2)$ (ii) $\lim\limits_{x\to 2} x(x-1)$

(iii) $\lim\limits_{x\to 2}\dfrac{x^2+4}{x+2}$ (iv) $\lim\limits_{x\to 0}\cos(\sin x)$

(v) $\lim\limits_{x\to 1}\dfrac{x^2+3x+2}{x^2-1}$

Solution:

(i) x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim\limits_{x\to 2}(x+2) = 2 + 2 = 4$.

(ii) Again x(x – 1) being a polynomial in x, its limit as $x \to 2$ is given by

$\lim\limits_{x\to 2} x(x-1) = 2(2-1) = 2$.

(iii) We have $\lim\limits_{x\to 2}\dfrac{x^2+4}{x+2} = \dfrac{(2)^2+4}{2+2} = 2$.

(iv) $\lim\limits_{x\to 0}\cos(\sin x) = \cos\left(\lim\limits_{x\to 0}\sin x\right) = \cos 0 = 1$.

(v) Note that for x = 1, the numerator of the given expression is a nonzero constant 6 and the denominator is zero. Therefore, the given limit is of the form $\dfrac{6}{0}$. Hence, we conclude that $\lim\limits_{x\to 1}\dfrac{x^2+3x+2}{x^2-1}$ does not exist.

Example 2: Evaluate the following limits:

(i) $\lim\limits_{x\to 5}(2x^2 - 3x + 4)$ (ii) $\lim\limits_{x\to -2}\dfrac{x^3+2x^2-1}{5-3x}$

Solution:

(i) $\lim\limits_{x\to 5}(2x^2 - 3x + 4)$

$= \lim\limits_{x\to 5}(2x^2) - \lim\limits_{x\to 5}(3x) + \lim\limits_{x\to 5} 4$

$= 2\lim\limits_{x\to 5} x^2 - 3\lim\limits_{x\to 5} x + \lim\limits_{x\to 5} 4$

$= 2(5^2) - 3(5) + 4 = 39$

(ii) We start by using laws of limit, but their use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim\limits_{x\to -2}(x^3+2x^2-1)}{\lim\limits_{x\to -2}(5-3x)}$$

$$= \frac{\lim\limits_{x\to -2} x^3 + \lim\limits_{x\to -2} 2x^2 - \lim\limits_{x\to -2} 1}{\lim\limits_{x\to -2} 5 - 3\lim\limits_{x\to -2} x}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}.$$

Example 3: Find $\lim\limits_{x\to 3} \sqrt[3]{2x^2 - 10}$.

Solution: Because $\lim\limits_{x\to 3} (2x^2 - 10) = 2(3^2) - 10 = 8$ and $\lim\limits_{x\to 8} \sqrt[3]{x} = 2$ we can write $\lim\limits_{x\to 3} \sqrt[3]{2x^2 - 10} = 2$.

Direct Substitution

We learned that the limit of f(x) as x approaches a does not depend on the value of f at x = a. It may happen, however, that the limit is precisely f(a). In such cases, we say that the limit can be evaluated by direct substitution. That is

$$\lim_{x\to a} f(x) = f(a). \quad \text{(substitute a for x)}$$

Such well behaved functions are continuous at a – we will examine this concept more closely in the next chapter. Here we discuss some limits that can be evaluated by direct substitution. In the next section, we will discuss techniques for evaluating limits for which direct substitution fails.

Basic Elementary Functions

The following functions are called basic elementary functions :

(i) $y = x^p$ (ii) $y = a^x$

(iii) $y = \log_a x$ (iv) $y = \text{trigo } x$

(v) $y = \text{trigo}^{-1} x$.

Elementary Functions

Elementary function is one which may be represented by a single formula y = f(x), where f(x) is made up of basic elementary functions and constants by means of finite number of operations of addition, substraction, multiplication, division and composition.

Note that piecewise defined functions are non-elementary. For example, [x], {x}, sgn x are non-elementary functions.

Limit of Elementary Functions

Theorem For all elementary functions, limit at any point in the domain is equal to the function's value at that point.

Example 4: Find $\lim\limits_{x\to 0} \dfrac{x^3 + 3x^2 - x - 3}{x^2 + x - 6}$.

Solution: Since the given function is an elementary function and the polynomial in the denominator does not vanish at x = 0, we obtain the limit by substituting x = 0 in the function.

$$\lim_{x\to 0} \frac{x^3 + 3x^2 - x - 3}{x^2 + x - 6} = \frac{0^3 + 3.0^2 - 0 - 3}{0^2 + 0 - 6} = \frac{-3}{-6} = \frac{1}{2}$$

Study Tip

Although such substitution produces the correct answer in this particular case, in many limits involving non-elementary functions, it produces either an incorrect answer or no answer at all.

Also do not assume from these problems that $\lim\limits_{x\to a} f(x)$ is invariably f(a).

To evaluate the $\lim\limits_{x\to 0} \dfrac{\ell n(1+x) - x^2 + 2}{\cos(\sin x) + 1 + \sin^{-1} x}$, we notice that the function $\dfrac{\ell n(1+x) - x^2 + 2}{\cos(\sin x) + 1 + \sin^{-1} x}$ is elementary and x = 0 lies in the domain. Hence, limit is equal to the function's value at x = 0,

$$\text{i.e. } \lim_{x\to 0} \frac{\ell n(1+x) - x^2 + 2}{\cos(\sin x) + 1 + \sin^{-1} x} = \frac{0 - 0 + 2}{1 + 1 + 0} = 1.$$

To evaluate $\lim\limits_{x\to 2} [x]$ we cannot use direct substitution property as $\lim\limits_{x\to 2} [x] = [2] = 2$, since [x] is a non-elementary function. We know that this limit does not exist.

Sometimes, using direct substitution property on non-elementary functions may give correct results as in $\lim\limits_{x\to 0} [x^2] = [0]^2 = 0$, but it is not advisable to use in general.

$|x|$ can be treated as an elementary function since we can write $|x| = \sqrt{x^2}$.

Thus, $\lim\limits_{x\to 2} x|x-2| = 2|2-2| = 0$.

Example 5: Find $\lim\limits_{x\to 1} \dfrac{x^2 - 4}{\cos \frac{\pi}{4} x}$.

Solution: The given function is an elementary function and the point x = 1 lies in the domain, and thus

$$\lim_{x\to 1} \frac{x^2 - 4}{\cos \frac{\pi}{4} x} = \frac{1^2 - 4}{\cos \frac{\pi}{4} \cdot 1} = -3\sqrt{2}.$$

But the limit of this function as x → 2 can not be found by means of the direct substitution since the denominator turns into zero at the point x = 2.

Concept Problems B

1. Show that $\lim\limits_{x\to 0.01}\dfrac{1}{\sqrt{x}} = 10$.

2. Evaluate $\lim\limits_{x\to -1}\dfrac{3x^2}{2x-1}$.

3. Evaluate $\lim\limits_{x\to \pi}\dfrac{\cos x}{\sin x - \pi}$.

4. Evaluate $\lim\limits_{x\to 0}\dfrac{x}{\cos^{-1} x}$.

5. Show that $\lim\limits_{x\to 0}\dfrac{\cos x + 4\tan x}{2 - x - 2x^4} = \dfrac{1}{2}$.

6. Why doesn't the limit $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$ follow from the product law of limits with f(x) = x and g(x) = sin(1/x) ?

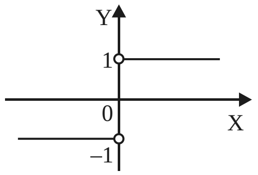
1.4 One-Sided Limits

For functions that are defined piecewise, a two-sided limit at some x, where the formula changes, is best obtained by first finding the one-sided limits at the number.

Consider the behaviour of the function

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}$$

The graph of y = f(x) is shown below :



As $x \to 0$, f(x) does not approach a specific number. However, as x approaches 0 through positive numbers, $f(x) \to 1$. Also, as x approaches 0 through negative numbers, $f(x) \to -1$. This behaviour illustrates the idea of one-sided limits, which will now be defined.

Left Hand Limit (L.H.L.)

Suppose that f is defined on the interval (c, a) immediately to the left of a. Then we say that the number ℓ is the left hand limit of f(x) as x approaches a from the left, and we write

$$f(a^-) = \lim_{x\to a^-} f(x) = \ell \qquad \text{...(1)}$$

provided that we can make the number f(x) as close to ℓ as we please merely by choosing the point x in (c, a) sufficiently close to a.

We may describe the left hand limit in (1) by saying that $f(x) \to \ell$ as $x \to a^-$, that is, as x approaches a from the left. The symbol a^- denotes the left hand side of a.

Formally, $\lim\limits_{x\to a^-} f(x) = \ell$, if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x satisfying $a - \delta < x < a \Rightarrow |f(x) - \ell| < \varepsilon$ and we call it the left hand limit.

For instance, we see that $\lim\limits_{x\to 0^-}\dfrac{|x|}{x} = -1$ because |x|/x is equal to –1 for all x to the left of zero.

Right Hand Limit (R.H.L.)

Suppose that f is defined on the interval (a, c) immediately to the right of a. Then we say that the number ℓ is the right hand limit of f(x) as x approaches a from the right, and we write

$$f(a^+) = \lim_{x\to a^+} f(x) = \ell \qquad \text{...(2)}$$

provided that we can make the number f(x) as close to ℓ as we please merely by choosing the point x in (a, c) sufficiently close to a.

Formally, $\lim\limits_{x\to a^+} f(x) = \ell$, if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x satisfying $a < x < a + \delta \Rightarrow |f(x) - \ell| < \varepsilon$ and we call it the right hand limit.

For instance, we see that $\lim\limits_{x\to 0^+}\dfrac{|x|}{x} = 1$ because |x|/x is equal to 1 for all x to the right of zero.

One-Sided Limits and Two-Sided Limits

Suppose that the function f is defined for $x \neq a$ in a neighbourhood of the point a. Then the two-sided limit $\lim\limits_{x\to a} f(x)$ exists and is equal to the number ℓ if and only if the one-sided limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ both exist and are equal to ℓ.

This theorem is useful in showing that certain limits do not exist, frequently by showing that the left hand and right hand limits are not equal to each other. And conversely, if there exists a limit ℓ of a function at the point a, then there exist limits of the function at the point a both on the right and on the left and they are equal.

Since the one-sided limits are not equal, the corresponding two-sided limit

$$\lim_{x\to 0}\frac{|x|}{x} \text{ does not exist.}$$

$$\text{Let } f(x) = \begin{cases} -x^3 & x \le 1 \\ 2+x, & x > 1 \end{cases}$$

The given function is defined on the entire number line. Let us compute the one-sided limits of this function at the point x = 1.

$$f(1^-) = \lim_{x\to 1-0}(-x^3) = -1$$

$$f(1^+) = \lim_{x\to 1+0}(2+x) = 3$$

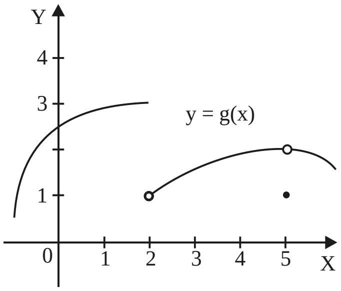
Thus, $f(1^-) \ne f(1^+)$. Hence, the given function has no limit at the point x = 1.

Example 1: The graph of a function g is shown in the figure. Use it to state the values (if they exist) of the following limits:

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2} g(x)$ (d) $\lim_{x\to 5^-} g(x)$

(e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$



Solution: From the graph we see that the values of g(x) approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore,

(a) $\lim_{x\to 2^-} g(x) = 3$ and

(b) $\lim_{x\to 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude that $\lim_{x\to 2} g(x)$ does not exist.

The graph also show that

(d) $\lim_{x\to 5^-} g(x) = 2$ and

(e) $\lim_{x\to 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, we have $\lim_{x\to 5} g(x) = 2$

Despite this fact, notice that $g(5) \ne 2$.

Study Tip

1. When a function is defined only on one side of a point a, then $\lim_{x\to a} f(x)$ is identical with the one-sided limit, if it exists. For example, if $f(x) = \sqrt{x}$, then f is defined only at and to the right of zero. Hence, $\lim_{x\to 0}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x} = 0$. Of course, $\lim_{x\to 0^-}\sqrt{x}$ does not exist, since $\sqrt{x}$ is not defined when x < 0.

2. A function cannot have an ordinary limit at an endpoint of its domain. It can have a one-sided limit.

   The function $f(x) = \sqrt{9-x^2}$ has the interval $-3 \le x \le 3$ as its domain. If a is any number on the open interval (–3, 3), then $\lim_{x\to a}\sqrt{9-x^2}$ exists and is equal to $\sqrt{9-a^2}$.

   Now consider a = 3. Let x approach 3 from the left; then $\lim_{x\to 3^-}\sqrt{9-x^2} = 0$. For x > 3, $\sqrt{9-x^2}$ is not defined.

   Hence, $\lim_{x\to 3}\sqrt{9-x^2} = \lim_{x\to 3^-}\sqrt{9-x^2} = 0$.

   Similarly, $\lim_{x\to -3}\sqrt{9-x^2} = \lim_{x\to -3^+}\sqrt{9-x^2} = 0$.

   Consider the limit $\lim_{x\to 0}\sin^{-1}(e^{|x|})$.

   Even if $f(0) = \pi/2$, the function is undefined on both sides of x = 0. Hence there is no question of limit at x = 0.

   In the limit $\lim_{x\to 3}\frac{[x^2]-2[x]-3}{[x^2]-4[x]+3}$, the function is undefined in the right neighbourhood of x = 3. Hence R.H.L. cannot be discussed. However, the limit is 1/3 since L.H.L. can be evaluated as 1/3.

3. There is no shorthand for the two-sided limit of $\lim_{x\to 0}\frac{1}{x}$ even though we write $\lim_{x\to 0^-}\frac{1}{x} = -\infty$ and $\lim_{x\to 0^+}\frac{1}{x} = \infty$. We may only say that the limit does not exist.

   Note that we do not write $\lim_{x\to 1}\frac{2x+1}{x-1} = \infty$. However, it would be correct to write $\lim_{x\to 1}\left|\frac{2x+1}{x-1}\right| = \infty$.

Example 2: Find $\lim_{x\to 1}\frac{x+3}{x^3-3x+2}$ and

$$\lim_{x\to 2}\frac{x+3}{(x-1)^2(x-2)}.$$

Solution: $\lim_{x\to 1}\frac{x+3}{x^3-3x+2}$

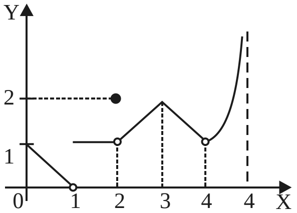
$$= \lim_{x\to 1}\frac{x+3}{(x-1)^2(x-2)} = -\infty$$

$$\text{Now, } \lim_{x\to 2^-}\frac{x+3}{x^3-3x+2}=-\infty$$

$$\lim_{x\to 2^+}\frac{x+3}{x^3-3x+2}=\infty$$

$$\therefore \quad \lim_{x\to 2}\frac{x+3}{(x-1)^2(x-2)}= \text{ does note exist}$$

4. In some cases a function is defined at x = a however limit does not exist at x = a and in some cases limit x → a exists but f(a) is not defined. Note these situations in the graph of the following function y = f(x) :



At x = 0 , $f(0^+) = 1$

At x = 1 , $f(1^-) = 0$; $f(1^+) = 1$

At x = 2 , $f(2^+) = f(2^-) = 1 \neq f(2)$

At x = 3 , $f(3^+) = f(3^-) = 2 = f(3)$

At x = 4 , $f(4^+) = f(4^-) = 1 \neq f(4)$

At x = 5 , $f(5^-) \to \infty$ (limit dne).

5. In functions involving the greatest integer function, fractional part function and signum function the following points must be noted :

(i) f (x) = [x] and f (x) = {x} has no limit at all integers.

(ii) y = [f(x)] and y = {f(x)} may not have limit at points where f(x) is an integer.

(iii) $f(x) = \dfrac{|x|}{x}$ has no limit at x = 0.

(iv) y = sgn(f(x)) may not have limit at points where f(x) = 0.

Non-Existence of Limit

Three of the most common types of behaviour associated with the non-existence of a limit.

1. f(x) approaches a different number from the right side of a than it approaches from the left side.
2. f(x) increase or decreases without bound as x approaches a.
3. f(x) oscillates too much as x approaches a.

Example 3: If $f(x) = \begin{cases}\sqrt{x-4} & \text{if} \quad x>4\\ 8-2x & \text{if} \quad x<4\end{cases}$ determine whether $\lim_{x\to 4} f(x)$ exists.

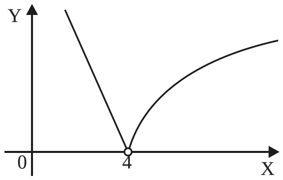
Solution: Since f(x) = $\sqrt{x-4}$ for x > 4, we have $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\sqrt{x-4} = \sqrt{4-4} = 0$.

Since f(x) = 8 – 2x for x < 4, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(8-2x) = 8-2.4 = 0.$$

The right and left hand limits are equal. Thus, the limit exists and $\lim_{x\to 4} f(x) = 0$.

The graph of f is shown in the figure.



Example 4: If $f(x) = \begin{cases}\dfrac{\sin[x]}{[x]}, & \text{for} \quad [x] \neq 0\\ 0, & \text{for} \quad [x] = 0\end{cases}$,

then find $\lim_{x\to 0} f(x)$ where [.] denotes the greatest integer function.

Solution:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\sin[x]}{[x]} = \frac{\sin(-1)}{(-1)} = \sin 1.$$

and $\lim_{x\to 0^+} f(x) = 0$ as it is given that f(x) = 0 for [x] = 0.

So, $\lim_{x\to 0} f(x)$ does not exist.

Substitution for One-sided Limits

$$\text{L.H.L.} = \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$$

Substitute x = a – h, assuming h as a small positive quantity. Note that as $x \to a^-$, $h \to 0^+$. But we often write $h \to 0$.

$$\text{R.H.L.} = \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

Substitute x = a + h, assuming h as a small positive quantity. As $x \to a^+$, $h \to 0^+$.

Example 5: Find $\lim_{x\to 1} x.\text{sgn}(x-1)$

Solution: L.H.L. = $\lim_{x\to 1^-} x.\text{sgn}(x-1)$ Put x = 1 – h

$$= \lim_{h\to 0}(1-h)\,\text{sgn}(1-h-1)$$

$$= \lim_{h\to 0}(1-h)\,\text{sgn}(-h)$$

$$= \lim_{h\to 0}(1-h)(-1) = -1.$$

R.H.L.= $\lim_{x\to 1^+} x.\ \text{sgn}(x-1)$ Put x = 1 + h

$$= \lim_{h\to 0}(1+h)\,\text{sgn}(1+h-1)$$

$$= \lim_{h\to 0}(1+h)\,\text{sgn}\, h = \lim_{h\to 0}(1+h).1 = 1$$

Since, L.H.L. ≠ R.H.L., $\lim_{x\to 1} x\ \text{sgn}(x-1)$ does not exist.

Example 6: Find $\lim\limits_{x\to 2}[x]$

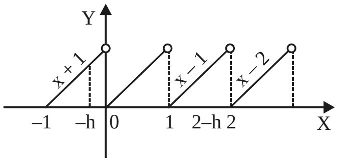
Solution: L.H.L. = $\lim\limits_{h\to 0}[2-h] = \lim\limits_{h\to 0} 1 = 1$.

R.H.L. = $\lim\limits_{h\to 0}[2+h] = \lim\limits_{h\to 0} 2 = 2$.

Since, L.H.L. ≠ R.H.L., $\lim\limits_{x\to 2}[x]$ does not exist.

Example 7: Evaluate $\lim\limits_{x\to 2}\{x\}$.

Solution:



{2–h} = 2 – h – 1 = 1 – h

We have {2 – h} = 2 – h – 1 = 1 – h

L.H.L. = $\lim\limits_{h\to 2}\{2-h\} = \lim\limits_{h\to 0}(1-h) = 1$.

R.H.L. = $\lim\limits_{h\to 0}\{2+h\} = \lim\limits_{h\to 0} h = 0$

Note that if h as a small positive quantity {–h} = 1 – h

Example 8: Find $\lim\limits_{x\to 2}\left(\left[x+\frac{1}{2}\right]+\{x^2\}\right)$

Solution: L.H.L. = [illegible]

$$= \lim_{h\to 0} 2+\{4-4h+h^2\}$$

$$= \lim_{h\to 0} 2+\{-4h+h^2\} = \lim_{h\to 0} 2+\{-h(4-h)\}$$

$= \lim\limits_{h\to 0} 2+1-(4h-h^2)$, since 4h – h² is a small positive quantity.

= 3

$$\text{R.H.L.} = \lim_{h\to 0}\left(\left[2+h+\frac{1}{2}\right]\right)+\{(2+h)^2\}$$

$$= \lim_{h\to 0} 2+\{4+h^2+4h\}$$

$$= \lim_{h\to 0} 2+\{4h+h^2\}$$

$$= \lim_{h\to 0} 2+4h+h^2 = 2$$

Since, L.H.L. ≠ R.H.L., the limit does not exist.

Example 9: Find $\lim\limits_{x\to\frac{5\pi}{4}}[\sin x+\cos x]$

Solution: L.H.L. = $\lim\limits_{x\to\frac{5\pi}{4}}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]$

$$\text{L.H.L.} = \lim_{h\to 0}\left[\sqrt{2}\sin\left(\frac{5\pi}{4}-h+\frac{\pi}{4}\right)\right]$$

$$= \lim_{h\to 0}\left[-\sqrt{2}\cos(h)\right] = -2$$

$$\text{R.H.L.} = \lim_{h\to 0}\left[\sqrt{2}\sin\left(\frac{5\pi}{4}+h+\frac{\pi}{4}\right)\right]$$

$$= \lim_{h\to 0}\left[-\sqrt{2}\cos(h)\right] = -2$$

Since, L.H.L. = R.H.L., the required limit is –2.

Example 10: Evaluate the left hand and right hand limits of the function

$$f(x) = \begin{cases} \dfrac{\sqrt{(x^2-6x+9)}}{(x-3)}, & x \neq 3 \\ 0, & x = 3 \end{cases} \quad \text{at } x = 3.$$

Solution: The given function can be written as

$$f(x) = \begin{cases} \dfrac{|x-3|}{(x-3)}, & x \neq 3 \\ 0, & x = 3 \end{cases}$$

∴ L.H.L. = $\lim\limits_{x\to 3-} f(x) = \lim\limits_{h\to 0}(3-h)$

$$= \lim_{h\to 0}\frac{|3-h-3|}{(3-h-3)} = \lim_{h\to 0}\frac{|-h|}{(-h)} = \lim_{h\to 0}\frac{h}{-h} = -1$$

and R.H.L. = $\lim\limits_{x\to 3+} f(x) = \lim\limits_{h\to 0}(3+h)$

$$= \lim_{h\to 0}\frac{|3+h-3|}{(3+h-3)} = \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

Hence the left hand limit and right hand limit of f(x) at x = 3 are –1 and 1 respectively.

Example 11: Let $f(x) = \begin{cases} \cos[x], & x \geq 0 \\ |x|+a, & x < 0 \end{cases}$

Find the value of a, given that $\lim\limits_{x\to 0} f(x)$ exists, where [.] denotes the greatest integer function.

Solution: Since $\lim\limits_{x\to 0} f(x)$ exists

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x).$$

$$\Rightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$$

$$\Rightarrow \lim_{h\to 0} |0-h| + a = \lim_{h\to 0} \cos[0+h]$$

⇒ a = cos 0 = 1

∴ a = 1

Study Tip

If f(x) is an even function, then

(i) $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x)$

(ii) $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to -a^+} f(x)$

If f(x) is an odd function then

(i) $\lim\limits_{x\to 0-} f(x) = -\lim\limits_{x\to 0+} f(x)$.

Further if $\lim\limits_{x\to 0} f(x)$ exists, then $\lim\limits_{x\to 0} f(x) = 0$.

(ii) $\lim\limits_{x\to a^-} f(x) = -\lim\limits_{x\to -a^+} f(x)$

Example 12: Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]$ where [.] denotes the greatest integer function.

Solution: Let P = $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]$

For $x > 0$, $\sin^{-1}x > x \Rightarrow \dfrac{\sin^{-1}x}{x} > 1$

$\therefore$ R.H.L. = $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right] = 1$

The function $\dfrac{\sin^{-1}x}{x}$ is even.

For $x < 0$, $\dfrac{\sin^{-1}x}{x} > 1$

$\therefore$ L.H.L.= $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right] = 1$. Hence P = 1.

Example 13: Find $\lim\limits_{x\to 0}\left(\tan^{-1}\dfrac{1}{x} + \cot^{-1}\dfrac{1}{x^2}\right)$

Solution: $\lim\limits_{x\to 0}\left(\tan^{-1}\dfrac{1}{x} + \cot^{-1}\dfrac{1}{x^2}\right)$

$$= \lim_{x\to 0}\tan^{-1}\frac{1}{x} + \lim_{x\to 0}\cot^{-1}\frac{1}{x^2}$$

$$= \lim_{x\to 0}\tan^{-1}\frac{1}{x} + 0 = \lim_{x\to 0}\tan^{-1}\frac{1}{x}$$

Now we evaluate the one sided limits :

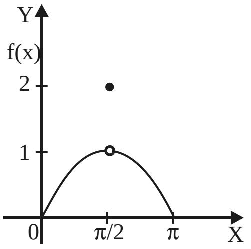
L.H.L. = $\lim\limits_{x\to 0^-}\tan^{-1}\dfrac{1}{x} = -\dfrac{\pi}{2}$

R.H.L. = $\lim\limits_{x\to 0^+}\tan^{-1}\dfrac{1}{x} = \dfrac{\pi}{2}$.

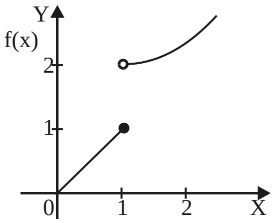
Finally, the limit does not exist.

Concept Problems C

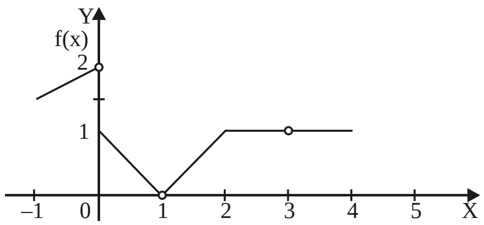
1. Find $\lim\limits_{x\to\pi/2} f(x)$



2. Find $\lim\limits_{x\to 1} f(x)$



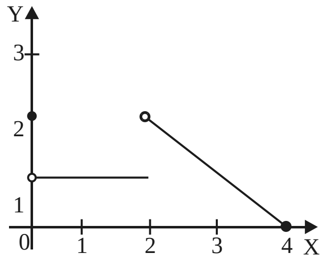
3. Find (i) $\lim\limits_{x\to 0} f(x)$, (ii) $\lim\limits_{x\to 1} f(x)$, (iii) $\lim\limits_{x\to 3} f(x)$



4. The following figure shows the graph of a function f. Decide which of the given limits exist and evaluate those which do.

(a) $\lim\limits_{x\to 0^+} f(x)$ (b) $\lim\limits_{x\to 1} f(x)$

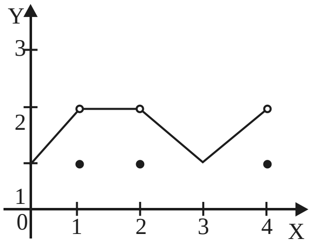
(c) $\lim\limits_{x\to 2^-} f(x)$ (d) $\lim\limits_{x\to 2^+} f(x)$



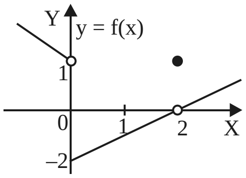
5. The following figure shows the graph of a function f. Decide which of the given limits exist and evaluate those which do.

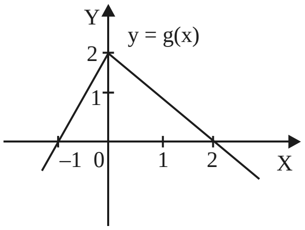
(a) $\lim\limits_{x\to 1} f(x)$ (b) $\lim\limits_{x\to 2} f(x)$

(c) $\lim\limits_{x\to 3} f(x)$ (d) $\lim\limits_{x\to 4^-} f(x)$



6. Use the graph of f and g to find the limits that exist.





(i) $\lim\limits_{x\to a} f(x) + g(x)$, a = 0, 2

(ii) $\lim\limits_{x\to 2} \dfrac{f(x)}{1+g(x)}$

7. Evaluate the following limits :

(i) $\lim\limits_{x\to 5^-} 10^{1/(x-5)}$ (ii) $\lim\limits_{x\to 1^+} \sin^{-1}\left(\log_3 \dfrac{x}{3}\right)$

(iii) $\lim\limits_{x\to 1^+} \dfrac{(x+1)}{(x+1)+2}$ (iv) $\lim\limits_{x\to 0} \tan^{-1}\dfrac{1}{x}$

8. Evaluate $\lim\limits_{x\to 0} \sin^{-1}(\sec x)$.

9. Evaluate $\lim\limits_{x\to 0^-} \cos^{-1}(1 + \tan x)$.

10. Given that $f(x) = \dfrac{x^2}{(x-1)(x-2)}$, show that

(i) $\lim\limits_{x\to 1^-} f(x) = \infty$, $\lim\limits_{x\to 1^+} f(x) = -\infty$,

(ii) $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to 2^+} f(x) = \infty$,

(iii) $\lim\limits_{x\to \infty} f(x) = 1 = \lim\limits_{x\to -\infty} f(x)$.

11. Find the left and right hand limits of the function $\psi(x) = \begin{cases} -x+1 & \text{for} \quad x \ge 1, \\ 2x-1 & \text{for} \quad x > 1 \end{cases}$ at the point x = 1.

12. Find the limit of $f(x) = \begin{cases} 2x-3, & x \ge 2 \\ 4-x^2, & x < 2 \end{cases}$ at x = 2.

13. Find $\lim\limits_{x\to 3} f(x)$ where $f(x) = \begin{cases} 2(x+1) & \text{if } x < 3 \\ 4 & \text{if } x = 3 \\ x^2-1 & \text{if } x > 3 \end{cases}$

14. Using graphs find the limit (if it exists):

(a) $(x) = \begin{cases} x^2+2, & x \ne 1 \\ 1, & x = 1 \end{cases}$, $\lim\limits_{x\to 1} f(x)$

(b) $\lim\limits_{x\to 5} \dfrac{|x-5|}{x-5}$

(c) $\lim\limits_{x\to \pi/2} \tan^2 x$.

15. Suppose that f is an odd function of x.

Does knowing that $\lim\limits_{x\to 0^+} f(x) = 3$ tell you anything about $\lim\limits_{x\to 0^-} f(x)$?

16. Suppose that f is an odd function of x. Does knowing that $\lim\limits_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim\limits_{x\to -2^-} f(x)$ or $\lim\limits_{x\to -2^+} f(x)$?

Practice Problems B

17. Evaluate (i) $\lim\limits_{x\to 1} f(x)$ (ii) $\lim\limits_{x\to -1} f(x)$

where $f(x) = \dfrac{x|x-3|}{(x^2-x-6)|x|}$.

18. Evaluate the following limits :

(i) $\lim\limits_{x\to 1} x\,\text{sgn}(x-1)$

(ii) $\lim\limits_{x\to \frac{\pi}{4}^+} \left[\tan\left(\dfrac{\pi}{8}+x\right)\right]^{\tan 2x}$

19. Evaluate the limits

(i) $\lim\limits_{x\to 0} \sin^{-1}[\sec x]$ (ii) $\lim\limits_{x\to 0} \dfrac{\sin[\cos x]}{1+[\cos x]}$

where [.] denotes the greatest integer function.

20. If $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x > 0 \\ x^2, & x \le 0 \end{cases}$, find $\lim\limits_{x\to 0} f(x)$

21. Evaluate $\lim\limits_{x\to -1^-} 1 - x + [x] - [1 - x]$,

where [.] denotes the greatest integer function.

22. Evaluate $\lim\limits_{x\to 0^-}\dfrac{\sum\limits_{r=1}^{2n+1}[x^r]+n+1}{1+[x]+|x|+2x}$

where [.] denotes the greatest integer function.

23. Find $\lim\limits_{x\to 8}\dfrac{\sin\{x-10\}}{\{10-x\}}$,

where {.} denotes the fractional part function.

24. Find the left and right hand limits of the function $\phi(x)=\dfrac{1}{x+2^{\frac{1}{x-2}}}$ at the point x = 2.

25. Evaluate the one-sided limits (where [.] denotes the greatest integer function):

(i) $\lim\limits_{x\to 1}\dfrac{1}{(x-1)^2}$ (ii) $\lim\limits_{x\to 1}\dfrac{[x^2]-1}{x^2-1}$

(iii) $\lim\limits_{x\to 0}\dfrac{1}{2-2^{1/x}}$ (iv) $\lim\limits_{x\to 0} x(-1)^{[1/x]}$

26. Evaluate the one-sided limits (where [.] denotes the greatest integer function) :

(i) $\lim\limits_{x\to 1}\dfrac{x^2-1}{|x-1|}$

(ii) $\lim\limits_{x\to 2}\dfrac{2\{x\}-4}{[x]-3}$

(iii) $\lim\limits_{x\to a}\left(\dfrac{|x^3|}{a}-\left[\dfrac{x}{a}\right]^3\right)$, a > 0

(iv) $\lim\limits_{x\to 1} f(x)$ where $f(x)=\begin{cases}x+1 & 0\le x<1\\ 3x+2 & 1<x<3\end{cases}$

27. In each of the following functions :

(a) Sketch the graph of the function f.

(b) For each integer n, evaluate the one-sided limits $\lim\limits_{x\to n^-} f(x)$ and $\lim\limits_{x\to n^+} f(x)$ in terms of n.

(c) Determine those values of a for which $\lim\limits_{x\to a} f(x)$ exists.

(i) $f(x)=\begin{cases}2 & \text{if } x \text{ is not an integer,}\\ 2+(-1)^x & \text{if } x \text{ is an integer,}\end{cases}$

(ii) f(x) = [10x]

(iii) $f(x)=x-[x]-\dfrac{1}{2}$.

1.5 Determinate and Indeterminate Forms

Until now we have found limits using graphs or the direct substitution property. We shall now classify the problems of limit in two ways:

(i) **Determinate form:** In which the limit can be predicted apriori without resorting to special methods. This is so named because two problems having the same form have equal limits.

For example, $\lim\limits_{x\to 1}\dfrac{2-x}{\tan\frac{\pi x}{2}}$ and $\lim\limits_{x\to 0}\dfrac{\cos x}{\ln|x|}$ are in the same form $\dfrac{1}{\infty}$ and we see that both of them have the limit 0.

Hence we call $\dfrac{1}{\infty}$ as a determinate form and we are sure that any problem in $\dfrac{1}{\infty}$ form will have limit 0.

If $\lim\limits_{x\to a} f(x)=b$ and $\lim\limits_{x\to a} g(x)=0$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is said to be of the form $\dfrac{b}{0}$. If b > 0 and $g(x)\to 0^+$ then the limit is ∞.

Note: Let $\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{x\to a} g(x)=\infty$, then we have

(i) $\lim\limits_{x\to a}(f(x)+g(x))=\infty$

(ii) $\lim\limits_{x\to a}(f(x)-g(x))=-\infty$

(iii) $\lim\limits_{x\to a} f(x)g(x)=\infty$ (for $b>0$)

(iv) $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)}=\infty$ (for $b>0$)

(v) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=0$.

Here we give a list of determinate forms. One should try to understand the meaning of these forms :

(i) $0+c=c$
(ii) $0+0=0$
(iii) $c+\infty=\infty$
(iv) $\infty-c=\infty$
(v) $c-\infty=-\infty$
(vi) $0+\infty=\infty$
(vii) $0-\infty=-\infty$
(viii) $\infty+\infty=\infty$
(ix) $0\times c=0$
(x) $\infty\times c=\infty$, $c>0$
(xi) $\infty\times\infty=\infty$
(xii) $0\times 0=0$
(xiii) $\dfrac{c}{0^+}=\infty$, $c>0$
(xiv) $\dfrac{0}{c}=0$, $c\neq 0$
(xv) $\dfrac{\infty}{0^+}=\infty$
(xvi) $\dfrac{0}{\infty}=0$
(xvii) $\infty^\infty=\infty$
(xviii) $0^\infty=0$

Consider some examples of determinate form :

(i) $\lim\limits_{x\to 0}\dfrac{1}{x^2}=\infty$

(ii) $\lim\limits_{x\to 0}\dfrac{1}{x^4}=\infty$,

(iii) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x^4}\right) = \lim\limits_{x\to 0}\dfrac{x^2-1}{x^4} = -\infty,$

(iv) $\lim\limits_{x\to 0}\left(2+\dfrac{1}{x^2}\right) = \infty,$

(v) $\lim\limits_{x\to 0}\left(1+x+\dfrac{1}{x^2}\right) = \infty,$

(vi) $\lim\limits_{x\to 0}\left[\left(2+\dfrac{1}{x^2}\right)-\left(1+x+\dfrac{1}{x^2}\right)\right] = \lim\limits_{x\to 0}(1-x)=1,$

(vii) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+2x}\right)^x = \lim\limits_{x\to\infty}\left(\dfrac{1}{2+1/x}\right)^x = \dfrac{1}{2^\infty}=0\,,$

(viii) $\lim\limits_{x\to 0}(1+\cos x)^{(x+2)} = \lim\limits_{x\to 0}(1+1)^2 = 4\,.$

(ii) Indeterminate form: In which the limit cannot be predicted apriori. Here special methods are required to find the limit. In this case two problems having the same form may not have equal limits.

For example, the limit of a quotient,

$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, is called an indeterminate form of type 0/0. Consider the following limits in 0/0 form:

$$\lim_{x\to 0}\frac{x}{x}=1,\quad \lim_{x\to 0}\frac{x^2}{x}=0\,,\quad \lim_{x\to 0}\frac{x}{x^3}=\infty$$

We notice that the result varies quite significantly even if they are in the same indeterminate form. The word "indeterminate" here refers to the fact that the limiting behaviour of the quotient cannot be determined without further study. The expression "0/0" is just a device to describe the circumstance of a limit of a quotient in which both the numerator and denominator approach 0.

The important indeterminate forms are :

$$\mathbf{\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^o,\ \infty^o}\ \textbf{and}\ \mathbf{1^\infty}$$

Note that '0' doesn't mean exact zero but it represents a value approaching towards zero.

The behaviour of '1' and '∞' are similar.

Indeterminate Products

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim\limits_{x\to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or – ∞). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an indeterminate form of type $0\cdot\infty$. We can deal with it by writing the product fg as a quotient :

$$fg = \frac{f}{1/g}\ \text{ or }\ fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of the type $\dfrac{0}{0}$ or ∞/∞ .

Several indeterminate forms arise from the limit $\lim\limits_{x\to a}[f(x)]^{g(x)}$

1. $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ type 0^0
2. $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = 0$ type ∞^0
3. $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

let $y = [f(x)]^{g(x)}$,

then $\ell n\ y = g(x)\ \ell n\ f(x)$

or by writing the function as an exponential

$$[f(x)]^{g(x)} = e^{g(x)\,\ell n\,f(x)}$$

In either method we are led to the indeterminate product $g(x)\ \ell n\ f(x)$, which is of type $0\cdot\infty$.

The reason why these forms carry the label "indeterminate" is not that they cannot be resolved, but rather that there is no predicting what the limit of the given function will be, or indeed whether it will exist at all, until the given function receives a thorough investigation.

It is tempting to argue that an indeterminate form of type 0 . ∞ has value 0 since "zero times anything is zero." However, this is fallacious since 0 . ∞ is not a product of numbers, but rather a statement about limits.

The form $(\infty+\infty)$, on the other hand, is determinate in the sense that unfailingly the sum function has limit ∞ if both summand functions have limit ∞ at the prescribed point of approach.

The following examples show how easy it is to construct quotient functions and sum functions with indeterminate forms having an arbitrary real number C as a limit, or having no limit whatever.

The form 0/0.

$\lim\limits_{x\to 0}\dfrac{C\sin x}{x} = C$ while $\lim\limits_{x\to 0}\dfrac{x\sin(1/x)}{x}$ does not exist.

The form (∞/∞).

$\lim\limits_{x\to\infty}\dfrac{Cx}{x} = C$ while $\lim\limits_{x\to\infty}\dfrac{x(\sin x+2)}{x}$ does not exist.

The form (∞ – ∞).

$\lim\limits_{x\to\infty}(x-(x-C)) = C$ while $\lim\limits_{x\to\infty}(x^2-x)$ does not exist.

The above examples were artificially framed to point that "indeterminate" means "unpredictable without special attention to the functions involved".

There is a myth that circulates among students which states that all indeterminate forms of types 0^o, ∞^o, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^o, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples show that such indeterminate forms can have any positive real value :

(a) $\lim\limits_{x\to 0^+}[x^{(\ln a)/(1+\ln x)}] = a$ (form 0^o)

(b) $\lim\limits_{x\to\infty}[x^{(\ln a)/(1+\ln x)}] = a$ (form ∞^o)

(c) $\lim\limits_{x\to 1} [x^{(\ln a)/(1+\ln x)}] = 1$ (form 1^∞)

We can easily verify these results.

Find whether the following limits are in determinate / indeterminate form. Also evaluate the limit in case of determinate form.

(i) $\lim\limits_{x\to 0} \dfrac{2}{1+2^{1/x^2}} = 0$

as $x \to 0$, $\dfrac{1}{x} \to 0$, $2^{1/x^2} \to \infty$, $\dfrac{2}{\infty} \to 0$.

(ii) $\lim\limits_{x\to 1^-} \dfrac{\sin^{-1} x}{\tan\frac{\pi}{2}x} = 0$, since $\left(\dfrac{\pi/2}{\infty}\right) \to 0$

(iii) $\lim\limits_{x\to\infty} (e^x + x) = \infty$, since ∞ + ∞ = ∞.

(iv) $\lim\limits_{x\to 0} (1+\sin x)^{1/x^2}$ (1^∞ indeterminate form)

(v) $\lim\limits_{x\to 0} (\underbrace{1+[\cos x]}_{\text{exact } 1})^{\frac{1}{x^2}}$ is not in 1^∞ indeterminate form

since $(\text{exact } 1)^\infty = 1$ (determinate form),

(vi) $\lim\limits_{x\to 0} (\underbrace{1+[\cos x]}_{\text{exact } 1})^{\frac{1}{x}} = 1.$

(vii) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$ ($\infty - \infty$ indeterminate form)

(viii) $\lim\limits_{x\to 1^+} \dfrac{[x^2]-1}{x-1} = 0$ $\left(\dfrac{\text{exact } 0}{\to 0}\right) = 0$

Let us try to deduce conclusion about the limits:

$\lim\limits_{x\to a} [f(x)+g(x)]$, $\lim\limits_{x\to a} [f(x).g(x)]$ and $\lim\limits_{x\to a} [f(x)/g(x)]$ based upon the individual behaviours of f(x) and g(x).

Assume that ℓ is a positive real number.

	$\lim\limits_{x\to a} f(x)$	$\lim\limits_{x\to a} g(x)$	$\lim\limits_{x\to a} [f(x) + g(x)]$
(i)	ℓ	∞	∞
(ii)	ℓ	$-\infty$	$-\infty$
(iii)	∞	∞	∞
(iv)	$-\infty$	$-\infty$	$-\infty$
(v)	∞	$-\infty$	indeterminate

	$\lim\limits_{x\to a} f(x)$	$\lim\limits_{x\to a} g(x)$	$\lim\limits_{x\to a} [f(x).g(x)]$
(i)	$\ell \neq 0$	∞	∞
(ii)	∞	∞	∞
(iii)	0	∞	indeterminate

	$\lim\limits_{x\to a} f(x)$	$\lim\limits_{x\to a} g(x)$	$\lim\limits_{x\to a} [f(x)/g(x)]$
(i)	$\ell \neq 0$	0	∞
(ii)	ℓ	∞	0
(iii)	∞	ℓ	∞
(iv)	0	0	indeterminate
(v)	∞	∞	indeterminate

These results are valid even when x tends to

a^-. a^+, ∞ or – ∞.

Example 1: If $\lim\limits_{x\to 2} \dfrac{f(x)-5}{x-2} = 3$ then find $\lim\limits_{x\to 2} f(x)$.

Solution: Since the denominator is approaching 0, we can have limit only through $\dfrac{0}{0}$ form.

Hence $\lim\limits_{x\to 2} f(x)$ must be 5.

Note that if $\lim\limits_{x\to 2} \dfrac{f(x)-5}{x-2} = 4$ then also $\lim\limits_{x\to 2} f(x)$ is 5.

Also if $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = 2$ then using $\dfrac{0}{0}$ form we have $\lim\limits_{x\to 0} f(x) = 0$ and further $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 0$.

Example 2: Find the left hand and right hand limits of the function $f(x) = \dfrac{1}{x + 2^{1/(x-3)}}$ as $x \to 3$.

Solution: If $x \to 3^-$, then

$1/(x-3) \to -\infty$ and $2^{1/(x-3)} \to 0$.

Consequently, $\lim\limits_{x\to 3^-} f(x) = 1/3$.

Now if $x \to 3^+$, $\lim\limits_{x\to 3^+} f(x) = 0$.

Example 3: Does the function

$y = \left(\sin\dfrac{1}{x}\right) \Big/ \left(\sin\dfrac{1}{x}\right)$ tend to a limit as x tends to 0?

Solution: The function is equal to 1 except when sin(1/x) = 0; i.e. when x = 1/π, 1/2π,.... For these values the formula for y assumes the meaningless expression 0/0, and y is therefore not defined for an infinitely many values of x near x = 0. Note that this is not the indeterminate form 0/0. Hence the limit does not exist.

Concept Problems

D

1. Find whether the following limits are in indeterminate form. Also indicate the form.

(i) $\lim\limits_{x\to 0} \dfrac{1}{x}$

(ii) $\lim\limits_{x\to 1} \dfrac{1-x}{1-x^2}$

(iii) $\lim\limits_{x\to 0} x\,\ell n\, x$

(iv) $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

(v) $\lim\limits_{x\to 0} (\sin x)^x$ (vi) $\lim\limits_{x\to 0} (\ell n\ x)^x$

(vii) $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ (viii) $\lim\limits_{x\to 0} (1)^{1/x}$

2. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Prove each statement.

(a) $\lim\limits_{x\to a} [f(x) + g(x)] = \infty$

(b) $\lim\limits_{x\to a} [f(x)\ g(x)] = \infty$ if $c > 0$

(c) $\lim\limits_{x\to a} [f(x)\ g(x)] = -\infty$ if $c < 0$

3. Let $\lim\limits_{x\to a} f(x) = 0$ with $f(x) \ne 0$ for $x \ne a$, $\lim\limits_{x\to a} g(x) = b \ne 0$.

Prove that $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = \infty$

4. Prove that the following limits donot exist :

(i) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan x}{\tan 2x}$ (ii) $\lim\limits_{x\to 0} (2)^{\frac{1}{x}}$

(iii) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{e^{\tan x}-1}{e^{\tan x}+1}$ (iv) $\lim\limits_{x\to 0} \tan^{-1}\dfrac{1}{x}$

5. Evaluate the following limits:

(i) $\lim\limits_{x\to 1} \dfrac{\sin^{-1} x}{\tan\dfrac{\pi x}{2}}$ (ii) $\lim\limits_{x\to 0} \dfrac{1}{\ln|x|}$

(iii) $\lim\limits_{x\to\infty} \dfrac{e^{x\ln 2}}{e^x}$ (iv) $\lim\limits_{x\to 0} x^2\left(1-e^{-\frac{1}{x^2}}\right)$

6. Show that $\lim\limits_{x\to\infty}\left(\dfrac{x}{e}\right)^x = \infty\cdot$

7. If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x\to 4} f(x)$.

8. If $\lim\limits_{x\to -2} \dfrac{f(x)}{x^2} = 1$, find

(i) $\lim\limits_{x\to -2} f(x)$ and (ii) $\lim\limits_{x\to -2} \dfrac{f(x)}{x}$

Concept Problems C

9. Find $\lim\limits_{x\to a} 2^{\frac{x}{|x-a|}}$

10. Find the following limits by inspection.

(a) $\lim\limits_{x\to 0^+} \dfrac{x}{\ln x}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{-x}}$

(c) $\lim\limits_{x\to(\pi/2)^-} (\cos x)^{\tan x}$

(d) $\lim\limits_{x\to 0^+} (\ln x)\cot x$

(e) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}-\ln x\right)$

(f) $\lim\limits_{x\to -\infty} (x + x^3)$

11. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+1}{2x^2-1}\right)^{\frac{x^3}{x-1}}$

12. If $f(x) = [x^2 + 1]^{[x+1]}$, where [.] denotes the greatest integer function, find $\lim\limits_{x\to 1} f(x)$.

METHODS OF EVALUATING LIMITS

1.6 Factorisation and Cancellation of Common Factors

If f(x) and g(x) are polynomials such that $f(a) = g(a) = 0$, then $(x-a)$ is a factor of both f(x) and g(x). Now to solve $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$, we cancel the common factor $(x-a)$ from both the numerator and denominator, and again put $x = a$ in the given expression. If we get a meaningful number, then the number is the limit of the given expression otherwise we repeat the process till we get rid of the indeterminate form (0/0).

Example 1: Find $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$.

Solution: Here, the denominator tends to zero as $x \to 3$ and the numerator also tends to zero. But since $x^2 - 9 = (x-3)(x+3)$, we have

$$\lim_{x\to 3}\frac{x-3}{x^2-9} = \lim_{x\to 3}\frac{x-3}{(x-3)(x+3)} = \lim_{x\to 3}\frac{1}{x+3} = \frac{1}{6}$$

In the solution of this problem we cancel $x-3$, and one may think that this is illegitimate since $x \to 3$, and the division by zero is not allowed. But this is not the case here : the functions $y = \frac{x-3}{x^2-9}$ and $y = \frac{1}{x+3}$ coincide identically for all $x \neq 3$, and the definition of a limit of a function for $x \to a$, does not involve the value of that function at the point $x = a$ itself, and therefore the limits of the above functions as $x \to 3$ are equal to each other. The essence of this transformation is that the limit of the new function is found easier than that of the original function.

Example 2: Evaluate $\lim_{x\to-3} \frac{x^2+x-6}{x+3}$

Solution: We cannot apply direct substitution because the limit of the denominator is 0.

$$\frac{x^2+x-6}{x+3}\begin{array}{l}\nearrow \lim\limits_{x\to-3}\left(x^2+x-3\right)=0\\ \searrow \lim\limits_{x\to-3}(x+3)=0\end{array}$$

$$\frac{x^2+x-6}{x+3}\begin{array}{l}\nearrow \lim\limits_{x\to-3}\left(x^2+x-6\right)=0\\ \searrow \lim\limits_{x\to-3}(x+3)=0\end{array}$$

Direct substitution fails here.

Because the limit of the numerator is also 0, the numerator and denominator have a common factor of (x + 3). Thus, for all x ≠ – 3, we can cancel this factor to obtain

$$\frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} = x-2,\ x \neq -3$$

It follows that

$$\lim_{x\to-3}\frac{x^2+x-6}{x+3} = \lim_{x\to-3}(x-2) = -5.$$

Although correct, the second equality in the preceding computation needs some justification, since cancelling the factor x + 3 alters the function by expanding its domain. However, the two functions are identical, except at x = – 3. From our previous discussions, we know that this difference has no effect on the limit as x approaches – 3.

Example 3: Find $\lim_{x\to2}\frac{x^6-24x-16}{x^3+2x-12}$

Solution: $\lim_{x\to2}\frac{x^6-24x-16}{x^3+2x-12}$

$$= \lim_{x\to2}\frac{(x-2)(x^5+2x^4+4x^3+8x^2+16x+8)}{(x^2+2x+6)(x-2)}$$

$$= \frac{168}{14} = 12$$

Substitution for Limits x → a

Suppose that $\lim_{x\to a} f(x) = \ell$

Let $x = a + t$, then $f(x) = f(a + t)$.

If $x \to a$ then $t \to 0$ and $f(a + t) \to \ell$, and we write

$$\lim_{x\to a} f(a+t) = \ell.$$

Example 4: Find $\lim_{x\to2}\frac{x^2-4}{\cos\frac{\pi}{4}x}$.

Solution: We put $x - 2 = z$, i.e. $x = z + 2$, and as $x \to 2$ we have $z \to 0$.

$$\lim_{x\to2}\frac{x^2-4}{\cos\frac{\pi}{4}x} = \lim_{z\to0}\frac{z(z+4)}{\cos\frac{\pi}{4}(z+2)} = \lim_{z\to0}\frac{z(z+4)}{-\sin\frac{\pi}{4}z}$$

$$= \lim_{z\to0}\frac{z(z+4)}{-\sin\frac{\pi}{4}z} = -\frac{4}{\pi}\lim_{z\to0}\frac{\frac{\pi}{4}z}{\sin\frac{\pi}{4}z}\cdot\lim_{z\to0}(z+4) = -\frac{16}{\pi}.$$

Note:

1. $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \ldots + xa^{n-2} + a^{n-1})$

   where n is even or odd positive integer.

2. $x^n + a^n = (x + a)(x^{n-1} - x^{n-2}a + x^{n-3}a^2 - x^{n-4}a^3 + \ldots + (-1)^{n-1}a^{n-1})$

where n is odd positive integer. This formula is not applicable when n is even.

Example 5: Find $\lim_{x\to0}\frac{\sqrt[5]{(1+x)^3}-1}{x}$.

Solution: Let us put $1 + x = y^5$. As $x \to 0$, $y \to 1$. Then we have

$$\lim_{x\to0}\frac{\sqrt[5]{(1+x)^3}-1}{x} = \lim_{y\to1}\frac{y^3-1}{y^5-1}$$

$$\lim_{y\to1}\frac{y^2+y+1}{y^4+y^3+y^2+y+1} = \frac{3}{5}$$

Example 6: Find $\lim_{x\to1}\frac{x^3-x^2-x+1}{x^3+x^2-x-1}$

Solution: Here we have an indeterminacy of the form 0/0. Let us factorize the numerator and the denominator of the function

$$\lim_{x\to1}\frac{x^3-x^2-x+1}{x^3+x^2-x-1} = \lim_{x\to1}\frac{x^2(x-1)-(x-1)}{x^2(x+1)-(x+1)}$$

$$= \lim_{x\to1}\frac{(x-1)^2(x+1)}{(x-1)(x+1)^2} = \lim_{x\to1}\frac{x-1}{x+1} = \frac{0}{2} = 0.$$

Example 7: Find $\lim_{x\to10}\frac{x^3-1000}{x^3-20x^2+100x}$

Solution: This is also an indeterminacy of the form 0/0. We have

$$\lim_{x\to10}\frac{x^3-1000}{x^3-20x^2+100x}$$

$$= \lim_{x\to10}\frac{(x-10)(x^2+10x+100)}{x(x-10)^2}$$

$$= \lim_{x\to 10} \frac{x^2 + 10x + 100}{x(x-10)}$$

The numerator of the fraction tends to 300 and the denominator tends to zero. Consequently, the fraction in question is an infinitely large quantity and

$$\lim_{x\to 10} \frac{x^3 - 1000}{x^3 - 2x^2 + 100x} \text{ does not exist.}$$

Example 8: Find $\lim\limits_{x\to -3} \dfrac{x^3+3x^2-x-3}{x^2+x-6}$

Solution: At the point x = – 3 both the numerator and the denominator turn into zero.
We have $x^2 + x - 6 = (x + 3)(x - 2)$ and $x^3 + 3x^2 - x - 3 = (x + 3)(x^2 - 1)$, and hence, on cancelling the factor (x + 3), we obtain

$$\lim_{x\to -3} \frac{x^3+3x^2-x-3}{x^2+x-6} = \lim_{x\to -3} \frac{x^2-1}{x-2} = \frac{(-3)^2-1}{-3-2} = -\frac{8}{5}$$

In the same manner we can find

$$\lim_{x\to 2} \frac{x^3+3x^2-x-3}{x^2+x-6} \text{ dne, } \lim_{x\to 1} \frac{x^3+3x^2-x-3}{x^2+x-6} = 0.$$

Example 9: Evaluate

$$\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$$

Solution: We have

$$\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$$

$$= \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

Example 10: Find $\lim\limits_{y\to 2} \left[\dfrac{1}{y-2}\left(\dfrac{1}{x+y-2} - \dfrac{1}{x}\right)\right]$

Solution: $\lim\limits_{y\to 2} \dfrac{x-x+2-y}{x(x+y-2)} \cdot \dfrac{1}{y-2}$

$$= -\lim_{y\to 2} \frac{1}{x(x+y-2)} = -\frac{1}{x^2}$$

Here, y is a variable, so that it might be thought that we are dealing with functions of two variables. However, the fact that x as a variable plays no role in this problem; for the moment, x can be considered a constant.

Example 11: Evaluate $\lim\limits_{x\to 2} \dfrac{2^x + 2^{3-x} - 6}{2^{-x/2} - 2^{1-x}}$

Solution: $\lim\limits_{x\to 2} \dfrac{2^x + 2^{3-x} - 6}{2^{-x/2} - 2^{1-x}} \quad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 2} \frac{2^{2x} + 8 - 6.2^x}{2^{x/2} - 2}$$

$$= \lim_{x\to 2} \frac{(2^x - 2)(2^x - 4)}{2^{x/2} - 2}$$

$$= \lim_{x\to 2} \frac{(2^x - 2)(2^{x/2} - 2)(2^{x/2} + 2)}{2^{x/2} - 2}$$

= 2 × 4 = 8.

Example 12: Evaluate $\lim\limits_{x\to 1} \dfrac{x^3 - x^2 \log x + \log x - 1}{x^2 - 1}$

Solution: The given limit $= \lim\limits_{x\to 1} \dfrac{(x^3-1)-(x^2-1)\log x}{x^2-1}$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)-(x-1)(x+1)\log x}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)\left[x^2+x+1-(x+1)\log x\right]}{(x-1)(x+1)}$$

$$= \frac{1^2+1+1-(1+1)\log 1}{(1+1)} = \frac{3}{2}.$$

Example 13: If the limit $\lim\limits_{x\to -2} \dfrac{3x^2+ax+a+3}{x^2+x-2}$ exists, find a and the limit.

Solution: We see that the denominator → 0. For the limit to exist, we must have the $\left(\dfrac{0}{0}\right)$ form. Hence the numerator must → 0, i.e.

$$\lim_{x\to -2} 3x^2 + ax + a + 3 = 0$$

$$\Rightarrow \quad 12 - 2a + a + 3 = 0 \Rightarrow a = 15.$$

Now $\lim\limits_{x\to -2} \dfrac{3x^2+ax+a+3}{x^2+x-2} = \lim\limits_{x\to -2} \dfrac{2x^2+15x+18}{x^2+x-2}$

$$= \lim_{x\to -2} \frac{3(x+2)(x+3)}{(x+2)(x-1)} = -1.$$

Example 14: If $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = 2$ then evaluate the following limits, giving explicit reasoning.

(i) $\lim\limits_{x\to0}[f(x)]$, (ii) $\lim\limits_{x\to0}\left[\dfrac{f(x)}{x}\right]$ where [.] denotes the greatest integer function.

Solution: (i) Let $l = \lim\limits_{x\to0}[f(x)] = \lim\limits_{x\to0}\left[\dfrac{f(x)}{x^2}\cdot x^2\right]$

Now argument of G.I.F is tending towards zero and from positive side as $\lim\limits_{x\to0}\dfrac{f(x)}{x^2} = 2$ and $x^2 \to 0^+$

we have $\dfrac{f(x)}{x^2}\cdot x^2 \to 0^+ \quad \Rightarrow l = 0.$

(ii) We write $\lim\limits_{x\to0}\left[\dfrac{f(x)}{x}\right] = \lim\limits_{x\to0}\left[\dfrac{f(x)}{x^2}\cdot x\right]$

and assume $A = \dfrac{f(x)}{x^2}\cdot x$

Now $x \to 0^+ \;\Rightarrow A \to 0^+$

and $\;\; x \to 0^- \;\Rightarrow A \to 0^-$

$\Rightarrow \quad \lim\limits_{x\to0^+}[A] = 0$ and $\lim\limits_{x\to0^-}[A] = -1$

$\Rightarrow$ The given limit does not exist.

Example 15: Discuss the behaviour of

$$\phi(x) = \frac{a_0x^m + a_1x^{m+1} + ...a_kx^{m+k}}{b_0x^n + b_1x^{n+1} + ... + b_\ell x^{n+\ell}}$$

where $a_0 \neq 0, b_0 \neq 0$ as x tends to 0 by positive or negative values.

Solution: If m > n, $\lim\limits_{x\to0}\phi(x) = 0.$

If m = n, $\lim\limits_{x\to0}\phi(x) = a_0/b_0.$

If m < n and n – m is even, $\phi(x) \to \infty$ or $\phi(x) \to -\infty$ according as $a_0/b_0 > 0$ or $a_0/b_0 < 0$.

If m < n and n – m is odd, then $\phi(x) \to \infty$ as $x \to 0^+$ and $\phi(x) \to -\infty$ as $x \to 0^-$, or $\phi(x) \to -\infty$ as $x \to 0$ and $\phi(x) \to \infty$ as $x \to 0^-$, according as $a_0/b_0 > 0$ or $a_0/b_0 < 0$.

Concept Problems

E

1. Evaluate the following limits:

(i) $\lim\limits_{x\to2}\dfrac{x^3+3x^2-9x-2}{x^3-x-6}$ (ii) $\lim\limits_{x\to0^+}\dfrac{(x+1)^2}{(x-1)^2-1}$

(iii) $\lim\limits_{h\to0}\dfrac{(x+h)^2-x^2}{h}$ (iv) $\lim\limits_{t\to-1^+}\dfrac{\sqrt{1+t}}{1+t^5}$

2. Evaluate the following limits:

(i) $\lim\limits_{y\to1^+}\dfrac{|y-1|-y+1}{|y-1|+y-1}$

(ii) $\lim\limits_{x\to3^-}\dfrac{x^3-27}{|x-3|(x-3)}$

3. Let $f(x) = \dfrac{x^2+x-2}{x^2+2x-3}$ find

(a) $\lim\limits_{x\to-2}f(x)$ (b) $\lim\limits_{x\to1}f(x)$

(c) $\lim\limits_{x\to-3}f(x)$ (d) $\lim\limits_{x\to0}f(x)$

4. Evaluate $\underset{x\to1^-}{\text{limit}}\dfrac{[x^2]-1}{x^2-1}$ where [.] denotes the greatest integer function.

5. Evaluate $\lim\limits_{x\to5}\dfrac{x^2-9x+20}{x-[x]}$ where [.] denotes the greatest integer function.

6. Evaluate the following limits where [.] denotes the greatest integer function :

(i) $\lim\limits_{x\to2^+}\dfrac{x-[x]}{x-2}$ (ii) $\lim\limits_{x\to2^-}\dfrac{x-[x]}{x-2}$

(iii) $\lim\limits_{x\to1^+}\dfrac{[x^2]-[x]^2}{x^2-1}$ (iv) $\lim\limits_{x\to10^-}\dfrac{[x^3]-x^3}{[x]-x}$

Practice Problems

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7. Evaluate the following limits:

(i) $\lim\limits_{x\to1}\dfrac{x^4-3x+2}{x^5-4x+3}$

(ii) $\lim\limits_{x\to1}\dfrac{x^3-4x^2+5x-2}{(x^2-1)^2}$

(iii) $\lim\limits_{x\to1}\dfrac{x^7-2x^5+1}{x^3-3x^2+2}$

(iv) $\lim\limits_{x\to-3}\dfrac{x^4-6x^2-27}{x^3+3x^2+x+3}$

8. Evaluate the following limits :

(i) $\lim\limits_{t\to1}\dfrac{\sqrt{t}}{t+1}-\dfrac{t^2-t}{t^4-1}$

(ii) $\lim\limits_{x\to\pi/6}\dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

(iii) $\lim\limits_{x\to\pi/4}\dfrac{1-\cot^3 x}{2-\cot x-\cot^3 x}$

9. Find a number c so that $\lim\limits_{x\to3}\dfrac{x^3+cx^2+5x+12}{x^2-7x+12}$ exists. Also find the corresponding limit.

10. Evaluate the one-sided limits in the following :

(i) $\lim\limits_{x\to\sqrt{2}}\dfrac{x^2[x^2]}{(1+|x-\sqrt{2}|)}$

(ii) $\lim\limits_{x\to0}\dfrac{x(e^{[x]+|x|}-2)}{[x]+|x|}$

where [.] denotes the greatest integer function.

1.7 Rationalization

If in any limit, the denominator or numerator involves the radical sign then we can rationalize the irrational expression by multiplying with their conjugates to remove the indeterminacy.

Example 1: Evaluate $\lim\limits_{x\to1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$.

Solution: $\lim\limits_{x\to1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

$$=\lim_{x\to1}\frac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$$

$$=\lim_{x\to1}\frac{(15-5x)}{(3-3x)}\times\frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}}=\frac{5}{6}$$

Example 2: Evaluate the following limits:

(i) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(ii) $\lim\limits_{x\to1}\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$

Solution: (i) The given limit takes the form $\dfrac{0}{0}$ when $x\to0$. Rationalizing the numerator, we get

$$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

$$=\lim_{x\to0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$=\lim_{x\to0}\left[\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$=\lim_{x\to0}\left[\frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$=\lim_{x\to0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{2}{2}=1.$$

(ii) We have $\lim\limits_{x\to1}\left[\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}\right]$

$$=\lim_{x\to1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)(x-1)}\right]$$

$$=\lim_{x\to1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}\right]$$

$$=\lim_{x\to1}\left[\frac{2x-3}{(2x+3)\left(\sqrt{x}+1\right)}\right]$$

$$=\frac{-1}{(5)(2)}=\frac{-1}{10}.$$

Example 3: Evaluate $\lim\limits_{x\to1}\sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

Solution: $\lim\limits_{x\to1}\sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

$$=\sin^{-1}\left(\lim_{x\to1}\frac{1-\sqrt{x}}{1-x}\right)$$

$$=\sin^{-1}\left(\lim_{x\to1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)$$

$$=\sin^{-1}\left(\lim_{x\to1}\frac{1}{1+\sqrt{x}}\right)=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}.$$

Example 4: Evaluate $\lim\limits_{x\to2^+}\dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$

Solution: $\lim\limits_{x\to 2^+} \dfrac{x-2}{\sqrt{x^2-4}-\sqrt{x-2}}$

$$= \lim_{x\to 2^+} \frac{x-2}{\sqrt{x^2-4}-\sqrt{x-2}} \cdot \frac{(\sqrt{x^2-4}+\sqrt{x-2})}{(\sqrt{x^2-4}+\sqrt{x-2})}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(\sqrt{x^2-4}+\sqrt{x-2})}{(x^2-4)-(x-2)}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(\sqrt{x^2-4}+\sqrt{x-2})}{x^2-x-2}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(\sqrt{x^2-4}+\sqrt{x-2})}{(x-2)(x+1)}$$

$$= \lim_{x\to 2^+} \frac{\sqrt{x^2-4}+\sqrt{x-2}}{x+1} = 0.$$

Note:

$$(x^{\frac{1}{n}}-a^{\frac{1}{n}})(x^{\frac{n-1}{n}}+x^{\frac{n-2}{n}}a^{\frac{1}{n}}+x^{\frac{n-3}{n}}a^{\frac{2}{n}}+.....+a^{\frac{n-1}{n}})=(x-a)$$

$$(x^{\frac{1}{n}}+a^{\frac{1}{n}})(x^{\frac{n-1}{n}}-x^{\frac{n-2}{n}}a^{\frac{1}{n}}+x^{\frac{n-3}{n}}a^{\frac{2}{n}}-...)=(x+a) \text{ if } n \text{ is odd}$$

Example 5: Evaluate $\lim\limits_{x\to -1} \dfrac{\sqrt[3]{(7-x)}-2}{(x+1)}$

Solution: $\sqrt[3]{(7-x)} - 2$

$$= \frac{(7-x)-8}{(7-x)^{2/3}+(7-x)^{1/3}.2+4}$$

$$= \frac{(x+1)}{(x+1)^{2/3}+(x+1)^{1/3}.2+4} \qquad ...(1)$$

$$\therefore \lim_{x\to -1} \frac{\sqrt[3]{(7-x)}-2}{(x+1)}$$

$$= -\lim_{x\to -1} \frac{(x+1)}{(x+1)(7-x)^{2/3}+(7-x)^{1/3}.2+4}$$

[from (1)]

$$= \lim_{x\to -1} \frac{1}{(7-x)^{2/3}+(7-x)^{1/3}.2+4}$$

$$= \frac{1}{4+4+4} = -\frac{1}{12}.$$

Example 6: Calculate $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$

Solution: We substitute $\sqrt[3]{x} = t$

Then, for the variable t, the expression under the limit sign can be written in the form

$$\frac{t^2-2t+1}{(t^3-1)^2}$$

The number to which the new variable t tends, as $x \to 1$, can be found as the limit of the function

$$t(x) = \sqrt[3]{x} \text{ as } x \to 1,$$

i.e. $\lim\limits_{x\to 1} t(x) = \lim\limits_{x\to 1} \sqrt[3]{x} = 1$

Thus we have

$$\lim_{t\to 1} \frac{t^2-2t+1}{(t^3-1)^2} = \lim_{t\to 1} \frac{(t-1)^2}{(t-1)^2(t^2+t+1)^2}$$

$$= \lim_{t\to 1} \frac{1}{(t^2+t+1)^2} = \frac{1}{9}.$$

Example 7: Evaluate $\lim\limits_{x\to 2a^+} \dfrac{\sqrt{x}-\sqrt{2a}+\sqrt{x-2a}}{\sqrt{x^2-4a^2}}$

Solution: $\lim\limits_{x\to 2a^+} \dfrac{\sqrt{x}-\sqrt{2a}+\sqrt{x-2a}}{\sqrt{x^2-4a^2}} \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 2a^+} \frac{\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}} + \lim_{x\to 2a^+} \frac{\sqrt{x-2a}}{\sqrt{x^2-4a^2}}$$

$$= \lim_{x\to 2a^+} \frac{(\sqrt{x}-\sqrt{2a})(\sqrt{x}+\sqrt{2a})}{\sqrt{x^2-4a^2}(\sqrt{x}+\sqrt{2a})}$$

$$+ \lim_{x\to 2a^+} \frac{(\sqrt{x-2a})}{(\sqrt{x-2a})(\sqrt{x+2a})}$$

$$= \lim_{x\to 2a^+} \frac{x-2a}{\sqrt{(x-2a)(x+2a)}.2\sqrt{2a}} + \frac{1}{\sqrt{4a}}$$

$$= \lim_{x\to 2a^+} \frac{\sqrt{x-2a}}{\sqrt{x+2a}\;2\sqrt{2a}} + \frac{1}{2\sqrt{a}}$$

$$= 0 + \frac{1}{2\sqrt{a}} = \frac{1}{2\sqrt{a}}.$$

Concept Problems F

1. Calculate the value of the function

$$f(x) = \frac{x}{\sqrt{x+1}-1}$$

at several points near x = 0 and hence estimate the limit of f(x) as $x \to 0$.

2. Evaluate the following limits:

(i) $\lim\limits_{x\to0}\dfrac{x}{\sqrt{5-x}-\sqrt{5+x}}$

(ii) $\lim\limits_{x\to9}\dfrac{3-\sqrt{x}}{4-\sqrt{2x-2}}$

(iii) $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x^2}-\sqrt[3]{x}}{(x-1)}$

(iv) $\lim\limits_{x\to1}\dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1}$

3. Evaluate $\lim\limits_{x\to-1}\dfrac{\sqrt{4+x+x^2}-2}{x+1}$

4. Evaluate $\lim\limits_{x\to-1}\dfrac{x+1}{\sqrt{6x^2+3}+3x}$

5. Find numbers a and b so that

$\lim\limits_{x\to0}\dfrac{\sqrt{ax+b}-1}{x}=1$.

Practice Problems E

6. Evaluate $\lim\limits_{x\to3}\left[\log_a\dfrac{x-3}{\sqrt{x+6}-3}\right]$

7. Evaluate the following limits :

(i) $\lim\limits_{x\to2}\dfrac{\sqrt{1+x+x^2}-\sqrt{7+2x-x^2}}{x^2-2x}$

(ii) $\lim\limits_{x\to2}\dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$

(iii) $\lim\limits_{x\to-1}\dfrac{x+1}{\sqrt[4]{x+17}-2}$

(iv) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}$

8. Evaluate $\lim\limits_{x\to3}\dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$.

9. Evaluate $\lim\limits_{x\to2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

10. Evaluate the following limits :

(i) $\lim\limits_{x\to1}\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}$

(ii) $\lim\limits_{x\to0}\dfrac{\sqrt[3]{8+3x}-2}{\sqrt[4]{16+5x}-2}$

(iii) $\lim\limits_{x\to0}\dfrac{\sqrt[5]{(1+x)^3}-1}{(1+x)\sqrt[3]{(1+x)^2}-1}$.

1.8 Limit Using Expansion Series of Functions

In this method basically we use the series expansion of sin x, cos x, tan x, log(1+x), a^x, e^x, etc. to evaluate the limit. Following are some of the frequently used series expansions:

(i) $a^x=1+\dfrac{x\ln a}{1!}+\dfrac{x^2\ln^2 a}{2!}+\dfrac{x^3\ln^3 a}{3!}+.........a>0$

(ii) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+............$

(iii) $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+.........\text{for } -1<x\le1$

(iv) $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+.......$

(v) $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+......$

(vi) $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+........$

(vii) $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+.......$

(viii) $\sin^{-1}x=x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+.......$

(ix) $\sec x=1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+......$

(x) $(1+x)^n=1+\dfrac{nx}{\underline{|1}}+n(n-1)\dfrac{x^2}{\underline{|2}}+....\text{ for } -1<x<1,\ n\in Q.$

(xi) $(1+x)^{1/x}=e\left(1-\dfrac{1}{2}x+\dfrac{11}{24}x^2+.....\right)$

Note: Using the above expansions, we can find other expansion series. For example to find the expansion series of sin^{2}x, we write, $\sin^2 x=\dfrac{1-\cos 2x}{2}$ and use the expansion series of cos x with x replaced by 2x.

Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\ldots\ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots\ldots$$

Example 1: Expand sin x in powers of x.

Solution: Here $f(x) = \sin x$, Hence $f(0) = 0$,

$f'(x) = \cos x$, $f'(0) = 1$,

$f''(x) = -\sin x$, $f''(0) = 0$,

$f'''(x) = -\cos x$, $f'''(0) = -1$

....

$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right) \quad f^n(0) = \sin\frac{n\pi}{2}$$

Thus $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots. + \dfrac{x^n \sin\frac{n\pi}{2}}{n!} + \ldots$

Example 2: Expand ln(cos x) in powers of x.

Solution: Here $f(x) = \ln(\cos x)$,

$f'(x) = -\tan x = -t$, say,

$f''(x) = -\sec^2 x = -(1 + t^2)$,

$f'''(x) = -2\tan x \sec^2 x = -2t(1 + t^2)$,

$f^{(4)}(x) = -2(1 + 3t^2)(1 + t^2) = -2(1 + 4t^2 + 3t^4)$,

$f^{(5)}(x) = -2(8t + 12t^3)(1 + t^2)$

$= -2(8t + 20t^3 + 12t^5)$,

$f^{(6)}(x) = -2(8 + 60t^2 + 60t^4)(1 + t^2)$

$= -2(8 + 68t^2 + 120t^4 + 60t^6)$

Hence,

$f(0) = 0$, and

$f'(0) = f^{(3)}(0) = f^{(5)}(0) = \ldots = 0$, also

$f''(0) = -1$, $f^{(4)}(0) = -2$, $f^{(6)}(0) = -16$.

Hence $\ln(\cos x) = -\dfrac{x^2}{2!} - 2\dfrac{x^4}{4!} - 16\dfrac{x^6}{6!}\ldots\ldots\ldots\ldots$

Example 3: Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

Solution: $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!}\ldots\ldots\right) - 1 - x}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!}\ldots\ldots\ldots\ldots = \frac{1}{2}.$$

Example 4: Evaluate $\dfrac{\tan x - \sin x}{x^3}$

Solution: $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3}\ldots\ldots\ldots\right) - \left(x - \frac{x^3}{3!}\ldots\ldots\ldots\right)}{x^3}$$

$$\lim_{x \to 0} \frac{\left(\frac{1}{3} + \frac{1}{3!}\right)x^3 + \ldots\ldots\ldots}{x^3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

Example 5: Evaluate $\lim\limits_{x \to 0} \dfrac{e^x + \ln\left(\frac{1 - x}{e}\right)}{\tan x - x}$

Solution: $\lim\limits_{x \to 0} \dfrac{e^x + \ln\left(\frac{1 - x}{e}\right)}{\tan x - x}$

$$= \lim_{x \to 0} \frac{e^x + \ln(1 - x) - \ln e}{\tan x - x} \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1}{\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) - x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{6}x^3(1 + \text{terms containing x and its higher powers})}{\frac{1}{3}x^3(1 + \text{terms containing x and its higher powers})}$$

$$= -\frac{1}{2}.$$

Example 6: Evaluate $\lim\limits_{x \to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{x^2 \sin x}$

Solution: $\lim\limits_{x \to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{x^2 \sin x}$

$$= \lim_{x \to 0} \frac{5\left(x - \frac{x^3}{3!} + \ldots.\right) - 7\left(2x - \frac{(2x)^3}{3!} + \ldots.\right) + 3\left(3x - \frac{(3x)^3}{3!} + \ldots.\right)}{x^2\left(x - \frac{x^3}{3!} + \ldots.\right)}$$

$$= \lim_{x \to 0} \frac{-\frac{5x^3}{3!} + \frac{56x^3}{3!} - \frac{81x^3}{3!}}{x^3\left(1 - \frac{x^3}{3!} + \ldots\right)}$$

$$= \frac{-5 + 56 - 81}{3!} = -5.$$

Example 7: Evaluate $\lim\limits_{x \to 0} \dfrac{\ell n(1 + x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$

Solution: $\lim\limits_{x \to 0} \dfrac{\ell n(1 + x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} .....\right) - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) + \frac{x^2}{2}}{x^3 . \frac{\tan x}{x} . \frac{\sin x}{x}}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

Example 8: Evaluate $\lim\limits_{x\to 0} \dfrac{\cos x^3 - 1 + \ell n(1+x^6)}{x^2(e^{x^2} - 1 - x^2)}$

Solution: $\lim\limits_{x\to 0} \dfrac{\cos x^3 - 1 + \ell n(1+x^6)}{x^2(e^{x^2} - 1 - x^2)}$

$$= \lim_{x\to 0} \frac{1 - \frac{x^6}{\underline{\lfloor 2}} + ... - 1 + x^6 - \frac{x^{12}}{2}}{x^2\left(1 + \frac{x^2}{\underline{\lfloor 1}} + \frac{x^4}{\underline{\lfloor 2}} .... - 1 - x^2\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\underline{\lfloor 2}} + ...}{\left(\frac{1}{\underline{\lfloor 2}} + ...\right)} = 1.$$

Example 9: Find $\lim\limits_{x\to 0} \dfrac{e^{\sin x} - \sin x - 1}{x^2}$

Solution: $\lim\limits_{x\to 0} \dfrac{e^{\sin x} - \sin x - 1}{x^2}$

$$= \lim_{x\to 0} \frac{\left(1 + \frac{\sin x}{\underline{\lfloor 1}} + \frac{\sin^2 x}{\underline{\lfloor 2}} + \frac{\sin^3 x}{\underline{\lfloor 3}} + ...\right) - \sin x - 1}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \left(\frac{1}{\underline{\lfloor 2}} + \frac{\sin x}{\underline{\lfloor 3}} + ...\right) = \frac{1}{\underline{\lfloor 2}} = \frac{1}{2}.$$

Example 10: Evaluate $\lim\limits_{x\to 1} \dfrac{(7+x)^{1/3} - 2}{x - 1}$

Solution: Put x → 1 + h

$$\lim_{h\to 0} \frac{(8+h)^{1/3} - 2}{h}$$

$$= \lim_{h\to 0} \frac{2 . \left(1 + \frac{h}{8}\right)^{1/3} - 2}{h}$$

$$= \lim_{h\to 0} \frac{2\left\{1 + \frac{1}{3} . \frac{h}{8} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)\left(\frac{h}{8}\right)^2}{1 . 2} + ... - 1\right\}}{h}$$

$$= \lim_{h\to 0} 2 \times \frac{1}{24} = \frac{1}{12}.$$

Example 11: Find $\lim\limits_{x\to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

Solution: This is of the form $\dfrac{0}{0}$ if we put x = 1.

Therefore we put x = 1 + h and expand.

$$\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{h\to 0} \frac{(1+h)^7 - 2(1+h)^5 + 1}{(1+h)^3 - 3(1+h)^2 + 2}$$

$$= \lim_{h\to 0} \frac{(1 + 7h + 21h^2 + ...) - 2(1 + 5h + 19h^2 + ...) + 1}{(1 + 3h + 3h^2 + ...) - 3(1 + 2h + h^2) + 2}$$

$$= \lim_{h\to 0} \frac{-3h + h^2 + ...}{-3h + ...}$$

$$= \lim_{h\to 0} \frac{-3 + h + ...}{-3 + ...} = 1.$$

Example 12: If $\lim\limits_{x\to 0} \dfrac{a\cos x + bx\sin x - 5}{x^4}$ exists, find a, b and the limit.

Solution: As x → 0, x^4 → 0 the limit must be in $\left(\dfrac{0}{0}\right)$ form. Hence $\lim\limits_{x\to 0}$ acos x + bx sin x – 5 = 0.

⇒ a – 5 = 0 ⇒ a = 5.

$$\text{Limit} = \lim_{x\to 0} \frac{5\left(1 - \frac{x^2}{\underline{\lfloor 2}} + \frac{x^4}{\underline{\lfloor 2}} ....\right) + bx\left(x - \frac{x^3}{\underline{\lfloor 3}} ......\right) - 5}{x^4}$$

$$= \lim_{x\to 0} \frac{\left(b - \frac{5}{2}\right)x^2 + \left(\frac{5}{24} - \frac{b}{6}\right)x^4 + ...}{x^4}$$

For limit to exist, x^4 must cancell from the numerator. Hence we assume the coefficients of all powers of x

less than 4 to be zero ⇒ $b - \dfrac{5}{2} = 0$ ⇒ $b = \dfrac{5}{2}$.

$$\text{Now the limit} = \lim_{x\to 0} \frac{\left(\frac{5}{24} - \frac{b}{6}\right)x^4 + ...}{x^4} = \frac{5}{24} - \frac{b}{6}$$

$$\therefore \text{ Limit} = \frac{5}{24} - \frac{5}{12} = \frac{-5}{24}.$$

Example 13: If $\lim\limits_{x\to 0} \dfrac{Ae^x - B\cos x + Ce^{-x}}{x\sin x} = 2$, find A, B and C.

Solution: The given limit is equal to

$$\lim_{x\to 0} \frac{A\left(1 + x + \frac{x^2}{\underline{\lfloor 2}} ....\right) - B\left(1 - \frac{x^2}{\underline{\lfloor 2}} + ...\right) + C\left(1 - x + \frac{x^2}{\underline{\lfloor 2}} ....\right)}{x\left(x - \frac{x^3}{\underline{\lfloor 3}} ....\right)}$$

$$= \lim_{x\to 0} \frac{(A - B + C) + (A - C)x + \left(\frac{A + B + C}{2}\right)x^2}{x^2\left(1 - \frac{x^2}{\underline{\lfloor 3}} ......\right)}$$

For limit to exist, x^2 must cancel from the numerator. Hence we assume $A - B + C = 0$ ...(1)

and $A - C = 0$ ...(2)

$\therefore$ Limit $= \dfrac{A+B+C}{2} = 2$. ...(3)

On solving these three equations, we get A = 1, B = 2, C = 1.

Example 14: Show that

$$\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2} = \frac{11e}{24}.$$

Solution: Let $y = (1 + x)^{1/x}$

$$\therefore \quad \ln y = \frac{1}{x}\ln(1+x)$$

$$= \frac{1}{x}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - ...\right)$$

$$= 1 - \frac{1}{2}x + \frac{1}{3}x^2 - ....., $$

Now $y = e^{1-\frac{1}{2}x+\frac{1}{2}x^2-......}$

$$= e.\,e^{1-\frac{1}{2}x+\frac{1}{2}x^2-......}$$

$$= e\left\{1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 - ......\right) + ... + \frac{1}{2}\left(-\frac{1}{2}x + \frac{1}{3}x^2 - ......\right)^2 + ...\right\}$$

$$= e\left\{1 - \frac{1}{2}x + \left(\frac{1}{3} + \frac{1}{8}\right)x^2 + ...\right\}$$

$$= e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 + ...\right)$$

Now $\displaystyle\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$

$$= \lim_{x\to 0}\frac{e(1 - \frac{1}{2}x + \frac{11}{24}x^2 + ...) - e + \frac{1}{2}ex}{x^2} = \frac{11e}{24}.$$

Example 15: Let f(x) be a function such that $\displaystyle\lim_{x\to 0}\frac{f(x)}{x} = 1$. Find the values of a and b such that

$$\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{\{f(x)\}^3} = 1.$$

Solution: Since, $\displaystyle\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{\{f(x)\}^3} = 1$

$$\Rightarrow \lim_{x\to 0}\frac{x\left(1 + a\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right) - b\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....\right)}{\{f(x)\}^3} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{x(1 + a - b) + x^3\left(\frac{a}{2!} + \frac{b}{3!}\right) + x^5\left(\frac{a}{4!} - \frac{b}{5!}\right) + ...}{\{f(x)\}^3} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{(1 + a - b)}{x^2} + \left(\frac{a}{2!} + \frac{b}{3!}\right) + x^2\left(\frac{a}{4!} - \frac{b}{5!}\right) + ...}{\left\{\frac{f(x)}{x}\right\}^3} = 1$$

Since the limit exists we must have 1 + a –b = 0 and $-\dfrac{a}{2!} + \dfrac{b}{3!} = 1 \Rightarrow \quad -3a + b = 6$

Solving these, we get a = –5/2 and b = –3/2.

Concept Problems

G

1. Evaluate the following limits :

(i) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+x} - 1 - \frac{1}{3}x}{x^2}$

(ii) $\displaystyle\lim_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2 + x\ln(1-x)}$

(iii) $\displaystyle\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^3}$

(iv) $\displaystyle\lim_{x\to 0}\frac{2x - \ln(1+2x)}{x^2}$

2. Evaluate the following limits :

(i) $\displaystyle\lim_{x\to 0}\frac{2x - \sin^{-1}x}{2x + \tan^{-1}x}$

(ii) $\displaystyle\lim_{x\to 0}\frac{3\sin x - \sin 3x}{x - \sin x}$

(iii) $\displaystyle\lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3}$

(iv) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+3x} - \sqrt{1+2x}}{x^2}$

3. Evaluate $\displaystyle\lim_{x\to 0}\frac{3\tan x - 3x - x^3}{x^5}$

4. Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x - \tan^{-1}x}{x^3}$

5. Evaluate $\displaystyle\lim_{x\to 0}\frac{(2-x)(e^x - x - 2)}{x^3}$

6. For what values of constants C and D is it true that

$$\lim_{x\to 0}(x^{-3}\sin 7x + Cx^{-2} + D) = -2$$

Practice Problems F

7. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x)}}{\sqrt{(1+x^3)}-\sqrt{(1+x)}}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-x^2}}{x+x^2}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-1}{(1+x)^{50}-1-50x}$

(iv) $\lim\limits_{x\to 0}\dfrac{\ln^2(1+x)-x^2}{x^3}$

8. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+ax}-\sqrt[n]{1+bx}}{x}$ (m, n ∈ N)

(ii) $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1}{x}$ (m, n ∈ N)

(iii) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+2x}-\sqrt[4]{1+9x}}{1-\sqrt{1-\dfrac{x}{2}}}$

(iv) $\lim\limits_{x\to 0}\dfrac{4\sqrt[4]{1-x^2}-4e^{x^2}}{\tan^{-1}x-x}$

9. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)-\dfrac{x^3}{2}+\dfrac{x^5}{8}}{x^7}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin x-x+x^3/6-x^5/120}{(\sin x)^7}$

(iii) $\lim\limits_{x\to 0}\dfrac{e^{x\cos x}-1-x}{\sin(x^2)}$

(iv) $\lim\limits_{x\to 0}\dfrac{x\cos^3 x-\ln(1+x)-\sin^{-1}\dfrac{x^2}{2}}{x^3}$

10. Evaluate $\lim\limits_{x\to 0}\dfrac{2\sin x+\dfrac{1}{2}\ln\dfrac{1+x}{1-x}-3x}{x^5}$

11. Evaluate $\lim\limits_{x\to 0}\dfrac{x^3e^{\frac{x^4}{4}}-\sin^{\frac{3}{2}}x^2}{x^7}$

12. Find the values of a and b so that

$\lim\limits_{x\to 0}\dfrac{(1+ax\sin x)-b\cos x}{x^4}$

may tend to a definite limit and also find limit.

13. For what values of the constants a, b is ,

$\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{x^3}+\dfrac{a}{x^2}+b\right)=0$?

14. If $\lim\limits_{x\to 0}\dfrac{1}{x^3}\left(\dfrac{1}{\sqrt{1+x}}-\dfrac{1+ax}{1+bx}\right)$ exists and has the value equal to ℓ, then find the value of $\dfrac{1}{a}-\dfrac{2}{\ell}+\dfrac{3}{b}$.

1.9 Standard Limits

Following are some basic limits which are used frequently in solving the limits.

(i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1=\lim\limits_{x\to 0}\dfrac{\tan x}{x}$

$=\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}$

[where x is measured in radians]

Also, $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$=\lim\limits_{x\to 0}\dfrac{(\sin^2 x)}{x^2(1+\cos x)}=\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^2\dfrac{1}{(1+\cos x)}$

$=\dfrac{1}{2}$.

(ii) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$ (a > 0).

In particular $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

(iii) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$

$\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e$, (a > 0, a ≠ 1)

(iv) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}=n$

$\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$.

Study Tip

(i) These limits help in finding limits in 0/0 form and most of them are based on x approaching 0. In case, if x approaches a, we use the substitution x = a + t so that t approaches 0 when x approaches a.

(ii) These formulae are also applicable when a function, say f(x) has been replaced for x. But it must be ensured that as x approaches a, f(x) must approach 0.

For instance, $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)} = 1$, if f(x) approaches 0 as x approaches a.

Caution

But $\lim\limits_{x\to a}\dfrac{\sin f(x)}{g(x)}$ where f(x) and g(x) approach 0 as x approaches a, should not be immediately taken as 1.

For example, $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} = \lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}\,.\,2 = 2$

Further, $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)}$ should not be taken as 1 when sin f(x) approaches 0 as x approaches a, but f(x) itself does not approach 0.

For example, $\lim\limits_{x\to 0}\dfrac{\sin(\pi - x)}{\pi - x} \neq 1$, since $\pi - x$ does not approach 0 as x approaches 0. In fact, the limit is equal to 0.

Why Radian Measure Is Used

Throughout calculus, angles are measured in radians, as is customary in calculus. If we measured angles in degrees instead, the formulas for the limits of the trigonometric functions would be more complicated. Each formula would have an extra factor, $\pi/180$, as we will now show.

Earlier, it was shown that when angles are measured in radians,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

When angles are measured in degrees, this limit is not 1. Let sin θ denote the sine of an angle of θ degrees. The following table suggests that the limit is much smaller (angles measured in degrees; data to four significant figures):

θ	10	5	1	0.1
sin θ	0.1736	0.08716	0.01745	0.001745
$\dfrac{\sin\theta}{\theta}$	0.01736	0.01743	0.01745	0.01745

The data suggest that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ is about 0.01745. We can find that the limit is precisely $\pi/180$.

Example 1: $\lim\limits_{x\to 0}\dfrac{\tan x^\circ}{x}$

Solution: $x^\circ = \dfrac{\pi}{180}x$ radian

Now $\lim\limits_{x\to 0}\dfrac{\tan x^\circ}{x} = \lim\limits_{x\to 0}\dfrac{\tan\dfrac{\pi x}{180}}{x}$

$$= \lim_{x\to 0}\frac{\dfrac{\tan\dfrac{\pi x}{180}}{\dfrac{\pi x}{180}}\cdot\dfrac{\pi x}{180}}{x} = \frac{\pi}{180}.$$

Example 2: Evaluate $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x/2}$

Solution: $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x/2} = \lim\limits_{x\to 0}\ 2\times 3\ \dfrac{e^{3x}-1}{3x} = -\,6.$

Example 3: Compute $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

Solution: We have $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

$$= \lim_{x\to 0}\left[\frac{\sin 2x}{2x}\cdot\frac{2x}{3x}\cdot\frac{3x}{\sin 3x}\right]$$

$$= \left[\lim_{2x\to 0}\frac{\sin 2x}{2x}\right]\cdot\frac{2}{3}\cdot\left[\lim_{3x\to 0}\frac{3x}{\sin 2x}\right],\ x\neq 0$$

$$= 1\,.\,\frac{2}{3}\,.\left[\lim_{3x\to 0}\frac{\sin 3x}{3x}\right]^{-1} = \frac{2}{3}\times 1 = \frac{2}{3}.$$

Example 4: Evaluate $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}$

Solution: The given expression is of the form

$$\frac{x^3-(2)^3}{x^2-(2)^2} = \frac{x^3-(2)^3}{x-2}\Big/\frac{x^2-(2)^2}{x-2}$$

$$\Rightarrow\quad \lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{x^3-(2)^3}{x-2}\Big/\lim_{x\to 2}\frac{x^2-(2)^2}{x-2}$$

$= 3(2^2)/(2(2^1))$ (using $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$)

$= 12/4 = 3.$

Example 5: Find $\lim\limits_{x\to -1}\dfrac{1+x^{1/3}}{1+x^{1/5}}$.

Solution: We have $\lim\limits_{x\to -1}\dfrac{\left(x^{1/5}\right)^5-(-1)}{x^{1/5}-(-1)} = 5(-1)^4 = 5$

and $\lim\limits_{x\to 1}\dfrac{(x^{1/3})^3-(-1)}{x^{1/3}-(-1)} = 3\,(-1)^2 = 3$

Hence the limit $= \lim\limits_{x\to -1}\dfrac{x^{1/3}-(-1)}{x^{1/5}-(-1)} = \dfrac{5}{3}.$

Example 6: Evaluate

$$\lim_{h\to 0}\frac{a^{x+h}+a^{x-h}-2a^x}{h^2},\qquad a>0$$

Solution: Limit $= \lim\limits_{h\to 0} a^x\left[\dfrac{a^h+a^{-h}-2}{h^2}\right]$

$$= \lim_{h\to 0} a^x \left[\frac{a^{2h} - 2a^h + 1}{a^h h^2}\right] = a^x \left(\frac{a^2 - 1}{h}\right)^2$$

$$= a^x \ln^2 a.$$

Example 7: Evaluate $\lim_{x\to a} \frac{a^x - a^a}{x - a}$, $a > 0$.

Solution: Put x = a + h

$$\text{Limit} = \lim_{h\to 0} \frac{a^{a+h} - a^a}{h} = \lim_{h\to 0} \frac{a^a\,(a^h - 1)}{h} = a^a \ln a$$

Example 8: Evaluate $\lim_{x\to 0} \left[\frac{\sin x}{x}\right]$.

Solution: Since, $0 < \frac{\sin x}{x} < 1$, $\left[\frac{\sin x}{x}\right] = 0$ in the neighbourhood of x = 0.

Hence $\lim_{x\to 0} \left[\frac{\sin x}{x}\right] = 0$.

With a similar reason, $\lim_{x\to 0} \left[\frac{\tan^{-1} x}{x}\right] = 0$.

Example 9: Evaluate $\lim_{x\to 0} \left[\frac{\tan x}{x}\right]$.

Solution: Since, $1 < \frac{\tan x}{x} < 2$ in the neighbourhood of x = 0, $\left[\frac{\tan x}{x}\right] = 1$. Hence, $\lim_{x\to 0} \left[\frac{\tan x}{x}\right] = 1$.

With a similar reason, $\lim_{x\to 0} \left[\frac{\sin^{-1} x}{x}\right] = 1$.

Example 10: Evaluate $\lim_{x\to 0} \left[\frac{-2x}{\tan x}\right]$, where [.] denotes greatest integer function.

Solution: We know when $x \to 0$

$$\Rightarrow \frac{x}{\tan x} < 1 \Rightarrow \frac{-x}{\tan x} > -1$$

$$\Rightarrow -2 < \frac{-2x}{\tan x} < -1$$

So, $\lim_{x\to 0} \left[\frac{-2x}{\tan x}\right] = -2$.

Example 11: Evaluate $\lim_{x\to e} \frac{\ln x - 1}{x - e}$.

Solution: Limit $= \lim_{x\to e} \frac{\ln x/e}{e\left(\frac{x}{e} - 1\right)}$

$$= \lim_{t\to 1} \frac{\ln t}{(t-1)\,e} \quad \text{(putting } x/e = t\text{)}$$

Put $t = 1 + y$ $= \lim_{y\to 0} \frac{\ln(1+y)}{ey} = \frac{1}{e}$

Example 12: Evaluate $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

Solution: $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

$$= \lim_{x\to 0} \frac{\tan x(1 - \cos x)}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan x \,.\, 2\sin^2 \frac{x}{2}}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \,.\, \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.$$

Example 13: Evaluate $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$

Solution: Here $\pi\cos^2 x \to \pi$ as $x \to 0$.

We change $\pi\cos^2 x$ to $\pi\sin^2 x$, so that it tends to 0.

$$\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin\{\pi(1 - \sin^2 x)\}}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin(\pi - \pi\sin^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to 0} \left\{\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{\pi}{1} \times \frac{\sin^2 x}{x^2}\right\}$$

$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \pi \times \lim_{x\to 0} \frac{\sin^2 x}{x^2}$$

$$= 1 \times \pi \times 1 = \pi.$$

Example 14: Evaluate $\lim_{x\to 1} \frac{x\sin\{x\}}{x - 1}$, where {x} denotes the fractional part of x.

Solution: As $x \to 1^-$, {x} can be replaced by 1– h, where h is a small positive quantity.

As $x \to 1^+$, {x} can be replaced by h, where h is a small positive quantity.

$$\therefore \quad \lim_{x\to 1^-} \frac{x\ \sin\{x\}}{x - 1} = \lim_{h\to 0^+} \frac{1 - h}{-h} \sin(1 - h) = -\infty$$

$$\lim_{x\to 1^+} \frac{x\ \sin\{x\}}{x - 1} = \lim_{h\to 0^+} (1 + h)\frac{\sin h}{h}$$

$$= 1 \times 1 = 1$$

Since, L.H.L. ≠ R. H.L. the limit does not exist.

Example 15: Evaluate $\lim_{x\to a} \frac{\cos x\ \ln(x - a)}{\ln(e^x - e^a)}$

Solution: $\lim_{x\to a} \frac{\cos x\ \ln(x - a)}{\ln(e^x - e^a)}$ Put x = a + h

$$= \cos a \cdot \lim_{h\to 0} \frac{\ln h}{\ln(e^{a+h}-e^{a})}$$

$$= \cos a \cdot \lim_{h\to 0} \frac{\ln h}{\ln\left(e^{a}\cdot\left(\frac{e^{h}-1}{h}\right)\cdot h\right)}$$

$$= \cos a \cdot \lim_{h\to 0} \frac{\ln h}{a+\ln\left(\frac{e^{h}-1}{h}\right)+\ln h}$$

$$= \cos a \cdot \lim_{h\to 0} \frac{1}{\frac{a}{\ln h}+\frac{\ln(e^{h}/h)}{\ln h}+1} = \cos a.$$

Example 16: Evaluate $\lim_{x\to 0}\frac{\sqrt{1-\cos 2x}}{x}$

Solution: $f(x) = \frac{\sqrt{1-\cos 2x}}{x}$

$$= \frac{\sqrt{2\sin^{2}x}}{x} = \frac{\sqrt{2}\,|\sin x|}{x},$$

$$\text{but } |\sin x| = \begin{cases} \sin x, & \text{if } 0<x<\pi/2, \\ -\sin x, & \text{if } -\pi/2<x<0 \end{cases}$$

Hence $f(0^{-}) = \lim_{x\to 0^{-}} f(x) = \lim_{x\to 0^{-}}\left(-\sqrt{2}\frac{\sin x}{x}\right) = -\sqrt{2}$

and $f(0^{+}) = \lim_{x\to 0^{+}} f(x) = \lim_{x\to 0^{+}}\left(\sqrt{2}\frac{\sin x}{x}\right) = \sqrt{2}$.

Example 17: Evaluate $\lim_{x\to 0}\frac{\ell n\cos x}{\sqrt[4]{1+x^{2}}-1}$.

Solution: Limit $= \lim_{x\to 0}\frac{\ell n(1+\cos x-1)(\cos x-1)}{(\cos x-1)\frac{\{(1+x^{2})^{1/4}-1\}}{x^{2}}x^{2}}$

$$= \frac{1}{\frac{1}{4}}\left(-\frac{1}{2}\right) = -2.$$

Example 18: Evaluate $\lim_{t\to 0}\frac{\ell n\cos(\sin t)}{t^{2}}$

Solution: $\lim_{t\to 0}\frac{\ell n\cos(\sin t)}{t^{2}}$

$$= \lim_{t\to 0}\frac{\ell n\left(1-2\sin^{2}\frac{(\sin t)}{2}\right)}{-2\sin^{2}\left(\frac{\sin t}{2}\right)}\cdot\frac{\left(-2\sin^{2}\left(\frac{\sin t}{2}\right)\right)}{t^{2}}$$

$$= \lim_{t\to 0} -2.\left(\frac{\sin\left(\frac{\sin t}{2}\right)}{\frac{\sin t}{2}}\right)^{2}.\frac{\sin^{2}t}{4.t^{2}} = -\frac{1}{2}.$$

Example 19: Find the limiting value of

$\frac{\tan 2x-2\sin x}{x^{3}}$ as x tends to zero.

Solution: $\lim_{x\to 0}\frac{\sin 2x-2\sin x\cos 2x}{x^{3}\cos 2x}$

$$= \lim_{x\to 0}\frac{2\sin x[\cos x-\cos 2x]}{x^{3}.1}$$

$$= 2\lim_{x\to 0}\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{x^{2}}$$

$$= 2\lim_{x\to 0}\frac{3\sin\frac{3x}{2}}{2\frac{3x}{2}}.\frac{\sin\frac{x}{2}}{\frac{x}{2}} = 3.$$

Example 20: Let a = min [$x^{2}+2x+3$, $x \in R$] and $b = \lim_{x\to 0}\frac{\sin x\cos x}{e^{x}-e^{-x}}$, then find the value of $\sum_{r=0}^{n}a^{r}b^{n-r}$.

Solution: a = minimum value of $(x+1)^{2}+2$

$\Rightarrow$ a = 2

$$b = \lim_{x\to 0}\frac{\sin 2x}{\frac{2(e^{2x}-1).2x}{2x}} = \frac{1}{2}.$$

$$\text{Now } \sum_{r=0}^{n}a^{r}b^{n-r} = \sum 2^{r}\left(\frac{1}{2}\right)^{n-r} = \frac{1}{2^{n}}\sum_{r=0}^{n}2^{2r}$$

$$= \frac{1}{2^{n}}\sum_{r=0}^{n}4^{r} = \frac{1}{2^{n}}[1+4+4^{2}+......+4^{n}]$$

$$= \frac{1}{2^{n}}\left[\frac{4^{n+1}-1}{3}\right] = \frac{4^{n+1}-1}{3.2^{n}}.$$

Example 21: Evaluate $\lim_{x\to 0}\frac{e^{x}-\ln(x+e)}{e^{x}-1}$.

Solution: $\lim_{x\to 0}\frac{(e^{x}-\ln\left(e\left(1+\frac{x}{e}\right)\right)}{x\left(\frac{e^{x}-1}{x}\right)}$

$$= \lim_{x\to 0}\left[\frac{(e^{x}-1)}{x}-\frac{\ln(1+(x/e))}{x}\right]$$

$$= 1 - \lim_{x\to 0} \frac{\ln\left(1+(x/e)\right)}{e\cdot\frac{x}{e}}$$

$$= 1 - \frac{1}{e}.$$

Example 22: Evaluate $\lim\limits_{x\to0}\left(\frac{\ell n(\cos 3x)}{x^2}\cdot\frac{2\sin x}{e^x-e^{-x}}\right)$.

Solution:

$$\lim_{x\to0}\frac{\ell n(\cos 3x).2\sin x}{x^2(e^x-e^{-x})} = \lim_{x\to0}\frac{\ell n(\cos3x).2\sin x.e^x}{x^2(e^{2x}-1)}$$

$$=\lim_{x\to0}\left(\frac{\ell n(1+\cos3x-1)}{(\cos3x-1)}\times(\cos3x-1)\times\frac{2\sin x}{x}\times\frac{1}{\left(\frac{e^{2x}-1}{2x}\right)2x^2}\right)$$

$$=\lim_{x\to0}\left(\frac{\ell n(1+\cos3x-1)}{(\cos3x-1)}\times\frac{\left(-2\sin^2\frac{3x}{2}\right)}{2x^2}\times\frac{2\sin x}{x}\times\frac{1}{\left(\frac{e^{2x}-1}{x}\right)}\right)$$

$$= 1 \times (-1) \times \frac{9}{4} \times 2 \times 1 = -\frac{9}{2}.$$

Example 23: Evaluate $\lim\limits_{x\to1^-}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}$.

Solution: Put $\cos^{-1} x = y$ and $x \to 1^- \Rightarrow y \to 0^+$

$$\lim_{x\to1^-}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2} = \lim_{y\to0}\frac{1-\sqrt{\cos y}}{y^2}$$

Now rationalizing the numerator, we get

$$=\lim_{y\to0}\frac{(1-\cos y)}{y^2(1+\sqrt{\cos y})}$$

$$=\lim_{y\to0}\frac{1-\cos y}{y^2}.\lim_{y\to0}\frac{1}{1+\sqrt{\cos y}} = \frac{1}{2}.\frac{1}{2}=\frac{1}{4}.$$

Example 24: Evaluate $\lim\limits_{x\to-1^+}\frac{\sin^{-1}\left(\sqrt{\pi}-\sqrt{\cos^{-1}x}\right)}{\sqrt{1-x^2}}$.

Solution:

$$\lim_{x\to-1^+}\frac{\sin^{-1}\left(\sqrt{\pi}-\sqrt{\cos^{-1}x}\right)\left(\sqrt{\pi}-\sqrt{\cos^{-1}x}\right)}{\left(\sqrt{\pi}-\sqrt{\cos^{-1}x}\right)\sqrt{1-x^2}}$$

$$=\lim_{x\to-1^+}\frac{\left(\sqrt{\pi}-\sqrt{\cos^{-1}x}\right)}{\sqrt{1-x^2}}\cdot\frac{\left(\sqrt{\pi}+\sqrt{\cos^{-1}x}\right)}{\left(\sqrt{\pi}+\sqrt{\cos^{-1}x}\right)}$$

$$=\lim_{x\to-1^+}\frac{1}{2\sqrt{\pi}}\frac{\left(\pi-\cos^{-1}x\right)}{\sqrt{1-x^2}}$$

$$=\lim_{x\to-1^+}\frac{1}{2\sqrt{\pi}}\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}$$

put $\cos^{-1}(-x) = \theta \Rightarrow x = -\cos\theta$

$$=\frac{1}{2\sqrt{\pi}}\lim_{\theta\to0^+}\frac{\theta}{|\sin\theta|}=\frac{1}{2\sqrt{\pi}}\lim_{\theta\to0^+}\frac{\theta}{\sin\theta}$$

$$=\frac{1}{2\sqrt{\pi}}.$$

Example 25: Solve $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$

Solution: Here, RHL

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x-\sin(x-\pi)}{\tan x+\cos^2(\tan x)}}$$

$\{\because\ \tan^{-1}(\tan x) = x-\pi, \text{ when } x > \frac{\pi}{2}\}$

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{1+\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1+0}{1+0}} = 1.$$

Now, LHL

$$=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x-\sin(x)}{\tan x+\cos^2(\tan x)}}$$

$\left\{\text{as } \tan^{-1}(\tan x) = x, \text{ when } x < \frac{\pi}{2}\right\}$

$$=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{1-\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}} = 1.$$

Example 26: Evaluate $\lim\limits_{h\to0}\dfrac{(x+h)^{\sqrt{x+h}}-x^{\sqrt{x}}}{h}$ $(x > 0)$.

Solution: $\lim\limits_{h\to0}\dfrac{e^{\sqrt{x+h}\,\ell n\,(x+h)}-e^{\sqrt{x}\,\ell n\,x}}{h}$

$$= \lim_{h\to0} e^{\sqrt{x}\,\ell n\,x}\left[\frac{e^{\sqrt{x+h}\,\ell n\,(x+h)-\sqrt{x}\,\ell n\,x}-1}{h}\right]$$

$$= x^{\sqrt{x}}\left[\lim_{s\to0}\frac{e^s-1}{s}\;\lim_{h\to0}\frac{\sqrt{x+h}\;\ell n\,(x+h)-\sqrt{x}\;\ell n\,x}{h}\right]$$

where $s = \sqrt{x+h}\;\ell n\,(x+h)-\sqrt{x}\;\ell n\,x$

$$= x^{\sqrt{x}}\,.\lim_{h\to0}\frac{\sqrt{x+h}\left(\ell nx+\ell n\left(1+\frac{h}{x}\right)\right)-\sqrt{x}\;\ell nx}{h}$$

$$= x^{\sqrt{x}}\left[\lim_{h\to0}\ell n\,x\,\frac{(\sqrt{x+h}-\sqrt{x})}{h}\right.$$

$$\left.+\lim_{h\to0}\frac{\sqrt{(x+h)}}{x}\;\ell n\left(1+\frac{h}{x}\right)^{x/h}\right]$$

$$= x^{\sqrt{x}}\left[\frac{\ell n\,x}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\right].$$

Example 27: Evaluate $\lim\limits_{x\to0}\dfrac{\sqrt{1-\sqrt{\cos x}}}{x}$

Solution: $\lim\limits_{x\to0}\dfrac{\sqrt{1-\sqrt{\cos x}}}{x}$

$$= \lim_{x\to0}\frac{\sqrt{1-\cos x}}{x}.\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

$$= \lim_{x\to0}\frac{\sqrt{2}\,|\sin(x/2)|}{2(x/2)}.\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

Now, we have

$$\text{LHL} = \lim_{x\to0^-}\frac{-1}{\sqrt{2}}.\frac{\sin(x/2)}{x/2}.\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

$$= \frac{-1}{\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{-1}{2}.$$

and $\text{RHL} = \lim\limits_{x\to0^+}\dfrac{-1}{\sqrt{2}}.\dfrac{\sin(x/2)}{x/2}.\dfrac{1}{\sqrt{1+\sqrt{\cos x}}}$

$$= \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{1}{2}.$$

Hence, limit does not exist.

Use of Substitution

Sometimes in solving limit problem we convert $\lim\limits_{x\to a} f(x)$ by substituting $x = a + h$ or $x = a - h$ as $\lim\limits_{h\to0} f(a + h)$ or $\lim\limits_{h\to0} f(a - h)$ according to the need of the problem.

Example 28: Compute $\lim\limits_{x\to3}\dfrac{e^x-e^3}{x-3}$

Solution: Put $y = x - 3$. So, as $x \to 3$, $y \to 0$.

Thus $\lim\limits_{x\to3}\dfrac{e^x-e^3}{x-3} = \lim\limits_{y\to0}\dfrac{e^{3+y}-e^3}{y}$

$$= \lim_{y\to0}\frac{e^3\,.\,e^y-e^3}{y}$$

$$= e^3\lim_{y\to0}\frac{e^y-1}{y} = e^3\;.\;1 = e^3\,.$$

Example 29: Find $\lim\limits_{x\to\pi/4}\dfrac{\ln\tan x}{1-\cot x}$

Solution: Put $x = t + \pi/4$

$$\lim_{t\to0}\frac{\ln\tan(t+\pi/4)}{1-\cot(t+\pi/4)}$$

$$= \lim_{t\to0}\frac{\ln\left[\frac{1+\tan t}{1-\tan t}\right]}{1-\left[\frac{\cot t-1}{\cot t+1}\right]}$$

$$= \lim_{t\to0}\frac{\ln(1+\tan t)}{\frac{2\tan t}{1+\tan t}}+\lim_{t\to0}\frac{\ln(1-\tan t)}{\frac{-2\tan t}{1+\tan t}}$$

$$= \frac{1}{2}\,[1.1 + 1.1] = \frac{2}{2} = 1.$$

Example 30: Evaluate $\lim\limits_{x\to\pi/3}\dfrac{\cos\left(x+\frac{\pi}{6}\right)}{(1-2\cos x)^{2/3}}$

Solution: Putting $x-\dfrac{\pi}{3} = z$

$$= \lim_{z\to0}\frac{\cos\left(\frac{\pi}{2}+z\right)}{(1-2\cos z+\sqrt{3}\sin z)^{2/3}}$$

$$= \lim_{z\to0}\frac{-\sin z}{(1-\cos z+\sqrt{3}\sin z)^{2/3}}$$

$$= \lim_{z\to 0} \frac{-2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)}{\left[2\sin\left(\frac{z}{2}\right)\right]^{2/3}\left[\sin\left(\frac{z}{2}\right)+\sqrt{3}\cos\left(\frac{z}{2}\right)\right]^{2/3}}$$

$$= \lim_{z\to 0} \frac{-2^{1/3}\left[\sin\left(\frac{z}{2}\right)\right]^{1/3}\cos\left(\frac{z}{2}\right)}{\left[2\sin\left(\frac{z}{2}\right)+\sqrt{3}\cos\left(\frac{z}{2}\right)\right]^{2/3}} = \frac{-2^{1/3}.0.1}{\left(\sqrt{3}\right)^{2/3}} = 0 .$$

Example 31: Evaluate $\lim_{\theta\to\pi/4} \frac{(1-\tan\theta)}{(1-\sqrt{2}\sin\theta)}$

Solution: Let $P = \lim_{x\to 0} \frac{(1-\tan\theta)}{(1-\sqrt{2}\sin\theta)}$ $\left(\text{form } \frac{0}{0}\right)$

Put $\theta = \frac{\pi}{4} + h$

$$\therefore \quad P = \lim_{h\to 0} \frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)}$$

$$= \lim_{h\to 0} \frac{1-\left(\frac{1+\tan h}{1-\tan h}\right)}{1-\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos h+\frac{1}{\sqrt{2}}\sin h\right)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{-2\tan h}{(1-\tan h)\left(\frac{(1-\cos h)(1+\cos h)}{(1+\cos h)}-\sin h\right)}$$

$$= -2 \lim_{h\to 0} \frac{\tan h}{(1-\tan h)\left(\frac{\sin^2 h}{1+\cos h}-\sin h\right)}$$

$$= -2 \lim_{h\to 0} \frac{\sin h}{(1-\cos h-\sin h)\left(\frac{\sin^2 h}{1+\cos h}-\sin h\right)}$$

$$= -2 \lim_{h\to 0} \frac{1}{(\cos h-\sin h)\left(\frac{\sin h}{1+\cos h}-1\right)}$$

$$= -2 . \frac{1}{(1-0)(0-1)} = 2.$$

Example 32: Let $\lim_{x\to a} \frac{x^x - a^x}{x-a} = l$, $a > 0$ and $\lim_{x\to a} \frac{a^x - x^a}{x-a} = m$, $a > 0$. If $l = m$ then find the value of 'a'.

Solution: $l = \lim_{x\to a} \frac{e^{x\ell n\,x} - e^{x\ell n\,a}}{x-a}$

$$= \lim_{x\to a} \frac{e^{x\ell n\,a}\left[e^{x(\ell n\,x-\ell n\,a)}-1\right]}{x(\ell n\,x-\ell n\,a)} . \frac{x(\ell n\,x-\ell n\,a)}{x-a}$$

$$= a^a . \lim_{h\to 0} \frac{(a+h)\,ln\left(1+\frac{h}{a}\right)}{h} = a^a.$$

$$m = \lim_{x\to a} \frac{e^{x\ell n\,a} - e^{a\ell n\,x}}{x-a}$$

$$= \lim_{x\to a} \frac{e^{a\ell n\,x}\left[e^{x\ell n\,a-a\ell n\,x}-1\right]}{x\ell n\,a-a\ell n\,x} . \frac{x\ell n\,a-a\ell n\,x}{x-a}$$

$$= a^a . \lim_{h\to 0} \frac{(a+h)\,\ell n\,a - a\ell n\,(a+h)}{h}$$

$$= a^a \lim_{h\to 0} \left[\ell n\,a - \ell n\left(1+\frac{h}{a}\right)^{a/h}\right] = a^a\ (ln\,a - 1)$$

Now $l = m \Rightarrow a = e^2$

Example 33: Let

$f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$, then find $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$, where $\{x\}$ denotes the fractional part of x.

Solution: We have f(x)

$$= \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$$

$$\therefore \quad \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-\{0+h\}).\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}.(1-\{0+h\})}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h).\cos^{-1}(1-h)}{\sqrt{2h}.(1-h)}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} . \lim_{h\to 0} \frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

(In second limit put $\cos^{-1}(1-h) = \theta$

$\Rightarrow 1 - h = \cos\theta$)

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} . \lim_{\theta\to 0} \frac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} . \lim_{\theta\to 0} \frac{\theta}{2\sin(\theta/2)} \quad (\because\ \theta > 0)$$

$= \sin^{-1} 1 . 1 = \pi/2$

and $\lim_{x\to -0} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-\{0-h\}).\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}.(1-\{0-h\})}}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1+h-1).\cos^{-1}(1+h+1)}{\sqrt{2(-h+1).(1+h-1)}}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}h}{h}\ .\ \lim_{h\to 0}\frac{\cos^{-1}h}{\sqrt{2(1-h)}}$$

$$= 1\ .\ \frac{\pi/2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

Example 34: Evaluate $\lim\limits_{x\to\pi/4}\dfrac{\sqrt{1-\sqrt{\sin 2x}}}{(\pi-4x)}$

Solution: Let L = $\lim\limits_{x\to\pi/4}\dfrac{\sqrt{1-\sqrt{\sin 2x}}}{(\pi-4x)}$

$$= \frac{1}{4}\lim_{x\to\pi/4}\frac{\sqrt{1-\sqrt{\sin 2x}}}{\left(\frac{\pi}{4}-x\right)}$$

Put x = $\dfrac{\pi}{4}+h$

$$\therefore\ L = \frac{1}{4}\lim_{h\to 0}\frac{\sqrt{1-\sqrt{\sin 2\left(\frac{\pi}{4}+h\right)}}}{\frac{\pi}{4}-\left(\frac{\pi}{4}+h\right)}$$

$$= -\frac{1}{4}\lim_{h\to 0}\frac{\sqrt{(1-\sqrt{(\cos 2h)})}}{h}$$

$$= -\frac{1}{4}\lim_{h\to 0}\frac{\sqrt{(1-\sqrt{(\cos 2h)})}}{h}.\frac{\sqrt{(1+\sqrt{(\cos 2h)})}}{\sqrt{(1+\sqrt{(\cos 2h)})}}$$

(form $\dfrac{0}{0}$)

$$= -\frac{1}{4}\lim_{h\to 0}\frac{\sqrt{1-\cos 2h}}{h\sqrt{(1+\sqrt{(\cos 2h)})}}$$

$$= -\frac{1}{4}.\lim_{h\to 0}\frac{\sqrt{2}\,|\sin h|}{h\sqrt{(1+\sqrt{(\cos 2h)})}}$$

$$= -\frac{1}{4}.\sqrt{2}\lim_{h\to 0}\frac{1}{\sqrt{(1+\sqrt{(\cos 2h)})}}\lim_{h\to 0}\frac{|\sin h|}{h}$$

$$= -\frac{1}{4}.\sqrt{2}.\frac{1}{\sqrt{2}}P = -\frac{1}{4}P$$

where P = $\lim\limits_{h\to 0}\dfrac{|\sin h|}{h}$.

We show that the limit P does not exist :

$$\text{L.H.L.} = \lim_{h\to 0^-}\frac{|\sin h|}{h} = \lim_{t\to 0^+}\frac{|\sin(0-t)|}{(0-t)}$$

$$= \lim_{t\to 0^+}\frac{\sin t}{-t} = -1$$

$$\text{and R.H.L.} = \lim_{h\to 0^+}\frac{\sin h}{h} = \lim_{t\to 0^+}\frac{|\sin(0+t)|}{(0+t)}$$

$$= \lim_{t\to 0^+}\frac{\sin t}{t} = 1$$

∵ L.H.L. ≠ R.H.L. Hence P does not exist.

∴ The limit L also does not exist.

Special Limits

Let us evaluate some limits with the help of the following special limits :

(i) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3} = \dfrac{1}{6}$ (ii) $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2} = \dfrac{1}{2}$

(iii) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x^3} = \dfrac{1}{3}$

(iv) $\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^3} = -\dfrac{1}{3}$

Example 35: Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6 2x}$

Solution: Let $l = \lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6 2x}$

$$= \lim_{x\to 0}\frac{e^{x^3}-1-x^3}{(2x)^6}$$

Now put $x^3 = t$

$$l = \frac{1}{64}\lim_{t\to 0}\frac{e^t-1-t}{t^2} \qquad \text{again put } t = 2y$$

$$= \frac{1}{64}\lim_{y\to 0}\frac{e^{2y}-1-2y}{4y^2}$$

$$= \frac{1}{64}\lim_{y\to 0}\frac{(e^y-1)^2+2e^y-2-2y}{4y^2}$$

$$= \frac{1}{64}\lim_{y\to 0}\left[\frac{1}{4}\left(\frac{e^y-1}{y}\right)^2+\frac{1}{2}\frac{e^y-y-1}{y^2}\right]$$

$$\therefore\quad l = \frac{1}{64}\left[\frac{1}{4}+\frac{1}{2}.64\ell\right]$$

$$\Rightarrow\quad \frac{l}{2} = \frac{1}{4.64}\ \Rightarrow\ l = \frac{1}{128}.$$

Example 36: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$

Solution: $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$

$$=\lim_{x\to 0}\frac{\dfrac{e^x-e^{-x}-2x}{x^3}}{\dfrac{x-\sin x}{x^3}}.x^3=\frac{l_1}{l_2}\text{ (say), where}$$

$$l_1=\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x^3}\quad\text{and}\quad l_2=\lim_{x\to 0}\frac{x\text{-}\sin x}{x^3}$$

Now $l_2=\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim\limits_{t\to 0}\dfrac{3t-\sin 3t}{27t^3}$ Put x =3t

$$=\lim_{t\to 0}\frac{3t-(3\sin t-4\sin^3 t)}{27t^3}$$

$$=\lim_{t\to 0}\frac{3(t-\sin t)}{27t^3}+\lim_{t\to 0}\frac{4\sin^3 t}{27t^3}=\frac{1}{9}l_2+\frac{4}{27}$$

$$\Rightarrow\ \frac{8l_2}{9}=\frac{4}{27}\quad\Rightarrow\ l_2=\frac{1}{6}.$$

Now $l_1=\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x^3}$ Put x = 3y

$$=\lim_{y\to 0}\frac{e^{3y}-e^{-3y}-6y}{27y^3}$$

$$=\lim_{y\to 0}\frac{\left(e^y-e^{-y}\right)^3+3\left(e^y-e^{-y}\right)-6y}{27y^3}$$

$$=\lim_{y\to 0}\frac{8}{27}\left(\frac{e^{2y}-1}{2y}\right)^3+\lim_{y\to 0}\frac{1}{9}\left(\frac{e^y-e^{-y}-2y}{y^3}\right)$$

$$=\frac{8}{27}+\frac{1}{9}l_1$$

$$\Rightarrow\ \frac{8l_1}{9}=\frac{8}{27}\quad\Rightarrow\ l_1=\frac{1}{3}$$

∴ The required limit $=\dfrac{1}{3}.\dfrac{6}{1}=2$.

Example 37: Evaluate $\lim\limits_{x\to\infty} x-x^2\ \ell n\left(1+\dfrac{1}{x}\right)$

Solution: $\lim\limits_{x\to\infty} x\left[1-\ell n\left(1+\dfrac{1}{x}\right)^x\right]$ Put x = 1/y

$$=\lim_{y\to 0}\left[\frac{1}{y}-\frac{\ell n(1+y)}{y^2}\right]$$

Put *ln*(1+y) = t ⇒ 1+y = e^t

⇒ y = e^t –1 ⇒ as y→0, t→0

$$=\lim_{t\to 0}\left[\frac{1}{e^t-1}-\frac{t}{(e^t-1)^2}\right]$$

$$=\lim_{t\to 0}\frac{\left(e^t-1\right)-t}{\left(\dfrac{e^t-1}{t}\right)^2.t^2}=\lim_{t\to 0}\frac{e^t-t-1}{t^2}=\frac{1}{2}$$

(using special limit(ii))

Caution

Doing $\lim\limits_{y\to 0}\left(\dfrac{1}{y}-\dfrac{\ln(1+y)}{y}.\dfrac{1}{y}\right)=\dfrac{1}{y}-\dfrac{1}{y}=0$ using $\lim\limits_{y\to 0}\dfrac{\ln(1+y)}{y}=1$, is not correct.

Example 38: Evaluate $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$

Solution: $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$

$$=\lim_{x\to 0}\frac{\left(e^{\frac{\ln(1+x)}{x}-1}-1\right)e}{x}\text{ , assume }M=\frac{\ln(1+x)}{x}-1$$

$$=\lim_{x\to 0}\frac{e^M-1}{M}.\frac{eM}{x}=\lim_{x\to 0}\frac{eM}{x}$$

$$=e.\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=-\frac{e}{2}$$

{using previous example}

Example 39: Evaluate $\lim\limits_{x\to 0}\dfrac{1}{(\sin^{-1}x)^2}-\dfrac{1}{x^2}$

Solution: Solving $\lim\limits_{x\to 0}\dfrac{1}{x^2}.\dfrac{x^2}{(\sin^{-1}x)^2}-.\dfrac{1}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{1}{x^2}-\dfrac{1}{x^2}$ = 0, is wrong.

The correct way is to put x = sin θ

$$\Rightarrow\ \lim_{x\to 0}\frac{1}{(\sin^{-1}x)^2}-\frac{1}{x^2}$$

$$=\lim_{\theta\to 0}\frac{1}{\theta^2}-\frac{1}{\sin^2\theta}=\lim_{\theta\to 0}\frac{\sin^2\theta-\theta^2}{\theta^2\sin^2\theta}$$

$$=\lim_{\theta\to 0}\frac{(\sin\theta-\theta)(\sin\theta+\theta)}{\theta^4}$$

$$=2.\lim_{\theta\to 0}\frac{\sin\theta-\theta}{\theta^3}=2\left(-\frac{1}{6}\right)=-\frac{1}{3}.$$

(using special limit(i))

Example 40: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x-x^2-\{x\}.\{-x\}}{x\cos x-x^2-\{x\}.\{-x\}}$

where {.} denotes the fractional part function.

Solution: $f(0^+) = \lim\limits_{h\to 0} \dfrac{\sin h - h^2 - \{h\}\{-h\}}{h\cos h - h^2 - \{h\}\{-h\}}$

$$= \lim_{h\to 0} \frac{\sin h - h^2 - h(1-h)}{h\cos h - h^2 - h(1-h)}$$

$$= \lim_{h\to 0} \frac{\sin h - h}{h\cos h - h}$$

$$= \lim_{h\to 0} \frac{h-\sin h}{h^3}\cdot\frac{h^3}{h(1-\cos h)} = \frac{1}{6}\cdot 2 = \frac{1}{3}.$$

Similarly $f(0^-) = \lim\limits_{h\to 0} \dfrac{-\sin h - h^2 - (1-h)h}{-h\cos h - h^2 - (1-h)h}$

$$= \lim_{h\to 0} \frac{-(\sin h + h)}{-h(1+\cos h)}$$

$$= \lim_{h\to 0} \frac{1+\dfrac{\sin h}{h}}{1+\cos h} = \frac{2}{2} = 1.$$

Since $f(0^-) \neq f(0^+)$, the limit does not exist.

Example 41: Evaluate $\lim\limits_{x\to 0} \dfrac{x + \ell n\left(\sqrt{1+x^2} - x\right)}{x^3}$

Solution: Let $\sqrt{1+x^2} - x = t$

$\Rightarrow \quad x \to 0 \;,\; t \to 1$

Also $1 + x^2 = t^2 + x^2 + 2tx \;\Rightarrow\; x = \dfrac{1-t^2}{2t}$.

The given expression $= \dfrac{\dfrac{1-t^2}{2t} + \ell n\, t}{(1-t^2)^3}\; 8t^3$

$$= \frac{2t\,\ell n\, t + 1 - t^2}{2t\,.\,(1-t^2)^3}\; 8t^3$$

$$= \frac{2t\,\ell n\, t + 1 - t^2}{(1+t)^3\;(1-t)^3}\;.\;4t^2$$

$$= \frac{1}{2}\;\frac{2t\,\ell n\, t + 1 - t^2}{(1+t)^3\;(1-t)^3}\;.\;4t^2$$

Put $t = 1 + y$; as $t \to 1\,,\; y \to 0$

Limit $= \dfrac{1}{2}\lim\limits_{y\to 0} \dfrac{2\,(1+y)\,\ell n\,(1+y) + 1 - (1+y)^2}{-\,y^3}$

Put $1 + y = e^z$; as $y \to 0,\; z \to 0$

$$= -\frac{1}{2}\lim_{z\to 0}\frac{2\,z\,e^z + 1 - e^{2z}}{\left(\dfrac{e^z-1}{z}\right).\,z^3}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{e^{2z} - 2z\,e^z - 1}{z^3}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{e^z\,(e^z - e^{-z} - 2z)}{z^3} = \frac{1}{2}\,.\,l$$

$\Rightarrow \quad l = \lim\limits_{u\to 0}\dfrac{e^{3u} - e^{-3u} - 6u}{27\,u^3}$ when $z = 3u$

$$= \lim_{u\to 0}\frac{(e^u - e^{-u})^3 + 3\,(e^u - e^{-u}) - 6u}{27\,u^3}$$

$$= \lim_{u\to 0}\left(\frac{e^{2u}-1}{2u}\right)^3 + \frac{1}{9}\,l$$

$\Rightarrow \quad \dfrac{8}{9}\,l = \dfrac{8}{27} \quad\Rightarrow\quad l = \dfrac{1}{3}$

$\therefore$ The required limit $= \dfrac{1}{6}$.

Concept Problems

H

1. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin x^{o}}{x}$ (ii) $\lim\limits_{x\to 0}\dfrac{\tan x^3}{x}$

2. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{5x}$ (ii) $\lim\limits_{x\to 0}\sin 8x\cot 3x$

(iii) $\lim\limits_{x\to \pi}\dfrac{\sin 3x}{\sin 2x}$ (iv) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{3x^2}$

3. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{e^{4x}-1}{\tan x}$

(ii) $\lim\limits_{x\to 1}\dfrac{x^n-1}{x^m-1}$ (m, n ∈ N)

(iii) $\lim\limits_{x\to 0}\dfrac{\log_2(1+x)}{x}$

(iv) $\lim\limits_{x\to 0}\dfrac{\ln(1+8x)}{\ln(1+7x)}$

4. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{3^x-1}$ (ii) $\lim\limits_{x\to 0}\dfrac{\ln \sec x}{x^2}$

5. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/m}-(1+x)^{1/n}}{x}$

(ii) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x\sqrt{1-x^2}}$

6. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\tan 3x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\cos x-\sec x}{x^2}$

(iii) $\lim\limits_{x\to 0}\dfrac{\cos 7x-\cos 9x}{\cos x-\cos 5x}$

(iv) $\lim\limits_{x\to 0}\dfrac{\ell n(1+\sin 4x)}{e^{\sin 5x}-1}$

7. Find whether the following limits exists:

(i) $\lim\limits_{x\to 0}\dfrac{1}{x}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(ii) $\lim\limits_{x\to 0}\dfrac{\cot^{-1}\dfrac{1}{x}}{x}$

(iii) $\lim\limits_{x\to 1}\dfrac{|x-1|}{\tan^{-1}(x-1)}$

(iv) $\lim\limits_{x\to 0^+} x^4\sin(2\sqrt{x})$

8. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\ell n\cos x}{\sqrt[4]{1+x^2}-1}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin 3x}-1}{\ln(1+\tan 2x)}$

(iv) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{\sin 4x}$

9. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-1}{x^2}$ (ii) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{1-\cos 3x}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{\ln(x+1)}$ (iv) $\lim\limits_{x\to 0}\dfrac{\ln(1+3x\sin x)}{\tan x^2}$

10. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{8^x-7^x}{6^x-5^x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{\ln^2(1+2x)}$

(iii) $\lim\limits_{\alpha\to 0}\dfrac{(5^\alpha-1)(4^\alpha-1)}{(3^\alpha-1)(6^\alpha-1)}$

(iv) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-1}{\tan 3x}$

11. Evaluate the following limits :

(i) $\lim\limits_{\alpha\to 0}\dfrac{(1-\cos\alpha)^2}{\tan^3\alpha-\sin^3\alpha}$

(ii) $\lim\limits_{x\to 0}\dfrac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}(\sin^{-1}x)^3}$

12. Find $\lim\limits_{x\to 0} f(x)$ if $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2} & \text{for } x<0\\[2ex] \dfrac{x}{2x+x^2} & \text{for } x>0\end{cases}$

13. Evaluate the following limits :

(i) $\lim\limits_{x\to\pi/4}\dfrac{\sin x-\cos x}{\pi-4x}$

(ii) $\lim\limits_{x\to 1}\dfrac{\sin(e^{x-1}-1)}{\ln x}$

(iii) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$

(iv) $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$

14. Evaluate the following limits :

(i) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$ (ii) $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$

(iii) $\lim\limits_{x\to 1}(1-x)\log_x 2$ (iv) $\lim\limits_{x\to 1}\dfrac{x^3+x^2-2}{\sin(x-1)}$

Practice Problems G

15. Show that $\lim\limits_{x\to 0}\dfrac{\sin(1+x)-\sin(1-x)}{x}=2\cos 1.$

16. Prove that $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}=\dfrac{1}{4}$

17. Evaluate $\lim\limits_{x\to 0}\dfrac{\cos(x\,e^{x})-\cos(xe^{-x})}{x^{3}}$

18. Evaluate $\lim\limits_{x\to 0}\dfrac{\tan([-\pi^{2}]x^{2})-\tan([-\pi^{2}])x^{2}}{\sin^{2}x}$ where [.] denotes the greatest integer function.

19. Evaluate $\lim\limits_{\alpha\to 0}\dfrac{\sin(\alpha^{n})}{(\sin\alpha)^{m}}$ (m and n positive integers)

20. Evaluate $\lim\limits_{h\to 0}\dfrac{\sin(a+2h)-2\sin(a+h)+\sin a}{h^{2}}$

21. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\ell n(1+a\sin x)}{\sin x}$

(ii) $\lim\limits_{x\to 0^{+}}\dfrac{1-\sqrt{\cos x}}{\sin^{4}3\sqrt{x}}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin 3x^{2}}{\ln\cos(2x^{2}-x)}$

(iv) $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^{3}}$

22. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{3\sin x-x^{2}+x^{3}}{\tan x+2\sin^{2}x+5x^{4}}$

(ii) $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin^{-1}x-\sin^{3}x}{3\tan^{-1}x-x^{2}+x^{5}}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin 2x+(\sin^{-1}x)^{2}-(\tan^{-1}x)^{2}}{3x}$

23. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin\sqrt[3]{x}\,\ln(1+3x)}{(\tan^{-1}\sqrt{x})^{2}\left(e^{5\sqrt[3]{x}}-1\right)}$

(ii) $\lim\limits_{x\to 0}\dfrac{e^{x^{2}}-3^{3x}}{\sin\left(\dfrac{x^{2}}{2}\right)-\sin x}$

(iii) $\lim\limits_{x\to 0}\dfrac{\log_{\sec x/2}\cos x}{\log_{\sec x}\cos x/2}$

24. Evaluate $\lim\limits_{x\to\pi}\dfrac{1-\cos 7\,(\pi-x)}{5(x-\pi)^{n}}$, $n=1,2$.

25. Evaluate the following limits :

(i) $\lim\limits_{x\to 3\pi/4}\dfrac{1+\sqrt[3]{\tan x}}{1-2\cos^{2}x}$ (ii) $\lim\limits_{x\to 0}\dfrac{x2^{x}-x}{1-\cos x}$

26. Evaluate the following limits :

(i) $\lim\limits_{\alpha\to\beta}\dfrac{\sin^{2}\alpha-\sin^{2}\beta}{\alpha^{2}-\beta^{2}}$

(ii) $\lim\limits_{y\to a}\left(\sin\dfrac{y-a}{2}\,.\,\tan\dfrac{\pi y}{2a}\right)$

27. Evaluate the one-sided limits and find whether the limit exists:

(i) $\lim\limits_{x\to 1}\dfrac{x\sin(x-[x])}{x-1}$

(ii) $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

28. Evaluate $\lim\limits_{x\to 2^{+}}\dfrac{\{x\}\sin(x-2)}{(x-2)^{2}}$, (where {.} denotes the fractional part function.)

29. Evaluate the following limits :

(i) $\lim\limits_{x\to 0^{+}}\dfrac{\cos^{-1}(1-x)}{\sqrt{x}}$

(ii) $\lim\limits_{x\to\pi}\dfrac{1-\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\left(\cos\dfrac{x}{4}-\sin\dfrac{x}{4}\right)}$

(iii) $\lim\limits_{x\to a}\dfrac{x^{x}-a^{a}}{x-a}$ $(a>0)$

30. Evaluate the following limits :

(i) $\lim\limits_{x\to 1}\dfrac{\ln(1+x-3x^{2}+2x^{3})}{\ln(1+3x-4x^{2}+x^{3})}$

(ii) $\lim\limits_{x\to 1}\dfrac{x^{x}-1}{x\ln x}$

31. Evaluate the following limits :

(i) $\lim\limits_{x\to -1^{+}}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}$

(ii) $\lim\limits_{x\to 1}\sqrt{\dfrac{\ln x}{x^{4}-1}}$

32. Evaluate the following limits :

(i) $\lim\limits_{x\to\pi/4}\dfrac{4\sqrt{2}-(\cos x+\sin x)^{5}}{1-\sin 2x}$

(ii) $\lim\limits_{x\to\pi/6}\dfrac{\sin(x-\pi/6)}{\sqrt{3}-2\cos x}$

33. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\log_{10}(1+2x-3x^{2}+4x^{3})}{\log_{10}(1-x+2x^{2}-7x^{3})}$

(ii) $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha}{\sqrt[3]{(1-\cos\alpha)^2}}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$

(iv) $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

1.10 Algebra of Limits

Earlier, we had seen the theorem on limits. But we face some difficulty in applying these theorems in several situations. We should take some precautions while applying the theorems.

1. $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)$

(i) Existent + Existent → Existent

$$\lim_{x\to 0}\left(\frac{\sin x+\tan x}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)+\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$$

$= 1 + 1 = 2.$

(ii) Existent + Non-existent → Non-existent

$$\lim_{x\to 0}\left(\frac{\sin^3 x+\tan x}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3+\lim_{x\to 0}\left(\frac{\tan x}{x^3}\right)$$

$= 1 + \infty = \infty.$

Also, $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}+\sin\dfrac{1}{x}\right)$ is non-existent.

(iii) Non-existent + Non-existent → No comment

For example, consider $\lim\limits_{x\to 0}([x] + [-x])$

L.H.L. $= -1 + 0 = -1$, RHL $= 0 + (-1) = -1$.

The limit exists even if both the individual limits do not exist. However, this may not always hold.

For example, consider $\lim\limits_{x\to 0}(2[x] + [-x])$

L.H.L. $= 2(-1) + 0 = -2$, RHL $= 2.0 + (-1) = -1$.

We see that the limit does not exist.

2. $\lim\limits_{x\to a}(f(x)\,.\,g(x)) = \left(\lim\limits_{x\to a} f(x)\right).\left(\lim\limits_{x\to a} g(x)\right)$

(i) Existent × Existent → Existent

$$\lim_{x\to 0}\left(\frac{\sin x.\tan^{-1}x}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}\left(\frac{\tan^{-1}x}{x}\right)$$

$= 1 \times 1 = 1.$

(ii) Existent × Non-existent → no comment

For example, consider $\lim\limits_{x\to 1}(x-1).[x]$

L.H.L. $= 0.(-1) = 0$, RHL $= 0.0 = 0$.

The limit exists even if one of the individual limits does not exist. However, this may not always hold. For example, the limit

$\lim\limits_{x\to 0}(\cos x).2^{1/x}$ does not exist.

In fact, all the limit questions in $0 \times \infty$ form cannot be determined in advance.

(iii) Non-existent × Non-existent → no comment

For example, consider $\lim\limits_{x\to 0}([x].[-x])$

L.H.L. $= (-1).0 = 0$, RHL $= 0.(-1) = 0$.

The limit exists even if both the individual limits do not exist. However, this may not always hold.

For example, the limit $\lim\limits_{x\to 1}([x].[-x])$ does not exist.

L.H.L. $= 0.(-1) = 0$, RHL $= 1.(-2) = -2$.

3. $\lim\limits_{x\to a}(f(x)/g(x)) = \dfrac{\left(\lim\limits_{x\to a} f(x)\right)}{\left(\lim\limits_{x\to a} g(x)\right)}$

$$\frac{\text{Existent}}{\text{Existent(non-zero)}} = \text{Existent}$$

In all other cases, we need to examine the limit carefully.

4. If $\lim\limits_{x\to a} f(x) = \ell$, then

$$\lim_{x\to a}(f(x))^{1/n} = \left(\lim_{x\to a} f(x)\right)^{\frac{1}{n}} = \lambda^{\frac{1}{n}}.$$

If n is an odd integer, then there is no concern. If n is even and ℓ is positive, again there is no concern. If n is even and ℓ is zero, we need to examine the domain of the function.

$\lim\limits_{x\to 0}(\sin x)^{1/3} = 0^{1/3} = 0$

$\lim\limits_{x\to 0}(\cos x)^{1/2} = 1^{1/2} = 1$

$\lim\limits_{x\to 0}(\sin x)^{1/2} = 0^{1/2} = 0$. Note that the function is defined only in the right neighbourhood of $x = 0$. Thus, the limit evaluated is actually the right hand limit $\lim\limits_{x\to 0^+}(\sin x)^{1/2} = 0$.

However, $\lim\limits_{x\to 0}(-\sin^2 x)^{1/2}$ is not equal to $0^{1/2}$ i.e. 0, since the function $(-\sin^2 x)^{1/2}$ is not defined either in the left or the right neighbourhood of $x = 0$.

$\lim\limits_{x\to a}(f(x))^{\frac{m}{n}} = \left(\lim\limits_{x\to a} f(x)\right)^{\frac{m}{n}}$, where m and n are integers.

Here also we need to examine the domain of the function.

Note:

A. If $\lim\limits_{x\to c}[f(x)+g(x)]$ exists then we can have the following cases:

(i) If $\lim\limits_{x\to c} f(x)$ exists, then $\lim\limits_{x\to c} g(x)$ must exist.

Since g = (f + g) – f, we have by the theorem on limits

$\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c}(f(x)+g(x)) - \lim\limits_{x\to 0} f(x)$ and this exists.

(ii) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ do not exist.

Consider $\lim\limits_{x\to 1}[x]$ and $\lim\limits_{x\to 1}\{x\}$.

where [.] and {.} represent greatest integer and fractional part function, respectively. Here both the limits do not exist but $\lim\limits_{x\to 1}([x]+\{x\}) = \lim\limits_{x\to 1} x = 1$ exists.

B. If $\lim\limits_{x\to c} f(x)g(x)$ exists, then we can have the following cases:

(i) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist. Obviously, then $\lim\limits_{x\to c} f(x)g(x)$ exists.

(ii) $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} g(x)$ does not exist.

Consider f(x) = x; g(x) = $\dfrac{1}{\sin x}$.

$\lim\limits_{x\to 0} f(x) = 0$ exists but $\lim\limits_{x\to 0} g(x)$ does not exist.

Here $\lim\limits_{x\to 0} f(x)\,.\,g(x)$ exists and equals 1 but this may not be always true.

(iii) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ do not exist.

Let $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$ and

$g(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$. Here $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist while $\lim\limits_{x\to 0} f(x).g(x)$ exists and equals 2. But this may not be always true.

Caution

(i) To prove $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2} = \dfrac{3}{2}$

we rationalize the numerator first and proceed.

However, solving as $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ using $\lim\limits_{x\to 0}\sqrt{\cos 2x} = 1$ is wrong. We should not calculate the limit of some term which is not as a separate factor to the problem.

(ii) We have $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{x^4} = \dfrac{1}{8}$ by proceeding with $\dfrac{1-\cos\left(2\sin^2\dfrac{x}{2}\right)}{x^4}$.

However, solving as $\lim\limits_{x\to 0}\dfrac{1-\cos\left(\dfrac{1-\cos x}{x^2}.x^2\right)}{x^4}$

$= \lim\limits_{x\to 0}\dfrac{1-\cos(x^2/2)}{x^4}$ [since $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$]

$= \dfrac{2\sin\dfrac{x^2}{4}.\sin\dfrac{x^2}{4}}{16.\dfrac{x^2}{4}.\dfrac{x^2}{4}} = \dfrac{1}{8}$ is wrong, although the answer may be correct.

(iii) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2} = \lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x-1+1}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{(e^{x^2}-1)}{x^2} + \lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = 1+\dfrac{1}{2} = \dfrac{3}{2}$

However, solving as $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x^2} = 1$ (as $\cos x \to 1$ as $x \to 0$) is wrong.

(iv) $\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-e^{\sin x}}{x}$

$\lim\limits_{x\to 0}\dfrac{e^{\sin 2x-\sin x}-1}{\sin 2x-\sin x}\cdot\dfrac{\sin 2x-\sin x}{x}$ = 1·[2 – 1] = 1

However, solving as

$\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-e^{\sin x}}{x} = \lim\limits_{x\to 0}\dfrac{e^{\frac{\sin 2x}{2x}.2x}-e^{\frac{\sin x}{x}.x}}{x}$

$= \lim\limits_{x\to 0}\dfrac{e^{2x}-e^{x}}{x} = \lim\limits_{x\to 0}\dfrac{e^{x}(e^{x}-1)}{x} = 1$

is wrong although the end result may be correct.

5. Composition Law

It is tempting to write

$$\lim_{x\to -4}\sqrt{x^2+9} = \sqrt{\lim_{x\to -4}(x^2+9)} \qquad ...(1)$$

$$= \sqrt{(-4)^2+9} = \sqrt{25} = 5\cdot$$

But can we simply "move the limit inside the radical" in (1) ? To analyze this question, let us write

$g(x) = \sqrt{x}$ and $f(x) = x^2 + 9$.

Then the function that appears in (1) is the composite function

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 9}\,.$$

Hence our question is whether or not

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))\,.$$

The next limit law answers this question in the affirmative, provided that the "outside" function" g meets a certain condition; if so, then the limit of the composite function g(f(x)) as x → a may be found by substituting into the function g the limit of f(x) as x → a.

A. If $\lim_{x\to a} f(x) = b$ and g(x) is continuous at x = b, then

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(b).$$

Proof: Since g is continuous at b, for every neighbourhood N of g(b) there exists a neighbourhood C of b such that g(y) is in N for every y in C.

In turn, since $\lim_{x\to a} f(x) = b$ there exists a deleted neighbourhood D of a such that f(x) is in C for every x in D. Combining these remarks, we see that for every neighbourhood N of g(b) there exists a deleted neighbourhood D of a such that f(x) is in C, and therefore g(f(x)) is in N, for every x in D. This proves the result.

A function such as $y = \dfrac{1}{x^2+3}$, is a composition of the polynomial function $f(x) = x^2 + 3$ and the reciprocal function $g(x) = \dfrac{1}{x}$

i.e., $g(f(x)) = \dfrac{1}{f(x)} = \dfrac{1}{x^2+3}$.

Thus, y = g(f(x)). Therefore, since $\lim_{x\to a} f(x) = f(a)$ and g is continuous at f(a), $\lim_{x\to a} y = \dfrac{1}{a^2+3}$.

Let $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & , x \neq 1 \\ 3 & , x = 1 \end{cases}$, $g(x) = \sin^{-1}\dfrac{x}{4}$.

To find $\lim_{x\to 1} g(f(x))$, we ask two questions.

Firstly, whether $\lim_{x\to 1} f(x)$ exists. The answer is yes : $\lim_{x\to 1} f(x) = 2$.

Secondly, whether g(x) is continuous at x = 2.

We know that $\sin^{-1}\dfrac{x}{4}$ is continuous.

Thus, using the above theorem,

$$\lim_{x\to 1} g(f(x)) = g(2) = \frac{\pi}{6}\,.$$

Now consider $\lim_{x\to 1} \ell n\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

$\lim_{x\to 1} \dfrac{1-\sqrt{x}}{1-x} = \dfrac{1}{2}$ and y = λnx is continuous at $x = \dfrac{1}{2}$.

$$\therefore \quad \lim_{x\to 1} \ell n\left(\frac{1-\sqrt{x}}{1-x}\right) = \ell n\frac{1}{2}\,.$$

But one should note that even if limit of f(x) does not exist, the limit of g(f(x)) may exist.

For example, let f(x) = sgn(x), whose limit does not exist at x = 0, and $g(x) = x^2$.

We see that $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} (\text{sgn}(x))^2 = 1$.

Also, the limit of g(f(x)) may exist even if g(x) is discontinuous. For example, let $f(x) = x^2 + 1$, whose limit at x = 0 is 1, and g(x) = [x]. We know that g(x) is discontinuous at x = 1.

However, $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} [x^2 + 1] = 1$.

Example 1: Evaluate $\lim_{x\to 1}\left[\dfrac{x^2-1}{4(x-1)}\right]$, [.] is G.I.F.

Solution: $\lim_{x\to 1} \dfrac{x^2-1}{4(x-1)} = \dfrac{1}{2}$

y = [x] is continuous at $x = \dfrac{1}{2}$.

Hence the limit is $\left[\dfrac{1}{2}\right] = 0$.

Study Tip

We know that the function |x| is continuous everywhere; thus, it follows that if $\lim_{x\to a} g(x)$ exists then

$$\lim_{x\to a} |g(x)| = \left|\lim_{x\to a} g(x)\right|$$

That is, a limit symbol can be moved through an absolute value sign, provided the limit of the expression inside the absolute value signs exists. For example,

$$\lim_{x\to 3} \left|5 - x^2\right| = \left|\lim_{x\to 3}(5 - x^2)\right| = |-4| = 4\,.$$

An useful application of composition law is the following:

Let $L = \lim_{x\to a} (f(x))^{g(x)}$

$\Rightarrow \quad \ln L = \lim_{x\to a} g(x).\ln f(x)$

$\Rightarrow \quad L = e^{\lim_{x\to a} g(x).\ell n f(x)}$

B. If $\lim_{x\to a} f(x) = b$, with f(x) ≠ b for every x in some neighbourhood of 'a' and if $\lim_{x\to b} g(x) = c$, then

$\lim\limits_{x\to a} g\left(f(x)\right) = c$. (This theorem is sometimes useful if g is discontinuous at b.)

Let $f(x) = \dfrac{\sin x}{x}$, $g(x) = \begin{cases} \dfrac{x^2-1}{x-1} &, x \neq 1 \\ 3 &, x = 1 \end{cases}$

To find $\lim\limits_{x\to 0} g(f(x))$, we ask two questions. Firstly, whether $\lim\limits_{x\to 0} f(x)$ exists. The answer is yes: $\lim\limits_{x\to 0} f(x) = 2$. Secondly, whether $\lim\limits_{x\to 1} g(x)$ exists. We have $\lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1} = \dfrac{x^2-1}{x-1} = 2$.

Thus, using the above theorem, $\lim\limits_{x\to 0} g(f(x)) = \lim\limits_{x\to 1} g(x) = 2$.

Note that the above limit has been evaluated even if g(x) is discontinuous. Further, $\lim\limits_{x\to 0} g(f(x)) \neq g\left(\lim\limits_{x\to 0} f(x)\right)$

Now let f(x) = 1, $g(x) = \begin{cases} \dfrac{x^2-1}{x-1} &, x \neq 1 \\ 3 &, x = 1 \end{cases}$.

Find $\lim\limits_{x\to 0} g(f(x))$.

The theorem is not applicable since the function f(x) attains the value 1 in the neighbourhood of '0'.

We see that g(f(x)) = 3. Hence $\lim\limits_{x\to 0} g(f(x)) = 3$.

Example 2: If $f(x) = \begin{cases} 1-x &, -1 \le x \le 2 \\ 2x-5, & 2 < x \le 3 \end{cases}$

$g(x) = \begin{cases} x^2-1 &, -2 \le x < -1 \\ 2x+3 &, -1 \le x < 1 \end{cases}$

find (i) $\lim\limits_{x\to 1} g\left(f(x)\right)$ and (ii) $\lim\limits_{x\to 2} g\left(f(x)\right)$.

Solution: (i) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} 1 - x = 0$.

Also, g(x) is continuous at x = 0.

Hence $\lim\limits_{x\to 1} g\left(f(x)\right) = g\left(\lim\limits_{x\to 1} f(x)\right) = g(0) = 3$.

(ii) $\lim\limits_{x\to 2} f(x)$ does not exist. The theorem is not applicable. We evaluate the one sided limits :

$$\begin{aligned} \text{L.H.L.} &= \lim_{x\to 2^-} g\left(f(x)\right) \\ &= \lim_{h\to 0} g\left(f(2-h)\right) \\ &= \lim_{h\to 0} g\left(1-(2-h)\right) \\ &= \lim_{h\to 0} g(-1+h) \\ &= \lim_{h\to 0} 2(-1+h)+3 = 1. \end{aligned}$$

$$\begin{aligned} \text{R.H.L.} &= \lim_{x\to 2^+} g\left(f(x)\right) \\ &= \lim_{h\to 0} g\left(f(2+h)\right) \\ &= \lim_{h\to 0} g\left(2(2+h)-5\right) = \lim_{h\to 0} g\left(4+2h-5\right) \\ &= \lim_{h\to 0} g\left(-1+2h\right) \\ &= \lim_{h\to 0} 2(-1 + 2h) + 3 = 1. \end{aligned}$$

Thus, $\lim\limits_{x\to 2} g\left(f(x)\right) = 1$.

6. Domination Law

If two function f(x) and g(x) satisfy the inequality

$f(x) < g(x)$ ...(1)

for all the values of x belonging to a neighbourhood of a point a except possibly at a, then

$\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$ ...(2)

provided that the limits of both functions, as $x \to a$, exist. Note that a strict inequality connecting functions may lead to a non-strict inequality for their limits. For example, if $0 < |x| < 1$ then $x^2 < |x|$. But nevertheless,

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} |x| = 0.$$

Now consider $f(x) = \dfrac{\sin x}{x}$ and $g(x) = \dfrac{\tan x}{x}$.

We have f(x) < g(x) in the neighbourhood of x = 0, $x \neq 0$. Since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$, we have $\lim\limits_{x\to 0} f(x) \le \lim\limits_{x\to 0} g(x)$.

Thus, we see the necessity of the equality sign in the domination law.

The theorem remains of course true if the strict inequality f(x) < g(x) entering into its formulation is replaced by the non-strict inequality $f(x) \le g(x)$.

In the case $x \to \infty$ we must require that inequality (1) should hold for x > N where N is a fixed number.

An important particular case of the theorem arises when one of the two functions g(x) and f(x) is a constant. For definiteness, let g(x) = constant. Then the theorem reads :

If $f(x) < M$ (or $f(x) \le M$) for $0 < |x - c| < \delta$ where δ is a fixed number then $\lim\limits_{x\to c} f(x) \le M$ provided that the limit of the function f(x) exists.

1.11 Limits when $x \to \infty$

Sometimes we will not be concerned with the behaviour of g(x) near a specific value of x, but rather with how the values of f(x) behave as x increases without bound or decreases without bound. This is sometimes called the **end behaviour** of the function because it describes how the function behaves for values of x that are far from the origin. To investigate the end behaviour of a function we find limits of the function as $x \to -\infty$ or as $x \to \infty$.

To discuss the end behaviour of a function we assume that the function is defined on an infinite interval.

Let the independent variable x of a function y = f(x) increase indefinitely. This means that x is made to take on the values becoming greater than any given positive number. In such a case we say that x becomes infinitely large or that x approaches infinity and write $x \to \infty$. If x decreases indefinitely, that is, becomes less than any given negative number, we say that x becomes negatively infinite or that it approaches minus infinity and write $x \to -\infty$.

Definition A number λ is said to be the limit of the function y = f(x) as $x \to \infty$ if for all sufficiently large values of x the corresponding values of the function f(x) become arbitrarily close to the number λ.

Limits as x Tends to – ∞

We can easily understand the meaning of the statement: f(x) approaches λ as $x \to -\infty$.

In fact, if x = – t and f(x) = f(– t) = g(t) then t tends to ∞ as x tends to – ∞, and the question of the behaviour of f(x) as x tends to – ∞ is the same as that of the behaviour of g(t) as t tends to ∞.

Example 1: Evaluate $\lim\limits_{x\to\infty}\dfrac{2x-1}{x+1}$.

Solution: Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\begin{tikzcd} & \lim\limits_{x\to\infty}(2x-1)\to\infty \\ \lim\limits_{x\to\infty}\dfrac{2x-1}{x+1} \arrow[ur] \arrow[dr] & \\ & \lim\limits_{x\to\infty}(x+1)\to\infty \end{tikzcd}$$

To resolve this problem, we divide both the numerator and the denominator by x. After dividing, the limit may be evaluated as follows:

$$\lim_{x\to\infty}\frac{2x-1}{x+1} = \lim_{x\to\infty}\frac{\dfrac{2x-1}{x}}{\dfrac{x+1}{x}}$$

$$= \lim_{x\to\infty}\frac{2x-(1/x)}{x+(1/x)} = \frac{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\dfrac{1}{x}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\dfrac{1}{x}}$$

$$= \frac{2-0}{1+0} = 2.$$

Example 2: Determine what happens to y as $x \to -\infty$ and $x \to \infty$.

(a) $y = \dfrac{x-1}{(x+3)(x-2)}$ (b) $\dfrac{(x+2)(1-x)}{x-3}$

Solution: (a) Divide numerator and denominator of $\dfrac{x-1}{(x+3)(x-2)}$ by x^2 (the highest power of x in the denominator), obtaining $\dfrac{1/x-1/x^2}{(1+3/x)(1-2/x)}$

Hence, as $x \to \infty$,

$$y \to \frac{0-0}{(1+0)(1-0)} = \frac{0}{1} = 0$$

As $x \to -\infty$, the factors x – 1, x + 3, and x – 2 are negative, and, therefore, $y \to 0^-$. As $x \to \infty$, those factors are positive, and, therefore, $y \to 0^+$.

(b) $\dfrac{(x+2)(1-x)}{x-3}$

$$= \frac{-x^2-x+2}{x-3} = \frac{-x-1+2/x}{1-3/x}\text{, after dividing}$$

numerator and denominator by x (the highest power of x in the denominator).

As $x \to \infty$, 2/x and 3/x approach 0, and –x – 1 approaches ∞. Thus, the denominator approaches 1 and the numerator approaches ∞.

As $x \to -\infty$, x + 2 and x – 3 are negative and 1 – x is positive ; so, $y \to \infty$. As x + 2 and x – 3 are positive and 1 – x is negative; so, $y \to -\infty$. It should be emphasized that the symbols ∞ and –∞ are not real numbers. The phrase "f(x) approaches ∞" is equivalent to saying that "f(x) approaches the unapproachable" or "f(x) increases without bound".

"The symbols ∞ and –∞ are used here to encapsulate a particular way in which limits fail to exist. It is important, to keep in mind that "∞" is not a number. The limit does not exist. To say, for example, that $f(x) \to \infty$ as $x \to a^+$ is to indicate that $\lim\limits_{x\to a^+} f(x)$ does not exist and that it fails to exist because the values of f(x) increase without bound as x approaches a from the right. Furthermore, since ∞ and –∞ are not numbers, it is inappropriate to manipulate these symbols using rules of algebra. For example, it is not correct to equate (∞ – ∞) with 0.

How Limits at Infinity Can Fail to Exist?

Limits at infinity can fail to exist for various reasons. One possibility is that the values of f(x) may increase or decrease without bound as $x \to \infty$ or as $x \to -\infty$. For example, the values of $f(x) = x^3$ increase without bound as $x \to \infty$ and decrease without bound as $x \to -\infty$. We denote this by writing

$$\lim_{x\to\infty} x^3 = \infty, \qquad \lim_{x\to-\infty} x^3 = -\infty,$$

$$\lim_{x\to\infty} (-x^3) = -\infty, \qquad \lim_{x\to-\infty} (-x^3) = \infty$$

If the values of f(x) increases without bound as $x \to \infty$ or as $x \to -\infty$, then we write

$$\lim_{x\to\infty} f(x) = \infty \text{ or } \lim_{x\to-\infty} f(x) = \infty$$

and if the values of f(x) decreases without bound as $x \to \infty$ or as $x \to -\infty$, then we write

$$\lim_{x\to\infty} f(x) = -\infty \text{ or } \lim_{x\to-\infty} f(x) = -\infty$$

Limits at infinity can also fail to exist because the graph of the function oscillates indefinitely in such a way that the values of the function do not approach a fixed number and do not increase or decrease without bound; the trigonometric functions sin x and cos x have this property. In such cases we say that the limit fails to exist because of oscillation.

Note: If $f(x) \to l$ as $x \to \infty$, and l is not zero, then $f(x)\cos x\pi$ and $f(x)\sin x\pi$ oscillate finitely as $x \to \infty$. If $f(x) \to \infty$ or $f(x) \to -\infty$, then they oscillate infinitely. The graph of either function is a wavy curve oscillating between the curves $y = f(x)$ and $y = -f(x)$.

End Behaviour of a Polynomial

The end behaviour of a polynomial matches the end behaviour of its highest degree term.

More precisely, if $c_n \neq 0$ then

$$\lim_{x\to-\infty} (c_0 + c_1x + \ldots + c_nx^n) = \lim_{x\to-\infty} c_nx^n$$

$$\lim_{x\to-\infty} (c_0 + c_1 x + \ldots. + c_nx^n) = \lim_{x\to-\infty} c_nx^n$$

We can get these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression.

$$\text{Thus, } c_0 + c_1x + \ldots. + c_nx^n = x^n\left(\frac{c_0}{x^n}+\frac{c_1}{x^{n-1}}+\ldots+c_n\right)$$

As $x \to -\infty$ or $x \to \infty$, it follows that all of the terms with positive powers of x in the denominator approach 0, so the above limits are certainly plausible.

For example $\lim_{x\to-\infty} (7x^5 - 4x^3 + 2x - 9)$

$$= \lim_{x\to-\infty} 7x^5 = -\infty$$

$$\text{and } \lim_{x\to-\infty} (-4x^8 + 17x^3 - 5x + 1)$$

$$= \lim_{x\to-\infty} -4x^8 = -\infty$$

Let us find $\lim_{x\to\infty} (x^2 - x)$.

Note that we cannot write

$$\lim_{x\to-\infty} (x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty.$$

The limit laws can't be applied to infinite limits because $(\infty - \infty)$ is indeterminate. However, we can write

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x - 1) = \infty$$

because both x and x – 1 become arbitrarily large and so their product does too.

Example 3: Discuss the behaviour of $2x^3 - 11x^2 + 12x$ when x is large.

Solution: First consider x positive and large. The three terms, $2x^3$, $-11x^2$, and $12x$, all become large in absolute value. To see how $2x^3 - 11x^2 + 12x$ behaves for large positive x, factor out x^3.

$$2x^3 - 11x^2 + 12x = x^3\left(2-\frac{11}{x}+\frac{12}{x^2}\right) \qquad \ldots(1)$$

Now, since $11/x$ and $12/x^2 \to 0$ as $x \to \infty$

$$\lim_{x\to\infty}\left(2-\frac{11}{x}+\frac{12}{x^2}\right) = 2$$

Moreover, as $x \to \infty$, $x^3 \to \infty$,

$$\text{Thus } \lim_{x\to\infty} x^3\left(2-\frac{11}{x}+\frac{12}{x^2}\right) = \infty$$

Hence $\lim_{x\to\infty} (2x^3 - 11x^2 + 12x) = \infty$

Now consider x negative and of large absolute value. The argument is similar. Using (1), and the fact that

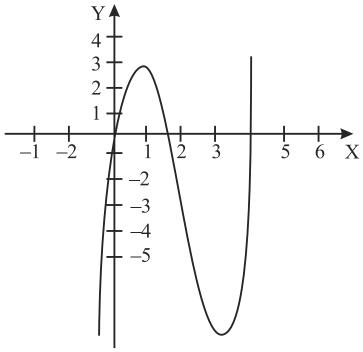
$$\lim_{x\to-\infty} x^3 = -\infty \text{ and } \lim_{x\to-\infty}\left(2-\frac{11}{x}+\frac{12}{x^2}\right) = 2$$

it follows that $\lim_{x\to-\infty} (2x^3 - 11x^2 + 12x) = -\infty$

This completes the discussion. It is interesting to graph $f(x) = 2x^3 - 11x^2 + 12x$ and see what is happening for |x| large. The figure is shown below.

Since $\lim_{x\to\infty} (2x^3 - 11x^2 + 12x) = \infty$ the graph rises arbitrarily high as $x \to \infty$.

Since $\lim_{x\to-\infty} (2x^3 - 11x^2 + 12x) = -\infty$, the graph goes arbitrarily far down as $x \to -\infty$,



The above example generalizes to any polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots.. + a_0$.

Let f(x) be a polynomial of degree atleast 1 and with the leading coefficient a_n positive. Then

$$\lim_{x\to\infty} f(x) = \infty$$

If the degree of f is even, then $\lim_{x\to-\infty} f(x) = \infty$

But if the degree of f is odd, then $\lim_{x\to-\infty} f(x) = -\infty$

End Behaviour of a Rational Function

Let us determine how f(x) = $\dfrac{x^3+6x^2+10x+2}{2x^3+x^2+5}$ behaves for arbitrarily large positive number x.

As x gets large, the numerator $x^3 + 6x^2 + 10x + 2$ grows large, influencing the quotient to become large. On the other hand, the denominator also grows large, influencing the quotient to become small. An algebraic step will help reveal what happens to the quotient. We have

$$f(x) = \frac{x^3+6x^2+10x+2}{2x^3+x^2+5} = \frac{x^3\left(1+\frac{6}{x}+\frac{10}{x^2}+\frac{2}{x^3}\right)}{x^3\left(2+\frac{1}{x}+\frac{5}{x^3}\right)}$$

$$= \frac{1+\frac{6}{x}+\frac{10}{x^2}+\frac{2}{x^3}}{2+\frac{1}{x}+\frac{5}{x^3}} \quad \text{for } x \neq 0$$

Now we can see what happens to f(x) when x is large.
As x increases, $6/x \to 0$, $10/x^2 \to 0$, $2/x^3 \to 0$, $1/x \to 0$ and $5/x^3 \to 0$.

Thus, $f(x) \to \dfrac{1+0+0+0}{2+0+0} = \dfrac{1}{2}$.

So, as x gets arbitrarily large through positive values, the quotient $\dfrac{x^3+6x^2+10x+2}{2x^3+x^2+5}$ approaches $\dfrac{1}{2}$.

The technique used in this example applies to any function that can be written as the quotient of two polynomials i.e. a rational function.

A useful technique for determining the end behaviour of rational function f(x) = p(x)/q(x) is to factor and cancel the highest power of x that occurs in the denominator q(x) from both p(x) and q(x). The denominator of the resulting fraction then has a (nonzero) limit equal to the leading coefficient of q(x), so the limit of the resulting fraction can be quickly determined. The following examples illustrate this technique.

Example 4: $\lim_{x\to-\infty} \dfrac{5x^3-2x^2+1}{3x+5}$

Solution: Divide the numerator and denominator by the highest power of x that occurs in the denominator, namely x.

We obtain

$$\lim_{x\to-\infty} \frac{5x^3-2x^2+1}{3x+5} = \lim_{x\to-\infty} \frac{5x^2-2x+1/x}{3+5/x} = \infty$$

where the final step is justified by the fact that

$$5x^2 - 2x \to \infty, \ \frac{1}{x} \to 0, \text{ and } 3 + \frac{5}{x} \to 3$$

as $x \to -\infty$.

Let f(x) be a polynomial and let ax^n be its term of highest degree. Let g(x) be another polynomial and let bx^m be its term of highest degree.

Then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{ax^n}{bx^m}$

and $\lim_{x\to-\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to-\infty} \dfrac{ax^n}{bx^m}$

The proofs of these facts are similar to the argument used in the previous example. In short, when working with the limit of a quotient of two polynomials as $x \to \infty$ or as $x \to -\infty$, disregard all terms except the one of highest degree in each of the polynomials. The next example illustrates this technique. Alternatively, divide each term of the numerator and denominator by x^k where k is the highest power of x in numerator and denominator. Then use the result $\lim_{x\to\infty} \dfrac{c}{x^k} = 0$ where c is a constant and k > 0.

(i) First of all simplify the given expression.

(ii) Then divide each term of the numerator and denominator by x^m, where x is the independent variable and m is the highest power of x.

(iii) Then expression of the form $\dfrac{c}{x^k}$ tends to 0, where c is a constant and k is a positive number since when $x \to \infty$ or $x \to -\infty$, $\dfrac{c}{x^k} \to 0$, k > 0.

Example 5: Examine the following limits:

(a) $\lim_{x\to\infty} \dfrac{3x^4+5x^2}{-x^4+10x+5}$

(b) $\lim_{x\to\infty} \dfrac{x^3-16x}{5x^4+x^3-5x}$

(c) $\lim_{x\to-\infty} \dfrac{x^4+x}{6x^3-x^2}$

Solution: By the preceding observations,

(a) $\lim_{x\to\infty} \dfrac{3x^4+5x^2}{-x^4+10x+5} = \lim_{x\to\infty} \dfrac{3x^4}{-x^4} = \lim_{x\to\infty}(-3) = -3$

(b) $\lim_{x\to\infty} \dfrac{x^3-16x}{5x^4+x^3-5x} = \lim_{x\to\infty} \dfrac{x^3}{5x^4} = \lim_{x\to\infty} \dfrac{1}{5x} = 0$

(c) $\lim_{x\to-\infty} \dfrac{x^4+x}{6x^3-x^2} = \lim_{x\to-\infty} \dfrac{x^4}{6x^3} = \lim_{x\to-\infty} \dfrac{x}{6} = -\infty$

In short, let $f(x) = a_0x^m + a_1x^{m-1} + ... + a_{m-1}x + a_m$, and $g(x) = b_0x^n + b_1x^{n-1} + ... + b_{n-1}x + b_n$ be two polynomial functions. Then

(i) $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \frac{a_0}{b_0}$ if m = n

(ii) $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = 0$ if m < n

(iii) $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \pm\infty$ if m > n.

(It is ∞ if and only if a_0 and b_0 have the same signs.

(iv) $\lim\limits_{x\to-\infty}\frac{f(x)}{g(x)} = \pm\infty$ if m > n.

(The correct sign is the sign of $(-1)^{m-n}\frac{a_0}{b_0}$.

$$\therefore \quad \lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1} + \ ....}{b_0x^n \ + \ b_1x^{n-1} + \ ...}$$

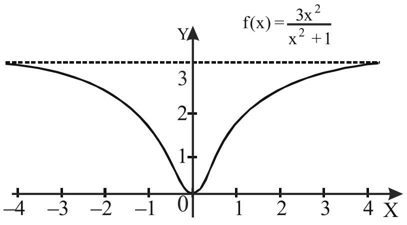
$$= \begin{cases} \frac{a_0}{b_0}, & \text{when } m = n \\ 0, & \text{when } m < n \\ \infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} > 0 \\ -\infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} < 0 \end{cases}$$

Thus, the limit of the ratio of two polynomials is one of the following : the ratio of leading coefficients of the polynomials, if their degrees coincide; infinity, if the degree of the numerator is greater than that of the denominator; zero, if the degree of the numerator is less than that of the denominator.

Definition of Horizontal Asymptote

The line y = L is a horizontal asymptote of the graph of f if $\lim\limits_{x\to-\infty} f(x) = L$ OR $\lim\limits_{x\to\infty} f(x) = L$.

Consider the graph of $f(x) = \frac{3x^2}{x^2+1}$.



The limit of f(x) as x approaches – ∞ or ∞ is 3.

As shown in the figure, we see that the value of f(x) approaches 3 as x increases without bound ($x \to \infty$). Similarly, f(x) approaches 3 as $x \to -\infty$. These limits at infinity are denoted by

$$\lim_{x\to-\infty} f(x) = 3 \text{ and } \lim_{x\to\infty} f(x) = 3.$$

We see that the line y = 3 is a horizontal asymptote of the graph of $f(x) = \frac{3x^2}{x^2+1}$.

Limits of Form $\left(\frac{\infty}{\infty}\right)$

Consider the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$. Here a may be finite or infinite.

This is called $\left(\frac{\infty}{\infty}\right)$ form.

One way to handle this indeterminate form is to write the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ as $\lim\limits_{x\to a}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$ which enters into $\left(\frac{0}{0}\right)$ form which can be solved by methods discussed earlier.

For example, $\lim\limits_{x\to 0}\frac{\cot x}{\cot 2x} \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0}\frac{\tan x}{\tan 2x} \left(\frac{0}{0}\right)$$

which can be evaluated using standard limits to get 1/2.

Another way is to find the largest term occurring in the numerator and denominator and divide them by this term and evaluate the limit using the basic result:

If a term $p(x) \to \infty$ as $x \to a$ then its reciprocal $\frac{1}{p(x)} \to \infty$ as $x \to a$.

For example, $\lim\limits_{x\to 0}\frac{x + 2\cot x}{\cos x + \cot x} \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0}\frac{\frac{x}{\cot x} + 2}{\frac{\cos x}{\cot x} + 1} = \lim_{x\to 0}\frac{x\tan x + 2}{\sin x + 1} = 2.$$

Example 6: Evaluate $\lim\limits_{x\to\infty}\frac{x^2 - 4x + 5}{x^3 - 3x^2 + 2x}$

Solution: Dividing the numerator and denominator by x3 which is the highest power of x.

$$= \lim_{x\to\infty}\frac{\frac{1}{x} - \frac{4}{x^2} + \frac{5}{x^3}}{1 - \frac{3}{x} + \frac{2}{x^2}}$$

$$= \frac{0-0+0}{1-0+0} = 0$$

Example 7: Evaluate $\lim\limits_{x\to\infty} \dfrac{x^3-4x^2+5}{x^2-3x+2}$

Solution: Dividing the numerator and denominator by x3 which is the highest power of x.

$$= \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{5}{x^3}}{\frac{1}{x}-\frac{3}{x^2}+\frac{2}{x^3}}$$

$$= \frac{1-0+0}{0-0+0} = \infty$$

Hence, the limit does not exist.

Example 8: Find $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+3x+4}{4x^3+3x^2+2x+1}$

Solution: This is an indeterminacy of the form ∞/∞. Let us divide the numerator and the denominator of the function by the leading power of x, i.e. by x^3.

$$\lim_{x\to\infty} \frac{x^3+2x^2+3x+4}{4x^3+3x^2+2x+1}$$

$$= \lim_{x\to\infty} \frac{1+2/x+3/x^2+4/x^2}{4+3/x+2/x^2+1/x^3} = \frac{1}{4}.$$

Example 9: $\lim\limits_{x\to\infty} \dfrac{3x^4-2}{\sqrt{x^8+3x+4}}$.

Solution: Let us divide the numerator and the denominator by x^4.

$$\lim_{x\to\infty} \frac{3x^4-2}{\sqrt{x^8+3x+4}}$$

$$= \lim_{x\to\infty} \frac{3-2/x^4}{\sqrt{1+3/x^7+4/x^8}} = \frac{3}{1} = 3.$$

Example 10: Find the value of $\lim\limits_{x\to\infty} \left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$

Solution: Since, $\lim\limits_{x\to\infty} \dfrac{x^2+2x-1}{2x^2-3x-2} = \dfrac{1}{2}$ and

$\lim\limits_{x\to\infty} \dfrac{2x+1}{2x-1} = 1$ we have $\lim\limits_{x\to\infty} \left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}} = \dfrac{1}{2}$.

Example 11: Find

(a) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2}}{3x-6}$ (b) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+2}}{3x-6}$

Solution: In both parts it would be helpful to manipulate the function so that the powers of x are transformed to powers of 1/x. This can be achieved in both cases by dividing the numerator and denominator by |x| and using the fact that $\sqrt{x^2} = |x|$.

(a) As $x \to \infty$, the values of x under consideration are positive, so we can replace |x| by x where helpful. We obtain

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+2/x^2}}{3-6/x} = \frac{\lim\limits_{x\to\infty}\sqrt{1+2/x^2}}{\lim\limits_{x\to\infty}(3-6/x)}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty}(1+2/x^2)}}{\lim\limits_{x\to\infty}(3-6/x)} = \frac{\sqrt{1+0}}{3-0} = \frac{1}{3}$$

(b) As $x \to -\infty$, the values of x under consideration are negative, so we can replace |x| by –x where helpful. We obtain

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{2}{x^2}}}{-3+6/x} = \frac{-1}{3}$$

Example 12: Evaluate $\lim\limits_{x\to\infty} \left\{ \dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}}}} \right.$ ....infinity $\left.\vphantom{\dfrac{x}{x}}\right\}$

Solution: Let $y = \dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}}}}$ ....infinity

Then $y = \dfrac{x}{x+\dfrac{y}{x^{2/3}}}$

$$\Rightarrow \quad y = \frac{x^{5/3}}{x^{5/3}+y}$$

$$\Rightarrow \quad y^2 + (x^{5/3})y - x^{5/3} = 0$$

$$\therefore \quad y = \frac{-x^{5/3} \pm \sqrt{x^{10/3} + 4x^{5/3}}}{2}$$

$$= y = \frac{-x^{5/3} + \sqrt{x^{10/3} + 4x^{5/3}}}{2} \qquad (\because\ y > 0)$$

$$= \frac{4x^{5/3}}{2(\sqrt{x^{10/3} + 4x^{5/3}} + x^{5/3})}$$

$$= \frac{2}{\sqrt{\left(1 + \frac{4}{x^{5/3}}\right)} + 1}$$

$$\therefore \quad \lim_{x\to\infty} y = \frac{2}{\sqrt{1+0}+1} = \frac{2}{2} = 1.$$

Study Tip

As $x \to \infty$ we get $\frac{1}{x} \to 0$. Hence in many problems we write the expression in terms of $\frac{1}{x}$ and apply $\frac{1}{x} \to 0$.

To present this in a simple form we use the substitution $x = 1/t$. Hence,

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t) \text{ and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f(1/t), \text{ if these limits exist.}$$

For the latter limit we may also write

$$\lim_{x\to-\infty} f(x) = \lim_{t\to 0^+} f(-1/t) \text{ using the}$$

substitution $x = -1/t$.

Example 13: Evaluate $\lim_{x\to\infty} x \sin\frac{1}{x}$

Solution: $\lim_{x\to\infty} x \sin\frac{1}{x}$

Put $x = \frac{1}{t}$ As $x \to \infty$, $t \to 0^+$

$$= \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$

Example 14: Find $\lim_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x-2}$

Solution: $\lim_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x-2}$

Put $x = \frac{-1}{t}$ As $x \to -\infty$, $t \to 0^+$

$$= \lim_{t\to 0^+} \frac{\sqrt{3+2t^2} \cdot \frac{1}{\sqrt{t^2}}}{\frac{1-2t}{t}}$$

$$= \lim_{t\to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)} \frac{t}{|t|}$$

$$= \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

Example 15: Evaluate $\lim_{x\to\infty} \frac{e^{1/x^2}-1}{2\arctan x^2 - \pi}$

Solution: $\lim_{x\to\infty} \frac{e^{1/x^2}-1}{2\arctan x^2 - \pi} \left(\frac{0}{0}\right)$

$$= \lim_{x\to\infty} \frac{e^{1/x^2}-1}{-2\left[\frac{\pi}{2} - \tan^{-1} x^2\right]} = \lim_{x\to\infty} \frac{e^{1/x^2}-1}{-2\cot^{-1} x^2}$$

$$= -\lim_{x\to\infty} \frac{1}{2} \frac{e^{1/x^2}-1}{\tan^{-1}\frac{1}{x^2}}$$

Put $x = \frac{1}{t}$ As $x \to \infty$, $t \to 0^+$

$$= -\lim_{t\to 0^+} \frac{1}{2} \frac{e^t - 1}{t} \cdot \frac{t}{\tan^{-1} t} = -\frac{1}{2}.$$

Example 16: Find $\lim_{x\to\infty} \left(\frac{1-2x}{(1+8x^3)^{1/3}} + 2^{-x^7}\right)$

Solution: $\lim_{\substack{x\to\infty \\ 1/x\to 0}} \frac{\left(\frac{1}{x} - 2\right)}{\left(\frac{1}{x^3} + 8\right)^{1/3}} + \frac{1}{2^{x^7}} = \frac{-2}{(8)^{1/3}} = -1.$

Example 17: Find $\lim_{x\to\infty} \frac{x+\cos x}{x+\sin x}$

Solution: Here sin x occurs in the given expression and when $x \to \infty$, sin x oscillates between –1 and 1.

Let $x = \frac{1}{t}$ $\therefore$ $t = \frac{1}{x}$ and when $x \to \infty$, $t \to 0$

Now $\lim_{x\to\infty} \frac{x+\cos x}{x+\sin x}$

$$= \lim_{t\to 0} \frac{\frac{1}{t} + \frac{\cos\frac{1}{t}}{1}}{\frac{1}{t} + \frac{\sin\frac{1}{t}}{1}} = \lim_{t\to 0} \frac{\frac{1 + t\cos\frac{1}{t}}{t}}{\frac{1 + t\sin\frac{1}{t}}{t}}$$

$$= \lim_{t\to 0}\frac{1+t\cos\frac{1}{t}}{1+t\sin\frac{1}{t}} = \frac{1+0}{1+0} = 1$$

$\because$ when $t \to 0$, $t\sin\frac{1}{t} \to 0$ and $t\cos\frac{1}{t} \to 0$

Note: $\lim\limits_{x\to\infty}\frac{\sin x}{x} = \lim\limits_{x\to\infty}\frac{\cos x}{x} = 0$.

Note: The graphs of $y = e^x$ and $y = \ln x$ suggest that as $x \to \infty$, $e^x \to \infty$ much more rapidly than $\ln x \to \infty$. Thus, when x is large positive, e^x is so much larger than ln x that the quotient $e^x/(\ln x)$ is large positive.

Hence $\lim\limits_{x\to\infty}\frac{e^x}{\ln x} = \infty$.

The exponential function is notable for its very rapid rate of increase with increasing x. In fact, e^x increases more rapidly as $x \to \infty$ than any fixed power of x.

Thus, $\lim\limits_{x\to\infty}\frac{x^2}{e^x} = 0$.

In fact, if n is a positive integer then

$$\lim_{x\to\infty}\frac{e^x}{x^n} = \infty, \qquad \lim_{x\to\infty} x^n e^{-x} = 0 .$$

Study Tip

For sufficiently large values of x we have

$$\begin{array}{lllll} \log_a x & << x^p & << a^x & << x! & \text{where} \\ a > 1 & p > 0 & a > 1 & x \in N. & \end{array}$$

What we mean by this is that the function x^p is much larger than $\log_a x$ for large values of x when $p > 0$ and $a>1$.
As $x\to\infty$, both $\log_a x$ $(a > 1)$ and x^p approach ∞, but x^p $(p > 0)$ is much larger than $\log_a x$.

Hence, $\lim\limits_{x\to\infty}\frac{\log_a x}{x} = 0$, for $a > 1$.

Specially, $\lim\limits_{x\to\infty}\frac{(\ln x)}{x} = 0$

Also, $\lim\limits_{n\to\infty}\frac{n^p}{a^n} = 0$ for $p > 0$ and $a > 1$, and $\lim\limits_{n\to\infty}\frac{a^n}{n!} = 0$.

For example, $\lim\limits_{n\to\infty}\frac{2^n}{n!} = 0$.

Example 18: Evaluate $\lim\limits_{x\to\infty}\frac{4^x - 2\ell nx}{x^2 - 2^{2x+1}}$

Solution Limit $= \lim\limits_{x\to\infty}\dfrac{1 - 2\frac{\ell nx}{4^x}}{\frac{x^2}{4^x} - 2} = \dfrac{1-2.0}{0-2} = -\dfrac{1}{2}$.

Concept Problems I

1. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\frac{x^4+5x^3+7}{2x^5+3x^4+1}$

(ii) $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}$

(iii) $\lim\limits_{x\to\infty}\frac{(2x^3+4x+5)(x^2+x+1)}{(x+2)(x^4+2x^3+7x^2+x-1)}$

(iv) $\lim\limits_{x\to\infty}\frac{2x^4+3x^2+5x-6}{x^3+3x^2+7x-1}$

2. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\frac{2x}{5^{x+3}}$

(ii) $\lim\limits_{x\to\infty}\cos\left\{\log\left(\frac{x-1}{x}\right)\right\}$

(iii) $\lim\limits_{x\to\infty}\frac{[x]-1}{x}$

(iv) $\lim\limits_{x\to\infty}\left(\frac{3x^2+2x+1}{x^2+x+2}\right)^{(6x+1)/(3x+2)}$

3. Evaluate the following limits:

(i) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}$ (ii) $\lim\limits_{x\to\infty}\frac{\tan^{-1}x}{x}$

(iii) $\lim\limits_{x\to\infty}\tan^{-1}(x^2 - x^4)$ (iv) $\lim\limits_{x\to\infty}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$

4. Show that $\lim\limits_{x\to\infty} x\tan^{-1}\frac{1}{x} = \lim\limits_{x\to\infty} x\cot^{-1}x = 1$

5. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\frac{\cos x}{\sqrt{x^2+1}}$ (ii) $\lim\limits_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

(iii) $\lim\limits_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$ (iv) $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2+1}}{3x-6}$

6. Show that $\lim\limits_{x\to\infty} x^n e^{-x} = 0$ for all positive integers. Hence find $\lim\limits_{x\to\infty}\frac{x^{1000}}{e^x}$.

Practice Problems H

7. Evaluate $\lim\limits_{x\to\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$ and $\lim\limits_{x\to-\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$

8. Assuming that m and n are positive integers, find

$$\lim_{x\to-\infty}\frac{2+3x^n}{1-x^m}.$$

9. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}+\sqrt[3]{x}+\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$

(ii) $\lim\limits_{x\to\infty} x^4\left(\cos\dfrac{1}{x}-1+\dfrac{1}{2x^2}\right)$

(iii) $\lim\limits_{x\to\infty}\dfrac{(x-3)^{40}(5x+1)^{10}}{(3x^2-2)^{25}}$

(iv) $\lim\limits_{x\to\infty}\dfrac{\sqrt[5]{x^7+3}+\sqrt[4]{2x^3-1}}{\sqrt[6]{x^8+x^7+1}-x}$

10. Find $\lim\limits_{x\to\infty}\ \log_{x-1}x\,.\,\log_x(x+1).\log_{x+1}(x+2).$

$\log_{x+2}(x+3)\ ....\ \log_{x^5-1}x^5.$

11. Find $\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+.......+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+....+\sqrt[n]{(2x-3)}}$

12. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\dfrac{x}{\ln^2 x}$ (ii) $\lim\limits_{x\to\infty}\dfrac{e^x}{x^5}$

(iii) $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt[3]{x}}$ (iv) $\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x}$

13. Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}\dfrac{x+\sin 2x}{x}$

(ii) $\lim\limits_{x\to\infty}\dfrac{x(2+\sin x)}{x^2+1}$

(iii) $\lim\limits_{x\to\infty}\dfrac{x(2+\sin 2x)}{x+1}$

(iv) $\lim\limits_{x\to\infty}\dfrac{\ln^{100}x}{x^5}$

14. Find $\lim\limits_{x\to\infty}\dfrac{(x+1)^{10}+(x+2)^{10}+....+(x+100)^{10}}{x^{10}+10^{10}}$

15. Find the limit of the sequence $\{x_n\}$ which is defined by the recursive relation:

$x_{n+1}=x_n(2-x_n)\ \forall\ n\ge 1$, where x_1 is an arbitrary number satisfying the inequalities $0<x_1<1$.

16. Evaluate $\lim\limits_{x\to-\infty}\dfrac{(3x^4+2x^2)\sin\dfrac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$.

1.12 Asymptotes

Infinite Limits

Consider the function f(x) = 1/x. As x approaches 0 from the right, 1/x, which is positive, becomes arbitrarily large. The notation for this is

$$\lim_{x\to 0^+}\frac{1}{x}=\infty.$$

As x approaches 0 from the left, 1/x, which is negative, has arbitrarily large absolute values. The notation for this is

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty.$$

The behaviour of 1/x, is quite different from that of $1/x^2$. Since x^2 is positive whether x is positive or negative and since $1/x^2$ is large when x is near 0, we have

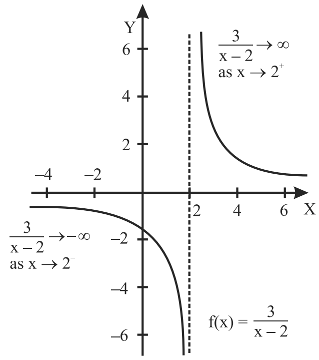
$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty \text{ and } \lim_{x\to 0^-}\frac{1}{x^2}=\infty$$

In this case we may write $\lim\limits_{x\to 0}\dfrac{1}{x^2}=\infty$, meaning that "as x → 0, both from the right and from the left, $1/x^2$ becomes arbitrarily large through positive values."

We can also write $\lim\limits_{x\to 0}\dfrac{1}{|x|}=\infty$ but there is no corresponding statement for $\lim\limits_{x\to 0}\dfrac{1}{x}$.

Let f be the function given by f(x) = 3/(x–2). From figure we can see that f(x) decreases without bound as x approaches 2 from the left, and f(x) increases without bound as x approaches 2 from the right. This behaviour is denoted as

$$\lim_{x\to 2^-}\frac{3}{x-2}=-\infty \text{ and } \lim_{x\to 2^+}\frac{3}{x-2}=\infty$$



f(x) increases and decreases without bound as x approaches 2.

If it were possible to extend the graphs toward infinity, we would see that the graph becomes arbitrarily close to the vertical line x = 2. This line is a vertical asymptote of the graph of f.

Example 1: Determine the points x = a for which each denominator is zero. Then see what happens to y as x → a– and as x → a+.

(a) $y = \dfrac{x - 1}{(x + 2)\,(x - 2)}$

(b) $y = \dfrac{(x + 2)\,(x - 1)}{(x - 3)^2}$

(c) $y = \dfrac{(x + 2)\,(1 - x)}{x - 3}$

Solution:

(a) The denominator is zero for x = –2 and x = 1.
As $x \to -2^-$, $y \to -\infty$; as $x \to -2^+$, $y \to \infty$.
As $x \to 1^-$, $y \to \infty$; as $x\ \infty\ 1^+$, $y \to -\infty$.

(b) The denominator is zero for x = 3
As $x \to 3^-$, $y \to \infty$; as $x \to 3^+$, $y \to \infty$.

(c) The denominator is zero for x = 3.
As $x \to 3^-$, $y \to \infty$; as $x \to 3^+$, $y \to -\infty$.

Vertical Asymptotes

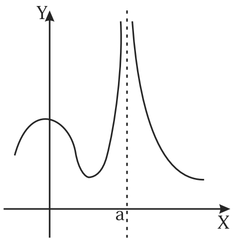
If f(x) approaches infinity (or negative infinity) as x approaches a from the right or the left, then the line x = a is a vertical asymptote of the graph of f.

The line x = a is called a vertical asymptote of the curve y = f(x) if atleast one of the following statements is true :

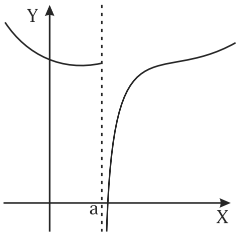
$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Note: If a function f has a vertical asymptote as x = a, then f is discontinuous at a.



$\lim_{x\to a} f(x) = \infty$



$\lim_{x\to a^+} f(x) = -\infty$

If the function is a quotient of two functions then the vertical asymptote occurs at a number where the denominator is 0 and the numerator is not 0.

Theorem Let f and g be continuous on an open interval containing a. If f(a) ≠ 0, g(a) = 0. and there exists an open interval containing a such that g(x) ≠ 0 for all x ≠ a in the interval, then the graph of the function given by $h(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at x = a.

Example 2: Determine all vertical asymptotes of the graphs of the functions:

(a) $f(x) = \dfrac{1}{2(x+1)}$ (b) $f(x) = \dfrac{x^2+1}{x^2-1}$ (c) f(x) = cot x

Solution:

(a) When x = – 1, the denominator is 0 and the numerator is not 0. Hence, we can conclude that x = – 1 is a vertical asymptote.

(b) By factoring the denominator as
$(x^2 - 1) = (x - 1)(x + 1)$
we can see that the denominator is zero at x = – 1 and x = 1. Moreover, because the numerator is not 0 at these two points, we conclude that the graph of f has two vertical asymptotes.

(c) By writing the function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

we conclude that vertical asymptotes occur at all values of x such that cos x ≠ 0 and sin x = 0. Hence, the graph of this function has infinitely many vertical asymptotes. These occur when x = nπ, where n is an integer.

The theorem requires that the value of the numerator at x = a be nonzero. If both the numerator and the denominator are 0 at x = a, then we obtain the indeterminate form 0/0, and we cannot determine the whether a vertical asymptote exists at x = a or not, without further investigation.

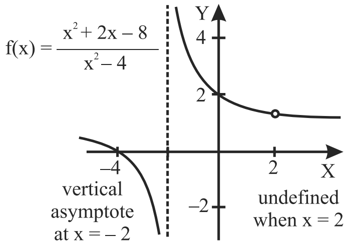
Example 3: Determine all vertical asymptotes of the graph $f(x) = \dfrac{x^2+2x-8}{x^2-4}$.

Solution: We simplify the expression, as follows :

$$f(x) = \frac{x^2+2x-8}{x^2-4} = \frac{(x+4)(x-2)}{(x+2)(x-2)} = \frac{x+4}{x+2},\ x \neq 0.$$

At all values of x other than x = 2, the graph of f coincides with the graph of g(x) = (x + 4)/(x + 2). Thus, we conclude that there is a vertical asymptote at x = – 2, as shown in the figure. From the graph, we can see that

$$\lim_{x\to 2^-}\frac{x^2+2x-8}{x^2-4}=-\infty \text{ and } \lim_{x\to 2^+}\frac{x^2+2x-8}{x^2-4}=\infty$$



f(x) increases and decreases without bound as x approaches – 2

Note that x = 2 is not a vertical asymptote.

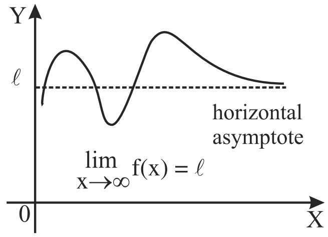
Horizontal Asymptotes

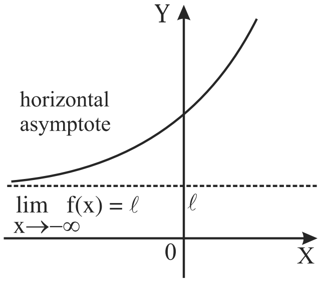
Bearing in mind the geometrical interpretation of the limit of a function, as x → ∞, we can say that if

$\lim_{x\to\infty} f(x) = \ell$, the curve y = f(x) has the straight line y = ℓ as its horizontal asymptote for x → ∞.

The line y = ℓ is called a horizontal asymptote of the curve y = f(x) if either

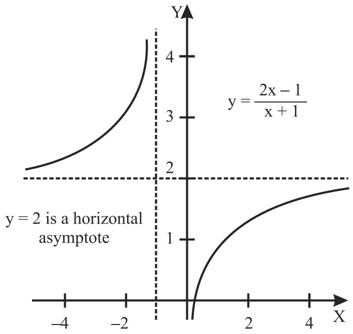
$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$





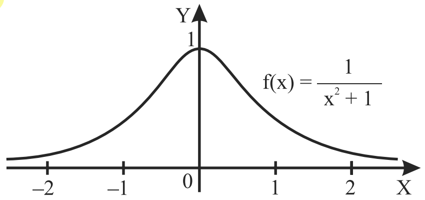
For example, the line y = 2 is a horizontal asymptote to the right in the curve $f(x) = \frac{2x-1}{x+1}$ since $\lim_{x\to\infty} f(x) = 2$.

By taking the limit as x → – ∞, we see that y = 2 is also a horizontal asymptote to the left. The graph of the function is shown in the figure.



Note: From the definition it follows that the graph of a function of x can have atmost two horizontal asymptotes – one to the right and one to the left.

We can see in the figure that the function f(x) = 1/(x² + 1) approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left.



f has a horizontal asymptote y = 0

In a number of cases the function f(x) behaves in a different manner as x → ∞ and as x → –∞ .

For instance, for the function $f(x) = \frac{\sqrt{9x^2+1}}{x-1}$ defined for all x ≠ 1, we have

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2+1}}{x-1}=-3,\ \lim_{x\to\infty}\frac{\sqrt{9x^2+1}}{x-1}=3$$

Therefore, when investigating for horizontal asymptotes of functions, we usually consider both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Example 4: Determine the limits

(a) $\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$ (b) $\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

Solution

(a) For x > 0, we can write $x = \sqrt{x^2}$. Thus, dividing both the numerator and the denominator by x produces

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$$

$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}.$$

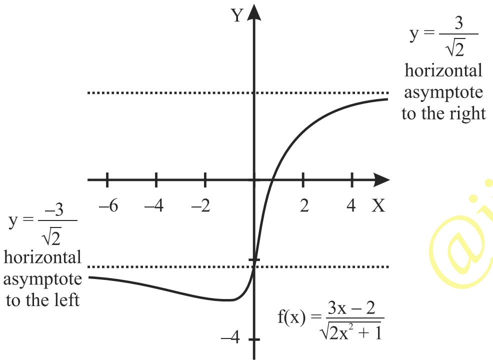
(b) For x < 0, you can write $x = -\sqrt{x^2}$. Thus, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\frac{2}{x}}{-\frac{\sqrt{2x^2+1}}{x^2}} = \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

and we can take the limit as follows.

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty} \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

$$= \frac{3-0}{-\sqrt{2+0}} = \frac{3}{-\sqrt{2}}.$$



Use of Asymptotes in Graphing

Some examples of graphing rational functions will show the usefulness of asymptotes.

Example 5: Draw the graph of $f(x) = \frac{1}{x(x-1)}$.

Solution: Note that when x = 0 or when x =1 the denominator x(x–1) is 0. Thus 0 and 1 are not in the domain of the function. More important, when x is near 0 or near 1 the quotient 1/[x(x–1)] has large absolute value. Thus the lines x = 0 and x = 1 are vertical asymptotes for the graph. To decide how the graph approaches these asymptotes, it is necessary to examine the sign of f(x) for x near 0 and for x near 1. Consider x near 0. If x is a very small positive number, then

$$f(x) = \frac{1}{x}\,\frac{1}{x-1}$$

is the product of 1/x, which is a large positive number, and 1/(x–1), which is near 1/(–1) = –1. So, for x small and positive, f(x) is negative and of large absolute value; that is, $\lim_{x\to0^+} f(x) = -\infty$. This fact is recorded in Fig. 1

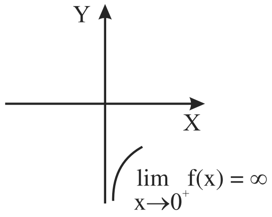


Fig. 1

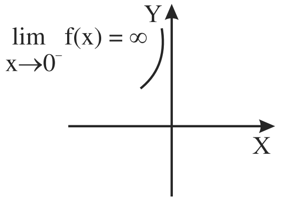


Fig. 2

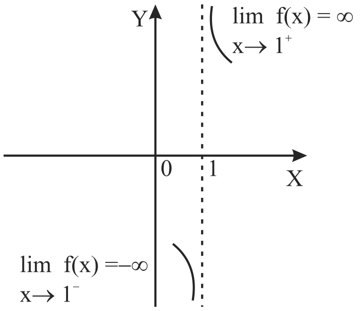

Y
$\lim_{x\to1^+} f(x) = \infty$
0
1
X
$\lim_{x\to1^-} f(x) = -\infty$

Fig. 3

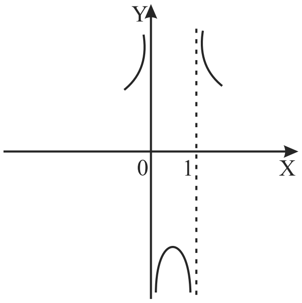


Fig. 4

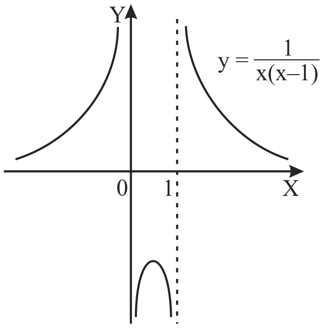


Fig. 5

If x is near 0, but negative, then 1/x is negative and of large absolute value. Again, 1/(x–1) is near –1. Thus,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x}\,\frac{1}{x-1} = \infty$$

This fact is recorded in Fig.2.

Next, how does f(x) behave near x = 1?

Consider first x near 1 but larger than 1.

Then $f(x) = \frac{1}{x}\,\frac{1}{x-1}$ is a large positive number, since 1/x is near 1 and x – 1 is a small positive number. Thus,

$$\lim_{x\to 1^+} f(x) = \infty$$

Similarly, $\lim\limits_{x\to 1^-} f(x) = -\infty$

These two facts are recorded in Fig. 3. Piecing together these three figures suggests that the graph of f for x in or near the interval (0,1) looks something like Fig. 4. How high the curve goes for x in (0,1) is the type of question answered in a later chapter with the aid of derivative.

How does f(x) behave when |x| is large?

Since $\lim\limits_{x\to\infty} \frac{1}{x(x-1)} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x(x-1)} = 0$

the x axis is a horizontal asymptote (both for x positive and for x negative). In both cases the graph approaches the x axis from above, not from below, since the function is positive when |x| is large. The graph of f must look something like Fig. 5.

Example 6: Discuss and sketch the graph of the equation $y = \frac{4x^2}{x^2+1}$.

Solution: The graph is symmetric w.r.t. the y axis, since x occurs with even powers only.

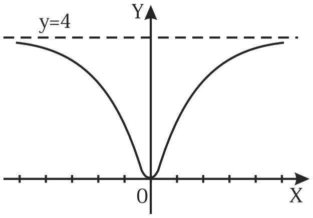
Since, $0 \le \frac{x^2}{x^2+1} < 1$, $0 \le y < 4$ for every number x.

Thus the graph lies between the lines y = 0 and y = 4.

Since, $\lim\limits_{x\to\pm\infty} \frac{4x^2}{x^2+1} = \lim\limits_{x\to\pm\infty} \frac{4}{1+1/x^2} = 4$,

the line y = 4 is an asymptote of the graph of the function. The graph may be drawn as shown in the figure.

$y = \frac{4x^2}{x^2+1}$



The notion of an asymptote can be extended to include curves as well as lines. Specifically, we say that f(x) is asymptotic to g(x) as x → ∞ if

$$\lim_{x\to\infty} [f(x) - g(x)] = 0$$

and that f(x) is asymptotic to g(x) as x → –∞ if

$$\lim_{x\to-\infty} [f(x) - g(x)] = 0$$

Informally stated, if f(x) is asymptotic to g(x) as x → ∞, then the graph of y = f(x) gets closer and closer to the graph of y = g(x) as x → ∞, and if f(x) is asymptotic to g(x) as x → –∞, then the graph of y = f(x) gets closer and closer to the graph of y = g(x) as x → –∞.

For example, if

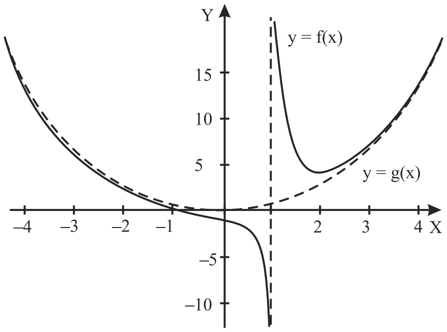
$$f(x) = x^2 + \frac{2}{x-1} \text{ and } g(x) = x^2$$

then g(x) is asymptotic to g(x) as x → ∞ and as x → –∞ since

$$\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} \frac{1}{x-1} = 0$$

$$\lim_{x\to-\infty} [f(x) - g(x)] = \lim_{x\to-\infty} \frac{1}{x-1} = 0$$

This asymptotic behaviour is illustrated in the following figure, which also shows the vertical asymptote of f(x) at x = 1.



Concept Problems

1. Determine the vertical and horizontal asymptotes of the graph of the given functions :

   (i) $f(x) = \frac{2x^2}{(x+2)^2}$ (ii) $f(x) = \frac{2x^2}{x^2+9}$

2. Sketch the graph of each of the following functions, indicating the vertical and horizontal asymptotes.

   (i) $f(x) = \frac{x^2-1}{x^2-4}$ (ii) $f(x) = \frac{x^2+1}{x^2-4}$

3. Use asymptotes to sketch the graphs.

   (i) $y = \frac{1}{x^2-x}$ (ii) $y = \frac{1}{x^4-x^2}$ (iii) $y = \frac{x(x-1)}{x^2+1}$

4. Sketch the graph of (i) $y = xe^{-x}$ (ii) y = x ln x.

1.13 Limit of a sequence

Here we consider limit of a sequence f(n) as n tends to ∞. We say that the sequence approaches the limit λ if for all sufficiently large integral values of n the corresponding values of f(n) become arbitrarily close to the number λ.

If $f(n) \to \infty$ we may say that f(n) diverges to ∞ . If f(n) does not tend to a limit or to ∞ , then it oscillates finitely or infinitely: in this case we say that the sequence oscillates finitely or infinitely.

The only difference between the 'tending of n to ∞' and 'tending of x to ∞', is that x assumes all values as it tends to ∞ , i.e. that the point x coincides in turn with every point of the number line to the right of its initial position, whereas n tended to ∞ by a series of jumps. We can express this distinction by saying that x tends continuously to ∞.

There is a very close correspondence between functions of x and functions of n. Every function of n may be regarded as a selection from the values of a function of x.

We can easily see that $[x] \to \infty$, x–[x] oscillates finitely, and $[x] + \sqrt{\{x-[x]\}} \to \infty$ as $x \to \infty$.

One simple remark may be inserted here. The function f(x) = x – [x] oscillates between 0 and 1, as is obvious from its graph. It is equal to zero whenever x is an integer, so that the function f(n) derived from it, is always zero and so it tends to the limit zero. The same is true if $f(x) = \sin x\pi$.

$$f(n) = \sin n\pi = 0.$$

It is evident that $f(x) \to l$ or $f(x) \to -\infty$ involves the corresponding property for f(n), but that the converse in often untrue.

Note: If f(n) is a sequence such that $\lim_{n\to\infty} f(n) = a$, then we have $\lim_{n\to\infty} f(n+1) = a$.

Example 1: Find $\lim_{n\to\infty} \dfrac{\sqrt{n^2+1}+\sqrt{n}}{\sqrt[4]{n^3+1}-\sqrt{n}}$

Solution: $\lim_{n\to\infty} \dfrac{n\left[\sqrt{1+1/n^2}+\sqrt{1/n}\right]}{n^{3/4}\left(\sqrt[4]{1+1/n^3}-\sqrt[4]{1/n}\right)}$

$$= \lim_{n\to\infty} n^{1/4} \frac{\left(\sqrt{1+1/n^2}+\sqrt{1/n}\right)}{\left(\sqrt[4]{1+1/n^3}-\sqrt[4]{1/n}\right)} = \infty .$$

Example 2: Find $\lim_{n\to\infty} \dfrac{n!}{(n+1)!-n!}$

Solution: $\lim_{n\to\infty} \dfrac{n!}{(n+1)!-n!}$

$$= \lim_{n\to\infty} \frac{\dfrac{n!}{(n+1)!}}{1-\dfrac{n!}{(n+1)!}} = \lim_{n\to\infty} \frac{\dfrac{1}{n+1}}{1-\dfrac{1}{n+1}} = 0 .$$

Example 3: Evaluate $\lim_{n\to\infty} \dfrac{\underline{|3(n+1)}}{(n+1)^3\,\underline{|3n}}$, $n \in N$

Solution: Limit = $\lim_{n\to\infty} \dfrac{(3n+1)(3n+2)(3n+3)}{(n+1)^3}$

$$= \lim_{n\to\infty} \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)^3} = 27$$

Example 4: Find

$$\lim_{n\to\infty} \frac{2n}{2n^2-1}\cos\frac{n+1}{2n-1} - \frac{n}{1-2n}\times\frac{n(-1)^n}{n^2+1}$$

Solution: $\lim_{n\to\infty} \left[\left(\dfrac{2}{2-1/n^2}\right)\dfrac{1}{n}\cos\left(\dfrac{1+1/n}{2-1/n}\right) - \dfrac{1}{(1/n-2)}\times\dfrac{(-1)^n}{(1+1/n^2)}\times\dfrac{1}{n}\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{2}{2-1/n^2}\right)\cos\left(\frac{1+1/n}{2-1/n}\right) - \frac{1}{(1/n-2)}\times\frac{(-1)^n}{(1+1/n^2)}\right]$$

$$= 0\times\left[\frac{2}{2}\times\cos\frac{1}{2}+\frac{1}{2}\times\frac{\pm 1}{1}\right] = 0.$$

Example 5: Evaluate

$$\lim_{n\to\infty} \frac{1.n+2.(n-1)+3(n-2).....+n.1}{n^3}$$

Solution: Limit = $\lim_{n\to\infty} \dfrac{\sum_{r=1}^{n} r(n-r+1)}{n^3}$

$$= \lim_{n\to\infty} \frac{(n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{\dfrac{(n+1)n(n+1)}{2} - \dfrac{n(n+1)(2n+1)}{6}}{n^3}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} .$$

Example 6: If the value of

$$\lim_{n\to\infty} n^3\left(1-\frac{3}{7}\right)\left(1-\frac{3}{8}\right)\left(1-\frac{3}{9}\right)....\left(1-\frac{3}{n}\right)$$

is k then find the sum of digits of k.

Solution: $\lim_{n\to\infty} n^3 \dfrac{4}{7}\times\dfrac{5}{8}\times\dfrac{6}{9}\times\dfrac{7}{10}\times......\dfrac{n-3}{n}$

$$= \lim_{n\to\infty} \frac{4\times 5\times 6n^3}{(n-2)(n-1)n} = 120 = k.$$

Thus the sum of digits of k = 3.

Example 7: Find the value of $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$.

Solution: We have $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$

$$= \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right)\right)$$

$$= \sum_{n=2}^{\infty}\left(\ln\left(\frac{n-1}{n}\right)+\ln\left(\frac{n+1}{n}\right)\right)$$

$$= \sum_{n=2}^{\infty}\left(\ln\left(\frac{n-1}{n}\right)-\ln\left(\frac{n}{n+1}\right)\right)$$

$$= \left(\ln\frac{1}{2}-\ln\frac{2}{3}\right)+\left(\ln\frac{2}{3}-\ln\frac{3}{4}\right) + \ldots..$$

$$= \ln\frac{1}{2} - \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln\frac{1}{2} - \ln 1 = - \ln 2 .$$

Example 8: Evaluate $\lim_{n\to\infty} \frac{x^n}{n!}$.

Solution: Since x is fixed number, there will be a positive integer N such that |x| < N.

Let $\frac{|x|}{N} = q \quad (0 < q < 1)$. Now, we have

$$\left|\frac{x^n}{n\,!}\right| = \left|\frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\ldots\ldots.\frac{x}{N-1}\cdot\frac{x}{N}\cdot\frac{x}{N+1}\ldots\frac{x}{n}\right|$$

$$< \frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\ldots\ldots..\frac{|x|}{N-1}.q.q\ldots..q$$

$$\left[\text{since } \frac{|x|}{N} \le q \text{ for } n \ge N\right]$$

$$= \frac{|x|^{N-1}}{(N-1)\,!}q^{n-N+1}$$

where $\frac{|x|^{N-1}}{(N-1)\,!}$ is a fixed quantity independent of n, while q^{n-N+1} approaches 0 as n approaches ∞.

Hence, we have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Example 9: Evaluate $\lim_{n\to\infty} \frac{2^{n+1}+n\underline{|n}}{\underline{|n+1}+2^n}$

Solution: Limit = $\lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}}{\underline{|n+1}}+\dfrac{n}{n+1}}{1+\dfrac{2^n}{\underline{|n+1}}} = \dfrac{0+1}{1} = 1$.

Example 10: Find

$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{n^2}\right)$$

Solution: We have limit

$$= \lim_{n\to\infty}\prod_{n=2}^{n}\frac{n^2-1}{n^2} = \lim_{n\to\infty}\prod_{n=2}^{n}\frac{n-1}{n}\cdot\prod_{n=2}^{n}\frac{n+1}{n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\ldots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\ldots..\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{n+1}{2} = \frac{1}{2}.$$

Example 11: Calculate $\lim_{n\to\infty}\sin(\pi\sqrt{n^2+1})$.

Solution: We have

$$\sqrt{n^2+1} = n\left(1+\frac{1}{n^2}\right)^{1/2} = n\left(1+\frac{1}{2}.\frac{1}{n^2}+\ldots\ldots..\right)$$

$$= n+\frac{1}{2n}+\ldots\ldots...$$

Hence $\sin(\pi\sqrt{n^2+1}) = \sin\left(\pi n+\frac{\pi}{2n}+\ldots\ldots.\right)$

$$= (-1)^n \sin\left(\frac{\pi}{2n}+\ldots\ldots.\right)$$

The sequence $\{(-1)^n\}$ is bounded and $\left\{\sin\left(\frac{\pi}{2n}+\ldots\ldots.\right)\right\}$ is infinitesimally small, and therefore the product of these two sequences is an infinitely small sequence. Thus,

$$\lim_{n\to\infty}\sin(\pi\sqrt{n^2+1}) = 0$$

.

Example 12: If $\lim_{n\to\infty}\frac{n\cdot 3^n}{n(x-2)^n+n\cdot 3^{n+1}-3^n} = \frac{1}{3}$

where n ∈ N, then find the number of integral values of 'x'.

Solution: We have

$$\lim_{n\to\infty}\frac{n\cdot 3^n}{n(x-2)^n+n\cdot 3^{n+1}-3^n} = \frac{1}{3}.$$

Dividing both numerator and denominator by n. 3^n,

$$\lim_{n\to\infty}\frac{1}{\left(\frac{x-2}{3}\right)^n+3-\frac{1}{n}} = \frac{1}{3}$$

Clearly for the limit to exist, we must have

$$-1 < \frac{x-2}{3} < 1 \quad\Rightarrow\quad -1 < x < 5$$

$\therefore$ The possible integral values of 'x' are 0, 1, 2, 3, 4
Hence there are 5 integral values of 'x'.

Behaviour of Two Functions as n Tends to Infinity

Results

1. If $f(n) \to \infty$ as n tends to infinity and $g(n) \geq f(n)$ for all values of n, then $g(n) \to \infty$ as n tends to infinity.
2. If $f(n) \to 0$, as n tends to infinity and $|g(n)| \leq |f(n)|$ for all values of n, then $g(n) \to 0$ as n tends to infinity.
3. If $\lim_{n\to\infty} |f(n)| = 0$, then $\lim_{n\to\infty} f(n) = 0$.
4. If f(n) tends to a limit or oscillates finitely as n tends to infinity , and $|g(n)| \leq |f(n)|$ when $n \geq N$, then g(n) tends to a limit or oscillates finitely as n tends to infinity.
5. If f(n) tends to ∞ or to $-\infty$ or oscillates infinitely as n tends to infinity , and $|g(n)| \geq |f(n)|$ when $n \geq N$, then g(n) tends to ∞ or to $-\infty$ or oscillates infinitely as n tends to infinity.

Behaviour of Sum of Two Functions as n Tends to Infinity

Theorem If f(n) and g(n) tends to limits a, b, then f(n) + g(n) tends to the limit a + b.
The argument is roughly like this : 'when n is large, f(n) is nearly equal to a and g(n) to b, and therefore their sum is nearly equal to a + b'.

Results

1. If f(n) tends to a limit, but g(n) tends to ∞ or to $-\infty$ or oscillates finitely or infinitely as n tends to infinity, then f(n) + g(n) behaves like g(n) as n tends to infinity.
2. If $f(n) \to \infty$ and $g(n) \to \infty$ or oscillates finitely as n tends to infinity, then $f(n) + g(n) \to \infty$ as n tends to infinity. In this statement we may obviously change ∞ into $-\infty$ throughout.
3. If $f(n) \to \infty$ and $g(n) \to -\infty$ as n tends to infinity, then f(n) + g(n) may tend either to a limit or to ∞ or to $-\infty$ or may oscillate either finitely or infinitely as n tends to infinity.
   These five possibilities are illustrated in order by
   (i) $f(n) = n$, $g(n) = -n$,
   (ii) $f(n) = n^2$, $g(n) = -n$,
   (iii) $f(n) = n$, $g(n) = -n^2$,
   (iv) $f(n) = n + (-1)^n$, $g(n) = -n$,
   (v) $f(n) = n^2 + (-1)^n n$, $g(n) = -n^2$.
4. If $f(n) \to \infty$ and g(n) oscillates infinitely as n tends to infinity, then f(n) + g(n) may tend to ∞ or oscillate infinitely as n tends to infinity, but cannot tend to a limit, or to $-\infty$, or oscillate finitely.
   For $g(n) = \{f(n) + g(n)\} - f(n)$; and, if f(n) + g(n) behaved in any of the three last ways, it would follow, from the previous results, that $g(n) \to -\infty$, which is not the case. As examples of the two cases which are possible, consider (i) $f(n) = n^2$, $g(n) = (-1)^n n$, (ii) $f(n) = n$, $g(n) = (-1)^n n^2$. Here again the signs of ∞ and $-\infty$ may be permuted throughout.
5. If f(n) and g(n) both oscillate finitely as n tends to infinity, then f(n) + g(n) must tend to a limit or oscillate finitely as n tends to infinity.
   For example
   (i) $f(n) = (-1)^n$, $g(n) = (-1)^{n+1}$,
   (ii) $f(n) = g(n) = (-1)^n$.
6. If f(n) oscillates finitely, and g(n) infinitely as n tends to infinity, then f(n) + g(n) oscillates infinitely as n tends to infinity.
7. If both f(n) and g(n) oscillate infinitely as n tends to infinity, then f(n) + g(n) may tend to a limit, or to ∞ , or to $-\infty$, or oscillate either finitely or infinitely as n tends to infinity.

Theorem If $\lim_{n\to\infty} f(n) = a$ and $\lim_{n\to\infty} g(n) = b$, then

$$\lim_{n\to\infty} f(n)g(n) = ab.$$

Theorem If $\lim_{n\to\infty} f(n) = a$ and $\lim_{n\to\infty} g(n) = b$ and b is not zero, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{a}{b}$.

Results

1. If $f(n) \to \infty$ and g(n) oscillates finitely as n tends to infinity, then f(n)g(n) must tend to ∞ and $-\infty$ or oscillate infinitely as n tends to infinity.
   Examples of these three possibilities may be obtained by taking f(n) to be n and g(n) to be one of the three functions $2 + (-1)^n$, $-2-(-1)^n$, $(-1)^n$
2. If f(n) and g(n) oscillate finitely as n tends to infinity, then f(n)g(n) must tend to a limit (which may be zero) or oscillate finitely as n tends to infinity.
   For example,
   (i) $f(n) = g(n) = (-1)^n$,
   (ii) $f(n) = 1 + (-1)^n$, $g(n) = 1 - (-1)^n$,
   (iii) $f(n) = \cos\frac{1}{3}n\pi$, $g(n) = \sin\frac{1}{3}n\pi$
3. A particular case of the theorem which is important is that in which g(n) is constant. The theorem then asserts simply that $\lim_{n\to\infty} kf(n) = ka$ if $\lim_{n\to\infty} f(n) = a$.
4. If $f(n) \to \infty$ as n tends to infinity then $kf(n) \to \infty$ or $kf(n) \to -\infty$, according as k is positive or negative, unless k = 0, when of course kf(n) = 0
   for all values of n and $\lim_{n\to\infty} kf(n) = 0$. And if f(n) oscillates finitely or infinitely as n tends to infinity, then so does kf(n), unless k = 0.

Note:

1. If f(n) is positive and $f(n + 1) \geq K.f(n)$, where $K > 1$, for all values of n, then $f(n) \to \infty$ as n tends to infinity.
   For $f(n) \geq K.f(n-1) \geq K^2 f(n-2) \geq ... \geq K^{n-1} f(1)$ from which the conclusion follows at once, since $K^n \to \infty$

Note that the same result is true if the conditions are satisfied only when $n \geq N$.

2. If $f(n)$ is positive and $f(n+1) < Kf(n)$, where $0 < K < 1$, then $\lim\limits_{n\to\infty} f(n) = 0$. This result is also true if the conditions are satisfied only when $n \geq N$.

3. If $|f(n+1)| \leq K\,|f(n)|$ when $n \geq N$ and $0 < K < 1$, then $\lim\limits_{n\to\infty} f(n) = 0$.

4. If $f(n)$ is positive and $\lim\limits_{n\to\infty} \frac{f(n+1)}{f(n)} = l > 1$, then $f(n) \to \infty$ as n tends to infinity.

   For we can determine N so that $\frac{f(n+1)}{f(n)} > K > 1$ when $n \geq N$; we may, for example take K half way between 1 and l and apply 1.

5. If $\lim\limits_{n\to\infty} \frac{f(n+1)}{f(n)} = l, \quad -1 < l < 1$,

   then $\lim\limits_{n\to\infty} f(n) = 0$.

Example 13: Determine $\lim\limits_{n\to\infty} n^r x^n$, where r is any positive integer.

Solution: Let $f(x) = n^r x^n$.

If $x = 0$ then $f(n) = 0$ for all values of n, and $\lim\limits_{n\to\infty} n^r x^n = 0$. In all other cases

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)x = x$$

First suppose x is positive.

Then $f(n) \to \infty$ if $x > 1$ and $f(n) \to 0$ if $x < 1$.

If $x = 1$, then $f(n) = n^r \to \infty$.

Secondly, suppose x is negative.

Then $|f(n)| = n^r\,|x|^n$ tends to ∞ if $|x| \geq 1$ and to 0 if $|x| < 1$.

$f(n)$ oscillates infinitely if $x \leq -1$

and $f(n) \to 0$ if $-1 < x < 0$.

Example 14: Prove that if x is positive then

$\sqrt[n]{x} \to 1$ as $n \to \infty$

Solution: Suppose, for example, $x < 1$.

Then $x, \sqrt{x}, \sqrt[3]{x}, \ldots$ is a decreasing sequence, and $\sqrt[n]{x} > 1$ for all values of n. Thus $\sqrt[n]{x} \to l$, where $l \geq 1$. But if $l > 1$ we can find values of n, as large s we please, for which $\sqrt[n]{x} \to l$ or $x > l^n$, and since $l^n \to \infty$ as $n \to \infty$, this is impossible.

Example 15: Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

Solution: $\sqrt[n+1]{(n+1)} < \sqrt[n]{n}$

if $(n+1)^n < n^{n+1}$ or $(1 + n^{-1})^n < n$, which is certainly satisfied if $n \geq 3$. Thus $\sqrt[n]{n}$ decreases as n increases from 3 onwards and since it is always greater than unity, it tends to a limit which is greater than or equal to unity. But if $\sqrt[n]{n} \to l$, where $l > 1$, then $n > l^n$, which is certainly untrue for sufficiently large values of n, since $l^n/n \to \infty$ with n.

Example 16: The sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies $\lim\limits_{n\to\infty} (x_{2n} + x_{2n}+1) = 315$ and $\lim\limits_{n\to\infty} (x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim\limits_{n\to\infty} (x_{2n}/x_{2n+1})$.

Solution: We have $\lim\limits_{n\to\infty} (x_{2n+1} - x_{2n-1})$

$= \lim\limits_{n\to\infty} [(x_{2n} + x_{2n+1}) - (x_{2n} + x_{2n-1})]$

$= 315 - 2003 = -1688$; it follows that $x_{2n+1} \to \infty$ as $n \to \infty$.

Then $\lim\limits_{n\to\infty} \frac{x_{2n}}{x_{2n+1}} = \lim\limits_{n\to\infty} \frac{x_{2n} + x_{2n+1}}{x_{2n+1}} - 1 = -1$

since $x_{2n} + x_{2n+1} \to 315$ while $x_{2n+1} \to -\infty$.

Concept Problems K

1. Evaluate the following limits :

   (i) $\lim\limits_{n\to\infty} \frac{2^n + 5.6^n}{3^n + 6^n}$

   (ii) $\lim\limits_{n\to\infty} \frac{(n+1)^4 - (n-1)^4}{3n^3 + 5}$

   (iii) $\lim\limits_{n\to\infty} \frac{n^3 + 3n^2 - n - 3}{n^2 + n - 6}$

   (iv) $\lim\limits_{n\to\infty} \frac{(\sqrt{n^2+1} + n)^2}{\sqrt[3]{n^6 + 1}}$

2. Evaluate the following limits:

   (i) $\lim\limits_{n\to\infty} n^2\left(1 - \cos\frac{1}{n}\right)$

   (ii) $\lim\limits_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$

   (iii) $\lim\limits_{n\to\infty} \frac{5^{n+1} + 3^n - 2^{2n}}{5^n + 2^n + 3^{n+3}}$

   (iv) $\lim\limits_{n\to\infty} \frac{2^{\frac{1}{n}}}{2^{\frac{1}{n}} + 1}$

3. Using the set $x_n = \frac{\pi}{2}(2n+1)$, show that the sequence $\{\sin x_n\}$ diverges. Hence prove that the function sin x does not have a limit as $x \to \infty$.

4. Prove that the following sequences tend to zero as n tends to infinity:

(a) $x_n = \frac{n}{2^n}$, (b) $x_n = \frac{3^n}{n!}$, (c) $x_n = \sqrt[n]{5^{-1}}$

5. Prove that $\sqrt[n]{(n!)} \to \infty$ as $n \to \infty$.

Practice Problems I

6. Evaluate the following limits:

(i) $\lim\limits_{n\to\infty} \frac{1^4+2^4+3^4+.....+n^4}{n^5}$

(ii) $\lim\limits_{n\to\infty} \frac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$

(iii) $\lim\limits_{n\to\infty} \frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}}$

(iv) $\lim\limits_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$

7. Find $\lim\limits_{n\to\infty} \frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$

8. Evaluate the following limits:

(i) $\lim\limits_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{4}+.....+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+.....+\frac{1}{3^n}}$

(ii) $\lim\limits_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+.......+\frac{1}{(2n-1)(2n+1)}\right)$

(iii) $\lim\limits_{n\to\infty}\left(\frac{1}{1.2.3}+\frac{1}{2.3.4}+......+\frac{1}{n(n+1)(n+2)}\right)$

9. For $x > 0$ and $\neq 1$ and $n \in N$, evaluate,

$$\lim_{n\to\infty}\left(\frac{1}{\log_x 2 \,.\, \log_x 4}+\frac{1}{\log_x 4 \,.\, \log_x 8}+......+\frac{1}{\log_x 2^{n-1} \,.\, \log_x 2^n}\right)$$

10. If $f(n+1) = \frac{1}{2}\left\{f(n)+\frac{9}{f(n)}\right\}$, $n \in N$ and $f(n) > 0$ for all $n \in N$, then find $\lim\limits_{n\to\infty} f(n)$

11. Evaluate the following limits:

(i) $\lim\limits_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+......+\frac{n}{1-n^2}\right]$

(ii) $\lim\limits_{n\to\infty}\left(\frac{1-2+3-4+.........-2n}{\sqrt{n^2+1}}\right)$

(iii) $\lim\limits_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{4}.......\cos\frac{x}{2^n}\right)$

1.14 Limits of Forms (0 x ∞) and (∞ – ∞)

(0 x ∞) Form

Consider the limit $\lim\limits_{x\to a} f(x).g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$. Here a may be finite or infinite.

This is called (0 x ∞) form.

One way to handle this indeterminate form is to write the limit $\lim\limits_{x\to a} f(x).g(x)$ as $\lim\limits_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$ which enters into $\left(\frac{0}{0}\right)$ form or $\lim\limits_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$ which is in $\left(\frac{\infty}{\infty}\right)$ form.

These can be solved by methods discussed earlier.

For example, $\lim\limits_{x\to 0}\sin 2x.\cot x$ is in (0 x ∞) form.

$\lim\limits_{x\to 0}\sin 2x.\cot x = \lim\limits_{x\to 0}\frac{\sin 2x}{\tan x}\left(\frac{0}{0}\right)$ which can be evaluated using standard limits to get 2.

and $\lim\limits_{x\to\infty} x(e^{\frac{1}{x}}-1) = \lim\limits_{x\to\infty}\frac{(e^{\frac{1}{x}}-1)}{1/x} = 1.$

Example 1: Evaluate $\lim\limits_{x\to\infty} x\tan\frac{1}{x}$

Solution: This is an ∞. 0 indeterminate form, but we convert it into a 0/0 form by using the substitution $u = 1/x$ and noting that $u \to 0$ as $x \to \infty$

$$\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{u\to 0}\frac{1}{u}\tan u = \lim_{u\to 0}\frac{\tan u}{u} = 1.$$

Example 2: Evaluate $\lim\limits_{x\to\infty} x(\ell nx-\ell n(x+2))$

Solution: Limit $= \lim\limits_{x\to\infty} x.\ell n\frac{x}{x+2}$ Put $x = \frac{1}{t}$

$$= \lim_{t\to 0^+}\frac{\ell n\frac{1}{1+2t}}{t} = \lim_{t\to 0^+}\frac{-\ell n(1+2t)}{2t}.2 = -2.$$

Example 3: Evaluate $\lim\limits_{x\to\infty} a^x\sin\left(\frac{b}{a^x}\right)$, $a > 1$

Solution: Limit $= \lim\limits_{x\to\infty}\left(\frac{\sin\left(\frac{b}{a^x}\right)}{\frac{b}{a^x}}\right).b = b$.

Example 4: Evaluate $\lim\limits_{x\to\infty} x\left(\tan^{-1}\dfrac{x+1}{x+2}-\dfrac{\pi}{4}\right)$.

Solution: Limit $= \lim\limits_{x\to\infty} x\left(\tan^{-1}\dfrac{x+1}{x+2}-\tan^{-1}1\right)$

$$= \lim_{x\to\infty} x.\tan^{-1}\left(\frac{\dfrac{x+1}{x+2}-1}{1+x+1}\right)$$

$$= \lim_{x\to\infty} x\left(\frac{\tan^{-1}\left(\dfrac{-1}{2x+3}\right)}{\dfrac{-1}{2x+3}}\right)\left(-\frac{1}{2x+3}\right)$$

$$= \lim_{x\to\infty}\frac{-1}{2+\dfrac{3}{x}} = -\frac{1}{2}.$$

Note: $\lim\limits_{x\to 0^+} x\ell nx = 0$

Proof: We have $\lim\limits_{t\to\infty}\dfrac{\ell nt}{t} = 0$. Put $t = \dfrac{1}{x}$

$$\Rightarrow \lim_{x\to 0^+}\frac{\ell n\dfrac{1}{x}}{\dfrac{1}{x}} = 0$$

$$\Rightarrow -\lim_{x\to 0^+} x\ell nx = 0$$

$$\Rightarrow \lim_{x\to 0^+} x\ell nx = 0.$$

Example 5: Evaluate $\lim\limits_{x\to 0^+}\sin x\ell n(\tan x)$

Solution: Limit $= \lim\limits_{x\to 0^+}\cos x(\tan x\ .\ \ell n(\tan x))$

$= 1 \times 0 = 0.$

Example 6: Evaluate $\lim\limits_{x\to 0^+}\dfrac{x^2\ell n(x+x^2)}{\tan x}$.

Solution: Limit $= \lim\limits_{x\to 0^+}\dfrac{x}{\tan x}.\dfrac{x.(x+x^2)}{(x+x^2)}\ell n(x+x^2)$

$$= \lim_{x\to 0^+}\frac{x}{\tan x}.\frac{1}{1+x}\left((x+x^2)\ell n(x+x^2)\right)$$

$= 1 \times 1 \times 0 = 0.$

Example 7: Evaluate $\lim\limits_{x\to 0^+} x^3(\ell nx)^2$

Solution: Limit $= \lim\limits_{x\to 0^+} x.(x\ell nx)^2 = 0 \times 0 = 0.$

Example 8: Evaluate $\lim\limits_{x\to 0^+} x^2(\ell nx)^3$

Solution: Put $x = \dfrac{1}{t}$. As $x\to 0$, $t\to\infty$.

$$\text{Limit} = \lim_{t\to\infty}\frac{\left(\ell n\dfrac{1}{t}\right)^3}{t^2}$$

$$= \lim_{t\to\infty}\frac{-(\ell nt)^3}{t^2}$$

$$= -\lim_{t\to\infty}\left(\frac{\ell nt}{t^{2/3}}\right)^3 = 0.$$

Example 9: Evaluate $\lim\limits_{n\to\infty} n^2(\sqrt[n]{a}-\sqrt[n+1]{a})$, $a > 0$.

Solution: Put $n = \dfrac{1}{y}$, $\lim\limits_{n\to\infty} n^2(\sqrt[n]{a}-\sqrt[n+1]{a})$

$$= \lim_{y\to 0}\frac{a^y - a^{\frac{y}{y+1}}}{y^2}$$

$$= \lim_{y\to 0} a^y\frac{\left[1-a^{\frac{y}{y+1}-1}\right]}{y^2}$$

$$= \lim_{y\to 0}\frac{a^y\left(1-a^{-\frac{y^2}{y+1}}\right)}{y^2}$$

$$= \lim_{y\to 0}\frac{a^{\frac{y^2}{y+1}}-1}{y^2} = \lim_{y\to 0}\frac{a^{\frac{y^2}{y+1}}-1}{\dfrac{y^2}{y+1}.(y+1)}$$

$= \ln a.$

Example 10: Evaluate:

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)....\infty}}}\right\}$$

Solution: Put $\dfrac{1}{n} = x$

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)....\infty}}}\right\}$$

$$= \lim_{x\to 0}\frac{\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x).....\infty}}}}{x^2}$$

$$= \lim_{x\to 0}\frac{(1-\cos x)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}-.....\infty}}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{(1-\cos x)}{x^2}.\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{(1+\cos x)} = (1)^2\cdot\frac{1}{1+1} = \frac{1}{2}.$$

One should transform the difference $f_1(x)–f_2(x)$ into the product $f_1(x)\left[1-\frac{f_2(x)}{f_1(x)}\right]$ and first evaluate the indeterminate form $\frac{f_2(x)}{f_1(x)}\left(\text{the form } \frac{\infty}{\infty}\right)$;

if $\lim\limits_{x\to a}\frac{f_2(x)}{f_1(x)} = 1$, then we reduce the expression to the form $\frac{1-\frac{f_2(x)}{f_1(x)}}{\frac{1}{f_1(x)}}\left(\text{the form } \frac{0}{0}\right)$.

Example 11: Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{x\sqrt{1+x}}\right)$

Solution: Limit = $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x\sqrt{1+x}}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\sqrt{1+x}\left(\sqrt{1+x}+1\right)}$$

$$= \lim_{x\to 0}\frac{1+x-1}{x\sqrt{1+x}.2} = \frac{1}{2}.$$

Example 12: Find

$$\lim_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right).$$

Solution: Here we have an indeterminacy of the form ∞–∞. Multiply and divide the given expression by

$$\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}\,.$$

$$\lim_{x\to\infty}\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}$$

$$= \lim_{x\to\infty}\frac{(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3})(\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3})}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}$$

$$= \lim_{x\to\infty}\frac{x^2+8x+3-x^2-4x-3}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}$$

$$= \lim_{x\to\infty}\frac{4x}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}$$

$$= \lim_{x\to\infty}\frac{4x}{\sqrt{1+8/x+3x^2}+\sqrt{1+4/x+3/x^2}} = \frac{4}{2} = 2.$$

Example 13: If $\lim\limits_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2$, find the values of a and b.

Solution: We have $\lim\limits_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)$

$$= \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+(1+b)}{(x+1)} = 2$$

Since, limit of the above expression is a finite nonzero number, degree of numerator = degree of denominator

⇒ $1-a=0$ ⇒ $a=1$

∴ Putting a = 1 in the limit we get,

$$\lim_{x\to\infty}\frac{-x(1+b)+(1+b)}{x+1}=2$$

⇒ $-(1+b)=2$ ⇒ $b=-3$

Hence, a = 1 and b = –3 .

Example 14: Find a and b if

$$\lim_{x\to-\infty}\left\{\sqrt{(x^2-x+1)}-ax-b\right\}=0$$

Solution: We have $\lim\limits_{x\to-\infty}\left\{\sqrt{(x^2-x+1)}-ax-b\right\}=0$

Put $x=-\frac{1}{t}$

$$\therefore \quad \lim_{t\to 0}\left\{\sqrt{\left(\frac{1}{t^2}+\frac{1}{t}+1\right)}+\frac{a}{t}-b\right\}=0$$

$$\Rightarrow \quad \lim_{t\to 0}\left\{\frac{\sqrt{(1+t+t^2)}+a-bt}{t}\right\}=0$$

As t → 0, numerator → 1 + a and denominator → 0

Since limit is finite, numerator must tend to 0.

∴ $1+a=0 \Rightarrow a=-1$

Substituting a = –1 in (1), we get

$$\lim_{t\to 0}\left\{\frac{\sqrt{(1+t+t^2)}-1-bt}{t}\right\}=0$$

$$\Rightarrow \quad \lim_{t\to 0}\left(\frac{\sqrt{(1+t+t^2)}-1}{t}\right)-b=0$$

$$\Rightarrow \quad \lim_{t\to 0}\frac{(\sqrt{(1+t+t^2)}-1)(\sqrt{(1+t+t^2)}+1)}{t(\sqrt{(1+t+t^2}+1)}=b$$

$$\Rightarrow \quad \lim_{t\to 0}\left\{\frac{1+t+t^2-1}{t(\sqrt{(1+t+t^2)}+1)}\right\}=b$$

$$\Rightarrow \quad \frac{1}{2}=b\,.$$

Hence a = –1 and b = $\frac{1}{2}$.

Example 15: Find $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

Solution: $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

$$= \lim_{x\to-\infty}\frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}$$

$$= \lim_{x\to-\infty} \frac{(x^2+4x)-(x^2-4x)}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$$

$$= \lim_{x\to-\infty} \frac{8x}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$$

$$= \lim_{x\to-\infty} \frac{-8}{\sqrt{\frac{x^2+4x}{x^2}}+\sqrt{\frac{x^2-4x}{x^2}}}$$

[When a negative number is to be written as square root of a number, negative sign is put before the sign of square root, for example – 4 = $-\sqrt{16}$].

$$\lim_{x\to-\infty} \frac{-8}{\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{4}{x}}} = \frac{-8}{2} = -4.$$

Example 16: Find $\lim_{x\to\infty}(\sqrt[3]{x^3+x^2}-\sqrt[3]{x^3-x^2})$.

Solution: $\lim_{x\to\infty}((x^3+x^2)^{1/3}-(x^3-x^2)^{1/3})$

$$= \lim_{x\to\infty} \frac{(x^3+x^2)-(x^3-x^2)}{(x^3+x^2)^{2/3}+(x^3+x^2)^{1/3}(x^3-x^2)^{1/3}+(x^3-x^2)^{2/3}}$$

$$= \lim_{x\to\infty} \frac{2x^2}{\left[\left(1+\frac{1}{x}\right)^{\frac{2}{3}}+\left(1+\frac{1}{x}\right)^{\frac{1}{3}}\left(1-\frac{1}{x}\right)^{\frac{1}{3}}+\left(1-\frac{1}{x}\right)^{\frac{2}{3}}\right]}$$

$$= \frac{2}{3}.$$

Example 17: Find p so that

$$\lim_{x\to\infty} x^p(\sqrt[3]{x+1}+\sqrt[3]{x-1}-2\sqrt[3]{x})$$

is some nonzero real number.

Solution: Make the substitution $t = \frac{1}{x}$. Then the limit equals to

$$\lim_{t\to0} t^{-p}\left(\sqrt[3]{\frac{1}{t}+1}+\sqrt[3]{\frac{1}{t}-1}-2\sqrt[3]{\frac{1}{t}}\right)$$

$$= \lim_{t\to0} t^{-p-\frac{1}{3}}\left(\sqrt[3]{1+t}+\sqrt[3]{1-t}-2\right).$$

We need the degree of the first nonzero term in the Maclaurin's expansion of $\sqrt[3]{1+t}+\sqrt[3]{1-t}-2$. We have

$$\sqrt[3]{1+t} = 1+\frac{1}{3}t-\frac{1}{9}t^2+\ldots\ldots\ldots,$$

$$\sqrt[3]{1-t} = 1-\frac{1}{3}t-\frac{1}{9}t^2+\ldots\ldots\ldots$$

It follows that $\sqrt[3]{1+t}+\sqrt[3]{1-t}-2 = -\frac{2}{9}t^2+\ldots\ldots$.

On considering the degree of the leading term, it follows that $-p-\frac{1}{3} = -2$. So $p = \frac{5}{3}$.

Example 18: Evaluate $\lim_{x\to0}\left(\operatorname{cosec}^3 x-\frac{1}{x^3}\right)$

Solution: We have $\lim_{x\to0}\left(\operatorname{cosec}^3 x-\frac{1}{x^3}\right)$, [form ∞-∞]

$$= \lim_{x\to0}\left(\frac{1}{\sin^3}-\frac{1}{x^3}\right) = \lim_{x\to0}\frac{x^3-\sin^3 x}{x^2\sin^3 x}$$

$$= \lim_{x\to0}\left\{\left(\frac{x^3-\sin^3 x}{x^6}\right).\left(\frac{x}{\sin x}\right)^3\right\}$$

$$= \lim_{x\to0}\frac{x^3-\sin^3 x}{x^6}$$

$$= \lim_{x\to0}\frac{(x^2+x\sin x+\sin^2 x)}{x^2}.\frac{(x-\sin x)}{x^4}$$

$$= \lim_{x\to0}\left(\left(1+\frac{\sin x}{x}\right)+\left(\frac{\sin x}{x}\right)^2\right).\left(\frac{x-\sin x}{x^4}\right)$$

$$= \lim_{x\to0} 3\frac{\left(x-x+\frac{x^3}{3!}-\frac{x^5}{5!}\ldots\ldots.\right)}{x^4}$$

$$= \lim_{x\to0}\frac{3\left(\frac{1}{3!}-\frac{x}{5!}\ldots\ldots.\right)}{x} = \infty.$$

Example 19: Evaluate $\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Solution: $\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

$$= \lim_{h\to0}\left(\frac{1}{\ln(1+h)}-\frac{1}{h}\right)$$

$$= \lim_{h\to0}\frac{h-\ln(1+h)}{h^2}\lim_{h\to0}\frac{h}{\ln(1+h)}$$

Put ln (1 + h) = t

$\Rightarrow$ 1 + h = e^t

$$\ell = \lim_{t\to0}\frac{e^t-1-t}{\left(\frac{e^t-1}{t}\right)^2\cdot t^2} = \lim_{t\to0}\frac{e^t-1-t}{t^2} \quad \ldots(1)$$

$$\text{Also, } \ell = \lim_{t\to0}\frac{e^{-t}-1+t}{t^2} \quad \ldots(2)$$

Adding (1) and (2)

$$2\ell = \lim_{t\to0}\frac{e^t+e^{-t}-2}{t^2} = \lim_{t\to0}\frac{e^{2t}-2e^t+1}{t^2}$$

$$= \lim_{t\to0}\left(\frac{e^t-1}{t}\right)^2 = 2\ell \Rightarrow \ell = \frac{1}{2}$$

Example 20: Evaluate $\lim\limits_{x \to \infty} (x - \ell n \ \cosh x)$

where $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Solution: $\lim\limits_{x \to \infty} \left[x - \ell n \left(\dfrac{e^x + e^{-x}}{2} \right) \right]$

$$= \lim_{x \to \infty} [\ \ell n\ e^x - \ell n\ (e^x + e^{-x}) + \ell n\ 2\ ]$$

$$= \lim_{x \to \infty} \ell n \frac{e^x}{e^x + e^{-x}} + \ell n\ 2$$

$$= \lim_{x \to \infty} \ell n \left(\frac{1}{1 + e^{-2x}} \right) + \ell n\ 2$$

$$= 0 + \ell n\ 2 = \ell n\ 2.$$

Example 21: Find

$$\lim_{n \to \infty} \left(\sqrt{n^2 + n + 1} - \left[\sqrt{n^2 + n + 1} \right] \right)$$

where [.] denotes the greatest integer function.

Solution: $n < \sqrt{n^2 + n + 1} < n + 1$

Hence $\left[\sqrt{n^2 + n + 1} \right] = n$

$$\therefore \quad \ell = \lim_{n \to \infty} \left(\sqrt{n^2 + n + 1} - n \right)$$

$$= \lim_{n \to \infty} \left(\frac{n + 1}{\sqrt{n^2 + n + 1} + n} \right) = \frac{1}{2}$$

Example 22: Evaluate $\lim\limits_{n \to \infty} \cos\left(\pi\sqrt{n^2 + n} \right)$, when n is an integer.

Solution: $\ell = \lim\limits_{n \to \infty} \cos\left(\pi\sqrt{n^2 + n} \right)$

$$= \lim_{n \to \infty} (-1)^n \cos\left(n\pi - \pi\sqrt{n^2 + n} \right)$$

$$= \lim_{n \to \infty} (-1)^n \cos\left(\pi\left(n - \sqrt{n^2 + n} \right) \right)$$

$$= (-1)^n \lim_{n \to \infty} \cos\left(\frac{n\pi}{n + \sqrt{n^2 + n}} \right)$$

$$= (-1)^n \lim_{n \to \infty} \cos\left(\frac{\pi}{1 + \sqrt{1 + \frac{1}{n}}} \right) = 0$$

Example 23: Evaluate

$$\lim_{n \to \infty} \left[\left\{ 4 \sum_{r=1}^{n} (r+1)(r+2)(r+3) \right\}^{\frac{1}{4}} - n \right]$$

Solution: Limit

$$= \lim_{n \to \infty} \left[\left\{ 4 . \frac{(n+1)(n+2)(n+3)(n+4)}{4} \right\}^{\frac{1}{4}} - n \right]$$

Put $n = \dfrac{1}{t}$

$$= \lim_{t \to 0} \frac{\{(1+t)(1+2t)(1+3t)(1+4t)\}^{\frac{1}{4}} - 1}{t}$$

$$= \lim_{t \to 0} \frac{(1 + 10t + ...)^{\frac{1}{4}} - 1}{t}$$

$$= \lim_{t \to 0} \frac{1 + \frac{10}{4}t + ... - 1}{t}$$

$$= \frac{10}{4} = \frac{5}{2}.$$

Concept Problems

L

1. Evaluate the following limits:
   (i) $\lim\limits_{x \to 0} x^{1/10} . \sin\dfrac{7}{x}$
   (ii) $\lim\limits_{x \to a} (a - x) \sec\dfrac{\pi x}{2a}$
   (iii) $\lim\limits_{x \to a} \left(\sin\dfrac{a - x}{2} \tan\dfrac{\pi x}{2a} \right)$
   (iv) $\lim\limits_{x \to \infty} 3^{-x} \sin(3^x)$
2. Evaluate the following limits:
   (i) $\lim\limits_{x \to 0^+} x.\ln(x + x^2)$
   (ii) $\lim\limits_{x \to 0} \sin x . \ln(\tan x)$
   (iii) $\lim\limits_{x \to 1} \ln x \ln(x - 1)$
3. Evaluate the following limits:
   (i) $\lim\limits_{x \to -2} \left(\dfrac{1}{x + 2} - \dfrac{12}{x^3 + 8} \right)$
   (ii) $\lim\limits_{x \to \infty} (\sqrt[3]{x + 1} - \sqrt[3]{x})$

(iii) $\lim\limits_{x\to\infty}(\sqrt{4x^2-9}-2x)$

(iv) $\lim\limits_{x\to\infty}(\sqrt{x+1}-\sqrt{x})\sqrt{x}$

4. Evaluate the following limits:

(i) $\lim\limits_{x\to 0}(\operatorname{cosec} x-\cot x)$

(ii) $\lim\limits_{x\to \pi}(\operatorname{cosec} x+\cot x)$

(iii) $\lim\limits_{x\to \pi/4}(\sin x-\cos x)\tan\left(\dfrac{\pi}{4}+x\right)$

(iv) $\lim\limits_{x\to 0}\dfrac{1}{\sin x}-\dfrac{1}{\tan x}$

5. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}(\sqrt{3x^2-1}+2x)$

(ii) $\lim\limits_{x\to\infty}(\sqrt[3]{x^3+2x^2+5}-x)$

(iii) $\lim\limits_{x\to\infty}(\sqrt[5]{x^5-3x^4+17}-x)$

(iv) $\lim\limits_{x\to\infty}(\sqrt{x^2+ax+b}-\sqrt{x^2+cx+d})$

6. Two citizens are arguing about

$$\lim_{x\to\infty}\left(\frac{3x^2+2x}{x+5}-3x\right)$$

The first claims, "For large x, 3x is small in comparison to $3x^2$, and 5 is small in comparison to x. So $(3x^2+3x)/(x+5)$ behaves like $3x^2/x=3x$. Hence the limit in question is 0." His companion replies, " Nonsense. After all,

$$\frac{3x^2+2x}{x+5}=\frac{3x+2}{1+(5/x)},$$

which clearly behaves like 3x + 2 for large x. Thus the limit in question is 2, not 0." Settle the argument.

7. Evaluate the following limits:

(i) $\lim\limits_{n\to\infty} n(\sqrt{n^2+1}-n)$

(ii) $\lim\limits_{n\to\infty}\left(\dfrac{1+2+3+......+n}{n+2}-\dfrac{n}{2}\right)$

(iii) $\lim\limits_{n\to\infty}\left[\dfrac{1+3+5+7+...+(2n-1)}{n+1}-\dfrac{2n+1}{2}\right]$

(iv) $\lim\limits_{n\to-\infty}\sqrt[3]{n^3+n}-\sqrt[3]{n^3+1}$

8. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}(x-\ln(x^2+1))$

(ii) $\lim\limits_{x\to\infty}x^{1/3}[(x+1)^{2/3}-(x-1)^{2/3}]$

(iii) $\lim\limits_{x\to\infty}\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}+x\tan\dfrac{1}{x}\right)$

9. Evaluate the following limits:

(i) $\lim\limits_{t\to\infty}t(\sqrt[t]{a}-1)\quad a>0$

(ii) $\lim\limits_{x\to\infty}x^n\sin\dfrac{a}{x}$, n > 0

(iii) $\lim\limits_{x\to\infty}\dfrac{2^x\tan\left(\dfrac{a}{2^x}\right)}{x\sin\left(\dfrac{1}{x}\right)}$

(iv) $\lim\limits_{x\to 0^+}x^{-2}(1.001)^{-\frac{1}{x}}$

10. Show that $\lim\limits_{x\to 0^+}x^n(\ell nx)^m=0$, m, n ∈ N

Practice Problems J

11. Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

(ii) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}\right)$

(iii) $\lim\limits_{x\to\infty}\left[\dfrac{3x^2}{2x+1}-\dfrac{(2x-1)(3x^2+x+2)}{4x^2}\right]$

(iv) $\lim\limits_{x\to 2}\left[\dfrac{1}{x(x-2)^2}-\dfrac{1}{x^2-3x+2}\right]$

12. Evaluate the following limits :

(i) $\lim\limits_{x\to 1}\left(\dfrac{1}{x^2-1}-\dfrac{2}{x^4-1}\right)$

(ii) $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right)$

(iii) $\lim\limits_{x\to 0}\left\{\dfrac{\pi}{4x}-\dfrac{\pi}{2x(e^{\pi x}+1)}\right\}$

(iv) $\lim\limits_{x\to\infty}\left(\sqrt[3]{(x+a)(x+b)(x+c)}-x\right)$

13. If $\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}+ax-b)=0$, then find the constants a & b.

14. Evaluate

$\lim\limits_{n\to\infty}(n^6+6n^5+12n^4+1)^{1/3}-(n^4+4n^3+6n+1)^{1/2}$

15. If $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$ find a and b.

16. For a certain value of B, the limit $\lim\limits_{x\to\infty}(x^4+5x^3+3)^B-x$ is finite and nonzero. Find B and then compute the limit.

17. Evaluate the following limits :

(i) $\lim\limits_{x\to\frac{\pi}{2}}\left(2x\tan x-\frac{\pi}{\cos x}\right)$

(ii) $\lim\limits_{x\to\infty}\left[\ln x-\ln\left(\sqrt{x^2+1}+x\right)\right]$

(iii) $\lim\limits_{n\to\infty}n.\cos\left(\frac{\pi}{4n}\right).\sin\left(\frac{\pi}{4n}\right)$

(iv) $\lim\limits_{n\to\infty}n^2\sin\left(\ln\sqrt{\cos\frac{\pi}{n}}\right)$

18. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}x^3\{\sqrt{x^2+\sqrt{1+x^4}}-x\sqrt{2}\}$

(ii) $\lim\limits_{x\to\infty}\sqrt{e^x}\,(\sqrt{e^x+1}-\sqrt{e^x-1})$

(iii) $\lim\limits_{x\to\infty}x^{3/2}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})$

19. Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$

(ii) $\lim\limits_{x\to\infty}x\left(\tan^{-1}\frac{x+1}{x+2}-\tan^{-1}\frac{x}{x+2}\right)$

(iii) $\lim\limits_{x\to\infty}x^5\left[\sin\left(\frac{1}{x}\right)-\frac{1}{x}+\frac{1}{6x^3}\right]$

1.15 Limits of Forms 0^0 and ∞^0

Calculating the Limits of Exponential Power Functions

We first note that

$$\lim_{x\to\infty}a^x=\begin{cases}0, & 0\le a<1\\ 1, & a=1\\ \infty, & a>1\\ \text{dne} & a<0\end{cases}$$

If $\lim\limits_{x\to a}f(x)=A>0$ and $\lim\limits_{x\to a}g(x)=B$, a finite quantity then $\lim\limits_{x\to a}(f(x))^{g(x)}=A^B$.

Also $\lim\limits_{x\to a}(f(x))^{g(x)}=e^{\lim_{x\to a}g(x)\,\ell n(f(x))}$

Consider the calculation of the limit, as x → a, of the exponential power function $[f(x)]^{g(x)}$, where the functions f(x) and g(x) are defined in a neighbourhood of the point a, and f(x) > 0.

The following cases are possible here :

1. If $\lim\limits_{x\to a}f(x)=b>0$, $\lim\limits_{x\to a}g(x)=c$, then $\lim\limits_{x\to a}f(x)^{g(x)}=b^c$.

2. If $\lim\limits_{x\to a}f(x)=0$, $\lim\limits_{x\to a}g(x)=c>0$ (or $+\infty$), then $\lim\limits_{x\to a}f(x)^{g(x)}=0$.

3. $\lim\limits_{x\to a}f(x)=0$, $\lim\limits_{x\to a}g(x)=c<0$ (or $-\infty$) then $\lim\limits_{x\to a}f(x)^{g(x)}=\infty$.

4. $\lim\limits_{x\to a}f(x)=0$, $\lim\limits_{x\to a}g(x)=0$, then $\lim\limits_{x\to a}f(x)^{g(x)}$ is an indeterminate form of type 0^0

5. If $\lim\limits_{x\to a}f(x)=\infty$, $\lim\limits_{x\to a}g(x)=c>0$ (or ∞), then $\lim\limits_{x\to a}f(x)^{g(x)}=\infty$.

6. If $\lim\limits_{x\to a}f(x)=\infty$, $\lim\limits_{x\to a}g(x)=c<0$ (or $-\infty$), then $\lim\limits_{x\to a}f(x)^{g(x)}=0$.

7. If $\lim\limits_{x\to a}f(x)=\infty$, $\lim\limits_{x\to a}g(x)=0$, then $\lim\limits_{x\to a}f(x)^{g(x)}$ is an indeterminate form of type ∞^0.

8. If $\lim\limits_{x\to a}f(x)=1$, $\lim\limits_{x\to a}g(x)=\infty$, then $\lim\limits_{x\to a}f(x)^{g(x)}$ is an indeterminate form of type 1^∞.

Note: If h > 0, then $\lim\limits_{n\to\infty}(1-h)^n=0$ and

$\lim\limits_{n\to\infty}(1+h)^n\to\infty$.

Example 1: Find $\lim\limits_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x}$

Solution: Here,

$\lim\limits_{x\to 0}\left(\frac{\sin 2x}{x}\right)=2$ and $\lim\limits_{x\to 0}(1+x)=1$.

Hence, $\lim\limits_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x}=2^1=2$.

Example 2: Find $\lim\limits_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2}$

Solution: We have

$$\lim_{x\to\infty}\frac{x+1}{2x+1}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{2+\frac{1}{x}}=\frac{1}{2}$$

and $\lim\limits_{x\to\infty} x^2=\infty$.

Therefore, $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{2x+1}\right)^{x^2}=0$

Limits of Form 0^0

Example 3: Evaluate $\lim\limits_{x\to 0^+}(x-x^2)^{x^2}$.

Solution: $\lim\limits_{x\to 0^+}(x-x^2)^{x^2}$ $\left(0^0\text{ form}\right)$

$$= e^{\lim\limits_{x\to 0^+} x^2\ell n(x-x^2)}$$

$$= e^{\lim\limits_{x\to 0^+}\frac{x}{1-x}\left[(x-x^2)\ell n(x-x^2)\right]}\ \left(\text{using } \lim_{x\to 0^+} x\,\ell n\, x = 0\right)$$

$= e^0 = 1.$

Example 4: Evaluate $\lim\limits_{x\to 1^-}(1-x^2)^{\frac{1}{\ell n(1-x)}}$

Solution: $\lim\limits_{x\to 1^-}(1-x^2)^{\frac{1}{\ell n(1-x)}}$ $\left(0^0\text{ form}\right)$

$$= e^{\lim\limits_{x\to 1^-}\frac{1}{\ell n(1-x)}\ell n(1-x^2)}$$

$$= e^{\lim\limits_{x\to 1^-}\left(\frac{\ell n(1-x)}{\ell n(1-x)}+\frac{\ell n(1+x)}{\ell n(1-x)}\right)}$$

$= e^{1+0} = e.$

Caution

$$\lim_{x\to 0}(\sin^2 x)^{\sqrt{x^2}-|x|}=1$$

Note that this is not in 0^0 indeterminate form. The exponent is exactly 0.

$$\lim_{x\to 0}\left(\sqrt{x^2}-|x|\right)^{\sin^2 x}=0$$

This is not in 0^0 indeterminate form. The base is exactly 0.

The limit $\lim\limits_{x\to 0}(\sin^2 x+\cos^2 x-1)^{\sqrt{x^2}-|x|}$ cannot be evaluated since the function is undefined as both base and exponent are exactly 0.

Limits of Form ∞^0

Example 5: Find the $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$.

Solution: The limit leads to the indeterminate form ∞^0.

We let f(x) = $x^{\frac{1}{x}}$ and find $\lim\limits_{x\to\infty}\ell n f(x)$

Since $\ell n\ f(x)=\ell n\, x^{\frac{1}{x}}=\dfrac{\ell n\, x}{x}$

$$\lim_{x\to\infty}\ell n f(x)=\lim_{x\to\infty}\frac{\ell n\ x}{x}=0$$

Therefore,

$$\lim_{x\to\infty} x^{1/x}=\lim_{x\to\infty}.f(x)=\lim_{x\to\infty}e^{\ell n f(x)}=e^0=1.$$

Example 6: Evaluate $\lim\limits_{x\to\infty} x^{\frac{\ell n 2}{1+\ell n x}}$.

Solution: $\lim\limits_{x\to\infty} x^{\frac{\ell n 2}{1+\ell n x}}$ $\left(\infty^0\text{ form}\right)$

$$= e^{\lim\limits_{x\to\infty}\frac{\ell n 2}{1+\ell n x}.\ell n x}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ell n 2}{\frac{1}{\ell n x}+1}}$$

$= e^{\ell n 2} = 2.$

Example 7: Evaluate $\lim\limits_{x\to\infty}(2^x+3^x)^{1/x}$.

Solution: $\lim\limits_{x\to\infty}(2^x+3^x)^{1/x}$ $\left(\infty^0\text{ form}\right)$

We choose the largest term in the base and divide the base with it. Here 3^x is the largest term.

$$\text{Limit} = \lim_{x\to\infty}\left(3^x\left(\left(\frac{2}{3}\right)^x+1\right)\right)^{1/x}$$

$$= \lim_{x\to\infty}3\left(1+\left(\frac{2}{3}\right)^x\right)^{1/x}$$

as x → ∞, $\left(\dfrac{2}{3}\right)^x\to 0$ since $0<\dfrac{2}{3}<1$.

$= 3(1+0)^0 = 3.$

Example 8: Evaluate $\lim\limits_{x\to\infty}\left(\left(\underline{|n}\right)^x+2^{nx}\right)^{1/x}$, $n\in N$.

Solution: For n = 1, 2, 3, we have $2^n > \underline{|n}$.

$$\text{Limit} = \lim_{x\to\infty}2^n\left(\left(\frac{\underline{|n}}{2^n}\right)^x+1\right)^{1/x}=2^n.$$

For $n\ge 4$, we have $2^n < \underline{|n}$.

$$\text{Hence limit} = \lim_{x\to\infty}\underline{|n}\left(\left(\frac{2^n}{\underline{|n}}\right)^x+1\right)^{1/x}=\underline{|n}.$$

Example 9: Find

$$\lim_{x\to 0}\left(1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+\ldots+n^{\operatorname{cosec}^2 x}\right)^{\sin^2 x}$$

Solution: Let $y=\operatorname{cosec}^2 x$. As $x\to 0$, $y\to\infty$

Limit = $\lim\limits_{y\to\infty}(1^y+2^y+\ldots n^y)^{1/y}$ (∞^0 form)

$$= \lim_{y\to\infty}\left(n^y\right)^{1/y}\left[\left(\frac{1}{n}\right)^y+\left(\frac{2}{n}\right)^y+\ldots+\left(\frac{n-1}{n}\right)^y+1\right]^{1/y}$$

$$= \lim_{y\to\infty}n\left[\left(\frac{1}{n}\right)^y+\left(\frac{2}{n}\right)^y+\ldots+\left(\frac{n-1}{n}\right)^y+1\right]^{1/y}$$

$= n.\ 1^0 = n.$

Concept Problems **M**

1. Evaluate $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$

2. Evaluate $\lim\limits_{n\to\infty}\sin^n\left(\dfrac{2\pi n}{3n+1}\right)$

3. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{2x-1}\right)^{x^2}$

4. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2}\right)^{\frac{2x}{x+1}}$

5. Evaluate $\lim\limits_{n\to\infty}(4^n+5^n)^{1/n}$

6. Evaluate $\lim\limits_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$

Practice Problems **K**

7. Evaluate $\lim\limits_{x\to 1}\left(\dfrac{x-1}{x^2-1}\right)^{x+1}$

8. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{(2x+1)}{x-1}}$

9. Evaluate $\lim\limits_{n\to\infty}\left(\cosh\dfrac{\pi}{n}-1\right)^{1/\ln n^{-1}}$

   where $\cosh x = \dfrac{e^x+e^{-x}}{2}$.

10. Evaluate $\lim\limits_{x\to 0^-}(1-\operatorname{cosec}^3 x)^{\tan x}$

11. Evaluate $\lim\limits_{x\to 0^+} x^{(x^x)}$

12. Evaluate $\lim\limits_{n\to\infty}\left((e^{\pi})^n+(\pi^e)^n\right)^{1/n}$

13. Evaluate $\lim\limits_{x\to\pi/2^-}(\tan x)^{\tan 2x}$

14. Evaluate $\lim\limits_{x\to\infty}(1^x+2^x+3^x+\ldots\ldots 99^x)^{1/x}$

1.16 Limits of Form 1^∞

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\,,$$

To prove $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

The limit is in the indeterminate form 1^∞.

We let $f(x) = (1+x)^{\frac{1}{x}}$ and find $\mathop{\text{Lim}}\limits_{x\to 0^+} \ell n\ f(x)$.

Since $\ell n\ f(x) = \ell n(1+x)^{\frac{1}{x}} = \dfrac{1}{x}\ell n(1+x)$

$$\lim_{x\to 0}\ell n\, f(x) = \lim_{x\to 0}\frac{\ell n(1+x)}{x}\ \left(\frac{0}{0}\right) = 1$$

$$\therefore\quad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ell n f(x)} = e\,\cdot$$

Generally, it is useful to remember that

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to 0}(1+x)^{a/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ax} = e^a\,.$$

Let $\lim\limits_{x\to a} f(x)=1$ and $\lim\limits_{x\to a} g(x)=\infty$ then

$$\lim_{x\to a}\left(f(x)\right)^{g(x)} = e^{\lim\limits_{x\to a} g(x)[f(x)-1]}$$

Proof: Let $\ell = \lim\limits_{x\to a}\left[1+f(x)-1\right]^{g(x)}$

$$= \lim_{x\to a}\left[\left\{1+\left(f(x)-1\right)\right\}^{\frac{g(x)}{f(x)-1}}\right]^{f(x)-1}$$

Put $f(x)-1 = y$, as $x \to a$, $y \to 0$

$$\therefore\quad \ell = \left[\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right]^{\lim\limits_{x\to a} g(x)\,[f(x)-1]}$$

$$= e^{\lim\limits_{x\to a} g(x)[f(x)-1]}$$

This is an important formula for solving 1^∞ form.

⊖ **Example 1:** Calculate $\lim\limits_{x\to 0^+}(1+\sin 4x)^{\cos x}$.

✓ **Solution:** First notice that as $x \to 0^+$, we have $1+\sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate.

Let $y = (1 + \sin 4x)^{\cos x}$

Then $\ell n\, y = \ell n[1 + \sin 4x)^{\cot x}] = \cot x\, \ell n\,(1 + \sin 4x)$

$$\lim_{x\to 0^+} \ell n\, y = \lim_{x\to 0^+} \frac{\ell n(1+\sin 4x)}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{\ell n(1+\sin 4x)}{\sin 4x}\cdot\frac{\sin 4x}{4x}\cdot\frac{4x}{\tan x}$$

So far we have computed the limit of $\ell n\, y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ell n\, y}$

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cos x} = \lim_{x\to 0^+} e^{\ell n\, y} = e^4$$

Example 2: Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+4}{x^2-3x+7}\right)^x$

Solution: By means of division of the numerator of the fraction by the denominator, we isolate the integral part:

$$\frac{x^2+5x+4}{x^2-3x+7} = 1+\frac{8x-3}{x^2-3x+7}.$$

Thus, when $x \to \infty$ the given function is a power whose base tends to unity and the exponent, to infinity (an indeterminacy of the form 1^∞). $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+4}{x^2-3x+7}\right)^x = \lim\limits_{x\to\infty}\left(1+\dfrac{8x-3}{x^2-3x+7}\right)^x$

$$= \lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{x(8x-3)}{x^2-3x+7}}$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{8-3/x}{1-3/x+7/x^2}}$$

Since $\dfrac{8x-3}{x^2-3x+7} \to 0$ as $x \to \infty$ and we have

$$\lim_{x\to\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}} = e\,\cdot$$

Taking into account that $\lim\limits_{x\to\infty}\dfrac{8-3/x}{1-3/x+7/x^2} = 8$,

we get $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+4}{x^2-3x+7}\right)^x = e^8$

Example 3: $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution: Limit $= e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x}$ (using formula)

$$= e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$$

$$= e^{2\times\frac{\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

Example 4: Evaluate $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Solution: $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

Put $x = a + h$

$$= \lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)}$$

$$= \lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)}$$

$$= e^{\lim\limits_{h\to 0}-\cot\frac{\pi h}{2a}\cdot\left(1+\frac{h}{a+h}-1\right)}$$

$$= e^{\lim\limits_{h\to 0}-\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right)\cdot\frac{\frac{2a}{\pi}}{a+h}} = e^{-2/\pi}.$$

Example 5: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{2/x}$ where a, b, c > 0.

Solution: We have $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{2/x}$ (form 1^∞)

$$= e^{\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right)\cdot\frac{2}{x}}$$

$$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)}$$

$$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)}$$

$$= e^{\frac{2}{3}\left\{\lim\limits_{x\to 0}\frac{a^x-1}{x}+\lim\limits_{x\to 0}\frac{b^x-1}{x}+\lim\limits_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= 3^{(2/3)\ln(abc)} = e^{\ln(abc)^{2/3}} = (abc)^{2/3}.$$

Example 6: Find the value of

$$\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\ln x\right)\right)^{\frac{1}{\ln x}}.$$

Solution: $\lim\limits_{x\to 1}\left(\tan\left(\dfrac{\pi}{4}+\ln x\right)\right)^{\frac{1}{\ln x}}$ (1^∞ form)

$$= \lim_{x\to 1}\left(1+\frac{2\tan(\ln x)}{1-\tan(\ln x)}\right)^{\frac{1}{\ln x}}$$

$$= e^{\lim\limits_{x\to 1}\frac{2\tan(\ln x)}{\{1-\tan(\ln x)\}}\cdot\frac{1}{\ln x}}$$

$$= e^{2\lim\limits_{x\to 1}\frac{\tan(\ln x)}{\ln x}\cdot\frac{1}{1-\tan(\ln x)}}$$

$$= e^{2\cdot(1)} = e^2$$

Example 7: Find the value of $\lim\limits_{n\to\infty}\left(\dfrac{a-1+\sqrt[n]{b}}{a}\right)^n$ (a , b > 0) .

Solution: $\lim\limits_{n\to\infty}\left(\dfrac{a-1+\sqrt[n]{b}}{a}\right)^n$ (1^∞ form)

$$= \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{b}-1}{a}\right)^n$$

$$= e^{\lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{b}-1}{a}\right).n}$$

$$= e^{\lim\limits_{n\to\infty}\frac{1}{a}\left(\frac{b^{1/n}-1}{1/n}\right)}$$

$$= e^{\frac{1}{a}\log_e b} = e^{\log_e b^{1/a}} = b^{1/a}.$$

Example 8: Evaluate $\lim\limits_{x\to 0}\dfrac{\ell n\left(1+x^2+x^4\right)}{\left(e^x-1\right)x}$

Solution: $\lim\limits_{x\to 0}\dfrac{\ell n\ \left(1+x^2+x^4\right)}{x^2\left(\dfrac{e^x-1}{x}\right)}$

$$= \lim_{x\to 0} l\, n\ \left(1+x^2+x^4\right)^{\frac{1}{x2}}$$

$$= l\, n \lim_{x\to 0} e^{\frac{1}{x2}\left(1+x^2+x^4-1\right)}$$

$$= \lim_{x\to 0}\frac{x^2+x^4}{x^2} = \lim_{x\to 0}\ (1+x^2) = 1$$

Example 9: Evaluate

$$\lim_{x\to 0}\frac{\ln\left[\tan\left(\frac{\pi}{4}+ax\right)\right]}{\sin bx}\ (b\neq 0).$$

Solution: $\lim\limits_{x\to 0}\dfrac{\ln\left[\tan\left(\frac{\pi}{4}+ax\right)\right]}{bx\ \dfrac{\sin bx}{bx}}$

$$= \lim_{x\to 0}\ln\left[\tan\left(\frac{\pi}{4}+ax\right)\right]^{\frac{1}{bx}}$$

$$= \lim_{x\to 0}\frac{1}{bx}\left[\tan\left(\frac{\pi}{4}+ax\right)-\tan\frac{\pi}{4}\right]$$

$$= \lim_{x\to 0}\frac{ax}{bx}.\frac{\tan ax}{ax}\left[1+\tan\left(\frac{\pi}{4}+ax\right).\tan\frac{\pi}{4}\right] = \frac{2a}{b}$$

Example 10: For a function

$f(x) = \dfrac{l\, n\left(\{\sin x\}\cdot\{\cos x\}+1\right)}{\{\sin x\}\{\cos x\}}$ where $\{x\}$ denotes fractional part function, prove that

(i) $f\left(\dfrac{\pi}{2}^-\right) = f\,(0^+)$ (ii) $f\,(0^-) = f\left(\dfrac{\pi}{2}^+\right)$

and show that $\lim\limits_{x\to 0} f(x)$ does not exist.

Solution: $f\,(0^+) = \lim\limits_{h\to 0}\dfrac{\ln(\sin h\cdot\cos h+1)}{\sin h\cdot\cos h}$

$$= \lim_{h\to 0}\frac{\ln(\sin h\cdot\cos h+1)}{\frac{\sin h}{h}\cdot h\cos h}$$

$$= \lim_{h\to 0}\ln(\sin h\cdot\cos h+1)^{1/h}$$

$$= \lim_{h\to 0}\frac{1}{h}(\sin h\cdot\cos h+1-1) = 1.$$

$$f\left(\frac{\pi}{2}^-\right) = \lim_{h\to 0}\frac{\ln(\cos h\cdot\sin h+1)}{\cos h\cdot\sin h} = f\,(0^+).$$

$$f\,(0^-) = \lim_{h\to 0}\frac{\ln\left((1-\sin h)\cos h+1\right)}{(1-\sin h)\cos h} = \ln 2$$

$\Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist.

As before $f\left(\dfrac{\pi}{2}^+\right)$ is same as $f\,(0^-)$

Example 11: Evaluate $L = \lim\limits_{n\to\infty}\left(\dfrac{3}{n^3}\sum\limits_{r=1}^{n} r^2\right)^n$

Solution: We first note that $\lim\limits_{n\to\infty}\dfrac{3}{n^3}\sum\limits_{r=1}^{n} r^2$

$$= \lim_{n\to\infty}\frac{3}{n^3}.\frac{n(n+1)(2n+1)}{6} = 1.$$

Hence the limit L is in 1^∞ form

$$\therefore\quad L = e^{\lim\limits_{n\to\infty}\left(\frac{(n+1)(2n+1)}{2n^2}-1\right)n}$$

$$= e^{\lim\limits_{n\to\infty}\frac{(2n^2+3n+1-2n^2)}{2n}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{3n+1}{2n}} = e^{\frac{3}{2}}.$$

Example 12: Evaluate

$$\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}.$$

Solution: $L = \lim\limits_{x\to 0}\left[\sin^2\left(\dfrac{\pi}{2-a\,x}\right)\right]^{\sec^2\frac{\pi}{2-b\,x}}$

$$= e^{-\lim\limits_{x\to0}\sec^2\frac{\pi}{2-bx}\cos^2\frac{\pi}{2-ax}} = e^{-\ell 2},$$

where $\ell = \lim\limits_{x\to0}\dfrac{\cos\frac{\pi}{2-ax}}{\cos\frac{\pi}{2-bx}}$

$$= \lim_{x\to0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-ax}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-bx}\right)}$$

$$= \lim_{x\to0}\frac{\frac{\pi}{2}-\frac{\pi}{2-ax}}{\frac{\pi}{2}-\frac{\pi}{2-bx}} = \lim_{x\to0}\frac{-ax}{2-ax}\ \frac{2-bx}{-bx}$$

$$= \frac{a}{b}. \qquad \therefore\ L = e^{-\frac{a^2}{b^2}}$$

Example 13: Find a polynomial of the least degree, such that $\lim\limits_{x\to0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$.

Solution: $\lim\limits_{x\to0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}$ exists only when it is in 1^∞ form i.e. $\lim\limits_{x\to0}\dfrac{x^2+f(x)}{x^2} = 0$.

So, the term of least degree in f(x) is of degree 2.

i.e., $f(x) = a_2x^2 + a_3x^3 + \ldots\ldots$

Now, $\lim\limits_{x\to0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$

$$\Rightarrow\quad e^{\lim\limits_{x\to0}\left(\frac{x^2+f(x)}{x^2}\right)\frac{1}{x}} = e^2 \qquad \Rightarrow\quad e^{\lim\limits_{x\to0}\frac{x^2+f(x)}{x^3}} = e^2$$

$$\therefore\quad \lim_{x\to0}\frac{x^2+f(x)}{x^3} = 2$$

$$\Rightarrow\quad \lim_{x\to0}\frac{x^2+a_2x^2+a_3x^3+\ldots..}{x^3} = 2$$

$\Rightarrow$ $a_2 = -1$, $a_3 = 2$ and a_4, a_5 .... are any arbitrary constants. Since, we want a polynomial of the least degree, we choose $f(x) = -x^2 + 2x^3$.

Concept Problems

N

1. Show that

$$\lim\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \begin{cases} 1/2 & \text{if } x\to0 \\ \sqrt{2/3} & \text{if } x\to1 \\ 1 & \text{if } x\to\infty \end{cases}$$

2. Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$

(ii) $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$

(iii) $\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$

(iv) $\lim\limits_{x\to\pi/2}\left(1+3\cot^2x\right)^{\tan^2x}$

3. Evaluate the following limits :

(i) $\lim\limits_{x\to1}(1+\sin\pi x)^{\cot\pi x}$

(ii) $\lim\limits_{\alpha\to0}(2-\cos\alpha)^{\operatorname{cosec}^2\alpha}$

(iii) $\lim\limits_{x\to0}(a^x+x)^{1/x}$

(iv) $\lim\limits_{x\to0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\frac{1}{\sin x}}$

4. Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

(ii) $\lim\limits_{x\to1}x^{\frac{3}{4+\ln x}}$

(iii) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^n}\right)^x\ (n>0)$

(iv) $\lim\limits_{x\to0}(\cos x)^{1/(e^x-1-x)}$

Practice Problems L

5. Evaluate the following limits:

(i) $\lim\limits_{x\to 0} (\cos m x)^{\frac{n}{x^2}}$, m, n ∈ N

(ii) $\lim\limits_{x\to\infty} \left[x \sin^{-1}\left(\frac{1}{x}\right)\right]^{x^2}$

(iii) $\lim\limits_{n\to\infty} \cos^n \frac{x}{\sqrt{n}}$

(iv) $\lim\limits_{x\to\infty} \left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + ...+ n^{1/x}}{n}\right)^{nx}$ n∈ N

6. Evaluate

$$\lim_{x\to\infty} \left(\frac{1}{x}.\frac{a^x - 1}{a-1}\right)^{1/x} \quad \text{where } a > 0,\ a \neq 1.$$

7. Evaluate the following limits:

(i) $\lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

(ii) $\lim\limits_{x\to 0} (\cos x)^{\cot^2 x}$

(iii) $\lim\limits_{x\to 0} \left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$

(iv) $\lim\limits_{x\to 0} \left(\frac{\tan^{-1} x}{x}\right)^{1/x^2}$

8. Evaluate the following limits:

(i) $\lim\limits_{x\to 0} \left(\frac{x-1+\cos x}{x}\right)^{1/x}$

(ii) $\lim\limits_{x\to 0} (1+\ln(1+x))^{1/\ln(x+1)^2}$

(iii) $\lim\limits_{x\to a} \left(\frac{x^n - a^n}{(x-a)na^{n-1}}\right)^{\frac{1}{x-a}}$

(iv) $\lim\limits_{x\to\infty} \left[\left(\frac{e}{1-e}\right)\left(\frac{1}{e} - \frac{x}{1+x}\right)\right]^x$

9. Evaluate the following limits :

(i) $\lim\limits_{n\to\infty} n^{\left(\frac{e^{\pi/n}+e^{-\pi/n}}{2}-1\right)}$

(ii) $\lim\limits_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{\alpha} + \sin\frac{1}{n}\right]^n$

(iii) $\lim\limits_{n\to\infty} \{\cos(\ell n(n-1) - \ell n(n+1))\}^{(n+1)^2}$

(iv) $\lim\limits_{n\to\infty} \tan^n\left[\frac{\pi-4}{4} + \left(1-\frac{1}{n}\right)^a\right]$

10. Find λ & μ if $\lim\limits_{x\to\infty} \left(1+\frac{l}{x}+\frac{m}{x^2}\right)^{2x} = e^2$.

11. For A and B positive constants, define $f(x) = (e^x + Ax)^{B/x}$.

(a) Compute $L_1 = \lim\limits_{x\to 0} f(x)$ and $L_2 = \lim\limits_{x\to\infty} f(x)$

(b) What is the largest value of A for which the equation $L_1 = B\,L_2$ has a solution ? What are L_1 and L_2 in this case ?

1.17 Sandwich Theorem/Squeeze Play Theorem

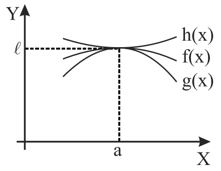
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. If $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(c) = \ell$ then $\lim\limits_{x\to a} f(x) = \ell$.

Note:

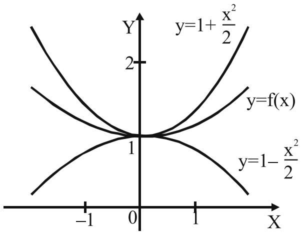
(i) This is true also when one or both of the above inequalities are strict.

(ii) This theorem is also applicable to one sided limits and limits at infinity.

(iii) The theorem holds even if ℓ is finite or infinite.



For example, if $1 - \frac{x^2}{4} \le f(x) \le 1 + \frac{x^2}{2}$ for all x ≠ 0, we can find $\lim\limits_{x\to 0} f(x)$, even if we donot have the formula of f(x). We can see in the figure that the graph of f(x) is squeezed between the graphs of the other two functions.

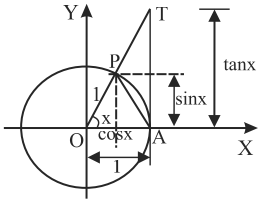


Since $\lim_{x\to0}(1-(x^2/4))=1$ and $\lim_{x\to0}(1+(x^2/2))=1$,

Sandwich Theorem implies that $\lim_{x\to0} f(x)=1$.

We shall now prove the standard limit $\lim_{x\to0}\frac{\sin x}{x}=1$ geometrically, with the help of Sandwich Theorem. We take a circle of unit radius and suppose that the central angle x expressed in radians is contained within the limits 0 and π/2. It will be enough to consider only x > 0, since $\frac{\sin x}{x}$ is an even function.

Proof: We shall compare the areas of three regions in the figure : two triangles and a sector.



On inspecting the figure we find that, area of ΔOAP < area of sector OAP < area of ΔOAT

$$\frac{\sin x}{2}<\frac{x}{2}<\frac{\tan x}{2} \Rightarrow \quad \sin x < x < \tan x$$

$$\Rightarrow \quad 1<\frac{x}{\sin x}<\frac{1}{\cos x} \qquad (0<x<\pi/2)$$

$$\Rightarrow \quad \cos x<\frac{\sin x}{x}<1$$

Since, $\lim_{x\to0}\cos x=1$, and $\lim_{x\to0^+}1=1$

Using Sandwich Theorem we have $\lim_{x\to0^+}\frac{\sin x}{x}=1$

Put x = – y, then $\lim_{y\to0^-}\frac{\sin y}{y}=1$.

Hence, $\lim_{x\to0}\frac{\sin x}{x}=1$.

Zero Limit Theorem

A function f approaches the limit 0 as x tends to a if and only if |f | approaches the limit 0 as x tends to a.

$\lim_{x\to a}|f(x)|=0 \Leftrightarrow \lim_{x\to a}f(x)=0$.

Proof: Since $-|f(x)|\le f(x)\le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, $\lim_{x\to a}f(x)=0$ by Sandwich Theorem.

Let $f(x)=x\cos^3\frac{1}{x}$. As $x\to0$, $f(x)\to0$.

Also $|f(x)|=\left|x\cos^3\frac{1}{x}\right|$ approaches zero because

$$-|x|\le\left|x\cos^3\frac{1}{x}\right|\le|x| \text{ and}$$

$-|x|$ and $|x|$ approach 0 as $x\to0$.

Example 1: Show that $\lim_{x\to0}x^2\sin\frac{1}{x}=0$.

Solution: First notice that we cannot use

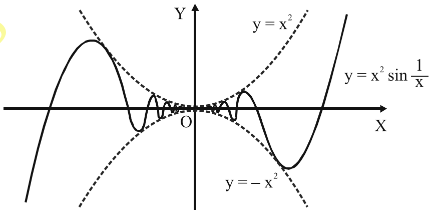
$$\lim_{x\to0}x^2\sin\frac{1}{x}=\lim_{x\to0}x^2\cdot\lim_{x\to0}\sin\frac{1}{x}$$

because $\lim_{x\to0}\sin(1/x)$ does not exist.

However, since $-1\le\sin\frac{1}{x}\le1$.

We have, as illustrated by the figure,

$$-x^2\le x^2\sin\frac{1}{x}\le x^2$$



We know that

$$\lim_{x\to0}x^2=0 \text{ and } \lim_{x\to0}(-x^2)=0$$

Taking $g(x)=-x^2$, $f(x)=x^2\sin(1/x)$, and $h(x)=x^2$ in the Sandwich theorem, we obtain

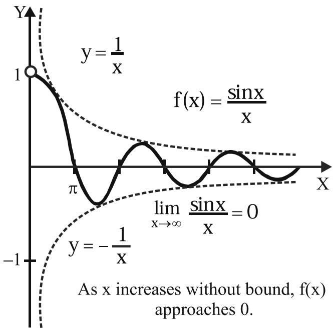
$$\lim_{x\to0}x^2\sin\frac{1}{x}=0.$$

Example 2: Determine the limit $\lim_{x\to\infty}\frac{\sin x}{x}$.

Solution: Since $-1\le\sin x\le1$, it follows that

$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x},\ x>0,$$

where $\lim_{x\to\infty}(-1/x)=0$ and $\lim_{x\to\infty}(1/x)=0$. Therefore, by Sandwich Theorem, we can obtain $\lim_{x\to\infty}\frac{\sin x}{x}=0$.



As x increases without bound, f(x) approaches 0.

Example 3: Evaluate $\lim\limits_{x\to\infty}\dfrac{[x]}{x}$.

Solution: We have $x - 1 < [x] \le x$,

$\Rightarrow 1-\dfrac{1}{x}<\dfrac{[x]}{x}\le 1$ for $x > 0$.

Now $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)=1$. Therefore by Sandwich theorem

$\lim\limits_{x\to\infty}\dfrac{[x]}{x}=1$.

Example 4: Let a function f(x) be such that $|f(x)| \le M$ for any $x \ne 0$. Prove that $\lim\limits_{x\to 0} x\,f(x) = 0$.

Solution: We have $|x\,f(x)| \le |x|\,.\,M$

$\Rightarrow \quad -M|x| \le x\,f(x) \le M|x|$ for any $x \ne 0$.

Since $\lim\limits_{x\to 0} M|x| = 0$ and $\lim\limits_{x\to 0} -M|x| = 0$,

by Sandwich theorem, $\lim\limits_{x\to 0} x\,f(x) = 0$.

Example 5: Evaluate $\lim\limits_{x\to\infty}\dfrac{x+7\sin x}{-2x+13}$ using Sandwich Theorem.

Solution: We have $-1 \le \sin x \le 1$ for all x.

$\Rightarrow \quad -7 \le 7\sin x \le 7$

$\Rightarrow \quad x - 7 \le x + 7\sin x \le x + 7$

Dividing throughout by $-2x + 13$ (a negative quantity for large values of x), we get

$$\frac{x-7}{-2x+13}\ge\frac{x+7}{-2x+13}\ge\frac{x+7}{-2x+13}.$$

We evaluate the limits of the extreme functions as x tends to infinity. Since both the limits are equal to $-1/2$, by Sandwich theorem the limit of the desired function is also $-1/2$.

Example 6: Evaluate $\lim\limits_{x\to\infty}\dfrac{x^2(2+\sin^2 x)}{x+10}$

Solution: $\dfrac{2x^2}{x+10}\le\dfrac{x^2(2+\sin^2 x)}{x+10}\le\dfrac{3x^2}{x+10}$

$\lim\limits_{x\to\infty}\dfrac{2x^2}{x+10}=\infty$ and $\lim\limits_{x\to\infty}\dfrac{3x^2}{x+10}=\infty$

Hence, by Sandwich Theorem $\lim\limits_{x\to\infty}\dfrac{x^2(2+\sin^2 x)}{x+10}=\infty$.

Example 7: Find $\lim\limits_{x\to 0} f(x)$ if $f(x)=\begin{cases}1+2x^2 & , \ x\in Q\\ 1+x^4 & , \ x\notin Q\end{cases}$

Solution: Consider $|x| \le 1$.

$x^4 \le x^2 \le 2x^2$.

Now $1 \le f(x) \le 1 + 2x^2 \ \forall\ x$.

$\lim\limits_{x\to 1} 1 + 2x^2 = 1$. By Sandwich theorem, $\lim\limits_{x\to 0} f(x) = 1$

Definition of e

The number e is defined as the limit $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

The number e is irrational and therefore cannot be expressed precisely by a finite decimal. Its approximate value is $e \approx 2.718$

It can be shown that the function $y=\left(1+\dfrac{1}{x}\right)^x$ has a limit not only when its argument runs through natural numbers (that is x = n where n = 1, 2, 3, ...) but also when it varies continuously and approaches $-\infty$ or ∞. In all the cases the limit is the same number e.

Proof: Let $n \le x < n + 1$

$\Rightarrow \quad 1+\dfrac{1}{n+1}<1+\dfrac{1}{x}\le 1+\dfrac{1}{n}$

$\Rightarrow \quad \left(1+\dfrac{1}{n+1}\right)^n<\left(1+\dfrac{1}{x}\right)^x\le\left(1+\dfrac{1}{n}\right)^{n+1}$

$\Rightarrow \quad \left(1+\dfrac{1}{n+1}\right)^{n+1}\left(1+\dfrac{1}{n+1}\right)^{-1}<(1+x)^{\frac{1}{x}}$

$\le\left(1+\dfrac{1}{n}\right)^n\left(1+\dfrac{1}{n}\right)$

Now as $x \to \infty$, n also tends to ∞ and in such a case both the right hand and left hand side of the above inequality tend to e. Thus by Sandwich theorem.

$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e.$

Example 8: Evaluate

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}\right)$$

Solution: Let f (n)

$$=\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}$$

Note that f(n) has n terms which are decreasing Suppose h (n)

$$=\left(\frac{n}{n^2+1}+\frac{n}{n^2+1}+\frac{n}{n^2+1}+....+\frac{n}{n^2+1}\right)$$

$h(n)=\dfrac{n^2}{n^2+1}$ (obviously $f(n) < h(n)$) and g (n)

$$=\lim_{n\to\infty}\left(\frac{n}{n^2+n}+\frac{n}{n^2+n}+\frac{n}{n^2+n}+....+\frac{n}{n^2+n}\right)$$

$=\dfrac{n^2}{n^2+n}$ (obviously $g(n) < f(n)$)

Hence $g(n) < f(n) < h(n)$

Since $\lim\limits_{n\to\infty} g(n) = 1 = \lim\limits_{n\to\infty} h(n)$

Hence using Sandwich Theorem $\lim\limits_{n\to\infty} f(n) = 1$.

Example 9: Let $x_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$. Calculate $\lim\limits_{n\to\infty} x_n$.

Solution: We have

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}}$$

Thus $\frac{n}{\sqrt{n^2+n}} < x_n < 1$

Since $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$,

we have $\lim\limits_{n\to\infty} x_n = 1$ in accordance with Sandwich theorem.

Caution

On the other hand, the general term in the expression for x_n is equal to $\frac{1}{\sqrt{n^2+k}}$ (k = 1, 2,...., n).

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2+k}} = 0$, if follows that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ... + \frac{1}{\sqrt{n^2+n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2}} + ...$$

$$+ \lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0 + 0 + ... + 0 = 0$$

The latter method has error in the reasoning. When we add infinite number of small quantities, the sum may not always tend to zero.

Example 10: Using Sandwich Theorem, show that

$$\lim_{n\to\infty} \frac{1.3.5....(2n-1)}{2.4.8......2^n} = 0 \cdot$$

Solution: We can see that

$$\frac{1.3.5....(2n-1)}{2.4.8......2^n} > \frac{1}{2.4.8.....2^n}$$

$$\frac{1.3.5....(2n-1)}{2.4.8....2^n} < \frac{\left(\frac{n}{2}\,\frac{(1+2n-1)}{n}\right)^n}{2^{1+2+.....n}}$$

Using AM. – GM. inequality in the numerator.

$$\Rightarrow \quad \frac{1.3.5....(2n-1)}{2.4.8....2^n} < \frac{n^n}{2^{\frac{n(n+1)}{2}}}$$

Thus, we have

$$\frac{1}{2.4.8...2^n} < \frac{1.3.5....(2n-1)}{2.4.8....2^n} < \left(\frac{n^2}{2^{n+1}}\right)^{\frac{n}{2}}$$

Now, we have

$$\lim_{n\to\infty} \frac{1}{2.4.8.....2^n} = 0 \text{ and } \lim_{n\to\infty} = \left(\frac{n^2}{2^{n+1}}\right)^{\frac{n}{2}}$$

Hence, by Sandwich theorem, we have

$$\lim_{n\to\infty} \frac{1.3.5....(2n-1)}{2.4.8....2^n} = 0$$

Example 11: Find $\lim\limits_{n\to\infty} \frac{[x]+[2x]+[3x]+....+[nx]}{n^2}$

where [.] denotes the greatest integer function.

Solution: We know that, $x - 1 < |x| \le x$

$\Rightarrow \quad 2x - 1 < [2x] \le 2x$

$\Rightarrow \quad 3x - 1 < [3x] \le 3x$

....

....

$\Rightarrow \quad nx - 1 < [nx] \le nx$

$\therefore \quad (x + 2x + 3x + ... + nx) - n$

$< [x] + [2x] + ... +[nx] \le (x + 2x + ... + nx)$

$$\Rightarrow \quad \frac{xn(n+1)}{2} - n < \sum_{r=1}^{n} [r\,x] \le \frac{x.n(n+1)}{2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} \le \lim_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2}$$

$$\le \lim_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right)$$

$$\Rightarrow \quad \frac{x}{2} \le \lim_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} \le \frac{x}{2}$$

Thus, $\lim\limits_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} = \frac{x}{2}$

Example 12: Evaluate

$$\lim_{x\to0^+}\left(\lim_{n\to\infty} \frac{[1^2x^x]+[2^2x^x]+.....+[n^2x^x]}{n^3}\right)$$

where [ ] denotes the greatest integer function.

Solution: We have $1^2x^x - 1 < [1^2x^x] \le 1^2x^x$

$2^2x^x - 1 < [2^2x^x] \le 2^2x^x$

....

....

$n^2x^x - 1 < [n^2x^x] \le n^2x^x$

Adding the above inequations,

$$\frac{x^x\Sigma n^2 - n}{n^3} < \frac{\Sigma[n^2x^x]}{n^3} \le \frac{x^x\Sigma n^2}{n^3}$$

$$x^x\,\frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n^2} < \frac{\Sigma[n^2x^x]}{n^3} \le x^x\,\frac{n(n+1)(2n+1)}{6n^3}$$

Now, as n tends to infinity, we have

$$\frac{x^x}{3} \le \lim_{n\to\infty} \frac{\Sigma[n^2x^x]}{n^3} \le \frac{x^x}{3}$$

Hence, by Sandwich Theorem, we have

$$\lim_{n\to\infty} \frac{\Sigma[n^2x^x]}{n^3} = \frac{x^x}{3}$$

Now, the required limit

$$\lim_{x\to0^+}\left(\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3}\right) = \frac{1}{3}\lim_{x\to0^+} x^x$$

$$= \frac{1}{3}\lim_{x\to0^+} e^{\frac{\ln x}{1/x}} \text{ (since 1/x is much larger than lnx)}$$

$$= \frac{1}{3}e^0 = \frac{1}{3}.$$

Concept Problems O

1. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $x \in [-1, 1]$, find $\lim\limits_{x\to0} f(x)$

2. It can be shown that the inequalities $1-\frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$ hold for all values of x close to zero. What does this tell you about $\lim\limits_{x\to0}\frac{x\sin x}{2-2\cos x}$?

3. If $x^4 \le f(x) \le x^2$ for x in [–1, 1] and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what point c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limit at these points?

4. By considering the inequality

$$1-\frac{x^2}{2!} \le \cos x \le 1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

prove that $\lim\limits_{x\to0}\frac{1-\cos x}{x} = 0$.

5. Find formulae for function g and h such that $g(x) \to 0$ and $h \to 0$ as $x \to \infty$ and such that $g(x) \le \frac{1}{x^2}\sin(50\pi\sqrt[3]{x}) \le h(x)$ for all values of x except zero. What can you say about $\lim\limits_{x\to\infty}\frac{1}{x^2}\sin(50\pi\sqrt[3]{x})$?

6. If $b - |x-a| \le f(x) \le b + |x-a|$ in the neighbourhood of $x = a$ then find $\lim\limits_{x\to a} f(x)$.

7. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Use Sandwich theorem to prove that $\lim\limits_{x\to0} f(x) = 0$.

8. Show that $\lim\limits_{x\to0}\frac{x^2\cos\frac{1}{x}}{\sin x} = \lim\limits_{x\to0}\frac{x^2\sin\frac{1}{x}}{\sin x}$

9. Find $\lim\limits_{x\to\infty} f(x)$ if $\frac{4x-1}{x} < f(x) \le \frac{4x^2+3x}{x^2}$ for all $x > 5$.

Practice Problems M

10. Evaluate the following limits:

(i) $\lim\limits_{x\to0} x^2\left(\cos 2x + x^2\cos\frac{2}{x}\right)$

(ii) $\lim\limits_{n\to\infty}\frac{\sqrt{n}\cos(\pi e^n)}{2n+1}$

(iii) $\lim\limits_{x\to\infty}\frac{2e^x+e^{-x}-3x}{[e^x+x^2]}$ where [.] is G.I.F.

(iv) $\lim\limits_{x\to0^+}\frac{x\sin\frac{1}{x}}{\sin x}$

11. Show that $\frac{1}{2}-\frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$ for all x close to x = 0 and hence find $\lim\limits_{x\to0}\frac{1-\cos x}{x^2}$.

12. Prove that

$$\lim_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}} > \lim_{x\to-1/2}\frac{2x^2-5x-3}{4x^2-18x-10} + \lim_{x\to0}\frac{\sin^2\frac{x}{2}}{x^2}$$

13. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x+\cos x}$

(ii) $\lim\limits_{x\to\infty} \dfrac{x\sin x!}{1+x^2}$

(iii) $\lim\limits_{x\to 0} \sqrt{x^3+x^2}\,\sin\dfrac{\pi}{x}$

(iv) $\lim\limits_{x\to 0^+} \sqrt{x}\;\; e^{\cos\frac{1}{x}}$

14. Evaluate $\lim\limits_{n\to\infty} \dfrac{[1^2x]+[2^2x]+[3^2x]+....+[n^2x]}{n^3}$ where [.] denotes the greatest integer function and $x \in R$

15. Let the inequality $g(x) < f(x) < 1 - \cos x + g(x)$ hold for all x in the neighborhood of x = 0. If $\lim\limits_{x\to 0}\dfrac{g(x)}{x} = 1$, find $\lim\limits_{x\to 0}\dfrac{f(x)}{\sin x}$.

16. If a and b are positive, then prove that $\lim\limits_{x\to 0^+}\dfrac{x}{a}\left[\dfrac{b}{x}\right] = \dfrac{b}{a}$, $\lim\limits_{x\to 0^+}\dfrac{b}{x}\left[\dfrac{x}{a}\right] = 0$. How do the functions behave when $x \to 0^-$?

17. Find $\lim\limits_{x\to 0} f(x)$ where $f(x) = \begin{cases} x^3 & , x \le 0 \\ x\sin\dfrac{1}{x} & , x > 0 \end{cases}$

18. Evaluate $\lim\limits_{x\to 0}(x^2+x)\left[\dfrac{1}{x}\right]$ where [.] denotes the greatest integer function.

19. Use Sandwich theorem to evaluate the limit $\lim\limits_{n\to\infty}\sum\limits_{k=n^2}^{(n+1)^2}\dfrac{1}{\sqrt{k}}$.

1.18 L' Hospital's Rule

Suppose f and g are differentiable on an open interval containing a, except possibly at a itself. Assume that $g'(x) \ne 0$ for all x in the open interval containing a, except possibly at a. Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$

(In other words, we have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$)

Then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$, provided the limit on the right side exists or is ∞ or $-\infty$

Note: In many texts, the name L'Hospital is spelt as L' Hôpital.

L'Hospital's rule says that the limit of a quotient of function is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. In other words, suppose that $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = A$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = A$ where A may be a number, or ∞ or $-\infty$.

It is especially important to verify the conditions regarding the limits of f and g before using L'Hospital's Rule.

L'Hospital's rule is also valid for one sided limits and for limits at infinity or negative infinity : that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a$, $x \to \infty$ or $x \to -\infty$.

Special Case Proof

For the special case in which f(a) = g(a) = 0, f' and g' are continuous, and $g'(a) \ne 0$, it is easy to see why L'Hospital's rule is true. It is more difficult to prove the general version of L'Hospital's rule.

If f and g are differentiable and f(a) = g(a) = 0 and $g'(a) \ne 0$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

Proof : Using the definition of the derivative we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}}{\lim\limits_{h\to 0}\dfrac{g(a+h)-g(a)}{h}} = \lim_{h\to 0}\frac{f(a+h)}{g(a+h)}$$

since f(a) = g(a) = 0

$$= \lim_{x\to a}\frac{f(x)}{g(x)}.$$

Notice that we have assumed $g'(a) \ne 0$.

Comprehensive Proof of L'Hospital's Rule

We first present L'Hospital's Rule as a theorem about one-sided limits.

L'Hospital's Rule I. Let f and g be functions which are differentiable on a non-empty open interval (a, b) with $g'(x) \ne 0$ for every x in (a, b). If , in addition,

(i) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$,

(ii) $\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)} = L$.

Proof:

We may assume that f(a) = g(a) = 0. (If this is not the case to begin with, we simply define, or redefine, the values of f and g to be zero at a.) Thus we ensure that f and g are continuous on [a, x]. Let x be an arbitrary number in (a, b). Then f and g are continuous on [a, x] (recall that differentiability at a point implies continuity) and are differentiable on (a, x). Moreover, the derivative g′ does not take on the value zero in (a, x).

Moreover, the derivative g′ does not take on the value zero in (a, x). Hence, by the Generalized Mean Value Theorem and the fact that f(a) = g(a) = 0, we obtain

$$\frac{f'(y)}{g'(y)} = \frac{f(x)-0}{g(x)-0} = \frac{f(x)}{g(x)},$$

for some number y in (a, x). As x approaches a from the right, so also does y, and hence

$\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^+}\dfrac{f'(y)}{g'(y)} = L$. This completes the proof.

Generalized Mean Value Theorem

Assume that a < b, and let f and g be functions which are continuous on the closed interval [a, b] and differentiable on the open interval (a, b). If $g'(x) \neq 0$ for every x in (a, b) then there exists a real number c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Example 1: Compute $\lim\limits_{x\to 2^+}\dfrac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}}$.

Solution: Let $f(x) = \sqrt{x}-\sqrt{2}$ and $g(x) = \sqrt{x-2}$. Obviously, $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} g(x) = 0$, and since

$$f'(x) = \frac{1}{2\sqrt{x}} \quad\text{and}\quad g'(x) = \frac{1}{2\sqrt{x-2}},$$

f and g are differentiable, and g′ does not take on the value zero on any open interval with left endpoint equal to 2. We obtain

$$\lim_{x\to 2^+}\frac{f'(x)}{g'(x)} = \lim_{x\to 2^+}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x-2}}}$$

$$= \lim_{x\to 2^+}\frac{\sqrt{x-2}}{\sqrt{x}} = \frac{0}{\sqrt{2}} = 0.$$

And it follows by L'Hospital's Rule that $\lim\limits_{x\to 2^+}\dfrac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}} = 0$.

It is a simple matter to verify that L'Hospital's Rule I remains true if (a, b) is replaced throughout by (b, a), and $\lim\limits_{x\to a^+}$ is replaced throughout by $\lim\limits_{x\to a^-}$. This fact, significant in itself, also implies the following two-sided form of L'Hospital's Rule.

L'Hospital's Rule II. Consider an open interval containing the number a, and let f and g be functions differentiable and with $g'(x) \neq 0$ at every point of the interval except possibly at a. If

(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$,

(ii) $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = L$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

The hypotheses have been taken as weak as possible. If, as frequently happens, the functions f and g are also continuous at a, then (i) can be replaced by the simpler condition f(a) = g(a) = 0.

Example 2: Evaluate $\lim\limits_{x\to a}\dfrac{x^{1/2}-a^{1/2}}{x^{1/3}-a^{1/3}}$, where a > 0.

Solution: If $f(x) = x^{1/2} - a^{1/2}$ and if $g(x) = x^{1/3} - a^{1/3}$, then the derivatives are given by $f'(x) = \dfrac{1}{2x^{1/2}}$ and $g'(x) = \dfrac{1}{3x^{2/3}}$, and it is clear that f and g are differentiable (and hence continuous) and g′ is not zero on an open interval containing a. Moreover f(a) = g(a) = 0. Hence, by L'Hospital's Rule,

$$\lim_{x\to a}\frac{x^{1/2}-a^{1/2}}{x^{1/3}-a^{1/3}} = \lim_{x\to a}\frac{\frac{1}{2x^{1/2}}}{\frac{1}{3x^{2/3}}} = \frac{3a^{2/3}}{3a^{1/2}} = \frac{3a^{1/6}}{2}.$$

Caution

While applying L'Hospital's rule we are not supposed to differentiate f(x)/ϕ(x) as a fraction. The numerator and denominator must be differentiated separately.

Example 3: Compute $\lim\limits_{x\to a}\dfrac{x-a}{x^2-a^2}$ where $a \neq 0$.

Solution: The fact that

$$\frac{x-a}{x^2-a^2} = \frac{x-a}{(x-a)(x+a)} = \frac{1}{x+a} \quad\text{if } x \neq a,$$

immediately implies that

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} = \lim_{x\to a}\frac{1}{x+a} = \frac{1}{2a}.$$

Of course, the same answer is obtained by L'Hospital's Rule. If we let f(x) = x – a and $g(x) = x^2 - a^2$, then f(a) = g(a) = 0 and f′(x) = 1 and g′(x) = 2x. Hence

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} = \lim_{x\to a}\frac{1}{2x} = \frac{1}{2a}.$$

It is important to realize that L'Hospital's Rule II can be applied only if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$. For example, if $f(x) = x^2 + 3x - 10$ and g(x) = 3x, then

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2+3x-10}{3x} = \frac{0}{6} = 0,$$

but

$$\lim_{x\to 2}\frac{f'(x)}{g'(x)} = \lim_{x\to 2}\frac{3x+3}{3} = \frac{7}{3}.$$

Example 4: Evaluate $\lim\limits_{x\to 0}\dfrac{x+\tan 2x}{x-\tan 2x}$

Solution: Since we have the indeterminate form 0/0, we apply L'Hospital's rule to give:

$$\lim_{x\to 0}\frac{x+\tan 2x}{x-\tan 2x}=\lim_{x\to 0}\frac{1+2\sec^2 2x}{1-2\sec^2 2x}=\frac{1+2}{1-2}=-3.$$

Example 5: Evaluate $\lim\limits_{h\to 0}\dfrac{\ell n(1+2h)-2h}{h^2}$

Solution: $\lim\limits_{h\to 0}\dfrac{\ell n(1+2h)-2h}{h^2}\quad\left(\text{form }\dfrac{0}{0}\right)$

$$=\lim_{h\to 0}\frac{\dfrac{2}{1+2h}-2}{2h}\quad\text{[by L'Hospital's rule]}$$

$$=\lim_{h\to 0}\frac{1-(1+2h)}{h}=\lim_{h\to 0}\frac{-2h}{h}=-2.$$

Example 6: Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^4}$

Solution: $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^4}\quad\left(\dfrac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{\sin x}{4x^3}\quad\left(\frac{1}{0}\right)\quad\text{[by L'Hospital's rule]}$$

$$=\infty$$

Note: If the hypotheses of L'Hospital's Rule II are satisfied for the functions f′ and g′, that is for the derivatives of f and g, respectively, then we can conclude that $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=\lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$.

This fact suggests the possibility of applying L'Hospital's Rule more than once, and in some problems it is necessary to take second or higher derivatives to find the limit.

Example 7: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}$

Solution: $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}\quad\left(\dfrac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{\cos x-1}{3x^2}\quad\left(\frac{0}{0}\right)\quad\text{[by L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{-\sin x}{6x}=-\frac{1}{6}\quad\text{[by L'Hospital's rule]}$$

If $\quad f'(a)=f''(a)=\ldots\ldots=f^{n-1}(a)=0$
and $\quad g'(a)=g''(a)=\ldots\ldots=g^{n-1}(a)=0$

but $f^n(a)$ and $g^n(a)$ are not both zero, then by repeated application of L'Hospital's rule, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^n(x)}{g^n(x)}$$

Rule: If the limit of f(x)/ g(x) as x → a takes the form 0/0, differentiate the numerator and denominator separately w.r.t. x and obtain a new function f′(x)/g′(x). Now as x → a, if it again takes the form 0/0, differentiate the numerator and denominator again w.r.t. x and repeat the above process, till indeterminate form persists.

The following examples illustrate the repeated use of L'Hospital's rule.

Example 8: Evaluate $\lim\limits_{x\to 1}\dfrac{3x^{1/3}-x-2}{3x^2-6x+3}$.

Solution: Let $f(x)=3x^{1/3}-x-2$ and $g(x)=3x^2-6x+3$. Then $f(1)=g(1)=0$, and the derivatives are given by $f'(x)=x^{-2/3}-1$ and $g'(x)=6x-6$. However, the value of

$$\lim_{x\to 1}\frac{f'(x)}{g'(x)}=\lim_{x\to 1}\frac{x^{-2/3}-1}{6x-6}$$

is not obvious because $f'(1)=g'(1)=0$. Taking derivatives again, we get $f''(x)=-\dfrac{2}{3}x^{-5/3}$ and $g''(x)=6$, and it follows that

$$\lim_{x\to 1}\frac{f''(x)}{g''(x)}=\lim_{x\to 1}\frac{-\dfrac{2}{3}x^{-5/2}}{6}=-\frac{1}{9}.$$

Thus two applications of L'Hospital's Rule yield

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{f'(x)}{g'(x)}=\lim_{x\to 1}\frac{f''(x)}{g''(x)}=-\frac{1}{9}.$$

Example 9: Evaluate $\lim\limits_{t\to 0}\dfrac{e^{2t}-e^t-t}{t^2}$.

Solution: Recognizing the indeterminate form 0/0 we apply L'Hospital's rule to yield

$$\lim_{t\to 0}\frac{e^{2t}-e^t-t}{t^2}=\lim_{t\to 0}\frac{2e^{2t}-e^t-1}{2t}$$

$$=\lim_{t\to 0}\frac{2e^{2t}-e^t}{2}=\frac{3}{2}$$

in which we were obliged to use (and justified in using) the quotient of the second derivatives, since the quotient of the first derivatives turned out to be in the indeterminate form 0/0.

In point of logic one should understand that we properly applied L'Hospital's rule to the quotient of derivatives first and thereupon applied it again to the given quotient function. The fact that we write equalities form left to right should not be allowed to confuse the order of logical application.

⚠ Caution

Before applying L'Hospital's rule at any stage be sure that the form is 0/0. Do not go on applying this rule even if the form is not 0/0.

Example 10: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-x^2-2}{\sin^2 x-x^2}$

Solution: We obtain

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{\sin x\cos x-2x}=\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{\sin 2x-2x}$$

By repeated uses of L'Hospital's Rule, we get

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2}{2\cos 2x-2}=\lim_{x\to 0}\frac{e^x-e^{-x}}{-4\sin 2x}$$

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{-8\cos 2x}=\frac{1+1}{-8(1)}-\frac{2}{8}=-\frac{1}{4}$$

Caution

Criticize the following use of L'Hospital Rule :

$$\lim_{x\to 2}\frac{x^3-x^2-x-2}{x^3-3x^2+3x-2}$$

$$=\lim_{x\to 2}\frac{3x^2-2x-1}{3x^2-6x+3}$$

$$=\lim_{x\to 2}\frac{6x-2}{6x-6}=\lim_{x\to 2}\frac{6}{6}=1$$

The second equation is an incorrect use of L'Hospital's Rule, since $\lim\limits_{x\to 2}(3x^2-2x-1)=7$ and $\lim\limits_{x\to 2}(3x^2–6x+3)=3$. So, the correct limit should be $\frac{7}{3}$.

Example 11: Evaluate $\lim\limits_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$

Solution: Since the form is 0/0, we shall apply L'Hospital's rule.

$$\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\lim_{x\to 0}\frac{xe^x+e^x-\{1/(1+x)\}}{2x},$$

[form 0/0]

$$=\lim_{x\to 0}\frac{xe^x+e^x+e^x+\{1/(1+x)^2\}}{2}$$

$$=\frac{0+1+1+1}{2}=\frac{3}{2}$$

A variation of L'Hospital's Rule, not difficult to prove, is the following:

L'Hospital's Rule III. Let f and g be differentiable on an open interval (a, ∞) with $g'(x)\neq 0$ for $x > a$. If

(i) $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} g(x)=0,$

(ii) $\lim\limits_{x\to\infty}\dfrac{f'(x)}{g'(x)}=L,$

then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=L.$

An analogous theorem holds if (a, ∞) is replaced by (–∞, a) and if $\lim\limits_{x\to\infty}$ is replaced throughout by $\lim\limits_{x\to-\infty}$.

Proof:

The result is a corollary of L'Hospital's Rule I and the Chain Rule. Let $t=\dfrac{1}{x}$, and set

$$F(t)=f\left(\frac{1}{t}\right)=g(x)\ \text{ and } G(t)=g\left(\frac{1}{t}\right)=g(x).\ \text{Since}$$

t approaches 0 from the right if and only if x increases without bound,

$$\lim_{t\to 0^+}F(t)=\lim_{t\to 0^+}f\left(\frac{1}{t}\right)=\lim_{x\to\infty}f(x)=0,$$

$$\lim_{t\to 0^+}G(t)=\lim_{t\to 0^+}g\left(\frac{1}{t}\right)=\lim_{x\to\infty}f(x)=0.$$

By the Chain Rule, $F'(t)=f'\left(\dfrac{1}{t}\right)\left(-\dfrac{1}{t^2}\right)$ and

$$G'(t)=g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right).$$

Hence

$$\lim_{t\to 0^+}\frac{F'(x)}{G'(x)}=\lim_{t\to 0^+}\frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}=\lim_{t\to 0^+}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}$$

The last limit exists and is equal to L since

$$\lim_{t\to 0^+}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=L.$$

By L'Hospital's Rule I it follows that $\lim\limits_{t\to 0^+}\dfrac{F(t)}{G(t)}=L,$ and the proof is complete.

An important observation is that all the forms of L'Hospital's Rule developed so far are valid whether L is finite or not. This fact requires no new proof and has really already been established. The reason is that the basic conclusion of L'Hospital's Rule I is the

equation $\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$, and this holds good whether or not $\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$ is finite or infinite.

There is another significant variation of L'Hospital's Rule, whose proof requires the Generalized Mean Value Theorem.

It states that the condition (i) can be replaced by $\lim_{x\to a}|g(x)| = \infty$. The specific statement which we prove is the following:

L'Hospital's Rule IV. Let f and g be functions which are differentiable on a non-empty open interval (a, b) with $g'(x) \neq 0$ for every x in (a, b). If

(i) $\lim_{x\to a^+}|f(x)| = \lim_{x\to a^+}|g(x)| = \infty$,

(ii) $\lim_{x\to a^+}\dfrac{f'(x)}{g'(x)} = L$,

then $\lim_{x\to a^+}\dfrac{f(x)}{g(x)} = L$.

Proof:

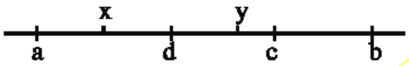
Let ε be an arbitrary positive number. By hypothesis (ii), there exists a real number c in (a, b) such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon, \qquad \text{for every x in (a, c).}$$

By hypothesis (i) there exists a real number d in (a, b), which we shall for convenience assume to be in (a, c), such that, for every x in (a, d), the following three inequalities hold :

$$g(x) \neq 0, \qquad \left|\frac{f(c)}{g(x)}\right| < \varepsilon, \qquad \left|\frac{g(c)}{g(x)}\right| < \varepsilon$$

(see Figure). It is a consequence of the last inequality that



$$\left|1 - \frac{g(c)}{g(x)}\right| < 1 + \varepsilon, \text{ for every x in (a, d).}$$

Now let x be an arbitrary real number in (a, d). By the Generalized Mean Value Theorem, there exists a real number y in (x, c) such that

$$\frac{f(x) - f(c)}{g(c) - g(c)} = \frac{f'(y)}{g'(y)}.$$

Hence

$$f(x) = \frac{f'(y)}{g'(y)}(g(x) - g(c)) + f(c).$$

Dividing by g(x), which cannot be zero, we get

$$\frac{f(x)}{g(x)} = \frac{f'(y)}{g'(y)}\left(1 - \frac{g(c)}{g(x)}\right) + \frac{f(c)}{g(x)}.$$

An equivalent equation is

$$\frac{f(x)}{g(x)} - L = \left(\frac{f'(y)}{g'(y)} - L\right)\left(1 - \frac{g(c)}{g(x)}\right) - L\frac{g(c)}{g(x)} + \frac{f(c)}{g(x)}.$$

From the general properties of the absolute value, it follows that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f'(y)}{g'(y)} - L\right| \left|1 - \frac{g(c)}{g(x)}\right| + |L|\left|\frac{g(c)}{g(x)}\right| + \left|\frac{f(c)}{g(x)}\right|.$$

Hence, the inequalities established in the first paragraph of the proof imply that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \varepsilon(1 + \varepsilon) + |L|\varepsilon + \varepsilon.$$

Since the right side of this inequality can be made arbitrarily small by taking ε sufficiently small, it follows that $\lim_{x\to a^+}\dfrac{f(x)}{g(x)} = L$, and the proof is complete.

It is not difficult to derive variations of the preceding theorem analogous to the modified versions described earlier. Thus, with the obvious changes in the hypotheses, this last form of L'Hospital's Rule also holds for two-sided limits and with a or L (or both) replaced by ± ∞.

Example 12: Compute $\lim_{x\to\infty}\dfrac{\sqrt[3]{x+1}}{x+4}$.

Solution: Let f and g be the functions defined by $f(x) = \sqrt[3]{x+1}$ and $g(x) = x + 4$, respectively. Since $f'(x) = \dfrac{1}{3(x+1)^{2/3}}$ and $g'(x) = 1$, we see that f and g are differentiable on the interval $(1, \infty)$ and that $g'(x) \neq 0$. Moreover, $\lim_{x\to\infty}|g(x)| = \lim_{x\to\infty}|x+4| = \infty$, and

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{\frac{1}{3(x+1)^{2/3}}}{1} = 0.$$

Example 13: Evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$

Solution: Since lnx approaches ∞ as x approaches ∞, L'Hospital's Rule implies that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0\cdot$$

Example 14: Evaluate $\lim_{x\to\infty}\dfrac{\ell n(x+1)}{x^2}$

Solution: $\lim_{x\to\infty}\dfrac{\ell n(x+1)}{x^2} \quad \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty}\frac{\frac{1}{x+1}}{2x} \qquad \text{[by L'Hospital's rule]}$$

$$= \lim_{x\to\infty}\frac{1}{(x+1)2x} \quad \left(\frac{1}{\infty}\right) = 0$$

Example 15: Evaluate $\lim_{x\to\infty}\dfrac{x}{e^x}$

Solution: Since e^x approaches ∞ as x approaches ∞, L'Hospital Rule implies that

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Note: $\lim\limits_{x\to\infty}\frac{\ln x}{x}=0$ and $\lim\limits_{x\to\infty}\frac{x}{e^x}=0$.

Example 16: Evaluate $\lim\limits_{x\to\infty}\frac{x^{100}}{e^x}$

Solution: $\lim\limits_{x\to\infty}\frac{x^{100}}{e^x}\ \left(\frac{\infty}{\infty}\right)$

we apply L'Hospital's rule repeatedly 100 times

$$=\lim_{x\to\infty}\frac{100.\,99.........2.\,1}{e^x}=0$$

Study Tip

While evaluating $\lim\limits_{x\to a}\{f(x)/g(x)\}$, when it is of the form ∞/∞, it is sometimes necessary to change it into the form 0/0, otherwise the process of differentiating the numerator and the denominator will never end.

Example 17: Evaluate $\lim\limits_{x\to0^+}\frac{\ln x}{\cot x}$

Solution: We have, $\lim\limits_{x\to0^+}\frac{\ln x}{\cot x}$, [form ∞/∞]

$$=\lim_{x\to0^+}\frac{1/x}{-\operatorname{cosec}^2 x},\qquad \text{[form }\infty/\infty]$$

$$=\lim_{x\to0^+}\frac{-\sin^2 x}{x},\qquad \text{[form 0/0]}$$

$$=\lim_{x\to0^+}\frac{-2\sin x\cos x}{1}=\frac{-2\times0\times1}{1}=0$$

Example 18: Evaluate $\lim\limits_{x\to\infty} x^m e^{-x}$.

Solution: We have $\lim\limits_{x\to\infty}x^m e^{-x}=\lim\limits_{x\to\infty}\frac{x^m}{e^x}$,

[form ∞/∞]

$$=\lim_{x\to\infty}\frac{mx^{m-1}}{e^x}\qquad \text{[form }\infty/\infty]$$

$$=\lim_{x\to\infty}\frac{m(m-1)x^{m-2}}{e^x}\qquad \text{[form }\infty/\infty]$$

....

$$=\lim_{x\to\infty}\frac{m(m-1)(m-2)\ldots3.2.1}{e^x}=\lim_{x\to\infty}\frac{m!}{e^x}=0,$$

[∵ $e^x\to\infty$ when $x\to\infty$]

Note:

If the quotient $\frac{f'(x)}{g'(x)}$ again yields an indeterminate form, at the point x = a, of one of the two above-mentioned types and f'(x) and g'(x) satisfy all the requirements that have been stated for f(x) and g(x), we can then pass to the ratio of second derivatives, etc. However, it should be borne in mind that the limit of the ratio $\frac{f(x)}{g(x)}$ may exist, whereas the ratios of the derivatives do not tend to any limit.

If $f(x)=x^2\sin\frac{1}{x}$, $g(x)=x$, then we have

$$\lim_{x\to0}\frac{f(x)}{g(x)}=0.$$

Here $\frac{f'(x)}{g'(x)}=2x\sin\frac{1}{x}-\cos\frac{1}{x}$

which oscillates when x → 0. Thus f/g may tend to a limit when f′/g′ does not. Our condition is sufficient but not necessary.

Example 19: Evaluate $\lim\limits_{x\to\infty}\frac{x+\sin x}{x-\cos x}$

Solution: This limit has the indeterminate form ∞/∞. If we try to apply L'Hospital's rule, we get

$$\lim_{x\to\infty}\frac{x+\sin x}{x-\cos x}=\lim_{x\to\infty}\frac{1+\cos x}{1+\sin x}$$

The limit on the right does not exist, because both sinx and cos x oscillate between –1 and 1 as x → ∞. Recall that L'Hospital's rule applies only if this limit exists. This does not mean that the limit of the original expression does not exist or that we cannot find it. It simply means that we cannot apply L'Hospital's rule. To find this limit, factor out an x from the numerator and denominator and proceed as follows :

$$\lim_{x\to\infty}\frac{x+\sin x}{x-\cos x}=\lim_{x\to\infty}\frac{x\left(1+\frac{\sin x}{x}\right)}{x\left(1-\frac{\cos x}{x}\right)}$$

$$=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}=\frac{1+0}{1-0}=1.$$

Example 20: Evaluate $\lim\limits_{x\to0}\frac{x^2\sin\frac{1}{x}}{\sin x}\ \left(\frac{0}{0}\right)$

Solution: Applying L'Hospital's rule, we get

$$\lim_{x\to0}\frac{2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)}{\cos x}$$

$$=\lim_{x\to0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}.\ \text{This limit does not exist.}$$

But this does not mean that the limit of the given function does not exist. We try another method.

$$\lim_{x\to0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to0}\frac{x^2\sin\frac{1}{x}}{\frac{\sin x}{x}}=\frac{0}{1}=0.$$

This implies that L'Hospital's rule is inapplicable in this question.

Example 21: Evaluate $\lim\limits_{x\to\infty}\dfrac{2+2x+\sin^2 x}{(2x+\sin 2x)e^{\sin x}}$

Solution: $\lim\limits_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ $\left(\dfrac{\infty}{\infty}\right)$

$$=\lim_{x\to\infty}\frac{2+2\cos 2x}{(2+2\cos 2x)e^{\sin x}+(2x+\sin 2x)e^{\sin x}.\cos x}$$

$$=\lim_{x\to\infty}\frac{4\cos^2 x}{e^{\sin x}(4\cos x+2x+\sin 2x)\cos x}$$

As x→ ∞, cosx becomes zero at several points. Hence, the function is undefined at several points. Therefore the limit cannot be evaluated. This implies that L'Hospital's rule is inapplicable. We try to find the limit by another method.

$$\lim_{x\to\infty}\frac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}=\lim_{x\to\infty}\left(1+\frac{2}{2x+\sin 2x}\right)e^{-\sin x}$$

The first bracket approaches 1 as x→ ∞, but $e^{-\sin x}$ oscillates between 1/e and e. Hence the limit does not exist.

Using L'Hospital's Rule with Other Limit Properties

Example 22: Evaluate $\lim\limits_{x\to0}\dfrac{(1-\cos x)\sin 4x}{x^3\cos x}$

Solution: This limit has the form 0/0, but direct application of L'Hospital's rule leads to a real mess. Instead, we compute the given limit by using the product rule for limits along with two simple applications of L'Hospital's rule. Specifically, using the product rule for limits (assuming the limits exist), we have

$$\lim_{x\to0}\frac{(1-\cos x)\sin 4x}{x^3\cos x}$$

$$=\left[\lim_{x\to0}\frac{1-\cos x}{x^2}\right]\left[\lim_{x\to0}\frac{\sin 4x}{x}\right]\left[\lim_{x\to0}\frac{1}{\cos x}\right]$$

$$=\left[\lim_{x\to0}\frac{\sin x}{2x}\right]\left[\lim_{x\to0}\frac{4\cos 4x}{1}\right]\left[\lim_{x\to0}\frac{1}{\cos x}\right]$$

$$=\left(\frac{1}{2}\right)(4)(1)=2.$$

Example 23: Evaluate $\lim\limits_{x\to0}\dfrac{\log(1-x^2)}{\log\cos x}$

Solution: Since the form is 0/0, we shall apply L´Hospital´s rule. $\lim\limits_{x\to0}\dfrac{\log(1-x^2)}{\log\cos x}$

$$=\lim_{x\to0}\frac{-2/(1-x^2)}{-\sin x/\cos x}=\lim_{x\to0}\left(\frac{2}{1-x^2}\cdot\frac{x}{\tan x}\right)$$

$$=\left(\lim_{x\to0}\frac{2}{1-x^2}\right)\left(\lim_{x\to0}\frac{x}{\tan x}\right)=2\times1=2$$

Example 24: Evaluate $\lim\limits_{x\to1}\dfrac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1}$

Solution: Since the form is 0/0, we shall apply L'Hospital's rule.

$$\text{Limit}=\lim_{x\to1}\frac{5x^4-6x^2-8x+9}{4x^3-6x^2+2},$$

[This is in 0/0 form. So, we again apply L´Hospital´s rule]

$$=\lim_{x\to1}\frac{20x^3-12x-8}{12x^2-12x},$$

[This is in 0/0 form. So, we again apply L´Hospital´s rule]

$$=\lim_{x\to1}\frac{60x^2-12}{24x-12}=\frac{60-12}{24-12}=\frac{48}{12}=4.$$

Example 25: Evaluate $\lim\limits_{x\to0}\dfrac{x^2+2\cos x-2}{x\sin^3 x}$.

Solution: $\lim\limits_{x\to0}\dfrac{x^2+2\cos x-2}{x\sin^3 x}$

$$=\lim_{x\to0}\left[\frac{x^2+2\cos x-2}{x^4}\cdot\frac{x^3}{\sin^3 x}\right]$$

$$=\lim_{x\to0}\left(\frac{x}{\sin x}\right)^3\lim_{x\to0}\frac{x^2+2\cos x-2}{x^4}$$

$$=\lim_{x\to0}\frac{x^2+2\cos x-2}{x^4},\quad\left[\because\lim_{x\to0}\frac{x}{\sin x}=1\right]$$

$$=\lim_{x\to0}\frac{2x-2\sin x}{4x^3}.$$

(by L´Hospital´s rule for the form 0/0)

$$=\lim_{x\to0}\frac{2-2\cos x}{12x^2}\qquad[\text{form } 0/0]$$

$$=\lim_{x\to0}\frac{2\sin x}{24x}=\frac{1}{12}\lim_{x\to0}\frac{\sin x}{x}=\frac{1}{12}.1=\frac{1}{12}.$$

Example 26: Evaluate $\lim\limits_{x\to1/2}\dfrac{\cos^2\pi x}{e^{2x}-2ex}$

Solution: $\lim\limits_{x\to1/2}\dfrac{\cos^2\pi x}{e^{2x}-2ex}$ $\left[\text{form }\dfrac{0}{0}\right]$

$$=\lim_{x\to1/2}\frac{2\cos\pi x.(-\pi\sin\pi x)}{2e^{2x}-2e}$$

$$= \lim_{x\to 1/2} \frac{-\pi \sin 2\pi x}{2e^{2x} - 2e}, \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 1/2} \frac{-2\pi^2 \cos 2\pi x}{4e^{2c}} = \frac{-2\pi^2(-1)}{4e} = \frac{\pi^2}{2e}.$$

Example 27: Evaluate the limit,

$$\lim_{x\to 0} \frac{(e^{2x^2} - 1 - 2x^2)(\cos x - 1)}{(\sin 3x - \ln(1+3x))x^4}.$$

Solution:

$$L = \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\lim_{x\to 0}\frac{(e^{2x^2}-1-2x^2)x^2}{x^2\cdot x^2(\sin 3x - \ln(1+3x))}$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{e^{2t}-1-2t}{t^2}\cdot\lim_{x\to 0}\frac{x^2}{\sin 3x - \ln(1+3x)}$$

where $x^2 = t$

$$= -\frac{1}{2}\lim_{y\to 0}\frac{4(e^y-1-y)}{y^2}\cdot\lim_{x\to 0}\frac{x^2}{\sin 3x - \ln(1+3x)}$$

where $2t = y$

$$= -\frac{1}{2}\cdot 2\cdot\lim_{x\to 0}\frac{x^2}{\sin 3x - \ln(1+3x)}$$

Now $l = \lim_{x\to 0}\frac{x^2}{\sin 3x - \ln(1+3x)}$.

Using L'Hospital's rule, $l = \frac{2}{9}$.

Hence, $L = -\frac{2}{9}$.

Example 28: Evaluate $\lim_{x\to\pi/2} \tan x . \log_e \sin x$.

Solution: limit $= \lim_{x\to\pi/2}\frac{\log_e \sin x}{\cot x}\left[\text{form }\frac{0}{0}\right]$

$$= \lim_{x\to\pi/2}\frac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x} \quad \text{[using L'Hospital's rule]}$$

$$= \lim_{x\to\pi/2}(-\sin x.\cos x) = 0$$

Example 29: Evaluate $\lim_{x\to 0}\frac{\ln\ \ln(1-x^2)}{\ln\ \ln\ \cos x}$

Solution: We have, $\lim_{x\to 0}\frac{\ln\ \ln(1-x^2)}{\ln\ \ln\ \cos x}$ [form ∞/∞]

$$= \lim_{x\to 0}\frac{\frac{1}{\ln(1-x^2)}\cdot\frac{1}{1-x^2}.(-2x)}{\frac{1}{\ln\cos x}\cdot\frac{1}{\cos x}.(-\sin x)}$$

$$= 2\lim_{x\to 0}\frac{x\cos x\ \ln\cos x}{\sin x\ .\ (1-x^2)\ \ln(1-x^2)}$$

$$= 2\lim_{x\to 0}\frac{x}{\sin x}.\lim_{x\to 0}\frac{\cos x}{1-x^2}.\lim_{x\to 0}\frac{\ln\cos x}{\ln(1-x^2)}$$

$$= 2\times 1\times 1\times\lim_{x\to 0}\frac{\ln\cos x}{\ln(1-x^2)}, \qquad \text{[form 0/0]}$$

$$= 2\lim_{x\to 0}\frac{\frac{1}{\cos x}.(-\sin x)}{\frac{1}{1-x^2}.(-2x)}$$

$$= 2\times\frac{1}{2}.\lim_{x\to 0}\left(\frac{\sin x}{x}.\frac{1-x^2}{\cos x}\right) = 1.$$

Example 30: Find the values of a and b in order that

$\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3}$ may be equal to 1.

Solution: We have $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$

[form 0/0, so we shall apply L'Hospital's rule]

$$= \lim_{x\to 0}\frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2} \quad ...(1)$$

Now the denominator of (1) → 0 as x → 0. Therefore if the numerator of (1) does not tend to 0 as x → 0, then the given limit cannot be equal to 1. Hence for the given limit to be equal to 1 the numerator of (1) must also → 0 as x → 0.

∴ $1 + a - b = 0$ or $a - b = -1$ ...(2)

Now if $1 + a - b = 0$, then (1) takes the form 0/0. Hence by applying L'Hospital's rule to (1), the given limit is equal to

$$\lim_{x\to 0}\frac{-a\sin x - a\sin x - ax\cos x + b\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{-ax\cos x + (b-2a)\sin x}{6x}, \text{ [form 0/0]}$$

$$= \lim_{x\to 0}\frac{-a\cos x + ax\sin x + (b-2a)\cos x}{6}$$

[by L'Hospital's rule]

$$= \frac{-a+b-2a}{6} = \frac{6-3a}{6} = 1 \qquad \text{[as given]}$$

∴ $b - 3a = 6$. ...(3)

Adding (2) and (3), we have $-2a = 5$ or $a = -5/2$.

∴ $b = a + 1 = (-5/2) + 1 = -3/2$.

Hence $a = -5/2$, $b = -3/2$.

Other Indeterminate Forms

Form $0 \times \infty$

This form can be easily reduced to the form 0/0 or to the form ∞/∞.

Let $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$

Then we can write $\lim\limits_{x\to a} f(x).g(x)$

$$= \lim_{x\to a} \frac{f(x)}{1/g(x)}, \qquad \text{[form 0/0]}$$

or $$= \lim_{x\to a} \frac{g(x)}{1/f(x)}, \qquad \text{[form } \infty/\infty]$$

Thus $\lim\limits_{x\to a} f(x).g(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L'Hospital's Rule or otherwise.

Example 31: Find $\lim\limits_{x\to\pi/2} \cot x \,.\, \log \sec x$

Solution: $\lim\limits_{x\to\pi/2} \cot x \log \sec x$ [0 . ∞ form]

$$= \lim_{x\to\pi/2} \frac{\log \sec x}{\tan x} \qquad \text{[form 0/0]}$$

$$= \lim_{x\to\pi/2} \frac{\frac{1}{\sec x}.\sec x \tan x}{\sec^2 x} = \lim_{x\to\pi/2} \sin x \cos x = 0$$

Example 32: Evaluate $\lim\limits_{x\to\infty} (\pi - 2\tan^{-1} x) \ln x$

Solution: $$\lim_{x\to\infty} \frac{(\pi - 2\tan^{-1} x)}{\frac{1}{\ell n x}} \qquad \left(\text{form } \frac{0}{0}\right)$$

Applying L' Hospital's Rule $= \lim\limits_{x\to\infty} \dfrac{-\dfrac{2}{1+x^2}}{\dfrac{1}{(\ln x)^2} \times \dfrac{1}{x}}$

$$= \lim_{x\to\infty} \frac{2x(\ln x)^2}{1+x^2} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2(\ln x)^2 + 2x.2\ln x.\frac{1}{x}}{2x}$$

$$= \lim_{x\to\infty} \frac{(\ln x)^2 + 2\ln x}{x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2\ln x.\frac{1}{x} + \frac{2}{x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} [2\ln x + 3] = 0.$$

Example 33: Evaluate $\lim\limits_{x\to 0^+} x^m (\ln x)^n$, where m, n are positive integers.

Solution: We have $\lim\limits_{x\to 0^+} x^m (\ln x)^n$, [form $0 \times \infty$]

$$= \lim_{x\to 0^+} \frac{(\ln x)^n}{x^{-m}}, \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x\to 0^+} \frac{n(\ln x)^{n-1}(1/x)}{-mx^{-m-1}} = \lim_{x\to 0^+} \left[-\frac{n}{m} . \frac{(\ln x)^{n-1}}{x^{-m}}\right],$$

[form ∞/∞, if n > 1]

$$= \lim_{x\to 0^+} \left(-\frac{n}{m}\right) . \frac{(n-1)(\ln x)^{n-2}.(1/x)}{-mx^{-m-1}}$$

$$= \lim_{x\to 0^+} (-1)^2 \frac{n(n-1)}{m^2} . \frac{(\ln x)^{n-2}}{x^{-m}}$$

$$= \lim_{x\to 0^+} (-1)^n . \frac{n(n-1)(n-2)\ldots\text{upto } n \text{ factors}}{m^n} . \frac{(\ln x)^{x-n}}{x^{-m}},$$

[by repeated application of the above process]

$$= \lim_{x\to 0^+} (-1)^n \frac{n!}{m^n}.x^m = (-1)^n \frac{n!}{m^n} . \lim_{x\to 0^+} x^m = 0 \cdot$$

Form $(\infty - \infty)$

This form can be reduced to the form 0/0 or to the form ∞/∞.

When $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} [f(x) - g(x)], \qquad [\text{form } \infty - \infty]$$

$$= \lim_{x\to a} \left[\frac{1}{1/f(x)} - \frac{1}{1/g(x)}\right]$$

$$= \lim_{x\to a} \frac{\{1/g(x)\} - \{1/f(x)\}}{\{1/f(x)\}.\{1/g(x)\}} \qquad \text{[form 0/0]}$$

Now this can be evaluated by applying L'Hospital's rule or otherwise.

Example 34: Find $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

Solution: If $x \to 0^+$, then $\sin x \to 0^+$

and $\dfrac{1}{\sin x} - \dfrac{1}{x} \to \infty - \infty$

Similarly, if $x \to 0^-$, $\sin x \to 0^-$

and $\dfrac{1}{\sin x} - \dfrac{1}{x} = \to -\infty - (-\infty) = -\infty + \infty$

Neither form reveals what happens in the limit. To find out, we first combine the fractions.

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

and then apply L'Hospital's rule to the result.

$$\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0\,.$$

Example 35: Evaluate $\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)$

Solution: $\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)\qquad(\infty-\infty)$

$$=\lim_{x\to0}\left(\frac{\tan^2 x-x^2}{x^2\,\frac{\tan^2 x}{x^2}.x^2}\right)\qquad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{(\tan x+x)(\tan x-x)}{x^4}$$

$$=\lim_{x\to0}\frac{(\tan x+x)(\tan x-x)}{x\;.\;x^3}$$

$$=2\lim_{x\to0}\frac{\tan x-x}{x^3}$$

$$=2\lim_{x\to0}\frac{\sec^2 x-1}{3x^2}\qquad\text{[by L'Hospital's rule]}$$

$$=2\lim_{x\to0}\frac{1-\cos^2 x}{\cos^2 x\;\;3x^2}$$

$$=\frac{2}{3}\lim_{x\to0}\frac{\sin^2 x}{1.x^2}=\frac{2}{3}\,.$$

Note:

The indeterminate forms 1^∞, ∞^0 are evaluated by first taking logarithms and then finding the limit of the logarithm of the power $[f_1(x)]^{f_2(x)}$ (which requires evaluating a form like $0.\infty$).

Example 36: Evaluate $\lim_{x\to0}(\text{cosec } x)^{1/\ln x}$

Solution: Let $L=\lim_{x\to0}(\text{cosec } x)^{1/\ln x}$ [form 1^∞]

$$\therefore\quad \ell n\,L=\lim_{x\to0}\frac{1}{\ell n\,x}(\ell n\,\text{cosec}\,x)\,,\qquad[\text{form }\infty/\infty]$$

$$=\lim_{x\to0}\frac{(1/\text{cosec } x)\;(\text{-cosec } x\;\cot x)}{1/x}$$

$$=\lim_{x\to0}\left(\frac{-x}{\tan x}\right),\qquad[\text{form }0/0]$$

$$=\lim_{x\to0}\left(\frac{-1}{\sec^2 x}\right)=-1$$

$\therefore\quad L = e^{-1} = 1/e.$

Example 37: Calculate $\lim_{x\to0^+}\left(x^{(x^x)}-x^x\right)$.

Solution: We first calculate $\lim_{x\to0^+}x^x$: it is just $\exp(\lim_{x\to0^+}x\ln x)$.

Now $\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x}$ by L'Hospital's Rule

$$=\lim_{x\to0^+}\frac{1/x}{-1/x^2}=\lim_{x\to0^+}-x=0$$

Hence $\lim_{x\to0^+}x^x = 1$.

Therefore $\lim_{x\to0^+}\left(x^{(x^x)}-x^x\right)=-1$.

Example 38: Find $\lim_{\theta\to0}\frac{e^\theta+e^{-\theta}+2\cos\theta-4}{\theta^4}$

Solution: $\lim_{\theta\to0}\frac{e^\theta+e^{-\theta}+2\cos\theta-4}{\theta^4}\quad\left[\text{form }\frac{0}{0}\right]$

$$=\lim_{\theta=0}\frac{e^\theta-e^{-\theta}-2\sin\theta}{4\theta^3}\qquad\text{(by L' Hospital's Rule)}$$

$$=\lim_{\theta=0}\frac{e^\theta+e^{-\theta}-2\cos\theta}{12\theta^2}$$

$$=\lim_{\theta=0}\frac{e^\theta-e^{-\theta}+2\sin\theta}{24\theta}$$

$$=\lim_{\theta=0}\frac{e^\theta+e^{-\theta}+2\cos\theta}{24}=\frac{4}{24}=\frac{1}{6}\,.$$

Example 39: Determine $\lim_{x\to0}(\cot x)^{1/\log x}$

Solution: Let $y = (\cot x)^{1/\log x}$.

$$\log y=\frac{1}{\log x}\log(\cot x)$$

$$\Rightarrow\quad\lim_{x\to0^+}\log y=\lim_{x\to0}\frac{\log\cot x}{\log x}\qquad\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to0^+}\frac{\frac{-\text{cosec}^2 x}{\cos x}}{1/x}\qquad\text{(by L' Hospital's Rule)}$$

$$=\lim_{x\to0^+}\frac{x}{\sin x}.\frac{1}{\cos x}=-1$$

$$\Rightarrow\quad\log\lim_{x\to0^+}y=-1\quad\Rightarrow\quad\lim_{x\to0^+}y=e^{-1}=1/e.$$

Example 40: Find $\lim\limits_{x\to0}\dfrac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$

Solution: The inconvenience of continuously differentiating the denominator, which involves $\tan^2 x$ as a factor, may be partially avoided as follows. We write

$$\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$$

$$=\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\cdot\left(\frac{x}{\tan x}\right)^2$$

so that $\lim\limits_{x\to0}\dfrac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$

$$=\lim_{x\to0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\lim_{x\to0}\left(\frac{x}{\tan x}\right)^2$$

$$=\lim_{x\to0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\ .\ 1$$

$$=\lim_{x\to0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$$

To evaluate the limit on the R.H.S., we notice that the numerator and denominator both become 0 for x = 0.

$$\therefore\quad\lim_{x\to0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$$

$$=\lim_{x\to0}\frac{\cos x-\sin x-[1/(1-x)]}{3x^2}$$

(by L' Hospital's Rule)

$$=\lim_{x\to0}\frac{-\sin x+\cos x-[1/(1-x)^2]}{6x}$$

$$=\lim_{x\to0}\frac{-\cos x-\sin x[2/(1-x)^3]}{6}=-\frac{3}{6}=-\frac{1}{2}.$$

Cycling

L'Hospital's rule does not help in evaluating the following limits. When we apply it we just keep on cycling. We need to find the limits using some other method.

Consider $L=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sec x}{\tan x}\quad\left(\dfrac{\infty}{\infty}\right)$ ...(1)

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sec x\tan x}{\sec^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x}\quad\left(\frac{\infty}{\infty}\right)\qquad ...(2)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{\sec x\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\tan x}=L\qquad ...(3)$$

Applying L'Hospital's rule again and again, we get the question back in (3).

However, in (2) we get $L=\dfrac{1}{L}$ from where we can get the limit. We have $L^2 = 1 \Rightarrow L = \pm 1$.
Since the function is positively valued, the limit is non-negative $\Rightarrow L = 1$.
Here are some more examples:

(i) $\lim\limits_{x\to\infty}\dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$ (ii) $\lim\limits_{x\to0^+}\dfrac{\sqrt{x}}{\sqrt{\sin x}}$

(iii) $\lim\limits_{x\to0^+}\dfrac{\cot x}{\operatorname{cosec} x}$

Leibniz's Formula for Differentiation of Integrals

To differentiate an integral function, one way is to find the integral and then differentiate as follows:

Let $f(x)=\displaystyle\int_{2x}^{x^2}t^2dt=\left.\frac{t^3}{3}\right|_{2x}^{x^2}=\frac{x^6}{3}-\frac{(2x)^3}{3}=\frac{x^6-8x^3}{3}$

Hence, $f'(x)=\dfrac{6x^5-24x^2}{3}=2x^5-8x^2$.

But a better approach is given by Leibniz rule:

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)}f(t)dt\right)=f(v(x))\frac{dv}{dx}-f(u(x))\frac{dv}{dx}.$$

The proof will be dealt in the chapter of definite integral. We can now differentiate the same function as follows:

$$\frac{d}{du}\left(\int_{2x}^{x^2}t^2dt\right)=(x^2)^2\ .\ 2x-(2x)^2\ .\ 2=2x^5-8x^2.$$

Example 41: Evaluate $\lim\limits_{x\to0}\dfrac{\int_{2x}^{x^2}t^2dt}{x^3}$

Solution: $\lim\limits_{x\to0}\dfrac{\int_{2x}^{x^2}t^2dt}{x^3}\quad\left(\dfrac{0}{0}\right)$

We apply L'Hospital's rule with the help of Leibniz rule

$$\text{Limit}=\lim_{x\to0}\frac{2x^5-8x^2}{3x^2}=\frac{2}{3}x^3-\frac{8}{3}=\frac{-8}{3}.$$

Example 42: Evaluate $\lim\limits_{x\to\infty}\dfrac{\int_{x^2}^{x^3}\ell nt\,dt}{x^4}$

Solution: $\lim\limits_{x\to\infty}\dfrac{\int_{x^2}^{x^3}\ell nt\,dt}{x^4}\quad\left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty}\frac{(\ell nx^3)3x^2-(\ell nx^2).2x}{4x^3}$$ [by L'Hospital's rule]

$$= \lim_{x\to\infty}\frac{9x\ell nx-4\ell nx}{4x^2}$$

$$= \lim_{x\to\infty}\frac{(9x-4)\ell nx}{4x^2}$$

$$= \lim_{x\to\infty}\left(9-\frac{4}{x}\right)\frac{\ell nx}{4x}$$

$= 9 \times 0 = 0.$

Example 43: Find the constants 'a' $(a > 0)$ and 'b' such that,

$$\lim_{x\to 0}\frac{\int\limits_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx-\sin x}=1.$$

Solution: $\frac{0}{0}$ form hence using L`Hospital rule

$$l = \lim_{x\to 0}\frac{\frac{x^2}{\sqrt{a+x}}}{b-\cos x}$$ for existence of limit

$\lim\limits_{x\to 0} b - \cos x = 0 \quad \Rightarrow b = 1$

hence $\lim\limits_{x\to 0}\frac{x^2}{1-\cos x}\,.\,\lim\limits_{x\to 0}\frac{1}{\sqrt{a+x}} = 1$

$\frac{2}{\sqrt{a}} = 1 \Rightarrow a = 4$

Example 44: Evaluate $\lim\limits_{x\to 0}\frac{x-\int\limits_0^x \cos t^2 dt}{x^3-6x}$.

Solution: $\lim\limits_{x\to 0}\frac{x-\int\limits_0^x \cos t^2 dt}{x^3-6x}$ [form $\frac{0}{0}$]

Applying L'Hospital's rule we get

$$= \lim_{x\to 0}\frac{1-\frac{d}{dx}\int\limits_0^x \cos t^2 dt}{3x^2-6} = \lim_{x\to 0}\frac{1-\cos(x^2)}{3(x^2-2)}$$

$$= \frac{1-\cos 0}{3(0-2)} = \frac{0}{-6} = 0.$$

Example 45: Evaluate $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}+\sqrt{x}+x\sqrt{x}-3}{x^3-1}$

Solution: $\lim\limits_{x\to 1}\frac{(\sqrt[3]{x}-1)+(\sqrt{x}-1)+(x^{3/2}-1)}{(x-1)(x^2+x+1)}$ $\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 1}\left\{\frac{\sqrt[3]{x}-1}{x-1}+\frac{(\sqrt{x}-1)}{x-1}+\frac{(x^{3/2}-1)}{x-1}\right\}\cdot\frac{1}{x^2+x+1}$$

$$= \left\{\left(\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}\right)+\left(\lim_{x\to 1}\frac{(\sqrt{x}-1)}{x-1}\right)+\left(\lim_{x\to 1}\frac{(x^{3/2}-1)}{x-1}\right)\right\}\cdot\lim_{x\to 1}\frac{1}{x^2+x+1}$$

(by L' Hospital's rule)

$$= \lim_{x\to 1}\left\{\frac{1}{3}(x)^{1/3-1}+\frac{1}{2}(x)^{1/2-1}+\frac{3}{2}(x)^{3/2-1}\right\}.\frac{1}{1^2+1+1}$$

$$= \left\{\frac{1}{3}(1)^{-2/3}+\frac{1}{2}(1)^{-1/2}+\frac{3}{2}(1)^{1/2}\right\}.\frac{1}{3}$$

$$= \left(\frac{1}{3}+\frac{1}{2}+\frac{3}{2}\right).\frac{1}{3} = \left(\frac{2+3+9}{6}\right).\frac{1}{3} = \frac{7}{9}.$$

Example 46: If $\lim\limits_{n\to\infty}\int_n^{2n}\frac{n^3x}{x^5+1}\,dx = k$ then find the value of $\left[\frac{1}{k}\right]$, where [.] is the greatest integer function.

Solution: $\lim\limits_{n\to\infty}\int_n^{2n}\frac{n^3x}{x^5+1}\,dx$ $(\infty \times 0)$

Put $n = \frac{1}{t}$

$$\lim_{t\to 0^+}\frac{\int_{\frac{1}{t}}^{\frac{2}{t}}\frac{x}{x^5+1}dx}{t^3}\quad\left(\frac{0}{0}\right)$$

Applying L' Hospital's rule

$$\lim_{t\to 0+}\frac{\frac{\frac{2}{t}}{\frac{32}{t^5}+1}\left(-\frac{2}{t^2}\right)-\frac{\frac{1}{t}}{\frac{1}{t^5}+1}\left(\frac{1}{t}\right)}{3t^2}$$

$$= \lim_{t\to 0+}\frac{\frac{-4}{32+t^5}+\frac{1}{1+t^5}}{3} = \frac{7}{24} = k.$$

$$\therefore \left[\frac{1}{k}\right] = \left[\frac{24}{7}\right] = 3.$$

Example 47: Evaluate $\lim\limits_{x\to\infty}\frac{\left(\int_0^x e^{x^2}dx\right)^2}{\left(\int_0^x e^{2x^2}dx\right)}$

Solution: If $y = \left(\int_0^x g(x)dx\right)^2$, then

$$\frac{dy}{dx} = 2\left(\int_0^x g(x)dx\right) . g(x)$$

Now $\lim\limits_{x\to\infty} \dfrac{(\int_0^x e^{x^2} dx)^2}{\int_0^x e^{2x^2} dx}$ $\quad \left[\dfrac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to\infty} \frac{2\left(\int_0^x e^{x^2} dx\right)(e^{x^2} . 1)}{e^{2x^2}} = \frac{2\int_0^x e^{x^2} dx}{e^{x^2}} \quad \left[\frac{\infty}{\infty} \text{form}\right]$$

$$= \lim_{x\to\infty} \frac{2e^{x^2} . 1}{e^{x^2} .2x} = \lim_{x\to\infty} \frac{1}{x} = 0 .$$

Example 48: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x^5}\int_0^x e^{-t^2} dt - \dfrac{1}{x^4} + \dfrac{1}{3x^2}\right)$

Solution: We have $\lim\limits_{x\to 0} \dfrac{3\int_0^x e^{-t^2} dt - 3x + x^3}{x^5}$ $\left(\text{form} \dfrac{0}{0}\right)$

Using L' Hospital's rule, we get

$$\lim_{x\to 0} \frac{3e^{-x^2} - 3 + 3x^2}{5x^4} = \frac{3}{5} \lim_{x\to 0} \frac{e^{-x^2} - 1 + x^2}{x^4} \left(\text{form} \frac{0}{0}\right)$$

Again using L' Hospital's rule, we get

$$= \frac{3}{5} \lim_{x\to 0} \frac{-e^{-x^2} .(-2x) - 0 + 2x}{4x^3}$$

$$= \frac{3}{5} . \frac{2}{4} \lim_{x\to 0} \frac{-e^{-x^2} + 1}{x^2}$$

Again using L' Hospital's rule, we get

$$= \frac{3}{10} \lim_{x\to 0} \frac{-e^{-x^2} .(-2x)}{(2x)} = \frac{3}{10} . 1 = \frac{3}{10} .$$

Example 49: Find $\lim\limits_{x\to\infty}\left(\dfrac{\int_x^{2x} (1+t^3)^{-1} dt}{\int_{3x}^{4x} (1+t^3)^{-1} dt}\right)$.

Solution: Both the numerator and denominator tend to zero as x → ∞, and hence L' Hospital's rule applies.

$$\frac{\int_x^{2x} \frac{1}{1+t^3} dt}{\int_{3x}^{4x} \frac{1}{1+t^3} dt}$$

$$= \frac{\dfrac{2}{1+8x^3} - \dfrac{1}{1+x^3}}{\dfrac{4}{1+64x^3} - \dfrac{3}{1+27x^3}}$$

$$= \frac{\dfrac{2}{1/x^3 + 8} - \dfrac{1}{1/x^3 + 1}}{\dfrac{4}{1/x^3 + 64} - \dfrac{3}{1/x^3 + 27}}$$

$$= \frac{\dfrac{2}{8} - 1}{\dfrac{4}{64} - \dfrac{3}{27}} = \frac{108}{7} .$$

Concept Problems

P

1. Evaluate the following limits:

(i) $\lim\limits_{x\to 0} \dfrac{a^x (x \ln a - 1) + 1}{x^2}$

(ii) $\lim\limits_{x\to 0} \dfrac{\ln(\tan^2 2x)}{\ln(\tan^2 x)}$

(iii) $\lim\limits_{x\to 0} \dfrac{\ln(\sin \alpha x)}{\ln(\sin \beta x)}$

2. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty} \dfrac{\ln (1+e^{-x})}{e^{-x}}$ (ii) $\lim\limits_{x\to\infty} \dfrac{\ln (1+e^{x})}{x}$

(iii) $\lim\limits_{x\to -\infty} \dfrac{\ln (1+e^{-2x})}{\ln (1+e^{-x})}$ (iv) $\lim\limits_{x\to\infty} \dfrac{a^{1/x} - b^{1/x}}{\ln\{x/(x-1)\}}$

3. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty} \dfrac{2x + x \sin 3x}{5x^2 - 2x + 1}$

(ii) $\lim\limits_{x\to\infty} \dfrac{2x - \sin x}{3x + \sin x}$

(iii) $\lim\limits_{x\to 1} \dfrac{x^x - x}{1 - x + \ln x}$

4. Show that $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^m}=0,\ m>0.$

5. Show that $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{2x+\sin x}$ exists even if

$\lim\limits_{x\to\infty}\dfrac{1+\cos x}{2+\cos x}$ does not exist.

6. Evaluate the following limits:

(i) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\sin\sqrt{x}\,dx}{x^3}$

(ii) $\lim\limits_{x\to\infty}\dfrac{\int_0^{x}(\tan^{-1}x)^2\,dx}{\sqrt{x^2+1}}$

(iii) $\lim\limits_{x\to 1}\dfrac{\int_1^{x}\sqrt{1+4t^2}\,dt}{x-1}$.

7. Evaluate the following limits:

(i) $\lim\limits_{x\to\infty}\dfrac{\int_0^{x}\sqrt{4+t^4}\,dt}{x^3}$

(ii) $\lim\limits_{x\to 1}\dfrac{\int_x^{x^2}e^{-t^2}\,dt}{x-1}$

(iii) $\lim\limits_{x\to-\infty}\dfrac{\int_0^{x}\sqrt{1+t^2}\,dt}{x^2}$

Practice Problems N

8. Evaluate the following limits:

(i) $\lim\limits_{t\to\infty}\dfrac{3t^2}{3(1+t)^{1/3}-t-3}$

(ii) $\lim\limits_{x\to\infty}\dfrac{x^{1/3}+2x+1}{x^{1/2}+3x-2}$.

9. Evaluate the following limits:

(i) $\lim\limits_{x\to 0}\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$

(ii) $\lim\limits_{x\to 0}\left[\dfrac{1}{x(1+x)}-\dfrac{\ln(1+x)}{x^2}\right]$

10. Prove that $\lim\limits_{n\to\infty}\dfrac{\log_a n}{n^\alpha}=0$ for $a>1,\ \alpha>1$

11. Show that $\lim\limits_{x\to\infty}\dfrac{(\ln x)^n}{n}=0$ and $\lim\limits_{x\to\infty}\dfrac{x^n}{e^x}=0$ for all positive integers n.

12. Prove that $\lim\limits_{x\to n}(x-n)\operatorname{cosec} x\pi=\dfrac{(-1)^n}{\pi}$, n being any integer.

13. Prove that $\lim\limits_{x\to n}\dfrac{1}{x-n}\left\{\operatorname{cosec} x\pi-\dfrac{(-1)^n}{(x-n)\pi}\right\}$

$=\dfrac{(-1)^n\pi}{6}$, n being any integer.

14. Evaluate the following limits:

(i) $\lim\limits_{x\to 0}\dfrac{1}{x^6}\left(\sin x.\sin^{-1}x-x^2\right)$

(ii) $\lim\limits_{x\to 0}\dfrac{1}{x^6}\left(\tan x.\tan^{-1}x-x^2\right)$

15. Suppose that F is a function differentiable on the open interval $(0,\infty)$ and such that $F'(x)=\dfrac{1}{x}$, for every $x>0$, Show that

(a) $\lim\limits_{x\to\infty}\dfrac{F(x)}{x}=0$ (b) $\lim\limits_{x\to\infty}\dfrac{F(x)}{x^2}=0$

(c) $\lim\limits_{x\to\infty}\dfrac{F(x)}{x^n}=0$, for every positive integer n.

16. Evaluate $\lim\limits_{x\to 0}\dfrac{\int_y^a e^{\sin^2 t}dt-\int_{y+x}^a e^{\sin^2 t}dt}{x}$.

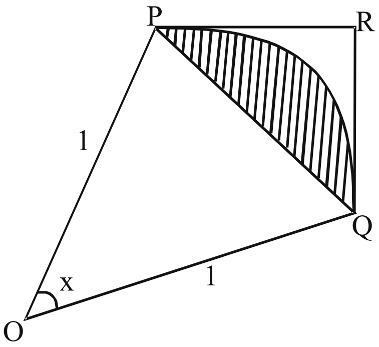
17. Evaluate

$$\lim_{h\to 0}\frac{\int_0^{\frac{\pi}{3}+h^4e^{1/h^2}}\cos^3 x\,dx-\int_0^{\pi/3}\cos^3 xdx}{h^4e^{1/h^2}}.$$

1.19 Geometrical Limits

Example 1: A circular arc of radius 1 subtends an angle of x radians, $0 < x < \pi/2$ as shown in the figure. The point R is the intersection of the two tangent line at P and Q. Let T(x) be the area of triangle PQR and let S(x) be the area of the shaded region then find

(i) T(x) (ii) S (x) (iii) $\lim\limits_{x\to0}\dfrac{T(x)}{S(x)}$



Solution: (i) In Δ OPR, $\tan\left(\dfrac{x}{2}\right) = \dfrac{PR}{1}$

$\therefore$ $PR = \tan\dfrac{x}{2} = RQ$

($\because$ Length of tangent from a point outside the circle are equal)

and $\angle PRQ = (\pi - x)$

$\therefore$ T(x) = Area of Δ PQR

$$= \frac{1}{2}.(PR)(RQ)\sin(\pi - x)$$

$$= \frac{1}{2}.\tan^2\left(\frac{x}{2}\right)\sin x \qquad ...(1)$$

$$= \tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}$$

(ii) S(x) = area of sector OPQ – area of Δ OPQ

$$= \frac{1}{2}(1)^2.x - \frac{1}{2}.(1)^2.\sin x$$

$$= \frac{(x-\sin x)}{2}$$

(iii) $$\lim_{x\to0}\frac{T(x)}{S(x)} = \lim_{x\to0}\frac{\tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}}{\frac{(x-\sin x)}{2}}$$

$$= \lim_{x\to0}\frac{\tan^2(x/2)\sin x}{(x-\sin x)}$$

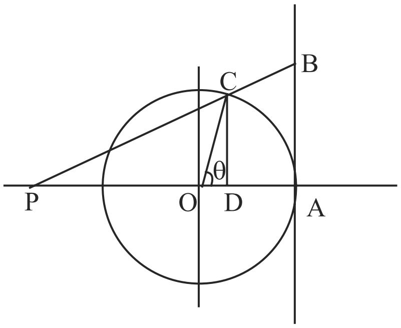
$$= \lim_{x\to0}\frac{\left(\frac{\tan(x/2)}{x/2}\right)^2\left(\frac{\sin x}{x}\right)}{(x-\sin x)}.\frac{x^3}{4}$$

$$= \frac{1}{4}\lim_{x\to0}\left(\frac{\tan(x/2)}{x/2}\right)^2\lim_{x\to0}\left(\frac{\sin x}{x}\right).$$

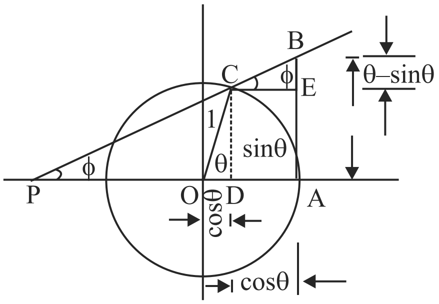
$$.\lim_{x\to0}\frac{x^3}{(x-\sin x)} = \frac{3}{2}.$$

Example 2: A tangent line is drawn to a circle of radius unity at the point A, and a segment AB is laid off whose length is equal to that of the arc AC . A straight line BC is drawn to intersect the extension of the diameter AO at the point P . Prove that:

(i) $PA = \dfrac{\theta(1-\cos\theta)}{\theta - \sin\theta}$ (ii) $\lim\limits_{\theta\to0} PA = 3$



Solution: See the labelled figure below:



(i) From the figure,

$$\tan\phi = \frac{\theta - \sin\theta}{1-\cos\theta} \qquad ...(1)$$

Now in Δ BPA , $\tan\theta = \dfrac{\theta}{PA}$

$$\Rightarrow \quad PA = \frac{\theta}{\tan\phi} = \frac{\theta(1-\cos\theta)}{\theta - \sin\theta}$$

(ii) $$\lim_{\theta\to 0}\frac{\theta\,(1-\cos\theta)}{\dfrac{\theta-\sin\theta}{\theta^3}.\,\theta^3}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\ \lim_{\theta\to 0}\frac{\theta^3}{\theta-\sin\theta}$$

Now $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$

Let $\displaystyle\ell = \lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3} = \lim_{t\to 0}\frac{3t-\sin 3t}{27\,t^3}$

where $\theta = 3t$

$$= \lim_{t\to 0}\frac{3t-\left(3\sin t-4\sin^3 t\right)}{27\,t^3}$$

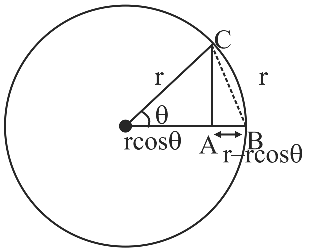
$$= \lim_{t\to 0}\frac{t-\sin t}{9\,t^3}+\frac{4}{27}$$

$$\ell = \frac{\ell}{9}+\frac{4}{27} \Rightarrow \frac{8\,\ell}{9} = \frac{4}{27} \qquad \Rightarrow \ell = \frac{1}{6}$$

Since $\displaystyle\lim_{\theta\to 0}\frac{\theta^3}{\theta-\sin\theta} = 6,$

$$\lim_{\theta\to 0} PA = \frac{1}{2}\,.\,6 = 3.$$

Example 3: In the figure shown, let f(θ) = area of Δ ABC and g(θ) = area of sector OBC – area of ΔOAC. Find the limit of ratio of f(θ) and g(θ) as C moves to B along the circumference.



Solution: We need to find

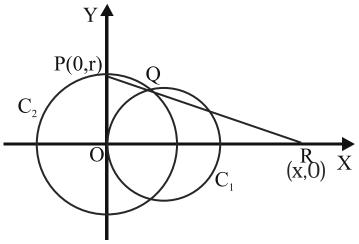
$$\lim_{\theta\to 0^+}\frac{f(\theta)}{g(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}(r-r\cos\theta)r\sin\theta}{\frac{1}{2}r^2\theta-\frac{1}{2}r\sin\theta.r\cos\theta}$$

$$= \lim_{\theta\to 0^+}\frac{\left(\dfrac{1-\cos\theta}{\theta^2}\right)\dfrac{\sin\theta}{\theta}}{\dfrac{\theta-\sin\theta\cos\theta}{\theta^3}} \qquad \left(\frac{0}{0}\right)$$

$$= \frac{1}{2}\lim_{\theta\to 0^+}\frac{1}{\dfrac{\theta-\frac{1}{2}\left(2\theta-\frac{(2\theta)^3}{\lfloor 3}\ldots.\right)}{\theta^3}}$$

$$= \frac{1}{2}\times\frac{12}{8} = \frac{3}{4}$$

Example 4: The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2 = 1$ and a shrinking circle C_2 with radius r and centre the origin. P is the point (0, r), Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r\to 0^+$?



Solution: We find the coordinates of the point Q by solving $(x-1)^2+y^2 = 1$, and $x^2+y^2 = r^2$. On substraction, we get $x = \dfrac{r^2}{2}$.

Points P, Q and R are collinear. Hence $m_{PR} = m_{PQ}$

$$\Rightarrow \quad \frac{r-0}{0-x} = \frac{r\sqrt{1-\frac{r^2}{4}}-r}{\frac{r^2}{2}-0}$$

$$\Rightarrow \quad x = \frac{\frac{r^2}{2}}{1-\sqrt{1-\frac{r^2}{4}}}$$

Now, $$\lim_{r\to 0^+} x = \lim_{r\to 0^+}\frac{\frac{r^2}{2}}{1-\left(1-\frac{r^2}{4}\right)^{1/2}}$$

$$= \lim_{r\to 0^+}\frac{\frac{r^2}{2}}{1-1+\frac{1}{2}.\frac{r^2}{4}+\ldots.} = 4.$$

Example 5: Let (tan α) x + (sin α) y = α and (α cosec α) x + (cos α) y = 1 be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. Find the limiting position of P when α → 0.

Solution: Here two straight lines, $(\tan\alpha)x + (\sin\alpha)y = \alpha$ and $(\alpha \operatorname{cosec}\alpha)x + (\cos\alpha)y = 1$ have their point of intersection as,

$$x = \frac{\alpha\cos\alpha - \sin\alpha}{\sin\alpha - \alpha} \text{ and } y = \frac{\alpha - x\tan\alpha}{\sin\alpha}$$

∴ when α → 0, we obtain the point P.

$$\text{Hence, } \lim_{\alpha\to 0} x = \lim_{\alpha\to 0} \frac{\alpha\cos\alpha - \sin\alpha}{\sin\alpha - \alpha} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{\alpha\to 0} \frac{-\alpha\sin\alpha + \cos\alpha - \cos\alpha}{\cos\alpha - 1}$$

(applying L' Hospital's rule)

$$= \lim_{\alpha\to 0} \frac{-\alpha\sin\alpha}{-2\sin^2\alpha/2} = \lim_{\alpha\to 0} \frac{\alpha\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}{2\sin^2\frac{\alpha}{2}}$$

$$= \lim_{\alpha\to 0} \frac{\alpha}{\tan\frac{\alpha}{2}} = \lim_{\alpha\to 0} \frac{\frac{2\alpha}{2}}{\tan\frac{\alpha}{2}} = 2$$

$$\text{Again as, } \lim_{\alpha\to 0} y = \lim_{\alpha\to 0} \frac{\alpha - x\tan\alpha}{\sin\alpha}$$

$$= \lim_{\alpha\to 0}\left(\frac{\alpha}{\sin\alpha} - \frac{x}{\cos\alpha}\right)$$

$$= \lim_{\alpha\to 0} \frac{\alpha}{\sin\alpha} - \lim_{\alpha\to 0} \frac{x}{\cos\alpha} = 1 - 2$$

$$\left[\text{as } \lim_{\alpha\to 0} x = 2\right]$$

$$\therefore \quad \lim_{\alpha\to 0} y = -1.$$

Hence, in limiting position, P is (2, –1).

Concept Problems

1. Show that the perimeter of a regular polygon of n sides is given by $P(\theta) = \frac{4\pi R}{\theta}\sin\frac{\theta}{2}$ where $\theta = \frac{2\pi}{n}$ is the central angle subtended by one side of the polygon. Hence show that a circle of radius R has circumference 2πR.

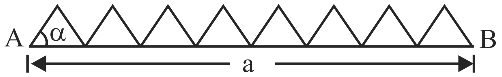
2. A straight line AB moves so as to include with two fixed straight lines OX, OY a triangle AOB of constant area. Prove that the limiting position of the intersection of two consecutive positions of AB is the middle point of AB.

3. Find the limit of the sum of the lengths of the ordinates of the curve $y = e^{-x}\cos\pi x$, drawn at the points x = 0, 1, 2, ..., n, as n → ∞.

4. Find the limit of the sum of the areas of the squares constructed on the ordinates of the curve $y = 2^{1-x}$ as on bases, where x = 1, 2, 3, ..., n, provided that n → ∞.

5. A straight line AB moves so that the sum of its intercepts OA, OB on two fixed straight lines OX, OY is constant. If P be the ultimate intersection of two consecutive positions of AB, and Q the point where AB is met by the bisector of the angle XOY, then prove that AP = QB.

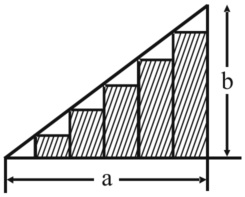
6. A segment AB = a (see fig.) is divided into n equal parts, each part serving as the base of an isosceles triangle with base angles α = 45°. Show that the limit of the perimeter of the broken line thus formed differs from the length of AB despite the fact that in the limit the broken line "geometrically merges with the segment AB".



Practice Problems

7. The point C_1 divides a segment AB = ℓ in half ; the point C_2 divides a segment AC_1 in half ; the point C_3 divides a segment C_2C_1 in half ; the point C_4 divides C_2C_3 in half, and so on. Determine the limiting position of the point C_n when n → ∞.

8. The side a of a right triangle is divided into n equal parts, on each of which is constructed an inscribed rectangle (see fig.). Determine the limit of the area of the step-like figure thus formed if n→∞.



9. A weight hanging by a spring is made to vibrate by applying a sinusoidal force, and the displacement at time t is given by

$$f(t) = \frac{C}{\beta^2-\alpha^2}(\sin\alpha t - \sin\beta t)$$

where C, α, β are constants such that $\alpha \neq \beta$. What happens to the displacement as $\beta \to \alpha$? You may assume that α is fixed.

10. A certain chemical process proceeds in such a fashion that the increase in quantity of a substance during each interval of time τ out of the infinite sequence of intervals $(i\tau, (i+1)\tau)$ $(i = 0, 1, 2, ...)$ is proportional to the quantity of the substance available at the commencement of each interval and to the length of the interval. Assuming that the quantity of substance at the initial time is Q_0, determine the quantity of substance $Q_t^{(n)}$ after the elapse of time t if the increase takes place each nth part of the time interval $\tau = \dfrac{t}{n}$.

Find $Q_t = \lim\limits_{n\to\infty} (Q)_t^{(n)}$.

11. A straight line AB of constant length moves with its extremities on two fixed straight lines OX, OY which are at right angles to one another. Prove that if P be the ultimate intersection of two consecutive positions of AB, and N the foot of the perpendicular from O on AB, then AP = NB.

12. If PCP' be any fixed diameter of an ellipse, and QV any perpendicular to this diameter ; and if the tangent at Q meet CP produced in T, prove that the limiting value of the ratio TP: PV, when PV is infinitely small, is unity.

1.20 Miscellaneous limits

Repeated Limits

The operations of proceeding to the limit zero with each of two variables x and y may or may not be commutative when applied to a function f(x,y). Thus

$$\lim_{x\to 0}\{\lim_{y\to 0}(x+y)\} = \lim_{x\to 0} x = 0,$$

$$\lim_{y\to 0}\{\lim_{x\to 0}(x+y)\} = \lim_{y\to 0} y = 0$$

but on the other hand

$$\lim_{x\to 0}\{\lim_{y\to 0}(\frac{x-y}{x+y})\} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\{\lim_{x\to 0}(\frac{x-y}{x+y})\} = \lim_{y\to 0}\frac{-y}{y} = \lim_{y\to 0}(-1) = -1$$

Hence the limits may or may not be commutative.

Example 1: Evaluate $\lim\limits_{x\to 0^+}\left(\lim\limits_{n\to\infty}\dfrac{nx}{1+nx}\right)$ and $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to 0^+}\dfrac{nx}{1+nx}\right)$

Solution: $\lim\limits_{x\to 0^+}\left(\lim\limits_{n\to\infty}\dfrac{nx}{1+nx}\right) = \lim\limits_{x\to 0^+}\left(\lim\limits_{n\to\infty}\dfrac{x}{\dfrac{1}{n}+x}\right)$

$$= \lim_{x\to 0^+}\left(\frac{x}{x}\right) = 1.$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0^+}\frac{nx}{1+nx}\right) = \lim_{n\to\infty}\left(\frac{0}{1+0}\right) = 0$$

Caution

Here we note that interchange of limits is giving different results. Hence it is advisable not to interchange the order of limits indiscriminately.

Representation of Functions Using Limits

Example 2: Simplify the function $y = \lim\limits_{n\to\infty} x^n$.

Solution: When $x^2 < 1$ i.e $-1 < x < 1$,

$$\lim_{n\to\infty} x^{2n} = \lim_{n\to\infty}(x^2)^n = 0$$

When $x^2 > 1$ i.e. $x < -1$ or $x > 1$,

$$\lim_{n\to\infty} x^{2n} = \lim_{n\to\infty}(x^2)^n = \infty.$$

When $x^2 = 1$ i.e. $x = \pm 1$,

$$\lim_{n\to\infty} x^{2n} = \lim_{n\to\infty}(x^2)^n = 1.$$

$$\therefore \quad \lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if} \quad x^2 < 1 \\ 1, & \text{if} \quad x^2 = 1 \\ \infty, & \text{if} \quad x^2 > 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } \quad -1<x<1 \\ 1, & \text{if } \quad x=\pm 1 \\ \infty, & \text{if } \quad x<-1 \text{ or}, x>1 \end{cases}$$

Here y is defined only when –1 < x ≤ 1. It is equal to 0 when –1 < x < 1 and to 1 when x = 1. Note that the points x = 1 and x = –1 are the points of change in definition.

Example 3: Evaluate $\lim\limits_{x\to 1}\lim\limits_{n\to\infty}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$

Solution: Let f(x) = $\lim\limits_{n\to\infty}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$

We first simplify this function. We have

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } \quad -1<x<1 \\ 1, & \text{if } \quad x=\pm 1 \\ \infty, & \text{if } \quad x<-1 \text{ or}, x>1 \end{cases}$$

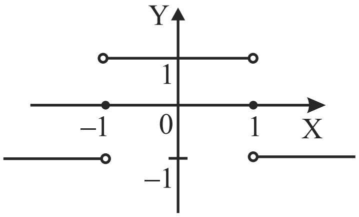
Hence, f(x)

$$= \begin{cases} \dfrac{1-0}{1+0} & ; \quad -1<x<1 \\ \dfrac{0-1}{0+1} & ; \quad x<-1, x>1 \\ \dfrac{1-1}{1+1} & ; \quad x=\pm 1 \end{cases}$$

using $\lim\limits_{n\to\infty}\left(\dfrac{\dfrac{1}{x^{2n}}-1}{\dfrac{1}{x^{2n}}+1}\right)$ = 0 when x < – 1, x > 1.

$$= \begin{cases} 1 & , \quad -1<x<1 \\ -1 & , \quad -1<x<1 \\ 0 & , \quad x=\pm 1 \end{cases}$$

The graph of f(x) can be drawn as follows:



Now we find the limit of f(x) as x → 1.

$$\lim_{x\to 1^-} f(x) = 1$$

$$\lim_{x\to 1^+} f(x) = -1.$$

Thus, the limit $\lim\limits_{x\to 1}\lim\limits_{n\to\infty}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$ does not exist.

Example 4: If $F(x) = \lim\limits_{n\to\infty}\dfrac{x^{2n}f(x)+g(x)}{x^{2n}+1}$, find F(x) in terms of f(x) and g(x).

Solution: $\therefore F(x) = \lim\limits_{n\to\infty}\dfrac{x^{2n}.f(x)+g(x)}{x^{2n}+1}$

$$= \begin{cases} \dfrac{0.f(x)+g(x)}{0+1}, & -1<x<1 \\ \dfrac{1.f(x)+g(x)}{1+1}, & x=\pm 1 \\ \dfrac{f(x)+\dfrac{g(x)}{x^{2n}}}{1+\dfrac{1}{x^{2n}}}, & x<-1 \text{ or } x>1 \end{cases}$$

$$\therefore \quad F(x) = \begin{cases} g(x), & \text{if } -1<x<1 \\ \dfrac{f(x)+g(x)}{2}, & \text{if } x=\pm 1 \\ f(x), & \text{if } x<-1 \text{ or } x>1 \end{cases}$$

Example 5: Evaluate $\lim\limits_{x\to 0}\lim\limits_{n\to\infty}\dfrac{\tan x^2+(x+1)^n.\sin x}{x^2+(x+1)^n}$

Solution: Let $f(x) = \lim\limits_{n\to\infty}\dfrac{\tan \pi x^2+(x+1)^n.\sin x}{x^2+(x+1)^n}$

$$f(x) = \begin{cases} \lim\limits_{n\to\infty}\dfrac{\tan \pi x^2+(x+1)^n.\sin x}{x^2+(x+1)^n}, \text{left nghd of } x=0 \\ \lim\limits_{n\to\infty}\dfrac{\dfrac{\tan \pi x^2}{(x+1)^n}+\sin x}{\dfrac{x^2}{(x+1)^n}+1}, \text{ right nghd of } x=0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{\tan \pi x^2}{x^2} & , \text{ left nghd of } x=0 \\ \sin x & , \text{ right nghd of } x=0 \end{cases}$$

Since $\lim\limits_{x\to 0^-} f(x) = \pi$, $\lim\limits_{x\to 0^+} f(x) = 0$, the required limit does not exist.

Example 6: Let f: R → R be defined by

$$f(x) = \lim_{m\to\infty}\left\{\lim_{n\to\infty}(\cos m!\pi x)^{2n}\right\}.$$

Prove that f(x) is 1 for rational values of x and, 0 for irrational values of x.

Solution: Let x be a rational number say p/q, where p, q are integers prime to each other. By taking m sufficiently large, m! πx can be made an even integral multiple of π, so that cos m! πx = 1.

Hence f(x) = $\lim\limits_{n\to\infty}(1)^{2n} = 1$ where x is rational.

If x is irrational, then cos m! πx will always lie between –1 and 1 for any integral value of m so that

$(\cos m!\,\pi x)^{2n} = (r_m)^{2n}$ where $|r_m| < 1$

for a fixed value of m.

Thus, f(x) = $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}(r_m)^{2n} = 0$, when x is irrational.

Hence, f(x) is 1 for rational values of x and, 0 for irrational values of x.

Example 7: Show that the function f: R → R defined by $f(x) = \lim\limits_{n\to\infty}\left[\lim\limits_{t\to 0}\frac{\sin^2(n!\,\pi x)}{\sin^2(n!\pi x)+t^2}\right]$ is equal to 0 when x is rational and to 1 when x is irrational.

Solution: Let x = p/q be a rational number. Then by taking n sufficiently large n! πx can be made an integral multiple of π so that sin n! πx = 0.

Hence $f(x) = \lim\limits_{t\to 0}\frac{0}{0+t^2} = 0$ when x is rational.

If x is irrational, then $0 < \sin^2 n!\,\pi x < 1$.

Then $f(x) = \lim\limits_{n\to\infty}\lim\limits_{t\to 0}\frac{1}{1+t^2/\sin^2(n!\pi)}$

$$= \frac{1}{1+0} = 1 \text{ when x is irrational.}$$

Thus, f(x)=0 when x is rational and 1 when x is irrational.

Summation of infinite series using definite integral as the limit of sum

From the definition of definite integral, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$$

where,

(i) Σ is replaced by ∫ sign,

(ii) $\frac{r}{n}$ is replaced by x,

(iii) $\frac{1}{n}$ is replaced by dx,

(iv) To obtain the limits of integration, we use

$a = \lim\limits_{n\to\infty}\frac{\phi(x)}{n}$ and $b = \lim\limits_{n\to\infty}\frac{\psi(x)}{n}$

The limiting value is obtained by finding the definite integral.

Special case of definite integral as limit of sum

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$$

Example 8: Evaluate

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots\ldots\frac{n}{2n^2}\right)$$

Solution: $\lim\limits_{n\to\infty}\frac{1}{n}\left(\frac{n^2}{n^2+1^2}+\frac{n^2}{n^2+2^2}+\ldots\ldots\frac{n^2}{2n^2}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\ldots..\frac{1}{1+\left(\frac{n}{n}\right)^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2} = \int_0^1\frac{1}{1+x^2}dx = \tan^{-1}x\Big|_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$$

Example 9: Evaluate $\lim\limits_{n\to\infty}\frac{\sqrt[n]{|\underline{n}}}{n}$

Solution: $\ell nL = \lim\limits_{n\to\infty}\frac{1}{n}\ell n\left(\frac{1.2.3\ldots\ldots..n}{n.n.n\ldots\ldots.n}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\ell n\frac{1}{n}+\ell n\frac{2}{n}+\ell n\frac{3}{n}+\ldots\ldots\ell n\frac{n}{n}\right]$$

$$= \int_0^1 \ell nx\,dx = -1.$$

Hence, L = e^{-1}.

Example 10: Evaluate $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$

Solution: On dividing numerator and denominator by n, we get $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}}$$

Now we find

$$a = \lim_{n\to\infty}\frac{1}{n} = 0 \text{ and } b = \lim_{n\to\infty}\frac{2n}{n} = 2$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}} = \int_0^2\frac{x}{\sqrt{1+x^2}}dx$$

$$= \sqrt{1+x^2}\Big|_0^2$$

$$= \sqrt{5}-1.$$

Example 11: If $A = [a_{ij}]_{n\times n}$, where $a_{ij} = i^{100} + j^{100}$, then find $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} a_{ii}}{n^{101}}$.

Solution: We have $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots\ldots + a_{nn}$

Now, $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} a_{ii}}{n^{101}} = \lim\limits_{n\to\infty} \dfrac{2\left(1^{100} + 2^{100} + \ldots\ldots + n^{100}\right)}{n^{101}}$

$$= 2\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{100}\left(\frac{r}{n}\right)^{100} = 2\int_0^1 x^{100}dx = \frac{2}{101}.$$

Example 12: Evaluate the limit

$$\lim_{x\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\left(1^1 \,.\, 2^2 \ldots\ldots n^n\right)^{\frac{1}{n^2}}.$$

Solution: Taking the logarithm of the expression inside the limit, we find that it is

$$-\frac{1}{2}\left(1+\frac{1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln k = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\left(\frac{k}{n}\right).$$

We can recognize this as the integral $\int_0^1 x\ln x\,dx$, Evaluating this integral using integration by parts, we find that

$$\int_0^1 x\ln dx = \frac{1}{2}x^2\ln x\Big|_0^1 - \int_0^1 \frac{x}{2}dx = -\frac{1}{4}.$$

Therefore, the original limit is $e^{-1/4}$.

Limit in Matrices and Determinants

Let $A = \begin{pmatrix} f(x) & g(x) \\ u(x) & v(x) \end{pmatrix}$ then we have

$$\lim_{x\to a} A = \begin{pmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) \end{pmatrix}$$

Note that the limit exists if and only if the limits of each of the functions exist simultaneously.

Let $\Delta = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$ then we have

$$\lim_{x\to a}\Delta = \lim_{x\to a}\,(f(x)v(x) - g(x)u(x)).$$

If the limits of each of the functions exist simultaneously then the above limit can be also be evaluated as

$$\lim_{x\to a}\Delta = \begin{vmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(u) \end{vmatrix} = \begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = \ell_1 m_2 - \ell_2 m_1.$$

Example 13: Evaluate $\lim\limits_{x\to 0}\begin{pmatrix} [x^2] & \sec x \\ (\sin^2 x)^{x^2} & x\cot x \end{pmatrix}$

Solution: $\lim\limits_{x\to 0}\begin{pmatrix} [x^2] & \sec x \\ (\sin^2 x)^{x^2} & x\cot x \end{pmatrix}$

$$= \begin{pmatrix} \lim\limits_{x\to 0}[x^2] & \lim\limits_{x\to 0}\sec x \\ \lim\limits_{x\to 0}(\sin^2 x)^{x^2} & \lim\limits_{x\to 0} x\cot x \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 14: Evaluate $\lim\limits_{x\to 0}\dfrac{1}{x^2}\begin{vmatrix} \sin x & \ell n(1+x) \\ \tan x & x\sec x \end{vmatrix}$

Solution: $\lim\limits_{x\to 0}\dfrac{1}{x^2}\begin{vmatrix} \sin x & \ell n(1+x) \\ \tan x & x\sec x \end{vmatrix}$

$$= \lim_{x\to 0}\begin{vmatrix} \dfrac{\sin x}{x} & \dfrac{\ell n(1+x)}{x} \\ \dfrac{\tan x}{x} & \dfrac{x\sec x}{x} \end{vmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 15: Evaluate $\lim\limits_{x\to 0}\begin{vmatrix} \sin 2x & -x\tan x^2 \\ \operatorname{cosec}^3 x & \cot x \end{vmatrix}$

Solution: $\lim\limits_{x\to 0}\begin{vmatrix} \sin 2x & -x\tan x^2 \\ \operatorname{cosec}^3 x & \cot x \end{vmatrix}$

Here the limit of each of the functions does not exist simultaneously. Therefore, the above limit can be evaluated by finding limit of the expansion of the determinant.

$$\therefore \text{ limit } = \lim_{x\to 0}\,(\sin 2x\cot x + x\tan x^2\operatorname{cosec}^3 x)$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{\tan x} + \frac{x\tan x^2}{\sin^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\dfrac{\sin 2x}{2x}}{\dfrac{\tan x}{x}} + \frac{\dfrac{x}{x}\dfrac{\tan x^2}{x^2}}{\dfrac{\sin^3 x}{x^3}}\right) = 2 + 1 = 3.$$

Concept Problems R

1. Define the function without limit

$y = \lim_{n\to\infty} (\cos^{2n} x)$.

2. Define the following functions without limit:

(i) $y = \lim_{n\to\infty} \sqrt[n]{1 + x^n}$ $(x \geq 0)$

(ii) $y = \lim_{x\to\infty} \frac{x}{1 + x^n}$ $(x \geq 0)$

(iii) $y = \lim_{a\to 0} \sqrt{x^2 + a^2}$

(iv) $y = \lim_{n\to\infty} (\text{arc}\cot nx)$

3. Let $f(x) = \lim_{n\to\infty} \frac{x^{2n}(x^2+1)+2^x}{1+x^{2n}}$, find $\lim_{x\to 1} f(x)$.

Practice Problems P

4. Define $f(x) = \lim_{n\to\infty} \frac{\tan \pi x^2 + (x+1)^n \sin x}{x^2 + (x+1)^n}$, and find $\lim_{x\to 0} f(x)$.

5. Sketch the graph of the following functions for $x \geq 0$

(a) $\phi(x) = \lim_{n\to\infty} x^{\left(\frac{1}{1+x^n}\right)}$

(b) $\Psi(x) = \lim_{n\to\infty} \frac{\log_e(1+x^n)}{\log_e n}$

6. Let $f(x) = \begin{cases} [x^x], & x > 0 \\ \lim_{t\to\infty} \left(\frac{\sin x}{x}\right)^t, & x < 0 \end{cases}$, find $\lim_{x\to 0} f(x)$

where [.] is G.I.F.

7. Evaluate $\lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{ax^2 + bxy + cy^2}{dx^2 + exy + fy^2} \right\}$

$- \lim_{y\to 0} \left\{ \lim_{x\to 0} \frac{ax^2 + bxy + cy^2}{dx^2 + exy + fy^2} \right\}$, $ad \neq 0$, $cf \neq 0$.

8. If $\lim_{x\to\infty} \frac{1}{(\sin^{-1} x)^n + 1} = 1$ then find the values of x.

9. Simplify the function $\phi(x) = \lim_{n\to\infty} (2/\pi) \arctan(nx)$.

10. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{n + 4k}$.

11. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n} \left(\frac{2k}{n} + 1\right)^3$

12. Find the limit $\lim_{x\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$

13. Evaluate $\lim_{x\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} \sin\left(\frac{i^2 \pi}{n^2}\right)$.

14. Evaluate $\lim_{n\to\infty} \left[\frac{(2n)!}{n!.n^n}\right]^{1/n}$

15. Evaluate

$$\lim_{k\to\infty} \left[\frac{e^{1/k}}{k^2} + 2\frac{(e^{1/k})^2}{k^2} + 3\frac{(e^{1/k})^3}{k^2} + \ldots\ldots + \frac{(e^{1/k})^k}{k}\right]$$

16. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{10n}{(n+2k)^2}$.

17. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{3n/2} \frac{\pi}{n} \sin\left(\frac{k\pi}{n}\right)$.

Target Problems for JEE Advanced

Problem 1: If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then evaluate

$$\lim_{x\to\frac{1}{\alpha}} \sqrt{\left\{\frac{1-\cos(cx^2+bx+a)}{2(1-\alpha x)^2}\right\}}$$

Solution: $ax^2 + bx + c = 0$ has roots α and β, then

$$\frac{a}{x^2}+\frac{b}{x}+c=0$$

i.e., $cx^2 + bx + a = 0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

$$\Rightarrow \quad c\left(x^2+\frac{b}{c}x+\frac{a}{c}\right)=c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)$$

$$\therefore \quad \lim_{x\to 1/\alpha} \sqrt{\left\{\frac{1-\cos(cx^2+bx+a)}{2(1-\alpha x)^2}\right\}}$$

$$= \lim_{x\to 1/\alpha} \sqrt{\left\{\frac{\sin^2\left(\frac{cx^2+bx+a}{2}\right)}{(1-\alpha x)^2}\right\}}$$

$$= \lim_{x\to 1/\alpha} \left|\frac{\sin\left(\frac{cx^2+bx+a}{2}\right)}{(1-\alpha x)}\right|$$

$$= \lim_{x\to 1/\alpha} \left|\frac{\sin\left(\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right)}{-\alpha\left(x-\frac{1}{\alpha}\right)}\right|$$

$$= \left|\lim_{x\to 1/\alpha} \frac{\sin\left(\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right)}{\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)} \cdot \lim_{x\to 1/\alpha} \frac{\frac{c}{2}\left(x-\frac{1}{\beta}\right)}{-\alpha}\right|$$

$$= \left|1.\frac{\frac{c}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)}{-\alpha}\right| = \left|\frac{c}{2\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right|.$$

Problem 2: Evaluate $\lim_{x\to 2} \frac{(\cos\alpha)^x+(\sin\alpha)^x}{x-2}$

Solution: Limit

$$= \lim_{x\to 2} \frac{(\cos\alpha)^x+(\sin\alpha)^x-\cos^2\alpha-\sin^2\alpha}{x-2}$$

$$= \lim_{x\to 2} \frac{\cos^2\alpha((\cos\alpha)^{x-2}-1)+\sin^2\alpha((\sin\alpha)^{x-2}-1)}{x-2}$$

Put $x - 2 = t$

$$= \lim_{t\to 0}\cos^2\alpha\left(\frac{\cos\alpha)^t-1}{t}\right)+\lim_{t\to 0}\sin^2\alpha\frac{((\sin\alpha)^t-1)}{t}$$

$= \cos^2\alpha\ \ell n\cos\alpha + \sin^2\alpha\ \ell n(\sin\alpha)$.

Problem 3: Evaluate $\lim_{x\to a} \frac{x^x-a^a}{a^x-a^a}$

Solution: Limit $= \lim_{x\to a} \frac{e^{\ell n x^x}-e^{\ell n a^a}}{a^a\frac{(a^{x-a}-1)}{(x-a)}(x-a)}$

$$= \lim_{x\to a} \frac{\left(e^{x\ell nx}-e^{a\ell na}\right)}{a^a\ell na\ (x-a)}$$

$$= \lim_{x\to a} \frac{e^{a\ell na}\left(e^{x\ell nx-a\ell na}-1\right)}{a^a\ell na.(x\ell nx-a\ell na)(x-a)}(x\ell nx-a\ell na)$$

$$= \lim_{x\to a} \frac{a^a\left(x\ell nx-a\ell na\right)}{a^a.\ell na.(x-a)}$$

put $x - a = t$

$$= \lim_{t\to 0} \frac{(t+a)\ell n(t+a)-a\ell na}{(\ell na)t}$$

$$= \lim_{t\to 0} \frac{a(\ell n(t+a)-\ell na)+t.\ell n(t+a)}{(\ell na)t}$$

$$= \lim_{t\to 0} \frac{\ell n\left(1+\frac{t}{a}\right)}{\ell na\left(\frac{t}{a}\right)}+\lim_{t\to 0}\frac{\ell n(t+a)}{\ell na}$$

$$= \frac{1}{\ell na}+1 = \log_a e + 1.$$

Problem 4: The integral value of n for which the

$$\lim_{x\to 0}\frac{\cos^2 x - \cos x - e^x \cos x + e^x - \dfrac{x^3}{2}}{x^n}$$

is a finite nonzero quantity.

Solution: The limit is equal to

$$\lim_{x\to 0}\frac{1}{x^n}\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots-1\right)\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-..\right)\right.\right.$$

$$\left.\left.-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-.....\right)\right]-\frac{x^3}{2}\right]$$

$$=\lim_{x\to 0}\frac{\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}\right)\left[-x-x^2-\dfrac{x^3}{3!}-\dfrac{2x^5}{5!}-.....\right]-\dfrac{x^3}{2}}{x^n}$$

$$=\lim_{x\to 0}\frac{\left(\dfrac{x^3}{2}+\dfrac{x^4}{2}+\dfrac{x^5}{12}-\dfrac{x^5}{24}+.....\right)-\dfrac{x^3}{2}}{x^n}$$

= nonzero if n = 4.

Problem 5: Evaluate

$$\lim_{x\to\infty}\sqrt[4]{(x+p)(x+q)(x+r)(x+s)}-x$$

Solution: $\lim_{x\to\infty}\sqrt[4]{(x+p)(x+q)(x+r)(x+s)}-x$

$$=\lim_{x\to\infty}\frac{(x+p)(x+q)(x+r)(x+s)-x^4}{(\sqrt[4]{(x+p)(x+q)(x+r)(x+s)}+x)}$$

$$\cdot\frac{1}{(\sqrt{(x+p)(x+q)(x+r)(x+s)}+x^2)}$$

[using $a-b=\dfrac{a^4-b^4}{(a+b)(a^2+b^2)}$]

$$\lim_{x\to\infty}\frac{Ax^3+Bx^2+Cx+D}{\left(\sqrt[4]{(x+p)(x+q)(x+r)(x+s)}+x\right)}$$

$$\cdot\frac{1}{(\sqrt{(x+p)(x+q)(x+r)(x+s)}+x^2)}$$

[where A = Σp, B = Σpq, C= Σpqr and D= Πp]

$$\lim_{x\to\infty}\frac{A+\dfrac{B}{x}+\dfrac{C}{x^2}+\dfrac{D}{x^3}}{\sqrt[4]{\left(1+\dfrac{p}{x}\right)\left(1+\dfrac{q}{x}\right)\left(1+\dfrac{r}{x}\right)\left(1+\dfrac{s}{x}\right)}+1}$$

$$=\frac{A}{(1+1)(1+1)}=\frac{1}{4}(p+q+r+s).$$

Problem 6: Evaluate a, b, c and d, if

$$\lim_{x\to\infty}\cdot(\sqrt{x^4+ax^3+3x^2+bx+2}$$

$$\sqrt{x^4-2x^3-cx^2+3x-d})=4$$

Solution: Here, $\lim_{x\to\infty}(\sqrt{x^4+ax^3+3x^2+bx+2}$

$-\sqrt{x^4+2x^3-cx^2+3x-d})=4$ (∞ – ∞ form)

Rationalizing

$$\lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$$

Since, limit is finite, so the degree of the numerator must be 2. So, a – 2 = 0
i.e., a = 2.

$$\lim_{x\to\infty}\frac{(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d'}}$$

Dividing numerator and denominator by x^2.
We get

$$\lim_{x\to\infty}\frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\dfrac{a}{x}+\dfrac{3}{x^2}+\dfrac{b}{x^3}+\dfrac{2}{x^4}}+\sqrt{1+\dfrac{2}{x}-\dfrac{c}{x^2}+\dfrac{3}{x^3}-\dfrac{d}{x^4}}}$$

$$\Rightarrow\quad\frac{3+c}{2}.$$

It is given that $\dfrac{3+c}{2}=4\ \Rightarrow\ c=5$

Hence, a = 2, c = 5 and b, d are real numbers.

Problem 7: Evaluate $\lim_{x\to 0}\dfrac{x+\ln\left(\sqrt{x^2+1}-x\right)}{x^3}$

Solution: Put $\ln\left(\sqrt{x^2+1}-x\right)=t$ ...(1)

$$\sqrt{x^2+1}-x=e^t\quad\text{or}\quad\sqrt{x^2+1}+x=e^{-t}$$

On subtraction, $2x=e^{-t}-e^t$

or $x=-\left(\dfrac{e^t-e^{-t}}{2}\right)$

As $x\to 0,\ t\to 0$ from (1).

$$\therefore\ \ell=\lim_{t\to 0}\frac{t-\dfrac{e^t-e^{-t}}{2}}{-\left(\dfrac{e^t-e^{-t}}{2}\right)^3}=\lim_{t\to 0}\frac{e^t-e^{-t}-2t}{2.t^3\left(\dfrac{e^{2t}-1}{2t}\right)^3}$$

$$\ell = \lim_{t\to 0}\frac{e^{t}-e^{-t}-2t}{2t^{3}} \qquad \left(\text{as } \lim_{t\to 0}\frac{e^{2t}-1}{2t}=1\right)$$

(put t = 3y)

$$= \lim_{y\to 0}\frac{e^{3y}-e^{-3y}-6y}{2.27y^{3}}$$

$$= \lim_{y\to 0}\frac{\left(e^{y}-e^{-y}\right)^{3}+3\left(e^{y}-e^{-y}\right)-6y}{54y^{3}}$$

$$\ell = \lim_{y\to 0}\left[\frac{\left(e^{y}-e^{-y}\right)^{3}}{54y^{3}}+\frac{3(e^{y}-e^{-y}-2y)}{54y^{3}}\right]$$

$$= \lim_{y\to 0}\left[\frac{8}{54}\left(\frac{e^{2y}-1}{2y}\right)^{3}+\frac{1}{9}\,\frac{e^{y}-e^{-y}-2y}{2y^{3}}\right]$$

or $\ell = \frac{4}{27}+\frac{\ell}{9} \Rightarrow \frac{8\ell}{9}=\frac{4}{27} \Rightarrow \ell=\frac{1}{6}$.

Problem 8: Evaluate $\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\frac{r^{3}-8}{r^{3}+8}$

Solution: $\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\frac{r^{3}-8}{r^{3}+8}$

$$= \lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\left(\frac{r^{2}+2r+4}{r^{2}-2r+4}\right)$$

$$= \lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\prod_{r=3}^{n}\left(\frac{r^{2}+2r+4}{r^{2}-2r+4}\right)$$

$$= \lim_{n\to\infty}\left\{\frac{1}{5}\,\frac{2}{6}\,\frac{3}{7}\,\frac{4}{8}\,\frac{5}{9}\ldots.\frac{(n-5)}{(n-1)}\,\frac{(n-4)}{(n)}\,\frac{(n-3)}{(n+1)}\,\frac{(n-2)}{(n+2)}\right\}\times$$

$$\left\{\frac{19}{7}\times\frac{28}{12}\times\frac{39}{19}\ldots.\frac{(n^{2}-2n+4)}{(n^{2}-6n+12)}\times\frac{(n^{2}+3)}{(n^{2}-4n+7)}\times\frac{(n^{2}+2n+4)}{(n^{2}-2n+4)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1.2.3.4}{(n-1)n(n+1)(n+2)}\times\frac{(n^{2}+3)(n^{2}+2n+4)}{7.12}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{(n^{2}+3)(n^{2}+2n+4)}{(n-1)n(n+1)(n-2)}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{\left(1+\frac{3}{n^{2}}\right)\left(1+\frac{2}{n}+\frac{4}{n^{2}}\right)}{\left(1-\frac{1}{n}\right)1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\right\}$$

$$= \frac{2}{7}\,\frac{(1+0)(1+0+0)}{(1-0)1(1+0)(1+0)}=\frac{2}{7}.$$

Problem 9: Find the value of

$$\lim_{x\to 0}\left(\frac{2}{x^{3}}\left(\sin^{-1}x-\tan^{-1}x\right)\right)^{\frac{2}{x^{2}}}.$$

Solution: $\sin^{-1}x = x+\frac{x^{3}}{6}+\frac{3}{40}x^{5}+\frac{5}{112}x^{7}\ldots..$

and $\tan^{-1}x = x-\frac{x^{3}}{3}+\frac{x^{5}}{5}-\frac{x^{7}}{7}\ldots\ldots...$

$$\lim_{x\to 0}\left(\frac{2}{x^{3}}\left(\sin^{-1}x-\tan^{-1}x\right)\right)^{\frac{2}{x^{2}}} \qquad (1^{\infty}\text{ form})$$

$$= e^{\overbrace{\lim\limits_{x\to 0}\frac{2}{x^{2}}\left(\frac{2}{x^{3}}(\sin^{-1}x-\tan^{-1}x)-1\right)}^{L\,(say)}}$$

$$\text{Now } L = 2\lim_{x\to 0}\frac{2\left(\sin^{-1}x-\tan^{-1}x\right)-x^{3}}{x^{5}}$$

$$= 2\lim_{x\to 0}\frac{2\left(x+\frac{x^{3}}{6}+\frac{3}{40}x^{5}+..-x+\frac{x^{3}}{3}-\frac{x^{5}}{5}\ldots..\right)-x^{3}}{x^{5}}$$

$$= -\frac{1}{2}$$

$$\text{Limit} = \frac{1}{\sqrt{e}} = e^{-1/2}.$$

Problem 10: A sequence of numbers x_n is determined by the equality $x_n = \frac{x_{n-1}+x_{n-2}}{2}$ and the values x_0 and x_1. Compute x_n in terms of x_0, x_1 and n. Also prove that $\lim\limits_{n\to\infty} x_n = \frac{x_0+2x_1}{3}$.

Solution: $x_n = \frac{x_{n-1}+x_{n-2}}{2}$

$$x_2 = \frac{x_1+x_0}{2};\ x_3 = \frac{x_2+x_1}{2},\ x_4 = \frac{x_3+x_2}{2} \text{ etc.}$$

$$x_2 - x_1 = \frac{x_0-x_1}{2}$$

$$x_3 - x_2 = \frac{x_1-x_2}{2} = -\frac{(x_0-x_1)}{2^{2}}$$

$$x_4 - x_3 = \frac{x_2-x_3}{2} = +\frac{(x_0-x_1)}{2^{3}}$$

$$x_5 - x_4 = \frac{x_3 - x_4}{2} = -\frac{(x_0 - x_1)}{2^4}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_n - x_{n-1} = \frac{x_{n-2} - x_{n-1}}{2} = (-1)^n \cdot \frac{(x_0 - x_1)}{2^n}$$

add

$$x_n - x_1$$

$$= \frac{x_0 - x_1}{2}\left[1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots\ldots + (-1)^n \frac{1}{2^{n-2}}\right]$$

$$\lim_{n\to\infty} (x_n - x_1) = \lim_{n\to\infty} \frac{x_0 - x_1}{2}\left[\frac{1}{1 + (1/2)}\right]$$

$$\lim_{n\to\infty} x_n - x_1 = \frac{x_0 - x_1}{2} \times \frac{2}{3}$$

$$\lim_{n\to\infty} x_n = \frac{x_0 - x_1}{3} + x_1 = \frac{x_0 + 2x_1}{3}$$

Problem 11: For $n \in N$, let x_n be defined as

$$\left(1 + \frac{1}{n}\right)^{n + x_n} = e, \text{ then find } \lim_{n\to\infty} x_n.$$

Solution: We have $\left(1 + \frac{1}{n}\right)^{n + x_n} = e$ ...(1)

Taking log on both sides of equation (1), we get

$$\Rightarrow \quad (n + x_n)\, ln\left(1 + \frac{1}{n}\right) = 1$$

$$\Rightarrow \quad n + x_n = \frac{1}{ln\left(1 + \frac{1}{n}\right)}$$

$$\Rightarrow \quad x_n = \frac{1}{l\,n\left(1 + \frac{1}{n}\right)} - n \qquad \text{...(1)}$$

Let $\frac{n+1}{n} = u \quad \Rightarrow \quad nu = n + 1$

$$\Rightarrow \quad n = \frac{1}{u - 1}$$

$$\therefore \quad \lim_{n\to\infty} x_n = \lim_{u\to1}\left(\frac{1}{\ell n\, u} - \frac{1}{u - 1}\right)$$

$$= \lim_{u\to1} \frac{(u-1) - \ln u}{(u-1)\ln u} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{u\to1} \frac{1 - \frac{1}{u}}{\frac{u-1}{u} + \ln u} = \lim_{u\to1} \frac{\frac{1}{u^2}}{\frac{1}{u^2} + \frac{1}{u}} = \frac{1}{2}.$$

Problem 12: If $\alpha_1, \alpha_2, \ldots., \alpha_n$ are the roots of equation $x^n + nax - b = 0$, show that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots.(\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$.

Solution: Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$ we have

$x^n + nax - b = (x - a_1)(x - a_2)\ldots(x - a_n)$

$$\Rightarrow \quad \frac{x^n + nax - b}{x - \alpha_1} = (x - \alpha_2)(x - \alpha_3)\ldots.(x - \alpha_n)$$

$$\Rightarrow \quad \lim_{x\to\alpha_1} \frac{x^n + nax - b}{x - \alpha_1} = \lim_{x\to\alpha_1} [(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n)]$$

$$\lim_{x\to\alpha_1} \frac{nx^{n-1} + na}{1} = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots(\alpha_1 - \alpha_n)$$

(using L' Hospital's rule on L.H.S.)

$$\Rightarrow \quad (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots.(\alpha_1 - \alpha_n) = na_1^{n-1} + na.$$

Problem 13: Find $\lim_{x\to1} \frac{nx^{n+1} - (n+1)x^n + 1}{(e^x - e)\sin \pi x}$ if $n = 100$.

Solution: $l = \lim_{x\to1} \frac{nx^n(x-1) - (x^n - 1)}{(e^x - e)\sin \pi x}$

Put $x = 1 + h$ so that as $x \to 1$, $h \to 0$

$$\therefore \quad l = -\lim_{h\to0} \frac{h \cdot n(1+h)^n - \left((1+h)^n - 1\right)}{e(e^h - 1)\sin \pi h}$$

$$l = -\lim_{x\to1}\Big[n \cdot h\left(1 + {}^nC_1\, h + {}^nC_2\, h^2 + {}^nC_3\, h^3 + \ldots\right)$$

$$-\left(1 + {}^nC_1\, h + {}^nC_2\, h^2 + {}^n C_3\, h^3 + \ldots. - 1\right)\Big]$$

$$\cdot \frac{1}{\pi e(h^2)\left(\frac{e^h - 1}{h}\right)\left(\frac{\sin \pi h}{\pi h}\right)}$$

$$= -\frac{n^2 - {}^nC_2}{\pi e} = -\left[\frac{2n^2 - n(n-1)}{2\pi e}\right]$$

$$= -\frac{n^2 + n}{2(\pi e)} = -\frac{n(n+1)}{2(\pi e)}$$

If $n = 100$ then $l = -\left(\frac{5050}{\pi e}\right)$.

Problem 14: Let f(n, θ)

$$= \left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)\ ....$$

to n factors. Show that $\lim\limits_{n\to\infty} f(n,\theta) = \dfrac{\theta}{\tan\theta}$.

Solution: $1-\tan^2\dfrac{\theta}{2} = \dfrac{\cos\theta}{\cos^2\frac{\theta}{2}}$;

$$1-\tan^2\frac{\theta}{2^2} = \frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}} \quad \text{and so on.}$$

Hence f(n, θ)

$$= \frac{\cos\theta}{\cos^2\frac{\theta}{2}}\cdot\frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}}\ ......\ \frac{\cos\frac{\theta}{2^{n-1}}}{\cos^2\frac{\theta}{2^n}}$$

$$= \frac{\cos\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}....\cos\frac{\theta}{2^n}}\cdot\frac{1}{\cos\frac{\theta}{2^n}}$$

But $\cos\dfrac{\theta}{2}\ .\ \cos\dfrac{\theta}{2^2}\ .\ \cos\dfrac{\theta}{2^3}\ ......\ \cos\dfrac{\theta}{2^n}$

$$= \frac{\sin\theta}{2^n\sin\frac{\theta}{2^n}}$$

Hence $f(n,\theta) = \cot\theta\ .\ 2^n\tan\dfrac{\theta}{2^n}$

$$= \cot\theta\ .\ \frac{\theta\tan\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}$$

Now $\lim\limits_{n\to\infty} f(n,\theta) = \theta\cot\theta$.

Problem 15: Evaluate

$$\lim_{n\to\infty}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)........\left(\frac{a_n+1}{a_n}\right),$$

where $a_1 = 1$ and $a_n = n\,(1 + a_{n-1})\ \forall\ n \ge 2$.

Solution: $\lim\limits_{n\to\infty}\left(\dfrac{a_1+1}{a_1}\right)\left(\dfrac{a_2+1}{a_2}\right)......\left(\dfrac{a^n+1}{a_n}\right)$

We know, $a_{n-1} + 1 = \dfrac{a_n}{n}$ ...(1)

$$\lim_{n\to\infty}\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right).....\left(\frac{a_{n+1}}{n+1}\right)\frac{1}{a_1.a_2.....a_n}$$

$$= \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!} = \lim_{n\to\infty}\frac{1+a_n}{n!} \qquad \text{[using (1)]}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{a_n}{n!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{a_{n-1}}{(n-1)!}\right) \qquad \text{[using (1)]}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+......+\frac{1}{(2)!}+\frac{1}{1!}+\frac{a_1}{1!}\right)$$

$\{a_1 = 1$ as given$\}$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+......+\frac{1}{(2)!}+\frac{1}{1!}+\frac{1}{1}\right) = e$$

$$\left\{\text{as, } e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+........\infty\right\}$$

Problem 16: Solve $\lim\limits_{n\to\infty} x_n$, when $x_n^2 = a + x_{n-1}$ and $x_0 = \sqrt{a}$.

Solution: Here, $x_0 = \sqrt{a}$, $x_1 = \sqrt{a+\sqrt{a}}$ which shows $x_1 > x_0$

$\therefore$ assuming $x_n > x_{n-1} \Rightarrow x_{n-1}^2 - a < x_{n-1}$

$$\Rightarrow \left\{x_{n-1}-\frac{\sqrt{4a+1}+1}{2}\right\}\left\{x_{n-1}+\frac{\sqrt{4a+1}-1}{2}\right\} < 0$$

$$\Rightarrow \quad x_{n-1} < \frac{\sqrt{4a+1}+1}{2}\ .$$

$$\therefore \quad \lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = l$$

$$\Rightarrow \quad l^2 - l - a = 0$$

$$\Rightarrow \quad l = \frac{1\pm\sqrt{1+4a}}{2}, \text{ since } l \ge 0$$

$$\therefore \quad l = \frac{1+\sqrt{1+4a}}{2}$$

$$\text{or} \quad \lim_{n\to\infty} x_n = \frac{\sqrt{1+4a}+1}{2}.$$

Problem 17: If $a_1 = \sqrt{ab_1}$, $b_1 = \dfrac{a+b}{2}$,

$$a_2 = \sqrt{a_1b_2},\ b_2 = \frac{a_1+b_1}{2}\ ...$$

and so on for (a > b > 0) then prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}.$$

Solution: Put b = a cos θ , we get $b_1 = a\cos^2\frac{\theta}{2}$ and

$$a_1 = a\cos\frac{\theta}{2}$$

$$b_2 = a\cos\frac{\theta}{2}\cos^2\frac{\theta}{4} \text{ and } a_2 = a\cos\frac{\theta}{2}\cos\frac{\theta}{4}$$

Similarly we get

$$a_n = a\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ \ldots\ldots\ \cos\frac{\theta}{2^n} \text{ and}$$

$$b_n = a\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ \ldots.\ \cos^2\frac{\theta}{2^n}$$

$$\Rightarrow \quad a_n = \frac{a\sin\theta}{2^n\sin\theta/2^n} \text{ and}$$

$$b_n = \frac{a\sin\theta\cos\theta/2^n}{2^n\sin\theta/2^n}$$

Now, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{a\sin\theta}{2^n\sin\theta/2^n} = \frac{a\sin\theta}{\theta}$.

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \frac{a\sqrt{(a^2-b^2)}}{a\cos^{-1}(b/a)} = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}.$$

and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{a\sin\theta\cos\theta/2^n}{2^n\sin\theta/2^n} = \frac{a\sin\theta}{\theta}$.

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}.$$

Problem 18: Let $x_0 = 2\cos\frac{\pi}{6}$ and $x_n = \sqrt{2+x_{n-1}}$, n = 1, 2, 3, ... find $\lim\limits_{n\to\infty} 2^{(n+1)}\cdot\sqrt{2-x_n}$.

Solution: Let $\frac{\pi}{6} = \theta$.

$$x_1 = \sqrt{2+x_0} = \sqrt{2+2\cos\left(\frac{\pi}{6}\right)} = 2\cos\frac{\pi}{12} = 2\cos\frac{\theta}{2}$$

$$x_2 = \sqrt{2+x_1} = \sqrt{2+2\cos\left(\frac{\pi}{12}\right)} = 2\cos\frac{\pi}{24} = 2\cos\frac{\theta}{2^2}$$

$$x_3 = \sqrt{2+x_2} = \sqrt{2+2\cos\left(\frac{\pi}{24}\right)} = 2\cos\frac{\pi}{48} = 2\cos\frac{\theta}{2^3}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$x_n = 2\cos\frac{\theta}{2^n}\ ; \quad \sqrt{2-x_n} = \sqrt{2-2\cos\frac{\theta}{2^n}} = 2\sin\frac{\theta}{2^{n+1}}$$

$$l = 2^{(n+1)}\, 2\sin\frac{\theta}{2^{n+1}} = \frac{2\theta\sin\frac{\theta}{2^{n+1}}}{\frac{\theta}{2^{n+1}}} = 2\theta = 2\cdot\frac{\pi}{6} = \frac{\pi}{3}.$$

Problem 19: Given $\lim\limits_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{1/n} = \frac{a}{b}$ where a and b are relatively prime, find the value of (a + b).

Solution: Consider $\frac{{}^{3n}C_n}{{}^{2n}C_n} = \frac{(3n)!}{n!(2n)!}\cdot\frac{n!n!}{(2n)!}$

$$= \frac{3n(3n-1)\ldots\ldots(2n+1)(2n)!}{(2n)!(2n)(2n-1)\ldots.(2n+1)n!}$$

$$= \frac{(2n+1)(2n+2)(2n+3)\ldots..(2n+n)}{(n+1)(n+2)(n+3)\ldots\ldots\ldots.(n+n)}$$

Let $P = \lim\limits_{n\to\infty}\left[\left(1+\frac{n}{n+1}\right)\left(1+\frac{n}{n+2}\right)\ldots\ldots\left(1+\frac{n}{n+n}\right)\right]^{1/n}$

Taking logarithm, *ln* P

$$= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{n}{n+1}\right)+\ln\left(1+\frac{n}{n+2}\right)+\ldots\ldots+\ln\left(1+\frac{n}{n+n}\right)\right]$$

where $T_r = \frac{1}{n}\ln\left(1+\frac{n}{n+r}\right) = \frac{1}{n}\ln\left(1+\frac{1}{1+(r/n)}\right)$

$$\therefore \quad \ln P = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{1}{1+(r/n)}\right)$$

$$= \int_0^1 \ln\left(1+\frac{1}{1+x}\right)dx = \int_0^1 \ln\left(\frac{x+2}{x+1}\right)dx$$

$$= \int_0^1 \ln(x+2)dx - \int_0^1 \ln(x+1)dx$$

$$= (x+2)\ln(x+2)-(x+2)\Big|_0^1$$

$$- (x+1)\ln(x+1)-(x+1)\Big|_0^1$$

= [3 *ln* 3 – 3]– [2 *ln* 2 – 2]– [(2 *ln* 2– 2) – (0 – 1)]
= [*ln* 27 – 3] – [*ln* 4 – 2] – [*ln* 4 – 1]
= [*ln* 27 – 1] – [2 *ln* 4 – 1]
= *ln* 27 – *ln* 16

$$\therefore \quad \ln P = \ln\frac{27}{16} \qquad \Rightarrow \quad P = \frac{27}{16}$$

∴ a = 27 ; b = 16 ⇒ a + b = 43.

Alternative:

$$\frac{{}^{3n}C_n}{{}^{2n}C_n} = \frac{(3n)!}{n!(2n)!}\cdot\frac{n!n!}{(2n)!}$$

$$= \left[(3n)(3n-1)(3n-2)\ldots\ldots\ldots\ldots(3n-(3n-1))\right]$$

$$\cdot\left[n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-(n-1))\right]$$

$$\cdot\frac{1}{[2n(2n-1)(2n-2)\ldots\ldots..2n-(2n-1)]^2}$$

$$\approx \frac{(3n)^{3n}(n)^n}{(2n)^{4n}} = \left(\frac{27}{16}\right)^n$$

$$\therefore \quad \lim_{n\to\infty}\left[\left(\frac{27}{16}\right)^n\right]^{1/n} = \frac{27}{16}.$$

Problem 20: Let $\displaystyle\sum_{r=1}^{n}\frac{r^4}{(2r-1)(2r+1)}$

$$= \frac{n^3}{A}+\frac{n^2}{B}+\frac{5n}{C}+\frac{f(n)}{D}, \ (A, B, C, D, \in N)$$

where f (n) is the ratio of two linear polynomials such that $\displaystyle\lim_{n\to\infty} f(n) = \frac{1}{2}$. Find the value of (A+B+C+D).

Solution: $T_r = \dfrac{r^4}{(2r-1)(2r+1)}$

$$= \frac{1}{16}\left[\frac{16r^4-1+1}{(2r-1)(2r+1)}\right]$$

$$T_r = \frac{1}{16}\left[\frac{(4r^2-1)(4r^2+1)+1}{4r^2-1}\right]$$

$$= \frac{1}{16}\left[(4r^2+1)+\frac{1}{(2r-1)(2r+1)}\right]$$

$$= \frac{1}{16}\left[(4r^2+1)+\frac{1}{2}\left(\frac{(2r+1)-(2r-1)}{(2r-1)(2r+1)}\right)\right]$$

$$= \frac{1}{16}\left[4r^2+1+\frac{1}{2}\left(\frac{1}{(2r-1)}-\frac{1}{(2r+1)}\right)\right]$$

$$\therefore \quad S_n = \frac{1}{4}\sum r^2+\frac{1}{16}\sum 1+\frac{1}{32}\sum\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$

$$= \frac{n(n+1)(2n+1)}{4\cdot 6}+\frac{n}{16}$$

$$+\frac{1}{32}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots..+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{2n^3+3n^2+n}{24}+\frac{n}{16}+\frac{1}{32}\left(1-\frac{1}{2n+1}\right)$$

$$= \frac{n^3}{12}+\frac{n^2}{8}+\frac{5n}{48}+\frac{1}{16}\left(\frac{n}{2n+1}\right)$$

$$= \frac{n^3}{A}+\frac{n^2}{B}+\frac{5n}{C}+\frac{f(n)}{D} \quad \text{(given)}$$

So A = 12, B = 8, C = 48 ; D = 16

Hence, A + B + C + D = 84

Problem 21: Evaluate

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\{\frac{1+\sqrt[n]{1^n+2^n}+\sqrt[n]{2^n+3^n}+\sqrt[n]{3^n+4^n}+\ldots\ldots+\sqrt[n]{(m-1)^n+m^n}}{m^2}\right\}$$

Solution:

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\{\frac{1+\sqrt[n]{1^n+2^n}+\sqrt[n]{2^n+3^n}+\sqrt[n]{3^n+4^n}+\ldots\ldots+\sqrt[n]{(m-1)^n+m^n}}{m^2}\right\}$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}\left\{\frac{1+2\sqrt[n]{\left(\frac{1}{2}\right)^n+1}+3\sqrt[n]{\left(\frac{2}{3}\right)^n+1}+\ldots\ldots+m\sqrt[n]{\left(\frac{m-1}{m}\right)^n+1}}{m^2}\right\}$$

$$= \lim_{m\to\infty}\frac{1+2+3+4+\ldots.+m}{m^2}$$

$$\left\{\because \left(\frac{1}{2}\right)^n \to 0 \text{ as } n\to\infty; \left(\frac{2}{3}\right)^n \to 0 \text{ as } n\to\infty;\right.$$

$$\left.\ldots..; \left(\frac{m-1}{m}\right)^n \to 0 \text{ as } n\to\infty\right\}$$

$$= \lim_{m\to\infty}\frac{m(m+1)}{2m^2} = \lim_{m\to\infty}\frac{1}{2}\left(1+\frac{1}{m}\right) = \frac{1}{2}.$$

Problem 22: Evaluate $\displaystyle\lim_{x\to 0}\left(\frac{(1+[x])^{1/\{x\}}}{e}\right)^{1/\{x\}}$

if it exist, where {x} denotes the fractional part of x.

Solution: L.H.L. $= \displaystyle\lim_{x\to 0-} f(x)$

$$= \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\left(\frac{1+\{0-h\})^{1/\{0-h\}}}{e}\right)^{1/\{0-h\}}$$

$$= \lim_{h\to 0}\left(\frac{(1+1-h)^{1/(1-h)}}{e}\right)^{1/(1-h)} = \left(\frac{2}{e}\right).$$

R.H.L. $= \displaystyle\lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\left(\frac{(1+\{h\})^{1/\{h\}}}{e}\right)^{1/\{h\}} = \lim_{h\to 0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}$$

$$= e^{\frac{1}{e}\lim\limits_{h\to 0}}\ \frac{((1+h)^{1/h}-e)}{h}$$

$$= e^{\frac{1}{e}\lim\limits_{h\to 0}}\ \left(\frac{e^{(1/h)\ln(1+h)}-e}{h}\right)$$

$$= e^{\lim\limits_{h\to 0}}\left[\frac{e^{\frac{\ln(1+h)-h}{h}}-1}{h}\right]$$

$$= e^{\lim\limits_{h\to 0}\frac{\left(e^{\frac{\ln(1+h)}{h}}-1\right)}{\left(\frac{\ln(1+h)-h}{h}\right)}\cdot\left(\frac{\ln(1+h)-h}{h^2}\right)} \qquad ...(1)$$

Now let $P = \lim\limits_{h\to 0}\dfrac{\ln(1-h)-h}{h^2}$ ...(2)

replacing h by –h then

$$P = \lim_{h\to 0}\frac{\ln(1-h)+h}{h^2} \qquad ...(3)$$

adding (2) and (3),

$$2P = \lim_{h\to 0}\frac{\ln(1-h^2)}{h^2} = -\lim_{h\to 0}\frac{\ln(1-h^2)}{(-h^2)} = -1$$

$\therefore \quad P = -\dfrac{1}{2}$ from (1),

R.H.L. $= e^{1.(-1/2)} = e^{-1/2}$.

L.H.L. ≠ R.H.L. Hence P does not exist.

Problem 23: Suppose that function f : R → R satisfies the inequality, $\left|\sum\limits_{k=1}^{n} 3^k\{f(x+ky)-f(x-ky)\}\right| \le 1$

for every positive integer n and for all x, y ∈ R.
Prove that f is a constant function.

Solution: Replacing n by (n – 1) in,

$$\left|\sum_{k=1}^{n} 3^k\{f(x+ky)-f(x-ky)\}\right| \le 1 \qquad ...(1)$$

we get:

$$\left|\sum_{k=1}^{n-1} 3^k\{f(x+ky)-f(x-ky)\}\right| \le 1 \qquad ...(2)$$

Subtracting (1) and (2) we get,

$$|3^n\ \{f(x+ny)-f(x-ny)\}| \le 2$$

$$\Rightarrow \quad |\{f(x+ny)-f(x-ny)\}| \le \frac{2}{3^n} \qquad ...(3)$$

We choose x and y such that x + ny = u and x – ny = v where u, v ∈ R and n ∈ N.

(3) becomes, $|f(u) - f(v)| \le \dfrac{2}{3^n}$ for arbitrary n ∈ N

i.e., as n → ∞ $\Rightarrow |f(u) - f(v)| \le \lim\limits_{n\to\infty}\dfrac{2}{3^n}$

$\Rightarrow \quad |f(u) - f(v)| \le 0 \quad \Rightarrow \quad f(u) = f(v)$

Hence, f is constant.

Problem 24: The function u_n takes on the following values : $u_1 = \dfrac{1}{4}$, $u_2 = \dfrac{1}{4} + \dfrac{1}{10}$ .....,

$$u_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \ldots\ldots\ \frac{1}{3^n+1}\ \ldots.$$

Prove that $\lim\limits_{n\to\infty} u_n < \dfrac{1}{2}$.

Solution: We have $\dfrac{1}{4} < \dfrac{1}{3}$; $\dfrac{1}{4} + \dfrac{1}{10} < \dfrac{1}{3} + \dfrac{1}{9}$; .....

Thus, $u_n < \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots\ldots \dfrac{1}{3^n} < \dfrac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$

$$u_n < \frac{1}{3}\left(1-\frac{1}{3}\right)^n \cdot \frac{3}{2} < \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

$$\lim_{n\to\infty} u_n < \lim_{n\to\infty}\frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right) = \frac{1}{2}.$$

Problem 25: Let $L = \lim\limits_{x\to 0}\dfrac{e^{x\sin(2007)x}-1}{x\ln(1+x)}$ and

$$M = \lim_{x\to\infty} x\left(2x-\left(\sqrt[3]{x^3+x^2+1}+\sqrt[3]{x^3-x^2+1}\right)\right)$$

then find the value of LM.

Solution: $L = \lim\limits_{x\to 0}\dfrac{e^{x\sin(2007)x}-1}{x\sin(2007x)}\cdot\dfrac{x\sin(2007x)}{x\ln(1+x)}$

$$= \lim_{x\to 0}\frac{\sin(2007x)}{(2007x)}\cdot\frac{(2007x)}{\ln(1+x)} = 2007.$$

$$M = \lim_{x\to\infty} -x[(x^3+x^2+1)^{1/3}+(x^3-x^2+1)^{1/3}-2x]$$

$$= \lim_{x\to\infty} -x\left[x\left(1+\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}+x\left(1-\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}-2x\right]$$

Put $x = \dfrac{1}{t}$

$$= \lim_{t\to 0}\frac{-[(1+t+t^3)^{1/3}+(1-t+t^3)^{1/3}-2]}{t^2}$$

$$= \lim_{t\to 0} -\frac{1}{t^2}\left[1+\frac{1}{3}(t+t^3)-(t+t^3)^2\right.$$

$$\left.1+\frac{1}{3}(-t+t^3)-\frac{1}{9}(-t+t^3)^2-2\right]$$

$$= \lim_{t\to 0}\frac{\frac{2}{9}t^2+\ldots\ldots}{t^2}$$

$$= \frac{2}{9}$$

$\therefore$ $\quad$ LM = $\frac{2}{9}$ × 2007 = 2 × 223 = 446.

Problem 26: Let $a_n = 2\cos\frac{\theta}{2^n}-1$ then show that $\lim\limits_{n\to\infty}$ $(a_1a_2a_3\ldots.a_{n-1}a_n) = \frac{2\cos\theta+1}{3}$, θ ∈ R.

Solution: Consider $a_n \cdot a_{n-1} \cdot a_{n-2} \ldots\ldots\ldots\ldots a_2 \cdot a_1$

$$= \left(2\cos\frac{\theta}{2^n}-1\right)\left(2\cos\frac{\theta}{2^{n-1}}-1\right)$$

$$\ldots\ldots\ldots\left(2\cos\frac{\theta}{2^2}-1\right)\left(2\cos\frac{\theta}{2}-1\right)$$

$$= \left(2\cos\frac{\theta}{2^n}+1\right)\left(2\cos\frac{\theta}{2^n}-1\right)\left(2\cos\frac{\theta}{2^{n-1}}-1\right)$$

$$\ldots\ldots\left(2\cos\frac{\theta}{2^2}-1\right)\left(2\cos\frac{\theta}{2}-1\right)\frac{1}{\left(2\cos\frac{\theta}{2^n}+1\right)}$$

Now $\left(2\cos\frac{\theta}{2^n}+1\right)\left(2\cos\frac{\theta}{2^n}-1\right)$

$$= 4\cos^2\frac{\theta}{2^n}-1 = 2\left(1+\cos\frac{\theta}{2^{n-1}}\right)-1$$

$$= 2\cos\frac{\theta}{2^{n-1}} + 1$$

Similarly, $\left(2\cos\frac{\theta}{2^{n-1}}+1\right)\left(2\cos\frac{\theta}{2^{n-1}}-1\right)$

$$= \left(4\cos^2\frac{\theta}{2^{n-1}}\right)-1 = 2\left(\cos\frac{\theta}{2^{n-2}}+1\right) - 1$$

$$= 2\cos\frac{\theta}{2^{n-2}}+1$$

Hence $\prod\limits_{i=1}^{n} a_i = \frac{(2\cos\theta+1)}{\left(2\cos\frac{\theta}{2^n}+1\right)}$

$$\therefore \quad \lim_{n\to\infty}\prod_{i=1}^{n} a_i = \frac{2\cos\theta+1}{3}.$$

Problem 27: Let f (x) = $\tan\frac{x}{2}\sec x + \tan\frac{x}{2^2}\sec\frac{x}{2} + \tan\frac{x}{2^3}\sec\frac{x}{2^2} + \ldots.. + \tan\frac{x}{2^n}\sec\frac{x}{2^{n-1}}$

and g (x) = f (x) + $\tan\frac{x}{2^n}$ where $x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and n ∈ N. Evaluate the following limits:

(a) $\lim\limits_{x\to 0}\left(\frac{g(x)}{x}\right)^{\frac{1}{x}}$ $\qquad$ (b) $\lim\limits_{x\to 0}\left(\frac{g(x)}{x}\right)^{\frac{1}{x^2}}$

(c) $\lim\limits_{x\to 0}\left(\frac{g(x)}{x}\right)^{\frac{1}{x^3}}$

Solution: $T_1 = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}\cos x} = \frac{\sin\left(x-\frac{x}{2}\right)}{\cos\frac{x}{2}\cos x}$

$$= \tan x - \tan\frac{x}{2}$$

Similarly, $T_2 = \tan\frac{x}{2} - \tan\frac{x}{2^2}$

$T_3 = \tan\frac{x}{2^2} - \tan\frac{x}{2^3}$

$\vdots \qquad \vdots \qquad \vdots$

$T_n = \tan\frac{x}{2^{n-1}} - \frac{x}{2^n}$

$$f(x) = \tan x - \tan\frac{x}{2^n}$$

$\Rightarrow$ f (x) + $\tan\frac{x}{2^n}$ = g (x) = tan x

Now (a) L = $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$$= e^{\frac{1}{x}\left(\frac{\tan x}{x}-x\right)} = e^{\operatorname*{Limit}\limits_{x\to 0}\frac{\tan x-x}{x^2}}$$

Consider $\quad l = \lim\limits_{x\to 0}\frac{\tan x - x}{x^2}$

$$= \lim_{x\to 0}\frac{\tan x - x}{x^3}\lim_{x\to 0} x = 0 \Rightarrow L = e^0 = 1.$$

(b) $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{\lim\limits_{x\to 0}\left(\frac{\tan x}{x}-1\right)\frac{1}{x^2}}$

$$= e^{\operatorname*{limit}\limits_{x\to 0}\frac{\tan x-x}{x^3}} = e^{1/3}.$$

(c) $\lim\limits_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^3}} = e^l$

where $l = \underset{x\to0}{\text{limit}}\frac{1}{x^3}\left(\frac{\tan x}{x}-1\right)$

$= \lim\limits_{x\to0}\frac{\tan x - x}{x^4} = \lim\limits_{x\to0}\frac{\tan x - x}{x^3}.\frac{1}{x} =$ dne.

$\Rightarrow$ limit does not exist.

Problem 28: Let $f(x) = \lim\limits_{n\to\infty}\sum\limits_{n=1}^{n}3^{n-1}\sin^3\frac{x}{3^n}$ and $g(x) = x - 4f(x)$. Evaluate $\lim\limits_{x\to0}(1+g(x))^{\cot x}$.

Solution: Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$T_1 = \sin^3\frac{x}{3} = \frac{1}{4}\left(3\sin\frac{x}{3}-\sin x\right)$$

$$T_2 = 3\sin^3\frac{x}{3^2} = \frac{3}{4}\left(3\sin\frac{x}{3^2}-\sin\frac{x}{3}\right)$$

and so on........

$$\therefore\quad T_1 = \frac{1}{4}\left(3\sin\frac{x}{3}-\sin x\right)$$

$$T_2 = \frac{1}{4}\left(3^2\sin\frac{x}{3^2}-3\sin\frac{x}{3}\right)$$

$$T_n = \frac{1}{4}\left(3^n\sin\frac{x}{3^n}-3^{n-1}\sin\frac{x}{3^{n-1}}\right)$$

$$\therefore\quad f(x) = \lim_{n\to\infty}\frac{1}{4}\left(3^n\sin\frac{x}{3^n}-\sin x\right)$$

$$= \frac{1}{4}\left(\lim_{n\to\infty}\frac{x\sin\frac{x}{3^n}}{\frac{x}{3^n}}-\sin x\right) = \frac{1}{4}(x-\sin x)$$

$g(x) = x - 4f(x) = \sin x$

Now $\lim\limits_{x\to0}(1+\sin x)^{\cot x} = e^{\lim_{x\to0}(\cot x)(\sin x)} = e$

Problem 29: If $a_1 = 1$ and $a_{n+1} = \frac{4+3a_n}{3+2a_n}$, $n \ge 1$, then show that $a_{n+2} > a_{n+1}$ and if a_n has a limit l as $n \to \infty$, then evaluate $\lim\limits_{n\to\infty}a_n$.

Solution: $a_1 = 1$

and $a_2 = \frac{4+3}{3+2} = \frac{7}{5} > 1$

$\therefore\quad a_2 > a_1$

Assuming $a_{n+1} > a_n$

$$\therefore\quad a_{n+2} - a_{n+1} = \frac{4+3a_{n+1}}{3+2a_{n+1}} - \frac{4+3a_n}{3+2a_n}$$

$$= \frac{(4+3a_{n+1})(3+2a_n)-(4+3a_n)(3+2a_{n+1})}{(3+2a_{n+1})(3+2a_n)}$$

$$= \frac{a_{n+1}-a_n}{(3+2a_{n+1})(3+2a_n)} > 0 \qquad \{\because a_{n+1} > a_n\}$$

$\therefore\quad a_{n+2} - a_{n+1} > 0$

$a_{n+2} > a_{n+1}$ whenever $a_{n+1} > a_n$

$\therefore$ The sequence of values a_n is increasing and since $a_1 = 1$, $a_n > 0$ for all n,

Now let, $l = \lim\limits_{n\to\infty}a_n = \lim\limits_{n\to\infty}a_{n+1}$

$$\lim_{n\to\infty}a_{n+1} = \lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow\quad l = \lim_{n\to\infty}\frac{4+3l}{3+2l}$$

$\Rightarrow\quad 3l + 2l^2 = 4 + 3l$

$l^2 = 2$

$\therefore\quad l = \sqrt{2}$.

Problem 30: Find the values of a, b and c so that

$$\lim_{x\to0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$$

Solution: We have $\lim\limits_{x\to0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ ...(1)

As $x \to 0$, numerator $\to a - b + c$

and denominator $\to 0$

Since limit is finite $a - b + c = 0$

or $\quad b = a + c$ ...(2)

From (1) and (2)

$$\lim_{x\to0}\frac{ae^x - (a+c)\cos x + ce^{-x}}{x\sin x} = 2$$

$$\Rightarrow \lim_{x\to0}\frac{a(e^x-1)+c(e^{-x}-1)+(a+c)(1-\cos x)}{x\sin x} = 2 \quad ...(3)$$

$$\Rightarrow \lim_{x\to0}\frac{\frac{a(e^x-1)}{x}+\frac{c(e^{-x}-1)}{-(-x)}}{\sin x} + \lim_{x\to0}\frac{(a+c)\sin^2 x}{(1+\cos x)x\sin x} = 2$$

$$\Rightarrow \lim_{x\to0}\frac{\frac{a(e^x-1)}{x}-\frac{c(e^{-x}-1)}{-x}}{\sin x} + \lim_{x\to0}\frac{(a+c)\sin^2 x}{(1+\cos x)x} = 2 \quad ...(4)$$

Since limit is finite, as $x \to 0$,

$\frac{a(e^x-1)}{x} - \frac{c(e^{-x}-1)}{(-x)} = a - c$ must be 0

$\therefore\quad a - c = 0 \quad \Rightarrow a = c$ ...(5)

$$\lim_{x\to0}\frac{a(e^x+e^{-x}-2)}{x\sin x} + \lim_{x\to0}\frac{2a\sin x}{(1+\cos x)x} = 2$$

$\Rightarrow \quad \lim\limits_{x\to 0} \dfrac{a(e^x-1)^2}{e^x x \sin x} + \lim\limits_{x\to 0} \dfrac{2a\sin x}{(1+\cos x)x} = 2$

$\Rightarrow \quad \lim\limits_{x\to 0} \dfrac{a\left(\dfrac{e^x-1}{x}\right)^2}{e^x.\left(\dfrac{\sin x}{x}\right)} + \lim\limits_{x\to 0} \dfrac{2a\sin x}{(1+\cos x)x} = 2$

$\Rightarrow \quad \dfrac{a(1)^2}{1.1} + \dfrac{2a.1}{2} = 2$

$\therefore \quad a = 1$

From (5), c = 1

And from (2), $\quad b = 1 + 1 = 2$

Alternative :

Here we use L' Hospital's rule

We have $\lim\limits_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ ...(1)

As $x \to 0$, numerator $\to a - b + c$ and denominator $\to 0$ $\quad\therefore\quad a - b + c = 0$

$\Rightarrow \quad b = a + c$ ...(2)

From (1) and (2)

$$\lim_{x\to 0} \frac{ae^x - (a+c)\cos x + ce^{-x}}{x\sin x} = 2 \quad \left(\text{form } \frac{0}{0}\right)$$

Using L' Hospital's rule

$$\lim_{x\to 0} \frac{ae^x + (a+c)\sin x - ce^{-x}}{(x\cos x + \sin x)} = 2 \qquad ...(3)$$

As $x \to 0$, numerator $\to a - b + c$

and denominator $\to 0$

Since limit is finite $a - c = 0$

$\Rightarrow \quad a = c$ ...(4)

From (3) and (4)

$$\lim_{x\to 0} \frac{ae^x + 2a\sin x - ae^{-x}}{x\cos x + \sin x} = 2 \quad \left(\text{form } \frac{0}{0}\right)$$

Again using L' Hospital's rule

$$\lim_{x\to 0} \frac{ae^x + 2a\cos x + ae^{-x}}{-x\sin x + \cos x + \cos x} = 2$$

$\Rightarrow \quad \dfrac{a + 2a + a}{0+1+1} = 2$

$\therefore \quad a = 1$

From (4), $\quad c = 1$ and from (2), $b = 1 + 1 = 2$.

Problem 31: Let $a_1, a_2, \ldots, a_n$ be sequence of real numbers with $a_{n+1} = a_n + \sqrt{1+a_n^2}$ and $a_0 = 0$. Prove that $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{2^{n-1}}\right) = \dfrac{4}{\pi}$.

Solution: Here, $a_{n+1} = a_n + \sqrt{1+a_n^2}$,

let $a_n = \cot(\alpha_n)$ $\Rightarrow a_{n+1} = \cot(\alpha_n) + \operatorname{cosec}(\alpha_n) \Rightarrow a_{n+1}$

$$\frac{\cos(\alpha_n)+1}{\sin(\alpha_n)} = \frac{2\cos^2(\alpha_n/2)}{2\sin(\alpha_n/2)\cos(\alpha_n/2)} = \cot\left(\frac{\alpha_n}{2}\right) = 1$$

Putting n = 1, $a_1 = \cot(\alpha_1)$ and $a_1 = a_0 +$

$= \sqrt{1+a_0^2} = 1$

$\Rightarrow \quad \cot(\alpha_1) = 1$ or $\alpha_1 = \dfrac{\pi}{4}$

Again, $a_2 = \cot\left(\dfrac{\alpha_1}{2}\right) = \cot\left(\dfrac{\pi}{8}\right)$

$a_3 = \cot\left(\dfrac{\alpha_2}{2}\right) = \cot\left(\dfrac{\pi}{4.2^2}\right)$

$a_4 = \cot\left(\dfrac{\alpha_3}{2}\right) = \cot\left(\dfrac{\pi}{4.2^3}\right)$ ....

$a_n \cot\left(\dfrac{\pi}{4.2^{n-1}}\right)$.

Put $\dfrac{1}{2^{n-1}} = x$

Hence, $\quad \lim\limits_{n\to\infty}\left(\dfrac{a_n}{2^{n-1}}\right) = \lim\limits_{n\to\infty} \dfrac{\cot\left(\dfrac{\pi}{4.2^{n-1}}\right)}{2^{n-1}}$

$= \lim\limits_{x\to 0} \dfrac{1}{\dfrac{\tan\left(\dfrac{\pi x}{4}\right)}{x}}$

$\therefore \quad \lim\limits_{n\to\infty}\left(\dfrac{a_n}{2^{n-1}}\right) = \dfrac{4}{\pi}$.

Things to Remember

1. Theorems on Limits

Let f and g be two functions such that $\lim\limits_{x\to a} f(x) = \ell$, $\lim\limits_{x\to a} g(x) = m$. Then

(i) $\lim\limits_{x\to a} k(f(x)) = k.\ell$,

(ii) $\lim\limits_{x\to a} [f(x) + g(x)] = \ell + m$

(iii) $\lim\limits_{x\to a} [f(x) - g(x)] = \ell - m$

(iv) $\lim\limits_{x\to a} [f(x) . g(x)] = \ell m$

(v) $\lim\limits_{x\to a} [f(x) / g(x)] = \ell / m, (m \neq 0)$

(vi) For any positive integer n,

$\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, provided $\lim\limits_{x\to c} f(x) > 0$ when n is even.

Also $\lim\limits_{x\to a}\left(f(x)\right)^{m/n} = \left(\lim\limits_{x\to a} f(x)\right)^{m/n}$ where m/n is a rational number and the limit on the right exists.

(vii) If $\lim\limits_{x\to a} f(x) = b$ and g(x) is continuous at x = b, then $\lim\limits_{x\to a} g(f(x)) = g\left(\lim\limits_{x\to a} f(x)\right) = g(b)$.

(viii) If $\lim\limits_{x\to a} f(x) = b$, with $f(x)^{1}$ b for every x in some neighbourhood of 'a' and if $\lim\limits_{x\to b} g(x) = c$, then $\lim\limits_{x\to a} g\left(f(x)\right) = c$. (This theorem is sometimes useful if g is discontinuous at b.

(ix) Domination Law : If two function f(x) and g(x) satisfy the inequality f(x) < g(x) for all the values of x belonging to a neighbourhood of a point a except possibly a then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$ provided that the limits of both functions, as x ⟶ a, exist.

2. For all elementary functions, limit at any point in the domain is equal to the function's value at that point.

3. The important indeterminate forms are :

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^o, ∞^o and 1^∞

4. Following are some of the frequently used series expansions :

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

(iii) ln (1+x) = $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(vi) tan x = $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots..$

(vii) $\tan^{-1}x$ = $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots.$

(viii) $\sin^{-1}x$ = $x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots.$

(ix) sec x = $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

(x) $(1 + x)^n = 1 + \frac{nx}{\lfloor 1} + n(n-1)\frac{x^2}{\lfloor 2} + \ldots.$ for $-1 < x < 1$

(xi) $(1 + x)^{1/x} = e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots..\right)$

5.

(i) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1 = \lim\limits_{x\to 0}\frac{\tan x}{x}$

$= \lim\limits_{x\to 0}\frac{\tan^{-1}x}{x} = \lim\limits_{x\to 0}\frac{\sin^{-1}x}{x}$

(ii) $\lim\limits_{x\to 0}\frac{a^x - 1}{x} = \ln a \quad (a > 0).$

(iii) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1$

$\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e, \; (a > 0, a \ne 1)$

(iv) $\lim\limits_{x\to 0}\frac{(1+x)^n - 1}{x} = n$

$\lim\limits_{x\to a}\frac{x^n - a^n}{x - a} = n a^{n-1}.$

(v) $\lim\limits_{x\to 0^+} x\ell n x = 0$

(vi) $\lim\limits_{x\to 0}\;(1+x)^{\frac{1}{x}} = \lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$

6. $\lim\limits_{x\to\infty}\frac{a_0x^m + a_1x^{m-1} + \;\ldots.}{b_0x^n \;+\; b_1x^{n-1} + \;\ldots}$

$$= \begin{cases} \frac{a_0}{b_0}, & \text{when } m = n \\ 0, & \text{when } m < n \\ \infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} > 0 \\ -\infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} < 0 \end{cases}$$

7. (i) $\lim\limits_{x\to\infty}\frac{\sin x}{x} = \lim\limits_{x\to\infty}\frac{\cos x}{x} = 0$

(ii) In fact, if n is a positive integer then

$\lim\limits_{x\to\infty}\frac{e^x}{x^n} = \infty$, $\quad \lim\limits_{x\to\infty} x^n e^{-x} = 0$.

(iii) For sufficiently large values of x we have

$\log_a x \;<<\; x^p \;<<\; a^x \;<<\; x!$ where

a > 1 p > 0 a > 1 x ∈ N.

(iv) $\lim\limits_{x\to\infty}\frac{\log_a x}{x} = 0$, for a > 1.

(v) $\lim\limits_{n\to\infty}\frac{n^a}{a^n} = 0$ for a > 0,

(vi) $\lim\limits_{n\to\infty}\frac{a^n}{n!} = 0$

8. The line x = a is called a vertical asymptote of the curve y = f(x) if atleast one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \quad \lim_{x\to a^{\square}} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \quad \lim_{x\to a^{\square}} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty$$

9. The line y = L is called a horizontal asymptote of the curve y = f(x) if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

10. $$\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{dne} & a < 0 \end{cases}$$

11. If $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} f(x) = B$, a finite quantity

then $\lim\limits_{x\to a} (f(x))^{g(x)} = A^B$.

Also, $\lim\limits_{x\to a} (f(x))^{g(x)} = e^{\lim\limits_{x\to a} g(x)\, \ell n(f(x))}$

12. Let $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$ then

$$\lim_{x\to a} (f(x))^{g(x)} = e^{\lim\limits_{x\to a} g(x)[f(x)-1]}$$

13. Sandwich Theorem : Suppose that g(x) ≤ f(x) ≤ h(x) for all x in some open interval containing c, except possibly at x = c itself.

If $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(c) = l$, then $\lim\limits_{x\to c} f(x) = l$.

14. L'Hospital's rule: Suppose f and g are differentiable on an open interval containing a, except possibly at a itself. Assume that $g'(x) \neq 0$ for all x in the open interval containing a, except possibly at a.

Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$.

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$).

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, provided the limit on the right side exists or is ∞ or – ∞.

15. Leibnitz's formula for differentiation of integrals

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)dt\right) = f(v(x))\frac{dv}{du} - f(u(x))\frac{dv}{dx}.$$

16. $$\lim_{n\to\infty} x^{2n} \begin{cases} 0, & \text{if} \quad -1 < x < 1 \\ 1, & \text{if} \quad x = \pm 1 \\ \infty, & \text{if} \quad x < -1 \text{ or, } x > 1 \end{cases}$$

17. $$\lim_{n\to\infty} \frac{1}{n} \sum_{r=\varphi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$$

where, $a = \lim\limits_{n\to\infty} \dfrac{\varphi(x)}{n}$ and $b = \lim\limits_{n\to\infty} \dfrac{\psi(x)}{n}$.

18. Let $A = \begin{pmatrix} f(x) & g(x) \\ u(x) & v(x) \end{pmatrix}$ then we have

$$\lim_{x\to a} A = \begin{pmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) \end{pmatrix}$$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. The value of $\lim\limits_{x\to 0}\left[\dfrac{|\sin x|}{|x|}\right]$, (where [.] denotes greatest integer function) is

(A) 0 (B) does not exists

(C) –1 (D) 1

2. The value of $\lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x \tan x}\right]$, (where [.] denotes the greatest integer function)

(A) 0 (B) 1

(C) does not exist (D) none of these

3. If $\lim\limits_{x\to\infty}(\sqrt{x^4 - x^2 + 1} - ax^2 - a)$ exists then a is equal to

(A) –1

(B) 1

(C) a cannot be determined

(D) none of these

4. $\lim\limits_{h\to 0} \dfrac{e^{\sin h} - (1 + \sin h)}{(\tan^{-1}(\sin h))^2}$ is

(A) 1 (B) 2

(C) $\frac{1}{2}$ (D) none of these

5. $\lim\limits_{x\to\infty} (1 + a^2)^x \cdot \dfrac{b}{(1 + a^2)^x}$ is (a, b ∈ R)

(A) $\sqrt{b}$ (B) b

(C) b^2 (D) none of these

6. $\lim\limits_{x\to\infty} x^2\left[\tan^{-1}\frac{2x^2+1}{x^2+2}-\tan^{-1}2\right]$ is

(A) $\frac{3}{5}$ (B) $-\frac{3}{5}$

(C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

7. If f(x) = 0 be a quadratic equation such that $f(-\pi)= f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right)=-\frac{3\pi^2}{4}$, then $\lim\limits_{x\to-\pi}\frac{f(x)}{\sin(\sin x)}$ is equal to

(A) 0 (B) π

(C) 2π (D) None of these

8. $\lim\limits_{n\to\infty}\sum\limits_{x=1}^{20}\cos^{2n}(x-10)$ is equal to

(A) 0 (B) 1

(C) 19 (D) 20

9. $\lim\limits_{x\to1}\frac{1+\sin\pi\left(\frac{3x}{1+x^2}\right)}{1+\cos\pi x}$ is equal to

(A) 0 (B) 1

(C) 2 (D) None of these

10. If f : (1, 2) → R satisfies the inequality

$\frac{\cos(2x-4)-33}{2}<f(x)<\frac{x^2|4x-8|}{x-2}$, $\forall\ x\in(1,2)$.

Then $\lim\limits_{x\to2^-}f(x)$ is

(A) 16

(B) cannot be determined from the given information

(C) –16

(D) does not exist

11. $\lim\limits_{x\to0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where {x} denotes the fractional part of x) is equal to

(A) e^2-7 (B) e^2-8

(C) e^2-6 (D) None of these

12. The value of

$\lim\limits_{x\to2}\left(\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{x+\sqrt{2x}}{x-2}-\frac{\sqrt2}{\sqrt x-\sqrt2}\right)^{-1}\right)$ is

(A) 1/2 (B) 2

(C) 1 (D) None of these

13. $\lim\limits_{x\to a^-}\sqrt{a^2-x^2}\ \cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$ is equal to

(A) $\frac{a}{\pi}$ (B) $\frac{2a}{\pi}$

(C) $-\frac{a}{\pi}$ (D) $\frac{4a}{\pi}$

14. $\lim\limits_{x\to0}\ \lim\limits_{x\to0}\frac{2(\tan x-\sin x)-x^3}{x^5}$ is equal to

(A) 1/4 (B) 1/2

(C) 1/3 (D) None of these

15. $\lim\limits_{x\to0}\frac{1-\cos x+2\sin x-\sin^3x-x^2+3x^4}{\tan^3x-6\sin^2x+x-5x^3}$ equals

(A) 1 (B) 2

(C) 3 (D) 4

16. $\lim\limits_{x\to0}\frac{\log\cos x}{\sqrt[p]{\cos x}-\sqrt[q]{\cos x}}$ is equal to

(A) $\frac{pq}{q+p}$ (B) $\frac{q-p}{pq}$

(C) $\frac{pq}{q-p}$ (D) $\frac{pq}{p-q}$

17. The limit

$\lim\limits_{n\to\infty}\left(1+\frac{1}{5}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{5^4}\right)....\left(1+\frac{1}{5^{2^n}}\right)$

is equal to

(A) 0 (B) 5/4

(C) 4/5 (D) 1/5

18. The value of $\lim\limits_{x\to0}\left(\frac{e^{x\ell n(2^x-1)}-(2^x-1)^x\sin x}{e^{x\ell nx}}\right)^{1/x}$ is equal to

(A) e (B) $\frac{1}{e}\ell n2$

(C) $e\ell n2$ (D) None of these

19. $\lim\limits_{n\to\infty}\left(\frac{n!}{(mn)^n}\right)^{1/n}$ $(m\in N)$ is equal to

(A) 1/em (B) m/e

(C) em (D) e/m

20. $\lim\limits_{y\to0}\left[\lim\limits_{x\to\infty}\frac{\exp\left(x\ell n\left(1+\frac{ay}{x}\right)\right)-\exp\left(x\ell n\left(1+\frac{by}{x}\right)\right)}{y}\right]$

(A) a + b (B) a – b

(C) b – a (D) – (a + b)

21. $\lim\limits_{x\to0}\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$ is equal to

(A) 1 (B) –1

(C) 2 (D) None

22. $\lim\limits_{x\to1}\frac{(\ell n(1+x)-\ell n2)(3.4^{x-1}-3x)}{[(7+x)^{1/3}-(1+3x)^{1/2}].\sin(x-1)}$ equals

(A) $\frac{9}{4}\ \ell n\ \frac{4}{e}$ (B) $\frac{9}{4}\ \ell n\ \frac{e}{4}$

(C) $\frac{4}{9}\ \ell n\ \frac{e}{4}$ (D) None of these

23. $\lim\limits_{x\to-\infty} \dfrac{x^5 \tan\left(\dfrac{1}{\pi x^2}\right) + 3|x|^2 + 7}{|x|^3 + 7|x| + 8}$ is equal to

(A) π (B) $\dfrac{1}{\pi}$

(C) $-\dfrac{1}{\pi}$ (D) None of these

24. $\lim\limits_{x\to 0} \dfrac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(.........\cos^2(x)\right)\right)\right)}{\sin\left[\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)\right]}$

(A) 2 (B) $\sqrt{2}$

(C) $-\sqrt{2}$ (D) None of these

25. If $\lim\limits_{x\to\infty} \left(\sqrt{a^2x^2+ax+1} - \sqrt{a^2x^2+1}\right)$

$= K.\lim\limits_{x\to\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right)$ then the value of K is

(A) 1 (B) a

(C) 2a (D) None of these

26. The value of $\lim\limits_{x\to\infty} (x^{-4} (\ell n\ x)^{10})$ is

(A) 1 (B) 0

(C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$

27. $\lim\limits_{x\to 1} \dfrac{\tan(x-1).\log_e x^{x-1}}{|x-1|^3}$ is equal to

(A) 1 (B) – 1

(C) 3 (D) None of these

28. $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\sqrt{\dfrac{1}{x^2}+1} - \dfrac{1}{x}\right) + x\ \ell n\left(1+a^{1/x}\right)$, $a > 0$, $a \neq 1$

(A) a (B) 1

(C) 1 + a (D) None of these

29. $\lim\limits_{n\to\infty} \left(\dfrac{n}{n^2-2} + \dfrac{4^n(-1)^n}{2^n-1}\right)^{-1}$ is equal to

(A) 2 (B) 1

(C) 0 (D) None of these

30. If $f(x) = \begin{cases} x & ; \quad x \le 0 \\ -x & ; \quad x > 0 \end{cases}$ and $g(x) = f(x) + |x|$. Then

$\lim\limits_{x\to 0^+} (\log_{|\sin x|} x)^{g(x)}$ is

(A) 0 (B) 1

(C) 1/2 (D) not exist

31. The value of the limit $\lim\limits_{n\to\infty} n^2 \left(\sqrt[n]{a} - \sqrt[n+1]{a}\right)$ $(a > 0)$ is

(A) ℓn a (B) e^a

(C) e^{-a} (D) none of these

32. The value of the limit $\lim\limits_{x\to 0} \left(\sin\dfrac{x}{m} + \cos\dfrac{3x}{m}\right)^{2m/x}$ is

(A) 1 (B) 2

(C) e^{6m} (D) ln 6m

33. $\lim\limits_{x\to-\infty} \left\{x + \sqrt{x^2 + 3x\cos\dfrac{1}{|x|}}\right\}$ is equal to

(A) 3/2 (B) –3/2

(C) –1 (D) none of these

34. $\lim\limits_{x\to-\infty} x + \sqrt{x^2 + x^2\sin(1/x)}$ is equal to

(A) 0 (B) 2

(C) –2 (D) none of these

35. If $\lim\limits_{x\to 0} \dfrac{x^{2n}\sin^n x}{x^{2n} - \sin^{2n} x}$ is a non zero finite number, then n must be equal to

(A) 1 (B) 2

(C) 3 (D) none of these

36. If $b < 0$, $b \neq -1$ and a is a positive constant then

$\lim\limits_{x\to-\infty} \dfrac{a+x}{|x| - \sqrt{b^2x^2 + x}}$ equals

(A) $\dfrac{1}{|b|-1}$ (B) $\dfrac{1}{-b-1}$

(C) $\dfrac{1}{b-1}$ (D) $\dfrac{1}{1-|b|}$

37. $\lim\limits_{n\to\infty} \left(\sum\limits_{r=1}^{m} r^n\right)^{1/n}$ is equal to, $(n \in N)$

(A) m (B) m/2

(C) e^m (D) $e^{m/2}$

38. If k is an integer such that

$\lim\limits_{n\to\infty} \left(\left(\cos\dfrac{k\pi}{4}\right)^n - \left(\cos\dfrac{k\pi}{6}\right)^n\right) = 0$, then

(A) k is divisible neither by 4 nor by 6

(B) k must be divisible by 12, but not necessarily by 24

(C) k must be divisible by 24

(D) either k is divisible by 24 or k is divisible neither by 4 nor by 6

39. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x}{\cos^{-1}\left[\dfrac{1}{4}(3\sin x - \sin 3x)\right]}$ where [.] denotes greatest integer function, is equal to

(A) $\dfrac{2}{\pi}$ (B) 1

(C) $\dfrac{4}{\pi}$ (D) does not exist

40. The value of $\lim\limits_{x\to\infty} \dfrac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x \log_x a)}$ $(a > 1)$ is equal to

(A) 1 (B) 0

(C) $\pi/2$ (D) does not exist

41. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{p}+\sqrt[n]{q}}{2}\right)^n$, p, q > 0 is equal to

(A) 1 (B) $\sqrt{pq}$

(C) pq (D) $\dfrac{pq}{2}$

42. $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}(\sqrt{x-1}-\sqrt{x})}{\sec^{-1}\left(\left(\dfrac{2x+1}{x-1}\right)^x\right)}$ is equal to

(A) 1 (B) 0

(C) $\dfrac{\pi}{2}$ (D) non-existent

43. The value of $\lim\limits_{x\to0}\dfrac{(\tan(\{x\}-1))\sin\{x\}}{\{x\}(\{x\}-1)}$, where { x } denotes the fractional part function, is

(A) is 1 (B) is tan 1

(C) is sin 1 (D) is non-existent

44. $\lim\limits_{x\to0^-}(\ell n(\{x\}+|[x]|))^{\{x\}}$ is equal to

(A) 0 (B) 1

(C) *ln* 2 (D) *ln* $\dfrac{1}{2}$

where [ ] is the greatest integer function and { } is the fractional part function.

45. The value of $\left(\lim\limits_{x\to0}\left[\dfrac{100x}{\sin x}\right]+\left[\dfrac{99\sin x}{x}\right]\right)$ is (where [.] denotes greatest integer function)

(A) 199 (B) 198

(C) 197 (D) None of these

46. The value of $\lim\limits_{n\to\infty}\prod\limits_{n=0}^{n}\left(1+\dfrac{1}{2^{2^n}}\right)$ is

(A) 1 (B) 2

(C) 4 (D) none

47. The value of

$$\lim_{n\to\infty}\frac{1^6+2^6+3^6.......n^6}{(1^2+2^2+3^2.......n^2)\left(1^3+2^3+3^3+.......n^3\right)}$$

(A) $\dfrac{14}{7}$ (B) $\dfrac{21}{8}$

(C) $\dfrac{132}{17}$ (D) $\dfrac{12}{7}$

48. Let a, b, c are non zero constant number then $\lim\limits_{r\to\infty}\dfrac{\cos\frac{a}{r}-\cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}}$ equals

(A) $\dfrac{a^2+b^2-c^2}{2bc}$ (B) $\dfrac{c^2+a^2-b^2}{2bc}$

(C) $\dfrac{b^2+c^2-a^2}{2bc}$ (D) independent of a,b,c

49. Let $P(x) = a_1x + a_2x^2 + a_3x^3 + ... + a_{100}x^{100}$, where $a_1 = 1$ and $a_i \in R\ \forall\ i = 2, 3, 4, ..., 100$.

then $\lim\limits_{x\to0}\dfrac{\sqrt[100]{1+P(x)}-1}{x}$ has the value equal to

(A) 100 (B) $\dfrac{1}{100}$

(C) 1 (D) 5050

50. Assume that $\lim\limits_{\theta\to1}f(\theta)$ exists and

$$\frac{\theta^2+\theta-2}{\theta+3}\le\frac{f(\theta)}{\theta^2}\le\frac{\theta^2+2\theta-1}{\theta+3}$$

holds for certain interval containing the point θ = –1 then $\lim\limits_{\theta\to1}f(\theta)$

(A) is equal to f(–1) (B) is equal to 1

(C) is non-existent (D) is equal to –1

51. $\lim\limits_{x\to0}\dfrac{6x^2(\cot x)(\operatorname{cosec}2x)}{\sec\left(\cos x+\pi\tan\left(\dfrac{\pi}{4\sec x}\right)-1\right)}$ has the value equal to

(A) 6 (B) –6

(C) 0 (D) –3

52. Suppose that a and b are real positive numbers then the value of $\lim\limits_{t\to0}\left(\dfrac{b^{t+1}-a^{t+1}}{b-a}\right)^{1/t}$ has the value equals to

(A) $\dfrac{a\ \ln b-b\ \ln a}{b-a}$ (B) $\dfrac{b\ \ln b-a\ \ln a}{b-a}$

(C) b ln b - a ln a (D) $\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}$

53. If $f(x)=\cot^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $g(x)=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, then $\lim\limits_{x\to a^+}\dfrac{f(x)-f(a)}{g(x)-g(a)}$, where $\left(0<a<\dfrac{1}{2}\right)$ is

(A) 1 (B) –1

(C) 2 (D) $\dfrac{1}{2}$

54. $\lim\limits_{x\to0}\dfrac{\ell n(1+x+x^2)+\ell n(1-x+x^2)}{\sec x-\cos x}$ is equal to

(A) 1 (B) – 1

(C) 0 (D) ∞

55. If $f(x)=\lim\limits_{n\to\infty}\dfrac{2x}{\pi}\tan^{-1}nx$, then value of $\lim\limits_{x\to0}[f(x)-1]$ is, where [.] represents greatest integer function

(A) 0 (B) –1

(C) 1 (D) does not exist

56. The value of $\lim\limits_{x\to1}\left(\dfrac{x^3+2x^2+x+1}{x^2+2x+3}\right)^{\frac{1-\cos(x-1)}{(x-1)^2}}$ is

(A) e (B) $e^{1/2}$

(C) 1 (D) none of these

57. If $f(x) = \frac{1}{1-x}$, $\lim\limits_{x\to 0} \frac{e^{\tan x} - e^{f\{f[f(x)]\}}}{\tan x - f\{f[f(x)]\}}$ is equal to

(A) 0 (B) 1

(C) – 1 (D) none of these

58. $\lim\limits_{n\to\infty} \left\{\frac{7}{10} + \frac{29}{10^2} + \frac{133}{10^3} + ... + \frac{5^n + 2^n}{10^n}\right\}$ equals

(A) 3/4 (B) 2

(C) 5/4 (D) 1/2

59. $\lim\limits_{x\to 0} \frac{\int_{\sqrt{x}}^{x^2} \tan^{-1}\left(\frac{t^2}{1+t^2}\right) dt}{\sin 2x}$ is equal to

(A) 0 (B) 1

(C) 1/2 (D) does not exist

60. $\sum\limits_{r=1}^{\infty} \frac{r^3 + (r^2+1)^2}{(r^4 + r^2 + 1)(r^2 + r)}$ is equal to

(A) 3/2 (B) 1

(C) 2 (D) infinite

61. If $\lim\limits_{x\to\infty} \frac{729^x - 243^x - 81^x + 9^x + 3^x - 1}{x^3} = k(\ln 3)^3$, then k is equal to

(A) 4 (B) 5

(C) 6 (D) none

62. $\lim\limits_{x\to\pi/2} \frac{\sin x - (\sin x)^{\sin x}}{1 - \sin x \ln \sin x}$ is equal to

(A) 1 (B) zero

(C) 2 (D) 2/3

63. $\lim\limits_{x\to\infty} \sqrt[x]{2\sum\limits_{n=0}^{x} \frac{x^n}{n!}}$ is equal to

(A) 1 (B) e

(C) $2e^{-1}$ (D) 0

64. If $\{t_n\}$ be a sequence such that $t_n = \frac{2n}{3n+1}$, S_n denote the sum of the first n terms and

$$\ell = \lim_{n\to\infty} \frac{n}{\sqrt{2}} \frac{S_{n+1} - S_n}{\sqrt{\sum\limits_{k=1}^{n} k}}, \text{ then}$$

$$\ell = \lim_{n\to\infty} \ell + 2\ell^2 + 3\ell^2 + ... + (n+1)\ell^{n+1} \text{ equals}$$

(A) 18 (B) 9

(C) 3 (D) 6

65. $\lim\limits_{n\to\infty} \frac{1.n + (n-1)(1+2) + (n-2)(1+2+3) + ..1.\sum\limits_{r=1}^{n} r}{n^4}$

is equal to

(A) 1/12 (B) 1/24

(C) 1/6 (D) 1/48

66. If $\lim\limits_{x\to 0} \frac{\int_0^{x^2} \sin x^2 dx}{x^n}$ is a non zero definite number, then value of n is

(A) 1 (B) 3

(C) 5 (D) 4

67. If a, b and c are real numbers then the value of

$$\lim_{t\to 0} \ln\left(\frac{1}{t}\int_0^t (1 + a\sin bx)^{c/x} dx\right) \text{ equals}$$

(A) abc (B) $\frac{ab}{c}$

(C) $\frac{bc}{a}$ (D) $\frac{ca}{b}$

68. If $C_0, C_1, C_2, ...C_n$ are binomial coefficients then $\lim\limits_{n\to\infty}$ $\left[C_n - \left(\frac{2}{3}\right)c_{n-1} + \left(\frac{2}{3}\right)^2 C_{n-2} + ...(-1)^n\left(\frac{2}{3}\right)^n C_0\right]$ is

(A) 0 (B) 1

(C) –1 (D) 2

69. If $f(x) = \begin{cases} \frac{\sin([x] + 2x)}{[x]} & \text{if } [x] \neq 0 \\ 0 & \text{if } [x] = 0 \end{cases}$, where [.] denotes the greatest integer function, then $\lim\limits_{x\to 0} f(x)$ is

(A) 0 (B) 1

(C) – 1 (D) none of these

70. Let $f(x) = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{3}{\pi}\tan^{-1} 2x\right)^{2n} + 5}$. Then the set of values of x for which f(x) = 0, is

(A) $|2x| > \sqrt{3}$ (B) $|(2x)| < \sqrt{3}$

(C) $|2x| \geq \sqrt{3}$ (D) $|2x| \leq \sqrt{3}$

71. The value of $\lim\limits_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{2n^4+1}{5n^5+1}}$ is equal to

(A) 1 (B) 0

(C) $\left(\frac{1}{e}\right)^{2/5}$ (D) $e^{2/5}$

72. The value of $\lim\limits_{n\to\infty} \frac{1}{n^3}$ $([1^2x + 1^2] + [2^2x + 2^2] + [2^2x + 2^2]$ $+ .... + [n^2x + n^2])$, is (where [.] denotes the greatest integer function)

(A) $\frac{x}{3}$ (B) $x + \frac{1}{3}$

(C) $\frac{x}{3} + \frac{1}{3}$ (D) none of these

73. The value of

$$\lim_{n\to\infty}\left(\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots\ldots+\frac{1}{nb}\right),\text{ is}$$

(A) $\log\left(\frac{a}{b}\right)$ (B) $\log\left(\frac{b}{a}\right)$

(C) log (ab) (D) none of these

74. Let f(x) be defined for all $x \in R$ such that

$$\lim_{x\to 0}\left[f(x)+\ln\left(1-\frac{1}{e^{f(x)}}\right)-\ln(f(x))\right]=0,$$

then f(0) is

(A) 0 (B) 1

(C) 2 (D) 3

75. A_0 is an equilateral triangle of unit area, A_0 is divided into four equal parts, each an equilateral triangle, by joining the mid points of the sides of A_0. The central triangle is removed. Treating the remaining three triangles in the same way of division as was done to A_0, and this process is repeated n times. The sum of the area of the triangles removed in S_n then $\lim_{n\to\infty} S_n$ is

(A) 1/2 (B) 1

(C) –1 (D) 2

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

76. Which of the following limits vanish?

$\lim_{x\to\infty} x^{\frac{1}{4}}$

(A) $\sin\frac{1}{\sqrt{x}}$ $\lim_{x\to\pi/2}$

(B) $(1-\sin x)\,.\,\tan x$

(C) $\lim_{x\to\infty}\frac{2x^2+3}{x^2+x-5}\,.\,\text{sgn}(x)$

(D) $\lim_{x\to 3^+}\frac{[x]^2-9}{x^2-9}$

77. The true statement(s) is / are

(A) If f(x) < g(x) for all x ≠ a, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

(B) If $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all x ≠ c, then $\lim_{x\to c} f(x)\cdot g(x) = 0$.

(C) If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c}|f(x)| = |L|$ and conversely if $\lim_{x\to c}|f(x)| = |L|$ then $\lim_{x\to c} f(x) = L$.

(D) If f(x) = g(x) for all real number other then x = 0 and $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} g(x) = L$.

78. Which of the following functions has two horizontal asymptotes

(A) $y = \frac{|x|}{x+1}$ (B) $y = \frac{2x}{\sqrt{x^2+1}}$

(C) $y = \frac{\sin x}{x^2+1}$ (D) $y = \cot^{-1}(2x+1)$

79. Which of the following functions has a vertical asymptote at x = – 1.

(A) $y = \frac{|x^2-1|}{x+1}$ (B) $y = \frac{x^2-6x-7}{x+1}$

(C) $y = \frac{x^2+1}{x+1}$ (D) $y = \frac{\sin(x+2)}{x+1}$

80. The true statement(s) is / are

(A) If $\lim_{x\to c} f(x) = 0$, then there must exist a number d such that f(d) < 0.001.

(B) $\lim_{x\to c} f(x) = L$, is equivalent to $\lim_{x\to c}(f(x)-L) = 0$.

(C) $\lim_{x\to a}(f(x)+g(x))$ may exist even if the limits $\lim_{x\to a}(f(x)$ and $\lim_{x\to a}(g(x)$ do not exist.

(D) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}(f(x)+g(x))$ does not exist, then $\lim_{x\to a} g(x)$ does not exist.

81. Which of the following functions have a graph which lies between the graphs of y = |x| and y = – |x| and have a limiting value as x → 0.

(A) y = x cos x (B) y = |x| sin x

(C) $y = x\cos\frac{1}{x}$ (D) $y = \left|x\sin\frac{1}{x}\right|$

82. The false statement(s) is / are

(A) If P(x) is a polynomial, then the function $f(x) = \frac{P(x)}{x-1}$ has a vertical asymptote at x = 1.

(B) A polynomial function has no vertical asymptote and a rational function has atleast one vertical asymptote.

(C) If f(x) has a vertical asymptote at x = 0, then f is undefined at x = 0.

(D) A function can have move than two horizontal asymptotes.

83. The function(s) which have a limit as n → ∞

(A) $\left(\frac{n-1}{n+1}\right)^2$ (B) $(-1)^n\left(\frac{n-1}{n+1}\right)^2$

(C) $\frac{n^2+1}{n}$ (D) $(-1)^n\frac{n^2+1}{n}$

84. The function(s) which have a limit as $x \to \infty$

(A) $\frac{\sin x\pi}{x}$ (B) $a\cos^2 x\pi + b\sin^2 x\pi$

(C) $x\sin x\pi$ (D) $\tan x\pi$

85. Which of the following limits exist? (where [.] indicates greatest integer function all throughout)

(A) $\lim\limits_{x\to 1}\frac{\sin[x]}{[x]}$ (B) $\lim\limits_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$

(C) $\lim\limits_{x\to 1}\left[\sin(\sin^{-1}x)\right]$ (D) $\lim\limits_{x\to\pi/2}\left[\sin^{-1}(\sin x)\right]$

86. Given $l_1 = \lim\limits_{x\to\frac{\pi}{4}} \cos^{-1}\left[\sec\left(x - \frac{\pi}{4}\right)\right]$;

$l_2 = \lim\limits_{x\to\frac{\pi}{4}} \sin^{-1}\left[\operatorname{cosec}\left(x + \frac{\pi}{4}\right)\right]$;

$l_3 = \lim\limits_{x\to\frac{\pi}{4}} \tan^{-1}\left[\cot\left(x + \frac{\pi}{4}\right)\right]$;

$l_4 = \lim\limits_{x\to\frac{\pi}{4}} \cot^{-1}\left[\tan\left(x - \frac{\pi}{4}\right)\right]$

where [x] denotes greatest integer function then which of the following limits exist

(A) l_1 (B) l_2

(C) l_3 (D) l_4

87. Give a real valued function f such that

$$f(x) = \begin{cases} \dfrac{\tan^2 x}{(x^2 - [x])^2} & \text{for} \quad x > 0 \\ 1 & \text{for} \quad x = 0 \\ \sqrt{\{x\}\cot\{x\}} & \text{for} \quad x < 0 \end{cases}$$

where, [.] is the integral part and {.} is the fractional part of x, then

(A) $\lim\limits_{x\to 0} f(x) = 1$

(B) $\lim\limits_{x\to 0^-} f(x) = \cot 1$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$

(D) none of these

88. $\lim\limits_{x\to c} f(x)$ does not exist when

where, [.] is the greatest integral part and {.} is the fractional part of x, then

(A) $f(x) = [\,[x]\,] - [2x - 1]$, $c = 3$

(B) $f(x) = [x] - x$, $c = 1$

(C) $f(x) = \{x\}^2 - \{-x\}^2$, $c = 0$

(D) $f(x) = \frac{\tan(\operatorname{sgn} x)}{\operatorname{sgn} x}$, $c = 0$

89. Let $f(x) = \lim\limits_{n\to\infty}\frac{2x^{2n}\sin\frac{1}{x} + x}{1 + x^{2n}}$ then which of the following alternative(s) is/are correct ?

(A) $\lim\limits_{x\to\infty} x\,f(x) = 2$

(B) $\lim\limits_{x\to 1} f(x)$ does not exist

(C) $\lim\limits_{x\to 0} f(x)$ does not exist

(D) $\lim\limits_{x\to -\infty} f(x)$ is equal to zero.

90. Consider the function $f(x) = \left(\frac{ax+1}{bx+2}\right)^x$ where $a^2 + b^2 \neq 0$ then $\lim\limits_{x\to\infty} f(x)$

(A) exists for all values of *a* and *b*

(B) is zero for a < b

(C) is non existent for a > b

(D) is $e^{-\left(\frac{1}{a}\right)}$ or $e^{-\left(\frac{1}{b}\right)}$ if a = b

91. Let $f(x) = \begin{cases} mx^2 + n & \text{for} \quad x < 0 \\ nx + m & \text{for } 0 \le x \le 1 \\ nx^3 + m & \text{for} \quad x > 1 \end{cases}$ where $m, n \in R$

then which of the following must be correct

(A) $\lim\limits_{x\to 0} f(x)$ exist for all values of *m* and *n*.

(B) $\lim\limits_{x\to 0} f(x)$ exists only if *m* = *n*.

(C) $\lim\limits_{x\to 0} f(x)$ exists for all values of *m* and *n*.

(D) $\lim\limits_{x\to 1} f(x)$ exists for no values of *m* and *n*.

92. $\lim\limits_{x\to 0^+}\int_x^{2x}\frac{\sin^m t}{t^n}$ (m, n, ∈ N) equals

(A) 0 if $m \ge n$ (B) *l*n 2 if $n - m = 1$

(C) $+\infty$ if $n - m = 1$ (D) None of these

93. If $\lim\limits_{x\to\infty}\left(\sqrt{x^2 - x + 1} - ax - b\right) = 0$, then for $k \ge 2$, $k \in N$ which of the following is/are correct ?

(A) $2a + b = 0$

(B) $a + 2b = 0$

(C) $\lim\limits_{n\to\infty}\sec^{2n}(k!\pi b) = 1$

(D) $\lim\limits_{n\to\infty}\sec^{2n}(k!\pi a) = 1$

94. Let $l_1 = \lim\limits_{x\to\infty}\sqrt{\frac{x - \cos^2 x}{x + \sin x}}$ and

$l_2 = \lim\limits_{h\to 0^+}\int_{-1}^{1}\frac{h\,dx}{h^2 + x^2}$. Then

(A) both l_1 and l_2 are less than $\frac{22}{7}$

(B) one of the two limits is rational and other irrational.

(C) $l_2 > l_1$

(D) l_2 is greater than 3 times of l_1.

95. In which one of the following cases, limit tends to e

(A) $\lim\limits_{x\to 1} x^{\frac{1}{x-1}}$

(B) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}$

(C) $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x+2}\right)^{x+3}$

(D) $\lim\limits_{x\to\infty}\left(1+f(x)\right)^{\frac{1}{f(x)}}$ when $\lim\limits_{x\to\infty} f(x) \to 0$

Assertion (A) and Reason (R)

Code:

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

96. Assertion (A) : If [x] denotes the greatest integer function then $\lim\limits_{x\to\infty}\dfrac{x}{[x]}$ does not exist.

Reason (R) : $\lim\limits_{x\to\infty}[x]$ does not exist.

97. Assertion (A) : The value of

$$\sum_{n=1}^{\infty}\frac{n+3}{2^n(n^2+3n+2)} \text{ is } \frac{1}{2}.$$

Reason (R) :

$$\sum_{n=1}^{\infty}\frac{n+3}{2^n(n^2+3n+2)}=\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{2}{n+1}-\frac{1}{n+2}\right)$$

98. Assertion (A) : $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3+x^2}-\sqrt[3]{x^3-x^2}\right)=\dfrac{2}{3}$

Reason (R) : $\lim\limits_{x\to\infty}\left(x+\dfrac{1}{3}-\sqrt[3]{x^3+x^2}\right)=0$ and

$$\lim_{x\to\infty}\left(x-\frac{1}{3}-\sqrt[3]{x^3-x^2}\right)=0$$

99. Assertion (A) : $\lim\limits_{x\to 0^+}\left(x^{x^x}-x^x\right)=-1$

Reason (R) : $\lim\limits_{x\to 0^+} x^x(x-1)=-1$

100. Assertion (A): An equilateral triangle is filled with n, rows of congruent circles. The limit of the ratio of area of circle to the area of triangle as

$n\to\infty$ is $\dfrac{\sqrt{3}\pi}{6}$.

Reason (R) : Let the triangle have side length 1 and radius of circles be r. Then $2(n-1)r+2r\sqrt{3}$ $=1$. There are $\dfrac{n(n-1)}{2}$ circles. the area ratio

$=\dfrac{\pi}{2\sqrt{3}}\dfrac{n(n-1)}{(n+(\sqrt{3}-1))^2}$ which approaches $\dfrac{\sqrt{3}\pi}{6}$

as $n\to\infty$.

101. Assertion (A) : Let $f:(0,\infty)\to R$ be a twice continuously differentiable function such that $|f''(x)+2x\,f'(x)+(x^2+1)\,f(x)|\le 1$ for all x.
Then $\lim\limits_{x\to\infty} f(x)=0$.

Reason (R) : Applying L'Hospital's rule twice

on the function $\dfrac{f(x)e^{\frac{x^2}{2}}}{e^{\frac{x^2}{2}}}$ we get $\lim\limits_{x\to\infty} f(x)=0$.

102. Assertion (A): A circle C_1 is inscribed in an equilateral triangle ABC with side length 2. Then circle C_2 is inscribed tangent to BC, CA and circle C_1. An infinite sequence of such circles is constructed, each tangent to BC, CA and the previous circle. The sum of areas of all the infinitely many circles is $\dfrac{5\pi}{8}$.

Reason (R) : Radius of C_1 is $\dfrac{1}{\sqrt{3}}$, that of C_2 is $\dfrac{1}{3\sqrt{3}}$ and radius of the remaining circle each shrink by a factor $\dfrac{1}{3}$.

103. Assertion (A): $\lim\limits_{x\to 0}[x]\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right)$ where [.] represents greatest integer function does not exist.

Reason (R) : $\lim\limits_{x\to 0}\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right)$ does not exist.

104. Assertion (A): $\lim\limits_{x\to\pi/2}\dfrac{\sin\left(\cot^2 x\right)}{(\pi-2x)^2}=\dfrac{1}{2}$

Reason (R) : $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ and $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1$, where θ is measured in radians.

105. Consider the function $f(x)=\cos^{-1}[\cot x]$ where [ ] indicates greatest integer function.

Assertion (A): $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ exists

Reason (R) : Both $\lim\limits_{x\to\frac{\pi}{2}^+} f(x)$ and $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)$ are finite.

Comprehension - 1

Let $f(x)=x[x[x]]$, $x\in[-1,3]$ where [.] represents the greatest integer function.

106. If the value of x so that f(x) = 13 is $\frac{a}{b}$, a, b ∈ N and g.c.d. (a, b) = 1 then the value of a + b is :

(A) 14
(B) 18
(C) 22
(D) None of these

107. The number of points of where limit of f(x) does not exist is :

(A) 3
(B) 4
(C) 5
(D) None of these

108. The value of $\lim\limits_{x \to \frac{\pi}{2}} f(2\sin x)$ is :

(A) 2
(B) 4
(C) 8
(D) None

Comprehension - 2

Consider a new definition for approximating the values of a function f(x) in the neighbourhood of a point x = a $\underset{x\to a}{\text{Approx}} f(x) = \lim\limits_{x\to a^+} f(x)$ whenever $\left[\,\left|\lim\limits_{x\to a^-} f(x) - \lim\limits_{x\to a^+} f(x)\right|\,\right] \le 1$, otherwise $\underset{x\to a}{\text{Approx}} f(x)$ is said to be non-existent. Here [.] denotes greatest integer function.

109. For which of the following functions, Approx f(x) exists :

(A) $\underset{x\to 1}{\text{Approx}} \dfrac{x^2 - 1}{|x-1|}$
(B) $\underset{x\to 2}{\text{Approx}} \dfrac{2\{x\} - 4}{[x] - 3}$
(C) $\underset{x\to 0}{\text{Approx}} \dfrac{1}{2 - 2^{\frac{1}{x}}}$
(D) None of these

110. The value of $\underset{x\to 0}{\text{Approx}} \dfrac{x(e^{[x]+|x|} - 2)}{[x] + |x|}$ is

(A) –1
(B) 0
(C) 1
(D) None

111. Let $f(x) = \begin{cases} [x^x] + a, & x > 0 \\ \lim\limits_{t\to\infty}\left(\dfrac{\sin x}{x}\right)^t, & x < 0 \end{cases}$. The complete set of the values of 'a' for which $\underset{x\to 0}{\text{Approx}}$ exists is

(A) (0, 2]
(B) (–2, 2)
(C) [–1, 1]
(D) None of these

Comprehension - 3

Consider the function $f(x) = \left[\dfrac{1}{x}\sin x\right]$ $x \neq 0$. Where [.] represents greatest integer function.

112. Which of the following function will be periodic

(A) f(x)
(B) f(|x|)
(C) f(–x)
(D) $f\left(|x| + \dfrac{\pi}{2}\right)$

113. If [.] represents greatest integer function, then which of the following, is identical to f of (x) for x ∈ (π, 2π)

(A) sgn (x)
(B) $\dfrac{x}{x}$
(C) sinπ[x]
(D) $\dfrac{\sin x}{\sin x}$

114. The limit of function g(x) = sgn f(x) as x tends to 0 is

(A) 0
(B) 1
(C) – 1
(D) does not exist

Comprehension - 4

Suppose $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \frac{t^2\,dt}{(a + t^r)^{1/p}}}{bx - \sin x} = \ell$ where p ∈ N, p ≥ 2, a > 0, r > 0 and b ≠ 0.

115. If ℓ exists and is non zero then

(A) b > 1
(B) 0 < b < 1
(C) b < 0
(D) b = 1

116. If p = 3 and ℓ = 1 then the value of 'a' is equal to

(A) 8
(B) 3
(C) 6
(D) 3/2

117. If p = 2 and a = 9, b = 1 and ℓ exists then the value of ℓ is equal to

(A) 3/2
(B) 2/3
(C) 1/3
(D) 7/9

Comprehension - 5

Let $a_1 > a_2 > a_3$ .... $a_n > 1$; $p_1 > p_2 > p_3$..... $> p_n > 0$; such that $p_1 + p_2 + p_3 + \ldots\ldots + p_n = 1$.

Also $F(x) = (p_1 a_1^{\,x} + p_2 a_2^{\,x} + \ldots\ldots. + p_n a_n^{\,x})^{1/x}$

118. $\lim\limits_{x\to 0^+} F(x)$ equals

(A) $p_1\,\ell n\, a_1 + p_2\,\ell n\, a_2 + \ldots.. + p_n\,\ell n\, a_n$
(B) $a_1^{p_1} + a_2^{p_2} + \ldots..a_n^{p_n}$
(C) $a_1^{p_1} \cdot a_2^{p_2} + \ldots..a_n^{p_n}$
(D) $\sum\limits_{r=1}^{n} a_r p_r$

119. $\lim\limits_{x\to\infty} F(x)$ equals

(A) $\ell n\, a_1$
(B) e^{a_1}
(C) a_1
(D) a_n

120. $\lim\limits_{x\to -\infty} F(x)$ equals

(A) $\ell n\, a_n$
(B) e^{a_1}
(C) a_1
(D) a_n

MATCH THE COLUMNS FOR JEE ADVANCED

121.	Column - I	Column - II
(A)	$\lim_{n\to\infty}\cos^2(\pi(\sqrt[3]{n^3+n^2+2n}))$ where n is an integer, equals.	(P) $\frac{1}{\sqrt{2}}$
(B)	$\lim_{n\to\infty} n\sin(2\pi\sqrt{1+n^2})$ ($n \in N$) equals.	(Q) $\frac{1}{4}$
(C)	$\lim_{n\to\infty}(-1)^n\sin(\pi\sqrt{n^2+0.5n+1})\left(\sin\frac{(n+1)\pi}{2n}\right)$ is (where $n \in N$).	(R) π
(D)	If $\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = e$ where 'a' is some real constant then the value of 'a' is equal to.	(S) non existent

122.	Column - I	Column - II
(A)	If $\lim_{x\to\infty}(\sqrt{(x^2-x-1)}-ax-b)=0$, where $a>0$, then there exists atleast one a and b for which point (a, 2b) lies on the line	(P) $y=-3$
(B)	If $\lim_{x\to 0}\frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}}=2$, then there exists atleast one a and b for which point (a, b^3) lies on the line	(Q) $3x-2y-5=0$
(C)	If $\lim_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$, then there exists atleast one a and b for which point (a, –2b) lies on the line	(R) $15x-2y-13=0$
(D)	If $\lim_{x\to -a}\frac{x^7+a^7}{x+a}=7$, where $a<0$, then there exists atleast one a for which point (a, 2) lies on the line.	(S) $y=2$

123.	Column-I	Column-II
(A)	If $f(x)=\lvert x-a\rvert+\lvert x-10\rvert+\lvert x-a-10\rvert$, where $a \in (0, 10)$, then the minimum value of f is	(P) 0
		(Q) 1
(B)	$\lim_{x\to 0}\frac{x(1-\cos 2x)^2-a(\sin x-\tan x)^2}{\tan^5 x+a\sin^8 x}$ is equal to	(R) 4
(C)	$\lim_{n\to\infty}\frac{n^a\sin^2(n!)}{n+1}$, $0<a<1$, $n \in N$, is equal to	(S) 10
(D)	The number of solutions of the equation $\log_a(2a-x)=\log_a^3 x$, $a>0$, $a \neq 1$ is	(T) depends on a

124.	Column - I	Column - II
(A)	$\lim_{x\to\infty}(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}})$ equals	(P) –2
(B)	The value of the limit, $\lim_{x\to 0}\frac{\sin 2x-2\tan x}{\ln(1+x^3)}$ is	(Q) –1
(C)	$\lim_{x\to 0^+}(\ln\sin^3 x-\ln(x^4+ex^3))$ equals	(R) 0
(D)	Let $\tan(2\pi\lvert\sin\theta\rvert)=\cot(2\pi\lvert\cos\theta\rvert)$, where $\theta \in R$ and $f(x)=(\lvert\sin\theta\rvert+\cos\theta\rvert)^x$. The value of $\lim_{x\to\infty}\left[\frac{2}{f(x)}\right]$ equals (Here [ ] represents greatest integer function)	(S) 1

125.

Column - I	Column - II
(A) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{e^{x^2}-e^x+x}$ equals	(P) 1
(B) If the value of $\lim\limits_{x\to 0^+}\left(\dfrac{(3/x)+1}{(3/x)-1}\right)^{1/x}$ can be expressed in the form of $e^{p/q}$, where p and q are relative prime then (p + q) is equal to	(Q) 2
(C) $\lim\limits_{x\to 0}\dfrac{\tan^3 x-\tan x^3}{x^5}$ equals	(R) 4
(D) $\lim\limits_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$	(S) 5

Review Exercises for JEE Advanced

1. Evaluate the following limits :

(i) $\lim\limits_{h\to 0}\dfrac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$

2. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{1-\cos x\cos 2x\cos 3x}{\sin^2 2x}$

(ii) $\lim\limits_{x\to 0}\dfrac{(\sin x-\tan x)^2-x(1-\cos 2x)^2}{8\tan^8 x+5\sin^5 x}$

3. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+x^2}\cos x}{\tan^4 x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2\dfrac{x}{2}}$

4. Evaluate the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{1}{x^8}\left[1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right]$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin\ \sin\ \tan\left(\dfrac{x^2}{2}\right)}{\ell n\ \cos 3x}$

5. Evaluate

$$\lim_{x\to 0}\frac{\sin(3x+a)-3\sin(2x+a)+3\sin(x+a)-\sin a}{x^3}$$

6. Evaluate $\lim\limits_{x\to 1}\dfrac{(1-x)(1-x^2).....(1-x^{2n})}{[(1-x)(1-x^2)......(1-x^n)]^2}$

7. Evaluate

$$\lim_{n\to\infty}\frac{1}{n^{n^2}}\left[(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)..........\left(n+\frac{1}{2^{n-1}}\right)\right]^n$$

8. Evaluate $\lim\limits_{x\to 0^+}(\log_{\sin x}\sin 2x)^{\log\sin 2x}$.

9. Evaluate

$$\lim_{x\to 0}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{..\infty}}}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{..........\infty}}}}$$

10. Find $\lim\limits_{n\to\infty}\dfrac{\sqrt{1-x_0^2}}{x_1x_2x_4.........x_n}$, where

$x_{r+1}=\sqrt{\dfrac{1+x_r}{2}}$.

11. Evaluate $\lim\limits_{n\to\infty}\dfrac{1.\sum\limits_1^n r+2.\sum\limits_1^{n-1} r+3.\sum\limits_1^{n-2} r+....+n.1}{n^4}$

12. If $T_r=\dfrac{1^2+2^2+3^2+...+r^2}{1^3+2^3+3^3+...+r^3}$ and $S_n=\sum\limits_{r=1}^{n}(-1)^r T_r$ then find $\lim\limits_{x\to\infty} S_n$.

13. Evaulate $\lim\limits_{x\to\infty}\ \ln_{\frac{x^2+1}{x+1}}\left(\sqrt{x^2+1}-x\right)$.

14. Prove that the graph of f(x) = $\dfrac{x^3+1}{x}$ is asymptotic to the graph of g(x) = x^2 using $\lim\limits_{x\to\pm\infty}$ (f(x) – g(x)).

15. Evaluate $\lim\limits_{x\to\infty} x\left[\tan^{-1}\left(\dfrac{x+1}{x+2}\right)-\tan^{-1}\left(\dfrac{x}{x+2}\right)\right]$

16. Evaluate $\lim\limits_{x\to 0}\dfrac{(\sin x\text{-}\tan x)^2+(1\text{-}\cos 2x)^4+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$

17. Find

(A) $\lim\limits_{x\to 1/2} f(x)$ (B) $\lim\limits_{x\to 3/2} f(x)$

where $f(x)=\dfrac{\dfrac{|x-1|}{2}+x|x-1|+2-\dfrac{2}{x}}{\sqrt{x-2+\dfrac{1}{x}}}$

18. Prove that

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}+\lim_{x\to\pi/4}\frac{\sqrt{2}-2\cos x}{\sin(x-\pi/4)}>\log 0.005.$$

19. Evaluate $\displaystyle\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex-\frac{11}{24}ex^2}{x^3}$

20. If $\lambda_{n+1}=\sqrt{2+\lambda_n}$, $\lambda_0=\sqrt{x}$, find $\displaystyle\lim_{x\to 4}\frac{\lambda_r-2}{x-4}$, for $r = 1$ and 3.

21. Simplify the function $\phi_n(x)=\dfrac{x^n f(x)+g(x)}{x^n+1}$ as $n\to\infty$.

22. Evaluate $\displaystyle\lim_{x\to 0}\left[\frac{a\sin x}{x}\right]+\left[\frac{b\tan x}{x}\right]$ where a, b are positive integers and [ ] denotes the greatest integer function.

23. If $f_1(x)=\dfrac{x}{2}+10$, $\forall\, x\in R$ and defined by $f_n(x)$ = $f_1\{f_{n-1}(x)\}$, $\forall \geq 2$. S then evaluate $\displaystyle\lim_{n\to\infty}f_n(x)$.

24. Evaluate $\displaystyle\lim_{x\to\infty}\left(\frac{1}{x}.\frac{a^x-1}{a-1}\right)^{1/x}$ where $a > 0$, $a \neq 1$.

25. Simplify the function

$y = 1-\displaystyle\lim_{m\to\infty}\left(\lim_{n\to\infty}\{\cos(m!\pi x)\}^{2n}\right)$

26. Let $f(x)=x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$.

Find $m=\displaystyle\lim_{x\to\infty}\frac{f(x)}{x}$ and $b=\displaystyle\lim_{x\to\infty}(f(x)-mx)$. Hence show that $\displaystyle\lim_{x\to\infty}(f(x)-mx-b)=0$.

27. Simplify the function $y=\displaystyle\lim_{n\to\infty}\frac{1}{1+n\sin^2\pi x}$

28. Simplify the function

$$y=\lim_{n\to\infty}\frac{\psi(x)+n\varphi(x)\sin^2\pi x}{1+n\sin^2\pi x}$$

29. Show that the function $f(x)=\dfrac{x\tan x+2x-1}{x+1}$ does not tend to a unique limit as $x\to\infty$, through all real values, but if x ranges through the sequence of values $x = n\pi+\dfrac{\pi}{4}$ $(n = 0, 1, 2, ...)$, then $f(x)\to 3$.

30. The function $f(x)$ and $\phi(x)$ are such that $f(0)=\phi(0)=0$. By considering the case where $f(x)=x^2\sin 1/x$, $\phi(x)=\tan x$, show that $\displaystyle\lim_{x\to 0}\frac{f(x)}{\varphi(x)}$ may exist when $\displaystyle\lim_{x\to 0}\frac{f'(x)}{\varphi'(x)}$ does not exist.

31. Show that the function

$$y=\lim_{n\to\infty}\frac{f(x)+n\varphi(x)\sin^2\pi x}{1+n\sin^2\pi x}$$

is equal to f(x) when x is an integer, but is equal to $\phi(x)$ in every other case.

32. Evaluate

$$\lim_{n\to\infty}\left[\frac{(n^3+1^3)(n^3+2^3)......(n^3+n^3)}{n^{3n}}\right]^{1/n}$$

33. If $\displaystyle\lim_{x\to\infty}\left(\frac{3x^2+1}{3x^2-1}\right)^{\frac{x^3}{1+x}}=k$, then find the value of $[6\,\ell nk]$, where [.] is greatest integer function.

34. If $\displaystyle\lim_{x\to 2}\frac{\sqrt{x^2+5}-\sqrt[3]{2x^3+3x+5}}{\sqrt[3]{x+6}-\sqrt[5]{4x^2+6x+4}}=b$, then find the value of 230b.

35. If $\displaystyle\lim_{x\to 0}\frac{\cos 4x+a\cos 2x+b}{x^4}$ = a finite quantity, then find the value of a^{b+1}

Target Exercises for JEE Advanced

1. Find $\displaystyle\lim_{x\to\infty}\left(\frac{x^{x+1}+x+1}{x^{x+1}}\right)^x$

2. $\displaystyle\lim_{x\to a}\left[\frac{(x^3-2ax^2-a^2x+2a^3)(x-2a)^{-1}+2a(x+a)}{x^3-a^2x}-\frac{2}{x-a}\right]$

3. Evaluate $\displaystyle\lim_{x\to a}\left(\frac{(a^{x-a}-1)(x^p-a^p)}{(x-a)^2(\ln a^p)a^{p-1}}\right)^{\frac{1}{x-a}}$

where $a > 0$, $p\in Q$

4. Evaluate: $\displaystyle\lim_{x\to\infty}x\,ln\left(e\left(1+\frac{1}{x}\right)^{1-x}\right)$

5. If $\displaystyle\lim_{x\to 1^-}\frac{(3+ax)^{5/2}-b\,\ell nx+c\sin(x-1)}{(x-1)^2}=2$, then find the value of the expression $a^2+b^2+c^2$.

6. If the $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$ exists and has the value equal to l, then find the value of $\dfrac{1}{a}-\dfrac{2}{l}+\dfrac{3}{b}$.

7. Evaluate $\lim\limits_{x\to\infty} x^2 (\sin x . \sin^3 \frac{1}{x} - (x \tan^3 \frac{1}{x})\text{sgn}(x))$

8. Find $\lim\limits_{n\to\infty} \dfrac{n\cos^2\frac{1}{2}n\pi + \sin^2\frac{1}{2}n\pi}{n\left(\cos^2\frac{1}{2}n\pi + n\sin^2\frac{1}{2}n\pi\right)}$

9. If $a_{n+1} = (a_n + x)^{-1}$, $a_0 = 0$, and $b_{n+1} = (b_n + y)^{-1}$, $b_0 = 0$, find $\lim\limits_{x\to y} \dfrac{a_n - b_n}{x^2 - y^2}$ for n = 2, 3.

10. Evaluate $\lim\limits_{x\to 0} \dfrac{\int_0^{\frac{\pi}{4}+x^4e^{1/x^2}} \cos^n t\, dt - \int\limits_0^{\frac{\pi}{4}} \cos^n t dt}{x^4 e^{\frac{1}{x^2}}}$, $n \in N$

11. Evaluate $\lim\limits_{x\to 0^+} \dfrac{\dfrac{\ell n\ x}{(1+x)^2} - \ell n\left(\dfrac{x}{1+x}\right)}{\ell n(1-x^2) + 2x}$

12. Let g(x) = f(x – 1) + f(x + 1) where

$f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ |x|, & |x| > 1 \end{cases}$ find $\lim\limits_{x\to\frac{\pi}{2}} g(\sin x)$.

13. Prove that $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \cot^{-1}\left(\dfrac{r^3 - r + \frac{1}{r}}{2}\right) = \dfrac{\pi}{2}$

14. Evaluate $\lim\limits_{x\to 0} \dfrac{(1-\cos x)(e^x - \cos x)}{x^n}$.

15. If α, β are two distinct real roots of the equation $ax^3 + x - 1 - a = 0$, $(a \ne -1, 0)$, which are not equal to unity then prove that

$$\lim_{x\to 1/\alpha} \frac{\ln((1+a)x^3 - x^2 + 1 - a)}{(e^{1-\alpha x} - 1)(x-1)} \text{ is a } \left(1 - \frac{\beta}{\alpha}\right).$$

16. $\lim\limits_{x\to 1}\left[\left(\dfrac{4}{x^2 - x^{-1}} - \dfrac{1-3x+x^2}{1-x^3}\right)^{-1} + 3\dfrac{x^4-1}{x^3-x^{-1}}\right]$

17. If $2f(\sin x) + \sqrt{2}\, f(\cos x) = \tan x$, then evaluate $\lim\limits_{x\to -1}$ $\sqrt{1-x}\ f(x)$ if $f\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\sqrt{2}-1}{\sqrt{2}}$.

18. Find

$$\lim_{x\to a}\left\{\left[\left(\frac{\sqrt{a}+\sqrt{x}}{\sqrt[4]{a}-\sqrt[4]{x}}\right)^{-1} - \frac{2\sqrt[4]{ax}}{x^{3/4} - a^{1/4}x^{1/2} + a^{1/2}x^{1/4} - a^{3/4}}\right]^{-1} - (\sqrt{2})^{\log_4 a}\right\}^8$$

19. Find values of α and β such that

$\lim\limits_{x\to\infty} \sqrt{(ax^2 + 2bx + c} - \alpha x - \beta = 0$ and prove that

$\lim\limits_{x\to\infty} x\left(\sqrt{(ax^2 + 2bx + c} - \alpha x - \beta\right) = \dfrac{ac - b^2}{2a\sqrt{a}}$

20. Show that the functions y = 1 – x – [x] – [1–x] and y = 1 – x – $\lim\limits_{n\to\infty} (\cos^{2n+1} 2\pi x)$ are identical.

21. If f(x + y) = f(x) + f(y) for all x, y ∈ R and f(1) = 1, then find $\lim\limits_{x\to 0} \dfrac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$

22. Show that $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{{}^nC_k}{n^k(k+3)} = e - 2$

23. Calculate the limit of the sequence $\{x_n\}$ if it is defined by the recursive relation :

$x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$ ∀n ≥ 1, a > 0,

where x_1 is an arbitrary negative number.

24. Show that if –1 < x < 1, then

$u_n = \dfrac{m(m-1)....(m-n+1)}{n!} x^n = \dbinom{m}{n} x^n$

tends to zero as $n \to \infty$.

25. If c_n, s_n denote the sums of the first n terms of the series

$\frac{1}{2}$ + cosθ + cos2θ +...., and

sin θ + sin 2θ + sin 3θ ....

and θ is not a multiple of 2π, then prove that

$\lim\limits_{n\to\infty} (c_1 + c_2 + ...+c_n)/n = 0$,

$\lim\limits_{n\to\infty} (s_1 + s_2 + ...\qquad + s_n)/n = \dfrac{1}{2}\cot\dfrac{\theta}{2}$

26. Draw a graph of the function y defined by the equation

$y = \lim\limits_{n\to\infty} \dfrac{x^{2n}\sin\frac{1}{2}\pi x + x^2}{x^{2n} + 1}$

27. Show that the graph of the function

$y = \lim\limits_{n\to\infty} \dfrac{x^n\phi(x) + x^{-n}\Psi(x)}{x^n + x^{-n}}$

is composed of parts of the graphs of φ(x) and Ψ(x), together with (as a rule) two isolated points.

28. Prove that the function y which is equal to 0 when x is rational and to 1 when x is irrational, may be represented in the form

$y = \lim\limits_{m\to\infty} \text{sgn}\{\sin^2(m!\pi x)\}$

where sgn x = $\lim\limits_{m\to\infty} \dfrac{2}{\pi}\arctan(nx)$.

29. Evaluate

$$\lim_{x\to 0}\frac{\cos^2(1-\cos^2(1-\cos^2(\ldots\cos^2(x)))\ldots)}{\left(\pi\dfrac{\sqrt{(x+4)}-2}{x}\right)}$$

30. Let the sequence x_n be defined as

$x_0 = \sqrt{a}$

$x_1 = \sqrt{a+\sqrt{a}}$

$x_2 = \sqrt{a+\sqrt{a+\sqrt{a}}}$

$x_3 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$

..................................., find $\lim\limits_{n\to\infty} x_n$.

31. Evaluate $\lim\limits_{n\to\infty}\dfrac{[x]+\frac{1}{2}[2x]+\frac{1}{3}[3x]+\ldots..+\frac{1}{n}[nx]}{1^2+2^2+3^2+\ldots\ldots n^2}$

where [.] denotes the greatest integer function

32. If $f(x) = \lim\limits_{n\to\infty}\{\sin x + 2\sin^2 x + 3\sin^3 x + \ldots. n\sin^n x\}$ then

evaluate $\lim\limits_{x\to\pi/2}\{(1-\sin x)^2 f(x)\}^{\frac{1}{(\sin x-1)}}$

33. The function u_n attains the values

$u_1 = \dfrac{1}{2}$; $u_2 = \dfrac{1}{2} + \dfrac{1}{2.4} + \dfrac{1}{2.4.6}$,

$u_n = \dfrac{1}{2} + \dfrac{1}{2.4} + \ldots.\quad \dfrac{1}{2.4\ldots.(2n)}$,......

Prove that $\lim\limits_{n\to\infty} u_n < 2$.

34. Prove that, if $k > 0$, $\lim\limits_{n\to\infty}\dfrac{1^k+2^k+\ldots.n^k}{n^{k+1}} = \dfrac{1}{k+1}$.

Hence evaluate $\lim\limits_{n\to\infty}\dfrac{F(n)}{(n+1)^{2k+2}}$, where F(n) is the coefficient of x^{n-2} in $(x-1^k)(x-2^k)\ldots.(x-n^k)$.

Previous Year's Questions (JEE Advanced)

A. FILL IN THE BLANKS

1. If f(x) = sin x, $x \neq n\pi$, n = 0, ± 1, ± 2, ± 3, ...

= 2, otherwise

and g(x) = $x^2 + 1$, $x \neq 0, 2$

= 4, x = 0

= 5, x = 2

then $\lim\limits_{x\to 0} g[f(x)]$ is............ **[IIT - 1986]**

2. $\lim\limits_{x\to-\infty}\left[\dfrac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$ = ... **[IIT - 1987]**

3. ABC is an isosceles triangle inscribed in a circle of radius r. IF AB = AC and h is the altitude from A to BC then the triangle ABC has perimeter

$P = 2(\sqrt{2hr-h^2} + \sqrt{2\,hr})$ and area

A = ... also $\lim\limits_{h\to 0}\dfrac{A}{P^3}$ = ... **[IIT - 1989]**

4. $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$ = ... **[IIT - 1990]**

5. $\lim\limits_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{1/x^2}$ = ... **[IIT - 1996]**

6. $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}$ = ... **[IIT - 1997]**

7. $\lim\limits_{h\to 0}\dfrac{\ln(1+2h)-2\ln(1+h)}{h^2}$ = ... **[IIT - 1997]**

B. TRUE / FALSE

8. If $\lim\limits_{x\to a}[f(x)g(x)]$ exists then both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist **[IIT - 1981]**

C. MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER

9. If $f(x)=\sqrt{\dfrac{x-\sin x}{x+\cos^2 x}}$, then $\lim\limits_{x\to\infty} f(x)$ is

(A) 0 (B) ∞

(C) 1 (D) None

[IIT - 1979]

10. $\lim\limits_{n\to\infty}\left\{\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\ldots.+\dfrac{n}{1-n^2}\right\}$ is equal to

(A) 0 (B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{2}$ (D) none of these

[IIT - 1984]

11. If $f(x) = \frac{\sin[x]}{[x]}$, $[x] \neq 0$

$= 0$, $[x] = 0$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0} f(x)$ equals

(A) 1 (B) 0
(C) –1 (D) none of these

[IIT - 1985]

12. $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$ for

(A) No value of n
(B) n is any whole number
(C) n = 0 only
(D) n = 2 only **[IIT - 1992]**

13. $\lim\limits_{x\to 0} \left[\frac{x}{\tan^{-1} 2x}\right]$ equals

(A) 0 (B) 1/2
(C) (D) ∞ **[IIT - 1992]**

14. $\lim\limits_{x\to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$ equals

(A) 1 (B) – 1
(C) e^2 (D) e

[IIT - 1993]

15. $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$ equals

(A) $1 + \sqrt{5}$ (B) $-1 + \sqrt{5}$
(C) $-1 + \sqrt{2}$ (D) $1 + \sqrt{2}$

[IIT - 1997]

16. $\lim\limits_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ is equal to

(A) 2 (B) – 2
(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ **[IIT - 1999]**

17. For $x \in R$, $\lim\limits_{x\to\infty} \left(\frac{x-3}{x+2}\right)^x$ equals

(A) e (B) e
(C) e^{-5} (D) e^5 **[IIT - 2000]**

18. $\lim\limits_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$ equals

(A) $-\pi$ (B) π
(C) $\pi/2$ (D) 1 **[IIT - 2001]**

19. The integer n for which

$\lim\limits_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number is

(A) 1 (B) 2
(C) 3 (D) 4 **[IIT - 2002]**

20. If $\lim\limits_{x\to 0} \frac{((a-n)nx - \tan x)\sin nx}{x^2} = 0$, where n is non zero real number, then a is equal to

(A) 0 (B) $\frac{n+1}{n}$

(C) n (D) $n + \frac{1}{n}$

[IIT - 2003]

21. The value of $\lim\limits_{x\to 0} ((\sin x)^{1/x} + (1 + x)^{\sin x})$, where $x > 0$ is

(A) 0 (B) –1
(C) 1 (D) 2 **[IIT - 2006]**

22. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \frac{\pi^2}{16}}$ equals

(A) $\frac{8}{\pi} f(2)$ (B) $\frac{2}{\pi} f(2)$

(C) $\frac{2}{\pi} f(1/2)$ (D) 4f(2) **[IIT - 2007]**

23. The value of $\lim\limits_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4} dt$ is

(A) 0 (B) $\frac{1}{12}$

(C) $\frac{1}{24}$ (D) $\frac{1}{64}$ **[IIT - 2010]**

24. If $\lim\limits_{x\to 0}\left[1 + x\ln(1 + b^2)\right]^{\frac{1}{x}} = 2b\sin^2\theta$, $b > 0$ and $\theta \in (-\pi, \pi)$, then the value of θ is

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$

(C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$ **[IIT - 2011]**

25. If $\lim\limits_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$ then,

(A) a = 1, b = 4 (B) a = 1, b = – 4
(C) a = 2, b = –3 (D) a = 2, b = 3

[IIT - 2012]

D. INTEGER ANSWER TYPE QUESTIONS

26. Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$ then the value of $\frac{m}{n}$ is

[2015]

27. Let α, β ∈ **R** be such that

$$\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x-\sin x}=1.$$ Then 6(α + β) equals **[2016]**

E. MULTIPLE CHOICE QUESTIONS WITH ONE OR MORE THAN ONE CORRECT ANSWER

28. The value of $\lim_{x\to 0^-}\sqrt{\frac{\frac{1}{2}(1-\cos 2x)}{x}}$

(A) 1
(B) –1
(C) 0
(D) none of these

[IIT - 1991]

29. $\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(A) exist and it equals $\sqrt{2}$
(B) exists and it equals – $\sqrt{2}$
(C) does not exist
(D) none of these **[IIT - 1998]**

30. Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$, a > 0. If L is finite, then

(A) a = 2
(B) a = 1
(C) $L=\frac{1}{64}$
(D) $L=\frac{1}{32}$

[IIT - 2009]

31. Let $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$, for x ≠ 1

then

(A) $\lim_{x\to 1-} f(x)$ does not exist
(B) $\lim_{x\to 1-} f(x)=0$
(C) $\lim_{x\to 1+} f(x)=0$
(D) $\lim_{x\to 1+} f(x)$ does not exist **[2017]**

F. SUBJECTIVE PROBLEMS

32. Evaluate $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{3x}},(a>0)$

[IIT - 1978]

33. Evaluate : $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

[IIT - 1980]

35. Use the formula

$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$ to find $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$ **[1982]**

35. Find the value of $\lim_{x\to\infty}\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n.$

[IIT - 2004]

Previous Year's Questions (JEE Main Papers)

1. $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) 1/2
(b) 1
(c) 2
(d) –1/4

[2013]

2. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to

(a) $\pi/2$
(b) 1
(c) $-\pi$
(d) π **[2014]**

3. If $\lim_{x\to 2}\frac{\tan(x-2)\{x^2+(k-2)x-2x\}}{x^2-4x+4}=5,$ then k is equal to

(a) 0
(b) 1
(c) 2
(d) 3 **[2014, online]**

4. If $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{\sin^2 x}$ is equal to

(a) 3
(b) $\frac{3}{2}$

(c) $\frac{5}{4}$ (d) 2 **[2015]**

5. If $\lim\limits_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) 4 (b) 3

(c) 2 (d) $\frac{1}{2}$ **[2015]**

6. Let $p = \lim\limits_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$, then log p is equal to

(a) 2 (b) 1

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$ **[2016]**

7. If $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}-\frac{4}{x^2}\right)^{2x} = e^3$. Then a is equal to

(a) 2 (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ (d) $\frac{2}{3}$ **[2016, online]**

8. $L = \lim\limits_{x\to 0}\frac{(1-\cos 2x)^2}{2x\tan x - x\tan 2x}$ is

(a) 2 (b) $-\frac{1}{2}$

(c) –2 (d) $\frac{1}{2}$ **[2016, online]**

9. $\lim\limits_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$ is equal to

(a) $\sqrt{3}$ (b) $1/\sqrt{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2\sqrt{2}}$ **[2017, online]**

10. If $\lim\limits_{n\to\infty} = \frac{1^a+2^a+\cdots+n^a}{[n+1]^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]} =$ $\frac{1}{60}$ for some positive real number a, then a is equal to

(a) 7 (b) 8

(c) $\frac{15}{2}$ (d) $\frac{17}{2}$ **[2017, online]**

11. $\lim\limits_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals

(a) 1/4 (b) 1/24

(c) 1/16 (d) 1/8 **[2017]**

12. For each $t \in \mathbf{R}$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x\to 0+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]\ldots+\left[\frac{15}{x}\right]\right)$$

(a) is equal to 0

(b) is equal to 15

(c) is equal to 120

(d) does not exist (in **R**) **[2018]**

13. $\lim\limits_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ equals

(a) –1/2 (b) 1/4

(c) 1/2 (d) 1 **[2018]**

14. $\lim\limits_{x\to 0}\frac{(27+x)^{1/3}-3}{9-(27+x)^{2/3}}$ equals

(a) $-\frac{1}{3}$ (b) $\frac{1}{6}$

(c) $-\frac{1}{6}$ (d) $\frac{1}{3}$ **[2018]**

15. $\lim\limits_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$

(a) exists and equals $\frac{1}{4\sqrt{2}}$

(b) does not exist

(c) exists and equals $\frac{1}{2\sqrt{2}}$

(d) exists and equal $\frac{1}{2\sqrt{2}\left(\sqrt{2}+1\right)}$ **[2019]**

16. For each $x \in \mathbf{R}$, let $[x]$ be the greatest integer less than or equal to x. then

$\lim\limits_{x\to 0-}\frac{x([x]+|x|)\sin[x]}{|x|}$ is equal to

(a) –sin 1 (b) 0

(c) 1 (d) sin 1 **[2019]**

17. For each $t \in \mathbf{R}$, Let $[t]$ be the greatest integer less than or equal to t, then

$$\lim_{x\to 1+} \frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$$

(a) equals –1 (b) equals 1

(c) does not exists (d) equals 0 **[2019]**

18. Let $[x]$ denote the greatest integer less than or equal to x. Then

$$\lim_{x\to 0} \frac{\tan(\pi \sin^2 x) + (|x| - \sin(x[x]))^2}{x^2}$$

(a) equal π (b) equals 0

(c) equals $\pi + 1$ (d) does not exist

19. $\lim_{x\to 0} \dfrac{x\cot(4x)}{\sin^2 x \cot^2(2x)}$ is equal to

(a) 2 (b) 0

(c) 4 (d) 1 **[2019]**

20. $\lim_{x\to \pi/4} \dfrac{\cot^3 x - \tan x}{\cos(x + \pi/4)}$ is

(a) 4 (b) $8\sqrt{2}$

(d) 8 (d) $4\sqrt{2}$ **[2019]**

21. $\lim_{x\to 1-} \dfrac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1-x}}$ is equal to

(a) $\dfrac{1}{\sqrt{2\pi}}$ (b) $\sqrt{\dfrac{\pi}{2}}$

(c) $\sqrt{\dfrac{2}{\pi}}$ (d) $\sqrt{\pi}$ **[2019]**

ANSWERS

PRACTICE PROBLEMS—A

1. (i) 1 (ii) does not exist
2. (i) 0 (ii) does not exist
(iii) –1
3. No, we donot conclude anything.
4. No, no, we donot conclude anything.
5. (a) No (b) No
(c) Yes (d) 0
7. π/2 **8.** 0
9. does not exist

CONCEPT PROBLEMS—A

1. (i) – ∞ (ii) –∞
(iii) 1 (iv) 1
2. (i) does not exist (ii) –5
(iii) 4
3. ∞ **4.** does not exist.
5. 0 **6.** 0
7. does not exist **8.** 0

CONCEPT PROBLEMS—B

2. –1 **3.** 1/π
4. 0
6. since limit of g(x) does not exist.

CONCEPT PROBLEMS—C

1. 1
2. L.H.L. = 1, R.H.L. = 2, $\lim_{x\to 1} f(x)$ does not exist.
3. (i) $\lim_{x\to 0} f(x)$ = does not exist
(ii) $\lim_{x\to 1} f(x) = 0$,
(iii) $\lim_{x\to 3} f(x) = 1$
4. (a) 1 (b) 1
(c) 1 (d) 2
5. (a) 2 (b) 2
(c) 1 (d) 2
6. (i) does not exist, 0 (ii) 0
7. (i) 0 (ii) $-\dfrac{\pi}{2}$
(iii) 1/2 (iv) does not exist
8. limit does not exist since function is undefined in the neighbourhood.
9. 0 **11.** 0 , 1
12. does not exist **13.** 8
14. (a) 3 (b) does not exist
(c) does not exist
15. –3
16. $\lim_{x\to 2^-} f(x)$ is not known but $\lim_{x\to -2^+} f(x) = -7$.

PRACTICE PROBLEMS—B

17. (i) –1/3 (ii) 1

18. (i) does not exist (ii) 0

19. (i) $\frac{\pi}{2}$ (ii) 0

20. 0 **21.** –2

22. 0 **23.** does not exist.

24. 1/2, 0

25. (i) ∞, ∞ (ii) ∞ , 0

(iii) 1/2, 0 (iv) 0, 0

26. (i) –2, 2 (ii) 1, 4

(iii) a^2, $a^2 - 1$ (iv) 2, 5

27. (i) 2, 2 (ii) 10n –1, 10n

(iii) $\frac{1}{2}, -\frac{1}{2}$

CONCEPT PROBLEMS—D

1. (i) determinate form (ii) $\frac{0}{0}$ form

(iii) (0 × ∞) form (iv) (∞ – ∞) form

(v) $(0)^0$ form (vi) $(\infty)^0$ form

(vii) $(1)^\infty$ form (viii) determinate form

5. (i) 0 (ii) 0

(iii) 0 (iv) 0

7. 7

8. (i) 4 (ii) –2

PRACTICE PROBLEMS–C

9. 0 if a < 0, dne if a ≥ 0

10. (a) 0 (b) ∞

(c) 0 (d) – ∞

(e) ∞ (f) – ∞

11. ∞ **12.** does not exist

CONCEPT PROBLEMS—E

1. (i) $\frac{15}{11}$ (ii) – ∞ (limit dne)

(iii) 2x (iv) ∞ (limit dne)

2. (i) 0 (ii) ∞ (limit dne)

3. (a) 0 (b) 3/4

(c) dne (d) 2/3

4. ∞ (limit dne).

5. $f(5^+) = 1$, $f(5^-) = 0$; limit dne

6. (i) 1 (ii) – ∞ (limit dne)

(iii) 0 (iv) 1

PRACTICE PROBLEMS—D

7. (i) 1 (ii) –1/4

(iii) 1 (iv) –7.2

8. (i) $\frac{1}{4}$ (ii) –3

(iii) 3/4 **9.** c = – 6 limit = 4.

10. (i) 2, 4 (ii) 0, –1

CONCEPT PROBLEMS—F

1. 2

2. (i) $-\sqrt{5}$ (ii) $\frac{2}{3}$

(iii) $\frac{1}{3}$ (iv) $\frac{1}{2}$

3. –1/4 **4.** 1

5. a = 2, b = 1

PRACTICE PROBLEMS—E

6. $\log_a 6$

7. (i) $\sqrt{7}/4$ (ii) $\frac{1}{8\sqrt{3}}$

(iii) 32 (iv) 1.2

8. $-\frac{1}{3}$ **9.** $\frac{34}{23}$

10. (i) 2/3 (ii) 1/6

(iii) 9/25

CONCEPT PROBLEMS—G

1. (i) –1/9 (ii) – 2/3

(iii) 0 (iv) 2

2. (i) 1/3 (ii) 24

(iii) 1/3 (iv) – 1/2

3. $\frac{2}{5}$ **4.** 1/2

5. limit does not exist.

PRACTICE PROBLEMS—F

7. (i) –1 (ii) 0

(iii) ∞ (iv) –1

8. (i) a/m–b/n (ii) a/m + b/n

(iii) –19/3 (iv) ∞

9. (i) 1/80 (ii) –1/5040

(iii) 1/2 (iv) – 11/6

10. 13/60 **11.** 1/2

12. $-\frac{1}{2}, 1, \frac{1}{24}$ **13.** $-3, \frac{9}{2}$

14. 72, $a = \frac{1}{4}$, $b = \frac{3}{4}$, $l = -\frac{1}{32}$

CONCEPT PROBLEMS—H

1. (i) $\frac{\pi}{180}$ (ii) 0

2. (i) 2/5 (ii) 8/3

(iii) – 3/2 (iv) 25/6

3. (i) 4 (ii) $\frac{n}{m}$

(iii) $\log_2 e$ (iv) $\frac{8}{7}$

4. (i) $\frac{1}{\ln 3}$ (ii) $\frac{1}{2}$

5. (i) $\frac{1}{m}-\frac{1}{n}$ (ii) $\ln\frac{a}{b}$

6. (i) $\frac{2}{3}$ (ii) -1

(iii) $\frac{4}{3}$ (iv) 4/5

7. (i) No (ii) No
(iii) No (iv) 0

8. (i) 1 (ii) –2
(iii) 3/4 (iv) 1/8

9. (i) 1/2 (ii) 25/9
(iii) 2 (iv) 3

10. (i) ln(8/7)/ ln(6/5) (ii) 9/4
(iii) (ln 5 .ln 4)/(ln 3 .ln 6) (iv) 1/3

11. (i) ∞ (ii) 1/2

12. 1/2

13. (i) $-\sqrt{2}/4$ (ii) 1

(iii) $-\frac{1}{4}$ (iv) ∞

14. (i) $\frac{2}{\pi}$ (ii) $\frac{1}{e}$

(iii) – *ln* 2 (iv) 5

PRACTICE PROBLEMS—G

17. – 2 **18.** tan10–10

19. 0, if n > m, 1 if n = m, ∞ if n < m.

20. – sin a

21. (i) a (ii) $\frac{1}{324}$

(iii) – 6 (iv) 1

22. (i) 3 (ii) 2/3
(iii) 2/3

23. (i) 3/5 (ii) 3 *ln* 3
(iii) 16

24. 0 for n = 1; 4.9 for n = 2

25. (i) –1/3 (ii) ln 4

26. (i) $\frac{\sin 2\beta}{2\beta}$ (ii) $-\frac{a}{\pi}$

27. (i) $-\infty$, 1 , No (ii) $-\sqrt{2}, \sqrt{2}$, No

28. 1

29. (i) $\sqrt{2}$ (ii) $\frac{1}{\sqrt{2}}$
(iii) $a^a \ln ea$

30. (i) 1/3 (ii) 1

31. (i) $\frac{1}{\sqrt{2\pi}}$ (ii) $\frac{1}{2}$

32. (i) $5\sqrt{2}$ (ii) 1

33. (i) –2 (ii) ∞
(iii) –1/12 (iv) 3/2

CONCEPT PROBLEMS—I

1. (i) 0 (ii) 8
(iii) 2 (iv) ∞

2. (i) 25 (ii) 1
(iii) 1 (iv) 9

3. (i) 3 (ii) 0

(iii) $-\frac{\pi}{2}$ (iv) 1

5. (i) 0 (ii) –3
(iii) 1 (iv) – 1/3

6. 0

PRACTICE PROBLEMS—H

7. $\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}$

8. $$\text{limit} = \begin{cases} 0, \text{ if } m > n \\ -3, \text{ if } m = n \\ \infty, \text{ if } m < n \text{ and } n-m \text{ odd} \\ -\infty, \text{ if } m < n \text{ and } n-m \text{ even} \end{cases}$$

9. (i) $\frac{2}{\sqrt{3}}$ (ii) $\frac{1}{24}$

(iii) $5^{10}/3^{25}$ (iv) ∞

10. 5 **11.** $\sqrt{2}$

12. (i) ∞ (ii) ∞
(iii) 0 (iv) 0

13. (i) 1 (ii) 0
(iii) does not exist (iv) 0

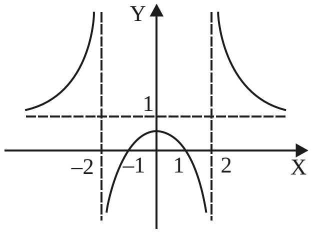
14. 100 **15.** 1

16. –2

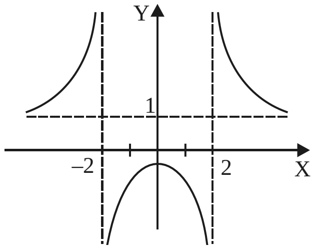
CONCEPT PROBLEMS—J

1. (i) x = –2, y = 2 (ii) y = 2

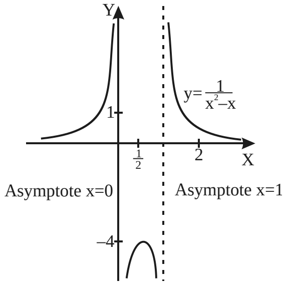
2. (i)



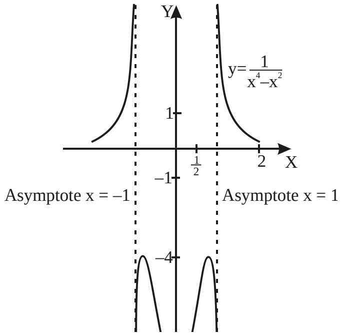
(ii)



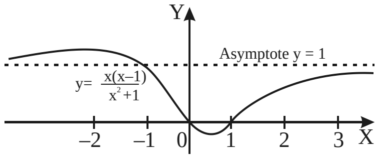
3. (i).



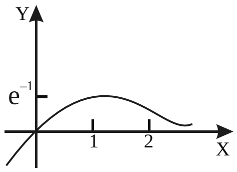
(ii)



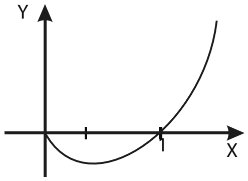
(iii)



4. (i)



(ii)



CONCEPT PROBLEMS—K

1. (i) 5 (ii) 8/3
(iii) ∞ (iv) 4

2. (i) 1/2 (ii) 1/3
(iii) 5 (iv) 1/2

PRACTICE PROBLEMS—I

6. (i) $\frac{1}{5}$ (ii) 3
(iii) 0 (iv) 1

7. 1 **8.** (i) $\frac{4}{3}$
(ii) $\frac{1}{2}$ (iii) $\frac{1}{4}$

9. $\log_2^2 x$ **10.** 3

11. (i) $-\frac{1}{2}$ (ii) –1
(iii) $\frac{\sin x}{x}$

CONCEPT PROBLEMS—L

1. (i) 0 (ii) $2a/\pi$
(iii) a/π (iv) 0

2. (i) 0 (ii) 0
(iii) 0

3. (i) – 1/2 (ii) 0
(iii) 0 (iv) 1/2

4. (i) 0 (ii) 0
(iii) $-\sqrt{2}$ (iv) 0

5. (i) ∞ (ii) 2/3
(iii) – 3/5 (iv) (a – c)/2

6. Both are wrong. The limit is –13.

7. (i) $\frac{1}{2}$ (ii) $-\frac{1}{2}$
(iii) $-\frac{3}{2}$ (iv) 0

8. (i) ∞ (ii) 4/3
(iii) 2

9. (i) ln a
(ii) ∞ for n > 1 ; a for n = 1 ; 0 for n < 1
(iii) a (iv) 0

PRACTICE PROBLEMS—J

11. (i) 2/9 (ii) $-\frac{5}{2}$
(iii) $-\frac{1}{2}$ (iv) ∞

12. (i) 1/2 (ii) 2
(iii) $\frac{\pi^2}{8}$ (iv) $\frac{a+b+c}{3}$

13. 1, $\frac{1}{2}$ **14.** 2

15. a = 1, b = –1. **16.** $\frac{5}{4}$

17. (i) -2 (ii) $\ln\left(\frac{1}{2}\right)$

(iii) $\frac{\pi}{4}$ (iv) $-\pi^2/4$

18. (i) $\frac{1}{4\sqrt{2}}$ (ii) 1

(iii) $-\frac{1}{4}$

19. (i) 0 (ii) 1/2

(iii) 1/120

Concept Problems—M

1. $\sqrt{\frac{2}{3}}$ **2.** 0

3. 0 **4.** 0

5. 5 **6.** $\frac{1}{e}$

Practice Problems—K

7. $\frac{1}{4}$ **8.** $\frac{1}{4}$

9. e^2 **10.** 1

11. 0 **12.** e^{π}

13. 1 **14.** 99

Concept Problems—N

2. (i) e^{-1} (ii) e

(iii) $e^{-\frac{2}{3}}$ (iv) e^3

3. (i) $\frac{1}{e}$ (ii) $\sqrt{e}$

(iii) ae (iv) 1

4. (i) e^5 (ii) 1

(iii) ∞ if $n < 1$; e if $n = 1$; 1 if $n > 1$.

(iv) 1/e

Practice Problems—L

5. (i) $e^{-\frac{m^2 n}{2}}$ (ii) $e^{1/6}$

(iii) $e^{-x^2/2}$ (iv) n!

6. 0 if $0 < a < 1$; lna if $a > 1$.

7. (i) 1 (ii) $e^{-1/2}$

(iii) 1 (iv) $e^{-1/3}$

8. (i) $e^{-1/2}$ (ii) $e^{1/2}$

(iii) $e^{1-\alpha}$ (iv) $e^{\frac{e}{1-e}}$

9. (i) 1 (ii) $e^{\frac{n-1}{2a}}$

(iii) e^{-2} (iv) $e^{2\alpha}$

10. $\lambda = 1$, $\mu \in R$

Concept Problems—O

1. $\sqrt{5}$ **2.** limit is 1

3. limit is 1 at $x = -1$ and 1; limit is 0 at $x = 0$.

5. $g(x) = -\frac{1}{x^2}$, $h(x) = \frac{1}{x^2}$, limit = 0

6. b

Practice Problems—M

8. 0 **9.** 4

10. (i) 0 (ii) 0

(iii) 2 (iv) dne

11. $\frac{1}{2}$

13. (i) 1 (ii) 0

(iii) 0 (iv) 0

14. $\frac{x}{3}$ **15.** 1

16. they have same limits as before.

17. 0 **18.** 1

19. 1

Concept Problems—P

1. (i) $\frac{1}{2}(\ln a)^2$ (ii) 1

(iii) 1

2. (i) 1 (ii) 1

(iii) 2 (iv) $\ln\frac{a}{b}$

3. (i) 0 (ii) 2/3

(iii) -2

5. 1/2

6. (i) 2/3 (ii) $\pi^2/4$

(iii) $\sqrt{5}$

7. (i) 1/3 (ii) 1/e

(iii) 1/2

Practice Problems—N

8. (i) -9 (ii) 2/3

9. (i) 4 (ii) $-\frac{1}{2}$

14. (i) 1/18 (ii) 2/3

16. $e^{\sin^2 y}$ **17.** 0

Concept Problems—Q

3. $\frac{e}{e-1}$ **4.** $1\frac{1}{3}$

PRACTICE PROBLEMS—O

7. $\lim\limits_{n\to\infty} AC_n = \frac{\ell}{3}$ **8.** $\frac{ab}{2}$

9. $\frac{-Ct}{2\alpha}\cos\alpha t$

10. $Q_t^{(n)} = Q_0\left(1+\frac{kt}{n}\right)^n$, where k is the proportionality factor (law of compound interest) ; $Q_t = Q_0e^{kt}$.

CONCEPT PROBLEMS—R

1. If $x \neq k\pi$ $(k\in I)$, then $\cos^2 x < 1$ and $y = 0$; but if $x = k\pi$, then $\cos^2 x = 1$ and $y = 1$.

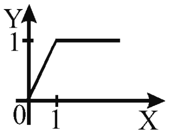
2. (i) $y = 1$ when $0 \le x \le 1$; $y = x$ when $1 < x < \infty$.
(ii) $x > 1$
(iii) $y = |x|$
(iv) $y = \pi$ when $x < 0$; $y = \frac{\pi}{2}$ when $x = 0$; $y = 0$ when $x > 0$.

3. 2

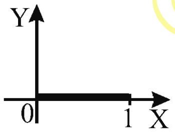
PRACTICE PROBLEMS—P

4. $f(x) = \begin{cases} \sin x & x > 0 \\ \frac{\tan \pi x^2}{x^2} & x < 0 \end{cases}$; $\lim\limits_{x\to 0^+} = 0$; $\lim\limits_{x\to 0^-} = \pi$; limit does not exist

5. (a)



(b)



6. 0 **7.** $\left(\frac{a}{d}-\frac{c}{f}\right)$

8. $x \in [0, \sin 1)$

9. $\phi(x) = 1$, $(x > 0)$; $\phi(x) = 0$, $(x = 0$; $\phi(x) = -1$, $(x < 0)$.

10. ln 5 **11.** 40

12. $\frac{14}{3}$ **13.** $1/\pi$

14. $\frac{4}{e}$ **15.** 1

16. 8 **17.** 1

OBJECTIVE EXERCISES

1. A **2.** A **3.** B
4. C **5.** B **6.** B
7. C **8.** B **9.** A
10. C **11.** A **12.** A
13. D **14.** A **15.** B
16. C **17.** B **18.** B
19. A **20.** B **21.** B
22. B **23.** C **24.** B
25. A **26.** B **27.** D
28. D **29.** C **30.** B
31. A **32.** C **33.** B
34. D **35.** B **36.** B
37. A **38.** D **39.** A
40. A **41.** B **42.** A
43. D **44.** C **45.** B
46. B **47.** D **48.** C
49. B **50.** D **51.** D
52. D **53.** B **54.** A
55. B **56.** D **57.** B
58. C **59.** A **60.** A
61. C **62.** B **63.** B
64. D **65.** B **66.** D
67. A **68.** A **69.** B
70. A **71.** C **72.** C
73. B **74.** A **75.** B
76. ABD **77.** BD **78.** ABD
79. CD **80.** ABCD **81.** ABCD
82. ABCD **83.** A **84.** A
85. ABCD **86.** AB **87.** BC
88. BC **89.** ABD **90.** BCD
91. BC **92.** ABC **93.** BCD
94. ABCD **95.** ABD **96.** D
97. A **98.** A **99.** B
100. C **101.** A **102.** D
103. B **104.** D **105.** D
106. B **107.** A **108.** A
109. C **110.** A **111.** B
112. D **113.** C **114.** A
115. D **116.** A **117.** B
118. C **119.** C **120.** D
121. (A)–(Q) ; (B)–(R) ; (C)–(P) ; (D)–(P)
122. (A), (Q) ; (B)–(P) ; (C)–(R); (D), (S)
123. (A)–(S) ; (B)–(R) ; (C)–(P) ; (D)–(Q)
124. (A)–(S) ; (B)–(P) ; (C)–(Q) ; (D)–(R)
125. (A)–(R) ; (B)–(S) ; (C)–(P) ; (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

1. (i) $\frac{2\sin a}{\cos^3 a}$ (ii) -1
2. (i) 7/4 (ii) $-\frac{4}{5}$
3. (i) 1/3 (ii) 6
4. (i) $\frac{1}{2^8}$ (ii) $-\frac{1}{9}$
5. $-\cos a$ **6.** $\frac{2n!}{(n!)^2}$
7. e^2 **8.** 2
9. (1/2) **10.** $\cos^{-1}x_0$

11. $\frac{1}{24}$ **12.** $-\frac{2}{3}$

13. – 1 **15.** $\frac{1}{2}$

16. 1/2

17. (a) $-\frac{3\sqrt{2}}{2}$ (b) $5\sqrt{\frac{2}{3}}$

19. $\frac{-7e}{16}$ **20.** $4^{-2}, 4^{-4}$

21. $\phi(x) = f(x), (|x| > 1); \phi(x) = g(x), (|x| < 1); \phi(x) = \frac{1}{2}\{f(x) + g(x)\}, (x = 1)$; and $\phi(x)$ is undefined when $x = -1$.

22. a + b – 1

23. 20

24. 1 if 0 < a < 1; a if a > 1

25. The function is equal to 0 when x is rational and to 1 when x is irrational.

27. The function is equal to 0 except when x is an integer, and then it is equal to 1.

26. m = 1, b = $-\frac{2}{3}$

28. The function is equal to $\phi(x)$ unless x is an integer, and then it is equal to $\Psi(x)$.

30. The former limit is 0, the second does not exist.

32. $4 \cdot e^{\frac{\pi}{\sqrt{3}} - 3}$ **33.** 4

34. 400 **35.** 256

TARGET EXERCISES for JEE ADVANCED

1. 1

2. –1/a for a ≠ 0, for a = 0 there is no limit.

3. $e^{\frac{p-1+a\ln a}{2a}}$ **4.** $\frac{3}{2}$

5. 41 **6.** –24

7. –1 **8.** does not exist

10. If n is even, limit does not exist. If n is odd limit is 0**)**

11. ∞ **12.** 3

14. $\begin{cases} 0 & \text{if} \quad n < 3 \\ 1/2 & \text{if} \quad n = 3 \\ \infty & \text{if} \quad n > 3 \end{cases}$ **16.** 3

17. $\frac{1}{\sqrt{2}}$ **18.** a^2

21. $\frac{1}{2}\ln 2$ **23.** $-\sqrt{a}$

24. If m is a positive integer, $u_n = 0$ for n > m.

Otherwise $\frac{u_{n+1}}{u_n} = \frac{m-n}{n+1}x \to -x$, unless x = 0

25. $\begin{cases} 1, & n \in I \\ 0, & n \notin I \end{cases}$ **26.** $\begin{cases} \psi(x), & n \in I \\ \phi(x), & n \notin I \end{cases}$

28. If x is rational then $\sin^2(m!nx)$, and therefore $\text{sgn}\{\sin^2(m!\pi x)\}$, is equal to zero from a certain value of m onwards; if x is irrational then $\sin^2(m!\pi x)$ is always positive, and so sgn $\{\sin^2(m!\pi x)\}$ is always equal to 1

29. $\frac{4}{\pi}$ **30.** $\frac{1+\sqrt{1+4a}}{2}$

31. 0 **32.** e

34. $(k + 1)^{-2}$

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. 1 **2.** –1 **3.** $\frac{1}{128r}$ **4.** e^5
5. e^2 **6.** 1 **7.** –1 **8.** F
9. C **10.** B **11.** D **12.** B
13. B **14.** C **15.** B **16.** C
17. C **18.** B **19.** C **20.** D
21. C **22.** A **23.** B **24.** D
25. B **26.** 7 **27.** 2 **28.** B
29. C **30.** AC **31.** B, D **32.** $\frac{2}{3\sqrt{3}}$
33. $a^2 \cos a + 2a \sin a$ **34.** 2 ln 2
35. $1 - \frac{2}{\pi}$

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. (c) **2.** (d) **3.** (d) **4.** (b)
5. (c) **6.** (c) **7.** (a) **8.** (a)
9. (b) **10.** (a) **11.** (c) **12.** (c)
13. (c) **14.** (c) **15.** (a) **16.** (a)
17. (d) **18.** (d) **19.** (d) **20.** (c)
21. (c)

HINTS & SOLUTIONS

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. $\lim\limits_{x\to 0}\left[\dfrac{|\sin x|}{|x|}\right]$

RHL as $x \to 0^{+}$, $\left|\dfrac{\sin x}{x}\right| \to 1^{-}$

So, $\lim\limits_{x\to 0^{+}}\left[\dfrac{|\sin x|}{|x|}\right] \to 0$

LHL as $x \to 0^{-}$, $\left|\dfrac{\sin x}{x}\right| \to 1^{-}$

So, $\lim\limits_{x\to 0^{-}}\left[\dfrac{|\sin x|}{|x|}\right] \to 0$

Hence, A is correct.

2. $\lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x.\tan x}\right]$

RHL as $x \to 0^{+}$, $\dfrac{x}{\sin x}\cdot\dfrac{x}{\tan x} \to 1^{-}$

LHL as $x \to 0^{-}$, $\dfrac{x}{\sin x}\cdot\dfrac{x}{\tan x} \to 1^{-}$

So, $\lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x \tan x}\right] \to 0$

Hence, A is correct.

3. $\lim\limits_{x\to\infty}\sqrt{x^4 - x^2 + 1} - ax^2 - a$

$$\lim_{x\to\infty}\frac{x^4 - x^2 + 1 - \left(ax^2 + a\right)^2}{\sqrt{x^4 - x^2 + 1} + ax^2 + a}$$

$$\lim_{x\to\infty}\frac{\left(1-a^2\right)x^4 - \left(2a^2+1\right)x^2 + \left(1-a^2\right)}{\sqrt{x^4 - x^2 + 1} + ax^2 + a}$$

For limit to exist, coeff of $x^4 = 0 \Rightarrow a^2 = 1$

$$\lim_{x\to\infty}\frac{-\left(2a^2+1\right)x^2 + \left(1-a^2\right)}{\sqrt{x^4 - x^2 + 1} + ax^2 + a} \Rightarrow \frac{-\left(2a^2+1\right)}{1+a}$$

$\Rightarrow$ It exist only for a = 1

Hence, B is correct.

4. $\lim\limits_{h\to 0}\dfrac{e^{\sin h} - (1+\sin\ h)}{\left(\tan^{-1}\sin\ h\right)^2}$

Apply L – Hospital Rule

$$\lim_{h\to 0}\frac{e^{\sin h}\cos h - \cos h}{2\tan^{-1}\sin h \times \dfrac{1}{1+\sin^2 h} \times \cos h}$$

$$\lim_{h\to 0}\frac{\left(e^{\sin h} - 1\right)\left(1+\sin^2 h\right)}{2\tan^{-1}\sin h}$$

$\Rightarrow \dfrac{1}{2}$

Hence, C is correct.

5. $\lim\limits_{x\to\infty}\left(1+a^2\right)^x \cdot \dfrac{b}{\left(1+a^2\right)^x} = b$

$\Rightarrow$ As $\left(1+a^2\right)^x \to \infty$ for $x \to \infty$

Hence, B is correct.

6. $\lim\limits_{x\to\infty} x^2\left[\tan^{-1}\left(\dfrac{2x^2+1}{x^2+2}\right) - \tan^{-1}2\right] \Rightarrow 0\times\infty$ form

$$\to \lim_{x\to\infty}\frac{\tan^{-1}\left(\dfrac{2x^2+1}{x^2+2}\right) - \tan^{-1}2}{\frac{1}{x^2}} \Rightarrow \frac{0}{0}\text{form}$$

Apply L – Hospital Rule.

$$\lim_{x\to\infty}\frac{1}{1+\left(\dfrac{2x^2+1}{x^2+2}\right)^2} \times \frac{\left(\left(x^2+2\right)(4x) - \left(2x^2+1\right)2x\right)}{\left(x^2+2\right)^2} \times \frac{1}{-2x^{-3}}$$

$$\lim_{x\to\infty} -\frac{1}{2} \times \frac{x^3(6x)}{\left(2x^2+1\right)^2 + \left(x^2+2\right)^2} \Rightarrow -3\times\frac{1}{5} = \frac{-3}{5}$$

Hence, B is correct.

7. $f(x) = a\left(x^2 - \pi^2\right)$

$f\left(\dfrac{\pi}{2}\right) = \dfrac{-3a\pi^2}{4} = \dfrac{-3\pi^2}{4}$

Hence, $a = 1$

$$\lim_{x\to -\pi}\frac{(x-\pi)(x+\pi)}{\sin\sin x}$$

Put x = $-\pi + h$

$$\lim_{h\to 0}\frac{(-2\pi + h)(h)}{-\sin\sinh} \times \frac{\sinh}{\sinh} = 2\pi$$

Hence, C is correct.

8. $\lim\limits_{n\to\infty}\sum\limits_{x=1}^{20}\cos^{2n}(x-10)$

$\Rightarrow \lim\limits_{n\to\infty}\left[\cos^{2n}9 + \cos^{2n}8 + ... + \cos^2 0 + \cos^2 1 + ... + \cos^2 10\right]$

All except $\cos^2 0$ will tend to zero & $\cos^2 0 = 1$

Hence, B is correct.

9. $$\lim_{x\to1}\frac{1+\sin\pi\left(\frac{3x}{1+x^2}\right)}{1+\cos\pi x}\Rightarrow\frac{0}{0}\text{ form}$$

Apply L – Hospital Rule,

$$\lim_{x\to1}\frac{\cos\pi\left(\frac{3x}{1+x^2}\right)\times3\pi\times\frac{\left(1+x^2\right)-x.2x)}{\left(1+x^2\right)^2}}{-\pi\sin\pi x}$$

$$\lim_{x\to1}-3\frac{(1-x)}{\sin\pi x}\times\frac{(1+x)}{\left(1+x^2\right)^2}\times\left(\cos\pi\left(\frac{3x}{1+x^2}\right)\right)\to0$$

Hence, A is correct.

10. $$\lim_{x\to2^-}\frac{\cos(2x-4)-33}{2}\to-16$$

$$\lim_{x\to2^-}\frac{x^2|4x-8|}{x-2}=\lim_{x\to2^-}\frac{4x^2|x-2|}{x-2}=-16$$

Applying Sandwich Theorem,

$$\lim_{x\to2^-}f(x)=-16$$

Hence, C is correct.

11. $$\lim_{x\to0}\left\{(1+x)^{2/x}\right\}$$

As $\lim\limits_{x\to0}(1+x)^{2/x}\to e^2\ \&\ e^2\approx7.29$

So, $\lim\limits_{x\to0}(1+x)^{2/x}=e^2-7$

Hence, A is correct.

12. $$\lim_{x\to2}\left[\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{x+\sqrt{2x}}{x-2}-\frac{\sqrt2}{\sqrt x-\sqrt2}\right)^{-1}\right]$$

$$\lim_{x\to2}\left[\frac{(x-2)\left(x^2+4+2x\right)}{x(x-2)(x+2)}-\frac{(x-2)\left(\sqrt x-\sqrt2\right)}{\sqrt x(x-2)-\sqrt2(x-2)}\right]=\frac{1}{2}$$

Hence, A is correct.

13. $$\lim_{x\to a^-}\sqrt{a^2-x^2}\cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$$

Put x = a cosθ

$$\lim_{\theta\to0^+}a\sin\theta\cot\left[\frac{\pi}{2}\tan\theta/2\right]$$

$$\lim_{\theta\to0^+}\frac{a\sin\theta}{\tan\left(\frac{\pi}{2}\tan\frac{\theta}{2}\right)}\times\frac{\frac{\pi}{2}\tan\theta/2}{\frac{\pi}{2}\tan\theta/2}\times\frac{\theta/2\times2}{\theta}$$

$$\Rightarrow\frac{4a}{\pi}$$

Hence, D is correct.

14. $$\lim_{x\to0}\frac{2(\tan x-\sin x)-x^3}{x^5}$$

Use series expansion,

$$\lim_{x\to0}\frac{\left[2\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\right)-\left(x-\frac{1}{6}x^3+\frac{x^5}{120}+....\right)\right]-x^3}{x^5}=\frac{1}{4}$$

Hence, A is correct.

15. $$\lim_{x\to0}\frac{1-\cos x+2\sin x-\sin^3x-x^2+3x^4}{\tan^3x-6\sin^2x+x-5x^3}$$

Apply L – Hospital Rule

$$\lim_{x\to0}\frac{\sin x+2\cos x-3\sin^2x\cos x-2x+12x^3}{3\tan^2x\sec^2x-12\sin x\cos x+1-15x^2}=2$$

Hence, B is correct.

16. $$\lim_{x\to0}\frac{\log\cos x}{\sqrt[p]{\cos x}-\sqrt[q]{\cos x}}\Rightarrow\frac{0}{0}\text{ form}$$

$$\lim_{x\to0}\frac{1}{\cos x}\times(-\sin x)\times\frac{1}{\frac{1}{p}(\cos x)^{\frac{1}{p}-1}(-\sin x)-\frac{1}{q}(\cos x)^{\frac{1}{q}-1}(-\sin x)}$$

$$\Rightarrow\frac{pq}{q-p}$$

Hence, C is correct.

17. $$\lim_{n\to\infty}\left(1+\frac{1}{5}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{5^4}\right)............\left(1+\frac{1}{5^{2^n}}\right)$$

multiply & divide by $\left(1-\frac{1}{5}\right)$

$$\lim_{n\to\infty}\frac{\left(1-\frac{1}{\left(5^{2^n}\right)^2}\right)}{4/5}\Rightarrow\frac{5}{4}$$

Hence, B is correct.

18. $$\lim_{x\to0}\left[\frac{e^{x\ln\left(2^x-1\right)}-\left(2^x-1\right)^x\sin x}{e^{x\ln x}}\right]^{1/x}\Rightarrow1^\infty\text{ form}$$

$$e^{\lim\limits_{x\to 0}\left[\frac{e^{x\ln\left(2^x-1\right)}-\left(2^x-1\right)^x\sin x-e^{x\ln x}}{xe^{x\ln x}}\right]}$$

$$e^{\lim\limits_{x\to 0}\left[\frac{\left(2^x-1\right)^x-\left(2^x-1\right)^x\sin x-x^x}{x.x^x}\right]}$$

$$e^{\lim\limits_{x\to 0}\left[\frac{\left(\frac{2^x-1}{x}\right)^x-1}{x}-\left(\frac{2^x-1}{x}\right)^x\frac{\sin x}{x}\right]}$$

$$e^{[\ln\ln 2-1]}=e^{\ln\left(\frac{\ln 2}{e}\right)}=\frac{\ln 2}{e}$$

Hence, B is correct.

19. $\lim\limits_{n\to\infty}\left(\dfrac{n!}{(mn)^n}\right)^{1/n}=\dfrac{1}{m}\lim\limits_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n}$

Let $y=\lim\limits_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^n$

$\Rightarrow \ln y=\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{r=1}^{n}\log\left(\dfrac{r}{n}\right)$

$=\int\limits_0^1 \log x\,dx$

$=x\log x-x\Big|_0^1=-1-\left(\lim\limits_{x\to 0}x\log x\right)$

$=-1-\lim\limits_{x\to 0}\dfrac{\log x}{1/x}$

$=-1-\lim\limits_{x\to 0}(-x)$

$=-1$

So, $y=\dfrac{1}{e}$

$\lim\limits_{n\to\infty}\left(\dfrac{n!}{(mn)^n}\right)^{1/n}=\dfrac{1}{me}$

Hence, A is correct.

20. $\lim\limits_{y\to 0}\left[\lim\limits_{x\to\infty}\dfrac{e^{x\ln\left(1+\frac{ay}{x}\right)}-e^{x\ln\left(1+\frac{by}{x}\right)}}{y}\right]$

$\lim\limits_{x\to\infty}\lim\limits_{y\to 0}\dfrac{\left(1+\dfrac{ay}{x}\right)^x-\left(1+\dfrac{by}{x}\right)^x}{y}$

$\lim\limits_{y\to 0}\dfrac{e^{ay}-e^{by}}{y}\Rightarrow\lim\limits_{y\to 0}\dfrac{\left(e^{ay}-1\right)}{y}-\lim\limits_{y\to 0}\dfrac{\left(e^{by}-1\right)}{y}$

$\Rightarrow a-b$

Hence, B is correct.

21. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$

Rationalise

$$\lim_{x\to 0}\frac{\tan^{-1}3x+\sin^{-1}3x}{-\left(\tan^{-1}2x+\sin^{-1}2x\right)}$$

$$\times\frac{\left(\sqrt{1-\sin^{-1}2x}+\sqrt{1+\tan^{-1}2x}\right)}{\left(1+\tan^{-1}3x\right)^{2/3}+\left(1-\sin^{-1}3x\right)^{2/3}+\left(\left(1+\tan^{-1}3x\right)\left(1+\sin^{-13x}\right)\right)^{1/3}}$$

$=-\dfrac{6\times 2}{4\times 3}=-1$

Hence, B is correct.

22. $\lim\limits_{x\to 1}\dfrac{\left(\ln(1+x)-\ln 2\right)3\left(4^{x-1}-x\right)}{\left[(7+x)^{1/3}-(1+3x)^{1/2}\right]\sin(x-1)}$

Put x = 1+h

$3\lim\limits_{h\to 0}\dfrac{\left(\ln(2+h)-\ln 2\right)\left(4^h-(1+h)\right)}{\left[(8+h)^{1/3}-(4+3h)^{1/2}\right]\sinh}$

$3\lim\limits_{h\to 0}\dfrac{\ln\left(1+{}^{h}\!/_{2}\right)\left(4^h-1-h\right)}{\left[(8+h)^{1/3}-(4+3h)^{1/2}\right]\sinh}$

$\dfrac{3\times\frac{1}{2}\times[\ln 4-1]}{\frac{1}{3}(8)^{-2/3}-3\times\frac{1}{2}(4)^{-1/2}}$

$\dfrac{3}{2}\dfrac{\ln 4/e}{\left(\frac{1}{12}-\frac{3}{4}\right)}=\dfrac{9}{4}\ln\dfrac{e}{4}$

Hence, B is correct.

23. $\lim\limits_{x\to-\infty}\dfrac{x^5\tan\dfrac{1}{\pi x^2}+3|x|^2+7}{|x|^3+7|x|+8}$

Put $x=-\dfrac{1}{y}$

$$\lim_{y\to 0}\frac{-\frac{1}{y^5}\tan\frac{y^2}{\pi}+\frac{3}{y^2}+7}{\frac{1}{y^3}+\frac{7}{y}+8}$$

$$\lim_{y\to 0}\frac{7y^5+3y^3-\tan\frac{y^2}{\pi}}{8y^5+7y^4+y^2}$$

$$\lim_{y\to 0}\frac{7y^3+3y-\frac{\tan y^2/\pi}{y^2}}{8y^3+7y^2+1}=-\frac{1}{\pi}$$

Hence, C is correct.

24. $$\lim_{x\to 0}\frac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(\text{.........}\cos^2 x\right)\right)\right)}{\sin\left[\pi\left(\frac{\sqrt{x+4}-2}{x}\right)\right]}$$

$$\lim_{x\to 0}\frac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(\text{.........}\cos^2 x\right)\right)\right)}{\sin\left[\frac{\pi\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}{x\left(\sqrt{x+4}+2\right)}\right]}$$

$$\lim_{x\to 0}\frac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(\text{.........}\cos^2 x\right)\right)\right)}{\sin\left(\frac{\pi}{\sqrt{x+4}+2}\right)}=\sqrt{2}$$

Hence, B is correct.

25. $$\lim_{x\to\infty}\frac{\left(\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right)\times\left(\sqrt{a^2x^2+ax+1}+\sqrt{a^2x^2+1}\right)}{\left(\sqrt{a^2x^2+ax+1}+\sqrt{a^2x^2+1}\right)}$$

$$=\lim_{x\to\infty}\frac{ax}{\sqrt{a^2x^2+ax+1}+\sqrt{a^2x^2+1}}$$

$$=\lim_{x\to\infty}\frac{a}{\sqrt{a^2+\frac{a}{x}+\frac{1}{x^2}}+\sqrt{a^2+\frac{1}{x^2}}}=\frac{1}{2}$$

Now, $$\lim_{x\to\infty}\frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\times\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{1+1/\sqrt{x}}}{\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}}+1}\Rightarrow\frac{1}{2}$$

So, $K = 1$

Hence, A is correct.

26. $$\lim_{x\to\infty}x^{-4}\left(\ln x\right)^{10}=\lim_{x\to\infty}\frac{\left(\ln x\right)^{10}}{x^4}=\frac{\infty}{\infty}\text{ form}$$

Apply L – Hospital Rule,

$$\lim_{x\to\infty}\frac{10\left(\ln x\right)^9}{4x^4}$$

Applying L-Hospital again and again, will reduce the power of ln x to zero while power of x remains 4.

Hence, finally, the limit tends to zero.

Hence, B is correct.

27. $$\lim_{x\to 1}\frac{\tan\left(x-1\right)\log_e^{x^{x-1}}}{\left|x-1\right|^3}$$

Put x = 1 + h

$$\lim_{h\to 0}\frac{\left(\tan h\right)h\left(\log\left(1+h\right)\right)}{\left|h\right|^3}$$

$$\text{RHL}\Rightarrow\ \lim_{h\to 0}\frac{\tan h}{h}.\log\frac{\left(1+h\right)}{h}=1$$

$$\text{LHL}\Rightarrow\ \lim_{h\to 0}\frac{-\tan h}{-h}\times\frac{-h}{h}\times\frac{\log\left(1-h\right)}{-h}=-1$$

Hence, D is correct.

28. $$\lim_{x\to 0}\frac{1}{x}\left(\sqrt{\frac{1}{x^2}+1}-\frac{1}{x}\right)+x\ln\left(1+a^{1/x}\right)$$

$$\Rightarrow\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x^2}+\lim_{x\to 0}x\ln\left(1+a^{1/x}\right)$$

$$\Rightarrow\lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{1+x^2}-1\right)}+\lim_{x\to 0}\frac{\ln\left(1+a^{1/x}\right)}{1/x}$$

If $a\in\left(0,1\right)\to\lim_{x\to 0}x\ln\left(1+a^{1/x}\right)\to 0$

Hence, Ans is $\frac{1}{2}$

If $a\in\left(1,\infty\right)$

$$\lim_{x\to 0}\frac{\ln\left(1+a^{1/x}\right)}{1/x}=\frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to 0}\frac{1}{1+a^{1/x}}\times\frac{a^{1/x}\ln a\times -1/x^2}{-1/x^2}$$

$$=\lim_{x\to 0}\frac{\ln a}{a^{-1/x}+1}=\ln a$$

Hence, Ans is $\frac{1}{2}+\ln a$

29. $\lim\limits_{n\to\infty}\left[\frac{n}{n^2-2}+\frac{4^n(-1)^n}{2^n-1}\right]=0$

Hence, C is correct.

30. $\lim\limits_{x\to 0^+}\left(\log_{|\sin x|}x\right)^{g(x)}$

As $x\to 0^+\to g(x)=0$

Hence $\lim\limits_{x\to 0^+}\left(\log_{|\sin x|}x\right)^0=1$

Hence, B is correct.

31. $\lim\limits_{n\to\infty}n^2\left[(a)^{1/n}-(a)^{1/n+1}\right]$

$$\lim_{n\to\infty}n^2a^{1/n+1}\left[a^{1/n-\frac{1}{n+1}}-1\right]$$

$$\lim_{n\to\infty}\frac{a^{\frac{1}{n+1}}\times n^2\times\left[a^{\frac{1}{n(n+1)}}-1\right]}{\frac{1}{n(n+1)}}\times\frac{1}{n(n+1)}$$

$$\lim_{n\to\infty}a^{\frac{1}{n+1}}\times\left[\frac{a^{\frac{1}{n(n+1)}}-1}{1/(n+1)}\right]\times\frac{1}{1+1/n}$$

$\Rightarrow \ln a$

Hence, A is correct.

32. $\lim\limits_{x\to 0}\left(\sin\left(\frac{x}{m}\right)+\cos\frac{3x}{m}\right)^{\frac{2m}{x}}$

$$e^{\lim\limits_{x\to 0}\left(\sin\frac{x}{m}+\cos\frac{3x}{m}-1\right)\frac{2m}{x}}$$

$$e^{2m\lim\limits_{x\to 0}\frac{\left(\sin\frac{x}{m}+\cos\frac{3x}{m}-1\right)}{x}}$$

Apply L – Hospital Rule

$$\Rightarrow e^{2m\lim\limits_{x\to 0}\frac{1}{m}\cos\frac{x}{m}-\frac{3}{m}\sin\frac{3x}{m}}$$

$$\Rightarrow e^2$$

33. $\lim\limits_{x\to -\infty}\left(x+\sqrt{x^2+3x\cos\frac{1}{|x|}}\right)$

Put $x=-\frac{1}{y}$

$$\lim_{y\to 0}\left(-\frac{1}{y}+\sqrt{\frac{1}{y^2}-\frac{3}{y}\cos y}\right)$$

$$\lim_{y\to 0}\frac{\sqrt{1-3y\cos y}-1}{y}$$

$$\lim_{y\to 0}\frac{-3y\cos y}{y\left(\sqrt{1-3y\cos y}+1\right)}=-\frac{3}{2}$$

Hence, B is correct.

34. $\lim\limits_{x\to -\infty}\left(x+\sqrt{x^2+x^2\sin\frac{1}{x}}\right)$

Put $x=-\frac{1}{y}$

$$\lim_{y\to 0}\left(-\frac{1}{y}+\sqrt{\frac{1}{y^2}-\frac{1}{y^2}\sin y}\right)$$

$$\lim_{y\to 0}\frac{\sqrt{1-\sin y}-1}{y}$$

$$\lim_{y\to 0}\frac{-\sin y}{y\left(\sqrt{1-\sin y}+1\right)}=-\frac{1}{2}$$

Hence, D is correct.

35. $\lim\limits_{x\to 0}\frac{x^{2n}\sin^n x}{x^{2n}-\sin^{2n}x}$

$$\lim_{x\to 0}\frac{x^{2n}\sin^n x}{\left(x^n+\sin^n x\right)\left(x^n-\sin^n x\right)}$$

$$\lim_{x\to 0}\frac{\sin^n x/x^n}{\left(1+\left(\frac{\sin x}{x}\right)^n\right)\left(\frac{x^n-\sin^n x}{x^n.x^n}\right)}$$

For limit to exist,

$\lim\limits_{x\to 0}\frac{x^n-\sin^n x}{x^{2n}}$ should exist and non zero.

$$\lim_{x\to 0}\frac{x^n-\left(x-\frac{1}{6}x^3+\frac{x^5}{120}+.......\right)^n}{x^{2n}}$$

For $n = 2$, this will be non zero.

Hence, B is correct.

36. $\lim\limits_{x\to-\infty}\dfrac{a+x}{|x|-\sqrt{b^2x^2+x}}$

Put $x=-\dfrac{1}{y}$

$$\lim_{y\to 0}\frac{a-\frac{1}{y}}{\frac{1}{y}-\sqrt{\frac{b^2}{y^2}-\frac{1}{y}}}=\lim_{y\to 0}\frac{ay-1}{1-\sqrt{b^2-y}}$$

$$\Rightarrow \frac{-1}{1-|b|}=\frac{-1}{1+b}\ \text{as } b<0$$

Hence, B & C are correct.

37. $\lim\limits_{n\to\infty}\left(\sum\limits_{r=1}^{m}r^n\right)^{1/n}$

$$=\lim_{n\to\infty}\left(1^n+2^n+............+m^n\right)^{1/n}$$

$$=\lim_{n\to\infty}\left(m^n\left(\left(\frac{1}{m}\right)^n+\left(\frac{2}{m}\right)^n+.......1\right)\right)^{1/n}$$

$$=m\lim_{n\to\infty}\left(\left(\frac{1}{m}\right)^n+\left(\frac{2}{m}\right)^n+.......1\right)^{1/n}$$

$= m$

Hence, A is correct.

38. $\lim\limits_{n\to\infty}\left(\left(\cos\dfrac{k\pi}{4}\right)^n-\left(\cos\dfrac{k\pi}{6}\right)^n\right)=0$

This will always hold true if $\cos\dfrac{k\pi}{4}$ or $\cos\dfrac{k\pi}{6}$ is not equal to ±1.

Hence $K\neq 4\,K_1$ or $6K_1$

So, K is not divisible by 4 & 6. But K can be a multiple of 24.

Hence, D is correct.

39. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x}{\cos^{-1}\left[\frac{1}{4}(3\sin x-\sin 3x)\right]}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos^{-1}\left[\sin^3 x\right]}=\frac{2}{\pi}$$

$\Rightarrow$ (Denominator will be exactly $\dfrac{\pi}{2}$ while numerator will tend to 1)

Hence, A is correct.

40. $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}\left(a^{-9}\log_a x\right)}{\sec^{-1}\left(a^x\log_x a\right)}$

As $\lim\limits_{x\to\infty}\dfrac{\log_a x}{x^a}\to 0$ & $\lim\limits_{x\to\infty}a^x\log_x a\to\infty$

Thus $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}\left(x^{-a}\log_a x\right)}{\sec^{-1}\left(a^x\log_x a\right)}=\dfrac{\pi/2}{\pi/2}=1$

Hence, A is correct.

41. $\lim\limits_{n\to\infty}\left(\dfrac{p^{1/n}+q^{1/n}}{2}\right)^n=1^\infty$ form

$$e^{\lim\limits_{n\to\infty}\frac{p^{1/n}-1+q^{1/n}-1}{2\times 1/n}}$$

$$e^{\frac{\ln p+\ln q}{2}}=\sqrt{pq}$$

Hence, B is correct.

42. $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}\left(\sqrt{x-1}-\sqrt{x}\right)}{\sec^{-1}\left(\left(\frac{2x+1}{x-1}\right)^x\right)}$

As $\lim\limits_{x\to\infty}\cot^{-1}\left(\sqrt{x-1}-\sqrt{x}\right)$

$$=\lim_{x\to\infty}\cot^{-1}\left(\frac{-1}{\sqrt{x-1}+\sqrt{x}}\right)$$

$\Rightarrow\dfrac{\pi}{2}$ and

$$=\lim_{x\to\infty}\sec^{-1}\left(\left(\frac{2x+1}{x-1}\right)^x\right)=\lim_{x\to\infty}\sec^{-1}\left(\left(\frac{2+1/x}{1-1/x}\right)^x\right)=\frac{\pi}{2}$$

Hence, $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}\left(\sqrt{x-1}-\sqrt{x}\right)}{\sec^{-1}\left(\left(\frac{2x+1}{x-1}\right)^x\right)}=1$

Hence, A is correct.

43. $\lim\limits_{x\to 0}\dfrac{\tan(\{x\}-1)\sin\{x\}}{\{x\}(\{x\}-1)}$

$$\text{RHL} = \lim_{x\to 0^+} \frac{\tan(x-1)\sin x}{x(x-1)} = \tan 1$$

$$\text{LHL} = \lim_{x\to 0^-} \frac{\tan(1+x-1)\sin(1+x)}{(1+x)(1+x-1)} = \sin 1$$

Hence, D is correct.

44. $\lim\limits_{x\to 0^-} \left(\ln\left(\{x\}+|[x]|\right)\right)^{\{x\}}$

$$\lim_{x\to 0^-} \left(\ln(1+x+1)\right)^{1+x}$$

$$\lim_{x\to 0^-} \left(\ln(2+x)\right)^{1+x} = \ln 2$$

Hence, C is correct.

45. $\lim\limits_{x\to 0}\left[\dfrac{100x}{\sin x}\right] + \left[99\dfrac{\sin x}{x}\right]$

$=100 + 98 = 198$ (As $\dfrac{\sin x}{x} < 1$ & $\dfrac{x}{\sin x} > 1$)

Hence, B is correct.

46. $\lim\limits_{n\to\infty} \prod\limits_{n=0}^{n} \left(1+\dfrac{1}{2^{2n}}\right)$

Multiple & divided by $\left(1-\dfrac{1}{2}\right)$

$$\lim_{n\to\infty} \frac{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\ldots\ldots\ldots\ldots\ldots\left(1+\frac{1}{2^{2n}}\right)}{\left(1-\frac{1}{2}\right)}$$

$$\lim_{n\to\infty} \frac{1-\left(\frac{1}{2^{2n}}\right)^2}{1-\frac{1}{2}} = 2$$

Hence, B is correct.

47. $\lim\limits_{n\to\infty} \dfrac{1^6+2^6+3^6+\ldots\ldots\ldots+n^6}{\left(1^2+2^2+\ldots\ldots\ldots\ldots+n^2\right)\left(1^3+2^3+\ldots\ldots\ldots+n^3\right)}$

$$\lim_{n\to\infty} \frac{1^6+2^6+3^6+\ldots\ldots\ldots+n^6}{\frac{n(n+1)(2n+1)}{6}\left(\frac{n(n+1)}{2}\right)^2}$$

$$\lim_{n\to\infty} 24\sum_{r=1}^{n} \frac{r^6}{n^3(n+1)^3(2n+1)}$$

$$24\lim_{n\to\infty} \frac{\frac{1}{n}\sum\left(\frac{r}{n}\right)^6}{\frac{n^3(n+1)^3(2n+1)}{n^7}} = 24\frac{\int_0^1 (x)^6\,dx}{2} = \frac{12}{7}$$

Hence, D is correct.

48. $\lim\limits_{r\to\infty} \dfrac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}} = \dfrac{0}{0}$ form

Apply L – Hospital Rule,

$$\lim_{r\to\infty} \frac{\frac{a}{r^2}\sin\frac{a}{r} - \frac{b}{r^2}\sin\frac{b}{r}\cos\frac{c}{r} - \frac{c}{r^2}\cos\frac{b}{r}\sin\frac{c}{r}}{-\frac{b}{r^2}\cos\frac{b}{r}\sin\frac{c}{r} - \frac{c}{r^2}\sin\frac{b}{r}\cos\frac{c}{r}}$$

$$\lim_{r\to\infty} \frac{a^2\frac{\sin a/r}{a/r} - b^2\frac{\sin b/r}{b/r}\cos c/r - c^2\cos\frac{b}{r}\frac{\sin c/r}{c/r}}{-bc\cos\frac{b}{r}\frac{\sin c/r}{c/r} - bc\cos\frac{c}{r}\frac{\sin b/r}{b/r}}$$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc}$$

Hence, C is correct.

49. $\lim\limits_{x\to 0} \dfrac{\sqrt[100]{1+P(x)}-1}{x}$

$$\lim_{x\to 0} \frac{\left(1+P(x)\right)^{1/100} - (1)^{1/100}}{P(x)} \times \frac{P(x)}{x}$$

$$\Rightarrow \frac{1}{100}$$

Hence, B is correct.

50. $\dfrac{\theta^2+\theta-2-\theta^2-2\theta+1}{\theta+3} \le 0 - \dfrac{(\theta+1)}{(\theta+3)} \ge 0$

51. $\lim\limits_{x\to 0} \dfrac{6x^2\cot x\,\text{cosec}\,2x}{\sec\left(\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right)\right) - 1}$

$$\lim_{x\to 0} \frac{6\frac{x}{\tan}.\frac{x}{\sin 2x}}{\sec\left(\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right)\right) - 1} \Rightarrow -3.$$

Hence, D is correct.

52. $$\lim_{t\to 0}\left(\frac{b^{t+1}-a^{t+1}}{b-a}\right)^{1/t}$$

$$e^{\lim\limits_{t\to 0}\left(\frac{\left(b^{t+1}-b\right)}{t}-\frac{\left(a^{t+1}-a\right)}{t}\right)\times\frac{1}{b-a}}$$

$$e^{\lim\limits_{t\to 0}\left(\frac{b\left(b^{t}-1\right)}{t}-\frac{a\left(a^{t}-1\right)}{t}\right)\times\frac{1}{b-a}}$$

$$e^{\frac{b\ln b-a\ln a}{b-a}}=e^{\ln\left(\frac{b^{b}}{a^{a}}\right)^{1/b-a}}=\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}$$

Hence, D is correct.

53. Put $x=a\tan\theta$

$$f\left(x\right)=\cot^{-1}\left(\frac{2x}{1-x^{2}}\right)=\cot^{-1}\tan 2\theta$$

$$=\frac{\pi}{2}-\tan^{-1}\tan 2\theta$$

$$g\left(x\right)=\cos^{-1}\left(\frac{1-x^{2}}{1+x^{2}}\right)=\cos^{-1}\left(\cos 2\theta\right)$$

$$\lim_{x\to a^{+}}\frac{f\left(x\right)-f\left(a\right)}{g\left(x\right)-g\left(a\right)}=\lim_{x\to\frac{\pi^{+}}{4}}\frac{\frac{\pi}{2}-\tan^{-1}\tan 2\theta-\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}{\cos^{-1}\cos 2\theta-\pi/2}$$

$$\lim_{x\to\frac{\pi^{+}}{4}}\frac{\frac{\pi}{2}-\tan^{-1}\tan 2\theta}{\cos^{-1}\cos 2\theta-\frac{\pi}{2}}=\lim_{x\to\frac{\pi^{+}}{4}}\frac{\frac{\pi}{2}-\left(2\theta\right)}{2\theta-\frac{\pi}{2}}=-1$$

Hence, B is correct.

54. $$\lim_{x\to 0}\frac{\ln\left(1+x+x^{2}\right)+\ln\left(1-x+x^{2}\right)}{\sec x-\cos x}$$

$$\lim_{x\to 0}\frac{\ln\left(1+x^{2}+x\right)+\ln\left(1+x^{2}-x\right)}{1-\cos^{2}x}\times\cos x$$

$$\lim_{x\to 0}\frac{\ln\left(\left(1+x^{2}\right)^{2}-x^{2}\right)}{1-\cos^{2}x}\times\cos x$$

$$=\lim_{x\to 0}\frac{\ln\left(1+x^{4}+x^{2}\right)}{x^{4}+x^{2}}\times\frac{x^{2}\left(1+x^{2}\right)}{1-\cos^{2}x}\times\cos x$$

$$=1$$

Hence, A is correct.

55. $$f\left(x\right)=\lim_{n\to\infty}\frac{2x}{\pi}\tan^{-1}nx=\left|x\right|$$

Then, $\lim\limits_{x\to 0}\left[f\left(x\right)-1\right]=\lim\limits_{x\to 0}\left[\left|x\right|-1\right]$

$$\text{RHL}=\lim_{x\to 0^{+}}\left[\left|x\right|-1\right]=-1$$

$$\text{LHL}=\lim_{x\to 0^{-}}\left[\left|x\right|-1\right]=-1$$

Hence, B is correct.

56. $$\lim_{x\to 1}\left(\frac{x^{3}+2x^{2}+x+1}{x^{2}+2x+3}\right)^{\frac{1-\cos\left(x-1\right)}{\left(x-1\right)^{2}}}$$

$$\Rightarrow\left(\frac{5}{6}\right)^{1/2}$$

Hence, D is correct.

57. $$f\left(x\right)=\frac{1}{1-x}$$

$$fof\left(x\right)=\frac{1}{1-\frac{1}{1-x}}=\frac{1-x}{-x}=\frac{x-1}{x}$$

$$fofof\left(x\right)=\frac{1}{1-\frac{\left(x-1\right)}{x}}=x$$

So, $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{fofof\left(x\right)}}{\tan x-fofof\left(x\right)}=\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x-x}$

$$\lim_{x\to 0}\frac{e^{x}\left(e^{\tan x-x}-1\right)}{\left(\tan x-x\right)}=1$$

Hence, B is correct.

58. $$s=\lim_{n\to\infty}\left[\frac{7}{10}+\frac{29}{10^{2}}+\ldots\ldots\ldots\frac{5^{n}+2^{n}}{10^{n}}\right]$$

$$T_{r}=\frac{5^{r}+2^{r}}{10^{r}}=\left(\frac{1}{2}\right)^{r}+\left(\frac{1}{5}\right)^{r}$$

$$S=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{2}\right)^{r}+\left(\frac{1}{5}\right)^{r}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{2^{r}}+\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{5^{r}}$$

$$=\frac{1/2}{1-1/2}+\frac{\frac{1}{5}}{1-1/5}$$

$$=1+\frac{1}{4}=\frac{5}{4}$$

Hence, C is correct.

59. $$\lim_{x\to\infty}\frac{\int_{\sqrt{x}}^{x^2}\tan^{-1}\left(\frac{t^2}{1+t^2}\right)dt}{\sin 2x}=\frac{0}{0}\text{ form}$$

Apply L – Hospital Rule.

$$\lim_{x\to\infty}\left(\tan^{-1}\frac{x^4}{1+x^4}\right)2x-\tan^{-1}\left(\frac{x}{1+x}\right)\times\frac{1}{2\sqrt{x}}\ \ldots\ldots\ldots\ldots\ \boxed{\text{I}}$$

$$\text{As }\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{x}{1+x}\right)}{\frac{x}{1+x}}\times\frac{x}{(1+x)\sqrt{x}}\to 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \boxed{\text{II}}$$

Using $\boxed{\text{II}}$ in $\boxed{\text{I}}$

$$\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{x^4}{1+x^4}\right)2x-\tan^{-1}\left(\frac{x}{1+x}\right)\times\frac{1}{2\sqrt{x}}}{2\cos 2x}=0$$

Hence, A is correct.

60. $$\sum_{r=1}^{\infty}\frac{r^3+\left(r^2+1\right)^2}{\left(r^4+r^2+1\right)\left(r^2+r\right)}$$

$$\sum_{r=1}^{\infty}\frac{r^4+r^2+1+r^3+r^2}{\left(r^4+r^2+1\right)\left(r^2+r\right)}$$

$$\sum_{r=1}^{\infty}\frac{1}{r^2+r}+\frac{r}{r^4+r^2+1}$$

$$\sum_{r=1}^{\infty}\frac{1}{r}-\frac{1}{r+1}+\frac{r}{\left(r^2+1\right)^2-r^2}$$

$$\sum_{r=1}^{\infty}\frac{1}{r}-\frac{1}{r+1}-\frac{1}{2}\left[\frac{1}{r^2+1+r}-\frac{1}{r^2+1-r}\right]$$

$$\Rightarrow\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{r^2+1+r}-\frac{1}{r^2+1-r}\right)$$

$$\Rightarrow 1-\frac{1}{2}(-1)=\frac{3}{2}$$

Hence, A is correct.

61. $$\lim_{x\to 0}\frac{(729)^x-(243)^x-(81)^x+9^x+3^x-1}{x^3}$$

$$\lim_{x\to 0}\frac{(243)^x\times\left(3^x-1\right)^x-9^x\left(9^x-1\right)+3^x-1}{x^3}$$

$$\lim_{x\to 0}\frac{\left(3^x-1\right)\left[(243)^x-9^x\left(3^x+1\right)+1\right]}{x^3}$$

$$\lim_{x\to 0}\frac{\left(3^x-1\right)\left[(243)^x-(27)^x-9^x+1\right]}{x^3}$$

$$\lim_{x\to 0}\frac{\left(3^x-1\right)\left(9^x-1\right)\left(27^x-1\right)}{x^3}=\ln 3\ln 9\ln 27=6\left(\ln 3\right)^3$$

Hence, C is correct.

62. $$\lim_{x\to\frac{\pi}{2}}\frac{\sin x-\left(\sin x\right)^{\sin x}}{1-\sin x\ln\sin x}$$

Apply L – Hospital Rule

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x-\left(\sin x\right)^{\sin x}\left[\cos x+\cos x\ln\sin x\right]}{-\left[\cos x+\cos x\ln\sin x\right]}=0$$

Hence, B is correct.

63. $$\lim_{x\to\infty}\sqrt[x]{2\sum_{n=0}^{x}\frac{x^n}{\lfloor n}}$$

$$\text{As }e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots\ldots\ldots\ldots\ldots\infty$$

$$\lim_{x\to\infty}2^{1/x}\left(\sum_{n=0}^{x}\frac{x^n}{\lfloor n}\right)^{1/x}=e$$

Hence, B is correct.

64. $$\text{As, }S_{n+1}-S_n=t_{n+1}=\frac{2(n+1)}{3n+4}$$

$$l=\lim_{n\to\infty}n\times\frac{2(n+1)}{(3n+4)\times\sqrt{n(n+1)}}=\frac{2}{3}$$

Now,

$$L=l+2l^2+3l^3+\ldots\ldots\ldots\ldots\ldots\infty$$

$$lL=\quad l^2+2l^3+\ldots\ldots\ldots\ldots\ldots\infty$$

$$(1-l)L=l+l^2+l^3+\ldots\ldots\ldots\ldots\ldots\infty$$

$$(1-l)L=\frac{l}{1-l}$$

$$L=\frac{l}{(1-l)^2}=\frac{2/3}{1/9}=6$$

Hence, D is correct.

65. $$\lim_{n\to\infty}\frac{(1.n)+(n-1)(1+2)+(n-2)(1+2+3)+\ldots\ldots\ldots.1\sum_{r=1}^{n}r}{n^4}$$

$$\lim_{n\to\infty}\frac{\sum_{r=0}^{n-1}(n-r)\frac{(r)(r+1)}{2}}{n^4}$$

$$\frac{1}{2}\lim_{n\to\infty}\frac{n\sum r^2+n\sum r-\sum r^3-\sum r^2}{n^4}$$

$$\frac{1}{2}\lim_{n\to\infty}\frac{\frac{n(n)(n-1)(2n-1)}{6}+\frac{n(n-1)(n)}{2}-\frac{n^2(n-1)^2}{4}-\frac{n(n-1)(2n-1)}{6}}{n^4}$$

$$\frac{1}{2\times2\times6}\lim_{n\to\infty}\frac{2n^2(n-1)(2n-1)+6n^2(n-1)-3n^2(n-1)^2-2n(n-1)(2n-1)}{n^4}$$

$$\frac{1}{24}\lim_{n\to\infty}\left[\begin{matrix}2\left[1-\frac{1}{n}\right]\left(2-\frac{1}{n}\right)+6\left(\frac{1}{n}-\frac{1}{n^2}\right)\\ -3\left(1-\frac{1}{n}\right)^2-2\left(\frac{1}{n}-\frac{1}{n^2}\right)\left(2-\frac{1}{n}\right)\end{matrix}\right]$$

$$\frac{1}{24}\times(4-3)=\frac{1}{24}$$

Hence, B is correct.

66. $\lim_{x\to\infty}\frac{\int_0^{x^2}\sin x^2dx}{x^n}=\frac{0}{0}$ form

Apply L – Hospital Rule,

$$\lim_{x\to\infty}\frac{\left(\sin x^4\right)2x}{nx^{n-1}}$$

For limit to exist & non zero

$x^{n-1}=x^5$

$\Rightarrow n-1=5$

$n=6$

67. $\lim_{t\to0}\ln\left(\frac{1}{t}\int_0^t 1+a\sin bx\right)^{c/x}dx$

$$\text{Now, }\lim_{t\to0}\frac{\int_0^t(1+a\sin bx)^{c/x}dx}{t}=\frac{0}{0}\text{ form}$$

Apply L – Hospital Rule.

$\lim_{t\to0}(1+a\sin bt)^{c/t}=1^{\infty}$ form

$e^{\lim_{t\to0}(a\sin bt)(c/t)}=e^{abc}$

$$\text{Hence, }\lim_{t\to0}\ln\left(\frac{1}{t}\int_0^t(1+a\sin bx)^{c/x}dx\right)=abc$$

Hence, A is correct.

68. $\lim_{n\to\infty}\left[C_n-\left(\frac{2}{3}\right)C_{n-1}+\left(\frac{2}{3}\right)^2C_{n-2}+........(-1)^n\left(\frac{2}{3}\right)^nC_0\right]$

$$\lim_{n\to\infty}\left[C_0-\frac{2}{3}C_1+\left(\frac{2}{3}\right)^2C_{n-2}+.................\right]$$

$$\lim_{n\to\infty}\left[\left(1-\frac{2}{3}\right)^n\right]=\lim_{n\to\infty}\left(\frac{1}{3}\right)^n\to0$$

Hence, A is correct.

69. As $x\to0^+,[x]=0\Rightarrow\lim_{x\to0^+}f(x)=0$

As $x\to0^-,[x]=-1\Rightarrow\lim_{x\to0^-}f(x)=\sin1$

Hence, D is correct.

70. If $\lim_{n\to\infty}\frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}=0$

Then $\frac{3}{\pi}\tan^{-1}2x>1$ or $\frac{3}{\pi}\tan^{-1}2x<-1$

$\tan^{-1}2x>\frac{\pi}{3}$ or $\tan^{-1}2x<-\frac{\pi}{3}$

$\frac{\pi}{2}\ge\tan^{-1}2x>\frac{\pi}{3}$ or $-\frac{\pi}{2}\le\tan^{-1}2x<-\frac{\pi}{3}$

$\infty\ge2x>\sqrt{3}$ or $-\infty\le2x<-\sqrt{3}$

$\Rightarrow|2x|>\sqrt{3}$

Hence, A is correct.

71. $I=\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{2n^4+1}{5n^5+1}}=0^0$ form

$$I=e^{\lim_{n\to\infty}\frac{2n^4+1}{5n^5+1}\ln\frac{n!}{n^n}}$$

$$=e^{\lim_{n\to\infty}\frac{2n^4+1}{5n^5+1}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)}$$

$$=e^{\lim_{n\to\infty}\frac{1}{n}\left[\frac{2+1/n^4}{5+1/n^5}\right]\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)}$$

$$=e^{\frac{2}{5}\int_0^1\ln x\,dx}$$

$$=e^{\frac{2}{5}(-1)}=\frac{1}{e^{2/5}}$$

Hence, C is correct.

72. $\lim_{n\to\infty}\frac{1}{n^3}\left(\left[1^2x+1^2\right]+\left[2^2x+2^2\right]+\left[3^2x+3^2\right]+........+\left[n^2x+n^2\right]\right)$

$$I = \lim_{n\to\infty} \frac{[1^2 x] + [2^2 x] + \ldots\ldots\ldots\ldots [n^2 x]}{n^3}$$

$$\downarrow$$

$$I_1$$

$$+ \lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2}{n^3} \text{ ------- } \boxed{1}$$

$$\downarrow$$

$$I_2$$

Using Sandwich Theorem,

$$\lim_{n\to\infty} \frac{1^2.x - 1 + 2^2 x - 1 + 3^2 x - 1 + \ldots\ldots\ldots\ldots + n^2 x - 1}{n^3} \le I_1 \le$$

$$\lim_{n\to\infty} \frac{\left(1^2 + 2^2 + \ldots\ldots\ldots + n^2\right) x}{n^3}$$

$$I_1 = \frac{x}{3} \ \& \ I_2 = \frac{1}{3}$$

$$\text{So, } I = \frac{x}{3} + \frac{1}{3}$$

Hence, C is correct.

73. $$\lim_{n\to\infty} \frac{1}{na} + \frac{1}{na+1} + \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{na + n(b-a)}$$

$$\lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=0}^{b-a} \frac{1}{a + r/n}\right]$$

$$\int_0^{b-a} \frac{1}{a+x} dx = \ln(a+x)\Big|_0^{b-a}$$

$$= \ln(b/a)$$

Hence, B is correct.

74. $$\lim_{x\to 0}\left[\left(f(x) + \ln\left(1 - \frac{1}{e^{f(x)}}\right) - \ln f(x)\right)\right] = 0$$

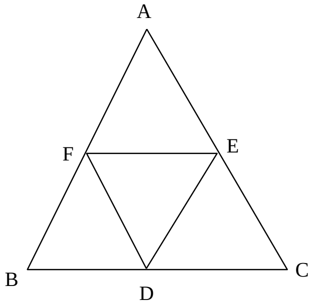
$$\lim_{x\to 0}\left[\left(f(x) + \ln\left(e^{f(x)} - 1\right) - f(x) - \ln f(x)\right)\right] = 0$$

$$\lim_{x\to 0} \ln\left(\frac{e^{f(x)} - 1}{f(x)}\right) = 0$$

Hence, $\lim_{x\to 0} f(x) \to 0$

So, $f(0) = 0$

Hence, A is correct.



75. If AC = a units

$a_1 = EC = \frac{a}{2}$ units

A = Areas of Δ Removed =

$$\frac{\sqrt{3}}{4} a_1^2 + 3 \times \frac{\sqrt{3}}{4} a_2^2 + 3^2 \frac{\sqrt{3}}{4} a_3^2 + \ldots\ldots\ldots\ldots$$

Where a_2 is side of Δ formed when process is repeated two times and so on

$$A = \lim_{n\to\infty} \frac{\sqrt{3}}{4}\left[\frac{a^2}{4} + 3\left(\frac{a}{4}\right)^2 + 3^2\left(\frac{a}{8}\right)^2 + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{\sqrt{3}}{4} \frac{a^2/4}{1 - 3/4} = \frac{\sqrt{3}}{4} a^2 \text{ ------------- } \boxed{1}$$

Now Area of ΔABC = 1 sq unit

$$\Rightarrow \frac{\sqrt{3}}{4} a^2 = 1 \text{ Then } A = 1$$

Hence, B is correct.

MULTIPLE CORRECT ANSWER TYPE

76. A, B, D

A) $$\lim_{x\to\infty} \frac{x^{1/4} \sin \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}} \times \sqrt{x}} = \lim_{x\to\infty} \frac{\sin\left(1/\sqrt{x}\right)}{1/\sqrt{x}} \times \frac{1}{x^{1/4}} = 0$$

B) $$\lim_{x\to\frac{\pi}{2}} (1 - \sin x) \tan x = \lim_{x\to\frac{\pi}{2}} \frac{(1 - \sin x)}{\cot x} = \frac{0}{0} \text{ form}$$

Use L – Hospital Rule

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\operatorname{cosec}^2 x} = 0$$

C) $$\lim_{x\to\infty} \frac{2x^2 + 3}{x^2 + x - 5} \operatorname{sgn} x$$

$$\lim_{x\to\infty} \frac{2x^2 + 3}{x^2 + x - 5} = 2$$

D) $$\lim_{x\to 3^+} \frac{[x]^2 - 9}{x^2 - 9} = 0$$

77. B, D

A) $\lim_{x\to a} f(x)$ may or may not exist.

B) $|g(x)| \le M \Rightarrow -M \le g(x) \le M$

Hence, $\lim_{x\to c} f(x).g(x) = 0$ as $\lim_{x\to c} g(x) = $ finite value

C) If $\lim\limits_{x\to c}|f(x)| = |L|$

Then, $\lim\limits_{x\to c} f(x)$ may or may not be L only

D) If $f(x) = g(x) \forall x \neq 0$

Then, $\lim\limits_{x\to c} f(x) = L$, Then $\lim\limits_{x\to 0} g(x)$ will also be L.

78. A, B, D

If $\lim\limits_{x\to\pm\infty} f(x) = a$, Then $y = a$ is an horizontal asymptote.

A) $\lim\limits_{x\to+\infty}\frac{|x|}{x+1} = 1$, Hence y = 1 is an asymptote

$\lim\limits_{x\to-\infty}\frac{|x|}{x+1} = -1$, Hence y = −1 is an asymptote

B) $\lim\limits_{x\to\infty}\frac{2x}{\sqrt{x^2+1}} = 2$

$\lim\limits_{x\to-\infty}\frac{2x}{\sqrt{x^2+1}} = -2$

C) $\lim\limits_{x\to\infty}\frac{\sin x}{x^2+1} = 0$

$\lim\limits_{x\to-\infty}\frac{\sin x}{x^2+1} = 0$

D) $\lim\limits_{x\to\infty}\cot^{-1}(2x+1) = 0$

$\lim\limits_{x\to-\infty}\cot^{-1}(2x+1) = \pi$

79. C, D

For vertical asymptote at $x = a$

$\lim\limits_{x\to a} f(x) = \pm\infty$

A) $\lim\limits_{x\to-1^+}\frac{|x^2-1|}{x+1} = 2$

B) $\lim\limits_{x\to-1}\frac{x^2-6x-7}{x+1} = -8$

C)
$$\lim_{x\to-1}\frac{x^2+1}{x+1} \nearrow \lim_{x\to-1^+}\frac{x^2+1}{x+1} = \infty$$
$$\lim_{x\to-1}\frac{x^2+1}{x+1} \searrow \lim_{x\to-1^-}\frac{x^2+1}{x+1} = -\infty$$

So, x = −1 is a vertical asymptote

D)
$$\lim_{x\to-1}\frac{\sin(x+2)}{x+1} \nearrow \lim_{x\to-1^+}\frac{\sin(x+2)}{x+1} = \infty$$
$$\lim_{x\to-1}\frac{\sin(x+2)}{x+1} \searrow \lim_{x\to-1^-}\frac{\sin(x+2)}{x+1} = -\infty$$

80. B, C, D

A) It may or may not exist

B) It is correct

C) It is correct eg $f(x) = \frac{1}{x}$ & $g(x) = 2 - \frac{1}{x}$ as $x \to 0$

D) It is correct

81. As

$$\lim_{x\to0} x\cos x = \lim_{x\to0}|x|\sin x = \lim_{x\to0} x\cos\frac{1}{x} = \lim_{x\to0}\left|x\sin\frac{1}{x}\right| = 0$$

Now $\cos x, \cos\frac{1}{x}$, $\sin x$ all lie between $[-1,1]$

so,

$$\Rightarrow -x < x\cos x \le x$$

Similarly $-|x| \le |x|\sin x \le |x|$

$$\& -x \le x\cos\frac{1}{x} \le x$$

for $y = \left|x\sin\frac{1}{x}\right|$

$$y = \left|x\sin\frac{1}{x}\right| \le |x|$$

Hence, all graphs will lie between $y = |x|$ & $-|x|$ and will have a limiting value at x →0.

Hence, A, B, C, D are correct.

82. A, B, C, D

A) It will depend whether P(x) = 0 has 1 as a root or not

B) A rational function may or may not have a vertical asymptote.

C) For vertical asymptote, there is no condition on value of f(x) at that point.

D) For a function, it can have at max two horizontal asymptote.

83. A, B, C

A) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n+1}\right)^2 = 1$

B) & D) as $(-1)^n$ is not defined when n →∞

C) $\lim\limits_{n\to\infty}\frac{n^2+1}{n} \to \infty$

84. A, B, C, D

A) $\lim\limits_{x\to\infty}\frac{\sin x\pi}{x} = 0$

B) $\lim\limits_{x\to\infty}\left(a\cos^2 x\pi + b\sin^2 x\pi\right) =$ Does not exist

C) $\lim\limits_{x\to\infty} x\sin x\pi =$ Does not exist

D) $\lim\limits_{x\to\infty}\tan(x\pi) =$ Does not exist

85. B, C, D

A) $$\lim_{x\to1}\frac{\sin[x]}{[x]}\begin{matrix}\text{RHL}\nearrow & \displaystyle\lim_{x\to1^+}\frac{\sin1}{1}=\sin1\\ \text{LHL}\searrow & \displaystyle\lim_{x\to1^-}\frac{\sin0}{0}=\text{ not defined}\end{matrix}$$

B) $$\lim_{x\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}=\lim_{x\to\infty}\frac{e}{(\pi)^n}=0$$

C) $$\lim_{x\to1}\left[\sin\sin^{-1}x\right]$$

As RHL will not exist &

$$\text{LHL}\to\lim_{x\to1^-}\left[\sin\sin^{-1}x\right]=0$$

D) $$\lim_{x\to\frac{\pi}{2}}\left[\sin^{-1}\sin x\right]=1$$

86. A, B

$$l_1=\lim_{x\to\frac{\pi}{4}}\cos^{-1}\left[\sec\left(x-\frac{\pi}{4}\right)\right]\begin{matrix}\text{RHL}\nearrow 0\\ \text{LHL}\searrow 0\end{matrix}$$

$$l_2=\lim_{x\to\frac{\pi}{4}}\sin^{-1}\left[\operatorname{cosec}\left(x+\frac{\pi}{4}\right)\right]\begin{matrix}\text{RHL}\nearrow \pi/2\\ \text{LHL}\searrow \pi/2\end{matrix}$$

$$l_3=\lim_{x\to\frac{\pi}{4}}\tan^{-1}\left[\cot\left(x+\frac{\pi}{4}\right)\right]\begin{matrix}\text{RHL}\nearrow -\pi/4\\ \text{LHL}\searrow 0\end{matrix}$$

$$l_4=\lim_{x\to\frac{\pi}{4}}\cot^{-1}\left[\tan\left(x-\frac{\pi}{4}\right)\right]\begin{matrix}\text{RHL}\nearrow \pi/2\\ \text{LHL}\searrow 3\pi/4\end{matrix}$$

87. B, C

$$f(x)=\begin{cases}\dfrac{\tan^2x}{\left(x^2-[x]^2\right)} & ,x>0\\ 1 & ,x=0\\ \sqrt{\{x\}\cot\{x\}} & ,x<0\end{cases}$$

A) $$\text{RHL}\to\lim_{x\to0^+}f(x)=1$$

$$\text{LHL}\to\lim_{x\to0^-}f(x)=\sqrt{\cot1}$$

B) $$\lim_{x\to0^-}f(x)=\sqrt{\cot1}$$

C) $$\cot^{-1}\left(\lim_{x\to0^-}f(x)\right)^2=\cot^{-1}\cot^1=1$$

88. B, C

A) $$\lim_{x\to3}[[x]]-[2x-1]\begin{matrix}\text{RHL}\nearrow -2\\ \text{LHL}\searrow -2\end{matrix}$$

B) $$\lim_{x\to1}[x]-x\begin{matrix}\text{RHL}\nearrow 0\\ \text{LHL}\searrow -1\end{matrix}$$

C) $$\lim_{x\to0}\{x\}^2-\{-x\}^2\begin{matrix}\text{RHL}\nearrow -1\\ \text{LHL}\searrow +1\end{matrix}$$

D) $$\lim_{x\to0}\frac{\tan(\operatorname{sgn}x)}{\operatorname{sgn}x}\begin{matrix}\text{RHL}\nearrow \tan1\\ \text{LHL}\searrow \tan1\end{matrix}$$

89. A, B, D

$$f(x)=\lim_{n\to\infty}\frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}$$

$$=\lim_{n\to\infty}\frac{2\sin\frac{1}{x}+\frac{1}{x^{2n-1}}}{1+\frac{1}{x^{2n}}}$$

A) $$\lim_{x\to\infty}xf(x)$$

$$=\lim_{x\to\infty}\lim_{n\to\infty}\frac{2x\sin\frac{1}{x}+\frac{1}{x^{2n}}}{1+\frac{1}{x^{2n}}}=2$$

B) $$\lim_{x\to1}f(x)=\lim_{x\to1}\lim_{n\to\infty}\frac{2\sin\frac{1}{x}+\frac{1}{x^{2n-1}}}{1+\frac{1}{x^{2n}}}$$

RHL = 2sin1

LHL = 1

C) $$\lim_{x\to0}f(x)=\lim_{x\to0}\lim_{n\to\infty}\frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}\begin{matrix}\text{RHL}\nearrow 0\\ \text{LHL}\searrow 0\end{matrix}$$

D) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\lim_{n\to\infty} \dfrac{2\sin\frac{1}{x}+\frac{1}{x^{2n-1}}}{1+\frac{1}{x^{2n}}} = 0$

90. B, C, D

$$L = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{ax+1}{bx+2}\right)^x$$

$$= \lim_{x\to\infty}\left(\frac{a+1/x}{b+2/x}\right)^x$$

L = 0 if *b* > *a*

L = ∞ if *b* < *a*

If *a* = *b*

L = $(1)^\infty$ form

$$L = e^{\lim_{x\to\infty}\left(\frac{ax+1}{bx+2}-1\right)x}$$

$$= e^{\lim_{x\to\infty}\left(\frac{ax+1-ax-2}{ax+2}\right)x}$$

$$= e^{\lim_{x\to\infty}\frac{-1}{a+2/x}} = e^{-1/a} \text{ or } e^{-1/b}$$

91. B, C

$$f(x) = \begin{cases} mx^2+n, x<0 \\ nx+m,\ 0\le x\le 1 \\ nx^3+m,\ x>1 \end{cases}$$

A) $\lim_{n\to 0^+} f(x) = m$

$\lim_{n\to 0^-} f(x) = \eta$

$\lim_{n\to 0} f(x)$ will exist only if $m = \eta$

$\lim_{n\to 1^-} f(x) = m+\eta$

C) $\lim_{n\to 1^+} f(x) = m+\eta$

$\lim_{n\to 1} f(x)$ will exist for all values of m & n.

92. A

$$I = \lim_{x\to 0^+}\frac{\int_x^{2x}\sin^m t\,dt}{x^n}$$

Applying L-Hospital Rule

$$I = \lim_{x\to 0^+}\frac{2\sin^m 2x - \sin^m x}{nx^{n-1}}$$

If m = n −1

$$I = \frac{2^{m+1}-1}{\eta} = \frac{2^\eta - 1}{n}$$

If $m \ge \eta$

$I = 0$

93. B, C, D

$$\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-ax-b\right)$$

$$\lim_{x\to\infty}\frac{x^2-x+1-(ax+b)^2}{\sqrt{x^2-x+1}+(ax+b)}$$

$$\lim_{x\to\infty}\frac{(1-a^2)x^2-(1+2ab)x+(1-b^2)}{\sqrt{x^2-x+1}+(ax+b)} = 0$$

$\Rightarrow a^2 = 1\ \&\ 2ab+1 = 0$

$\Rightarrow a = +1\ \&\ b = -\frac{1}{2}$ or $a = -1\ \&\ b = \frac{1}{2}$

$a = -1\ \&\ b = \frac{1}{2}$ is invalid as limit will not be zero

Hence, $a = 1,\ b = -1/2$

C) $\lim_{n\to\infty}\sec^{2n}(k!\pi b) = \lim_{n\to\infty}\sec^{2n}(\text{Multiple of }\pi)$
$= 1$

D) $\lim_{n\to\infty}\sec^{2n}(k!\pi a) = \lim_{n\to\infty}\sec^{2n}(\text{Multiple of }\pi)$
$= 1$

94. A, B, C, D

$$l_1 = \lim_{x\to\infty}\sqrt{\frac{x-\cos^2 x}{x+\sin x}} = 1$$

$$l_2 = \lim_{h\to 0^+}\int_{-1}^{1}\frac{hdx}{h^2+x^2}$$

$$= \lim_{h\to 0^+} h\times\frac{1}{h}\tan^{-1}\left(\frac{x}{h}\right)\Big|_{-1}^{1}$$

$$= \lim_{h\to 0^+}\tan^{-1}\left(\frac{1}{h}\right)-\tan^{-1}\left(-\frac{1}{h}\right)$$

$= \pi$

95. A, B, D

A) $\lim_{x\to 1} x^{\frac{1}{x-1}}$

$\Rightarrow e^{\lim_{x\to 1}(x-1)\times\frac{1}{x-1}} = e$

B) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$

$\Rightarrow e^{\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)x} = e$

$\lim\limits_{x\to\infty}\left(\frac{x+4}{x+2}\right)^{x+3}$

C) $\Rightarrow e^{\lim\limits_{x\to\infty}\left(\frac{x+4}{x+2}-1\right)(x+3)}$

$\Rightarrow e^{\lim\limits_{x\to\infty}\frac{2(x+3)}{x+2}} = e^{2}$

$\lim\limits_{x\to\infty}\left(1+f(x)\right)^{1/f(x)}$

D) $\Rightarrow e^{\lim\limits_{x\to\infty} f(x).\frac{1}{f(x)}}$

$= e$

Assertion-Reason Type

96. As $f(x)=\frac{x}{[x]}=1+\frac{\{x\}}{[x]}$

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\left(1+\frac{\{x\}}{[x]}\right)=1$

And $\lim\limits_{x\to\infty}[x]$ does not exist.

Hence, D is correct.

97. As $\sum\frac{n+3}{2^{n}\left(n^{2}+3n+2\right)}=\sum\limits_{n=1}^{\infty}\frac{1}{2n}\left(\frac{2}{n+1}-\frac{1}{n+2}\right)$

$=\sum\limits_{n=1}^{\infty}\left(\frac{1}{2^{n-1}(n+1)}-\frac{1}{2^{n}(n+2)}\right)$

$=\left(\frac{1}{2}-\frac{1}{2\cdot(3)}\right)+\left(\frac{1}{2\cdot 3}-\frac{1}{2^{2}\cdot 4}\right)+\left(\frac{1}{2^{2}\cdot 4}-\frac{1}{2^{3}\cdot 5}\right)+\ldots$

$=\frac{1}{2}-\frac{1}{2^{n}(n+2)}$

Now $\lim\limits_{n\to\infty}\sum\frac{n+3}{2^{n}\left(n^{2}+3n+2\right)}=\frac{1}{2}$

Hence, A is correct.

98. $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^{3}+x^{2}}-\sqrt[3]{x^{3}-x^{2}}\right)$

$$\lim_{n\to\infty}\frac{\left(2x^{2}\right)}{\left(\left(x^{3}+x^{2}\right)^{2/3}+\left(x^{3}-x^{2}\right)^{2/3}+\left(x^{6}-x^{4}\right)^{1/3}\right)}$$

$$\lim_{x\to\infty}\frac{2}{\left((1+1/x)^{2/3}+(1-1/x)^{2/3}+(1-1/x^{2})^{1/3}\right)}$$

$=\frac{2}{3}$

As $\lim\limits_{n\to\infty}\left(x+\frac{1}{3}-\sqrt[3]{x^{3}+x^{2}}\right)=0$ .... (I)

$$=\lim_{x\to\infty}\frac{(x+1/3)^{3}-(\sqrt[3]{x^{3}+x^{2}})^{3}}{\left(\left(x+\frac{1}{3}\right)^{2}+\left(x^{3}+x^{2}\right)^{2/3}+\left(x+\frac{1}{3}\right)\sqrt[3]{x^{3}+x^{2}}\right)}$$

$$\lim_{x\to\infty}\frac{1/27+x/3}{\left(x+\frac{1}{3}\right)^{2}+\left(x^{3}+x^{2}\right)^{2/3}+(x+1/3)\sqrt[3]{x^{3}+x^{2}}}=0$$

Similarly, $\lim\limits_{x\to\infty}\left(x-\frac{1}{3}-\sqrt[3]{x^{3}-x^{2}}\right)=0$ .... (II)

Subtracting, (I) from (II), we get

$\lim\limits_{x\to\infty}\sqrt[3]{x^{3}+x^{2}}-\sqrt[3]{x^{3}-x^{2}}=\frac{2}{3}$

Hence, A is correct.

99. As $\lim\limits_{x\to 0^{+}} x^{x}\to 1$

$\Rightarrow\lim\limits_{x\to 0^{+}}\left(x^{x^{x}}-x^{x}\right)=-1$

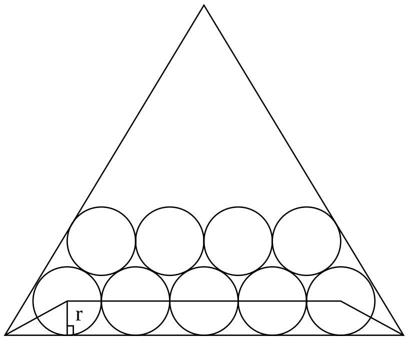
& $\lim\limits_{x\to 0^{+}} x^{x}(x-1)=-1$

Hence, B is correct.

100. Number of circles = 1 + 2 +-------+ n = $\frac{n(n+1)}{2}$

Let side of triangle is 'a'

$\Rightarrow a=2(n-1)r+2r\cot\frac{\pi}{6}=2r(n-1+\sqrt{3})$ .... (I)



$$\frac{\text{Area of circle}}{\text{Area of triangle}}=\frac{\frac{n(n+1)}{2}\pi\left(\frac{a}{2(n-1+\sqrt{3})}\right)^2}{\frac{\sqrt{3}}{4}a^2}$$

$$=\frac{\pi n(n+1)}{2\sqrt{3}(n-1+\sqrt{3})^2}$$

$$\lim_{n\to\infty}\frac{x(n)(n+1)}{2\sqrt{3}(n+\sqrt{3}-1)^2}=\frac{\pi}{2\sqrt{3}}=\frac{\sqrt{3}\pi}{6}$$

Hence, C is correct.

101. Let $f(x)=\frac{f(x)e^{x^2/2}}{e^{x^2/2}}$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{e^{x^2/2}f'(x)+x\,f(x)e^{x^2/2}}{xe^{x^2/2}}$$

$$=\lim_{x\to\infty}\frac{e^{x^2/2}f''(x)+xe^{x^2/2}f'(x)+(x\,f'(x)+f(x))e^{x^2/2}+x^2f(x)e^{x^2/2}}{x^2e^{x^2/2}+e^{x^2/2}}$$

$$=\lim_{x\to\infty}\frac{f''(x)+2xf'(x)+f(x)+x^2f(x)}{x^2+1}=0$$

$$\Rightarrow\lim_{x\to\infty}f(x)=0$$

Hence, A is correct.

102. Let side of $\Delta ABC=$ a units

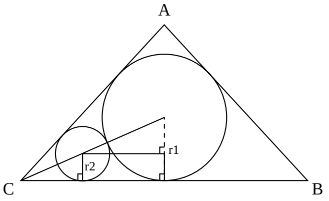
$$r_1=\frac{1}{3}\times\frac{\sqrt{3}}{2}a=\frac{a}{2\sqrt{3}}$$

Now, $(r_1+r_2)^2=(r_1-r_2)^2+\left(\sqrt{3}(r_1-r_2)\right)^2$

$\Rightarrow r_1+r_2=2(r_1-r_2)$

$3r_2=r_1$

$\Rightarrow\frac{r_1}{3}=r_2$



Sum of areas of all circles

$\pi\left(r_1^2+r_2^2+\ldots\right)$

$$=\pi\left(\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{3\sqrt{3}}\right)^2+\ldots\right)$$

$$=\frac{\pi/3}{1-1/9}=\frac{3\pi}{8}$$

Hence, D is correct.

103. RHL $\rightarrow\lim_{x\to0^+}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)=0$

LHL $\rightarrow\lim_{x\to0^-}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)=1$

RHL $\rightarrow\lim_{x\to0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)=1$

LHL $\rightarrow\lim_{x\to0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)=-1$

Hence, B is correct.

104. $\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(\cot^2x\right)}{(\pi-2x)^2}$

$$\lim_{n\to0}\frac{\sin\left(\tan^2h\right)}{4h^2}=\frac{1}{4}$$

Hence, D is correct.

105. RHL $\rightarrow\lim_{x\to\frac{\pi^+}{2}}\cos^{-1}[\cot x]=\pi$

LHL $\rightarrow\lim_{x\to\frac{x^-}{2}}\cos^{-1}[\cot x]=0$

So, $\lim_{x\to\frac{\pi}{2}}f(x)$ does not exist

Hence, D is correct.

Comprehension Type

Comprehension 1:

$f(x)=x[x[x]],\quad x\in[-1,3]$

$x=-1,\ f(x)=1$

For $x\in[-1,0),f(x)=0$
$x\in[0,1),f(x)=0$
$x\in[1,2),f(x)=x$
$x\in\left[2,\frac{5}{2}\right),f(x)=4x$
$x\in\left[\frac{5}{2},2\right),f(x)=5x$
$x=3,f(x)=9x$

106. As $f(x)=13$ is possible when $x=\frac{13}{5}$

$\Rightarrow a+b=18$

Hence, B is correct.

107. Limit of f(x) does not exist at $x = -1, 1, 2, 5, 3$

108. $\lim\limits_{x\to\frac{\pi^-}{2}} f(2\sin x) = \lim\limits_{h\to 0^-} f(+2\cos h) = 2$

Hence, A is correct.

Comprehension 2:

109. $A: \lim\limits_{x\to 1^+} \frac{x^2-1}{|x-1|} = 2, \qquad \lim\limits_{x\to 1^-} \frac{x^2-1}{|x-1|} = -2$

Approx f(x) does not exist.

$B: \lim\limits_{x\to 2^+} \frac{2\{x\}-4}{[x]-3} = 4, \qquad \lim\limits_{x\to 2^-} \frac{2\{x\}-4}{[x]-3} = 1$

Approx f(x) does not exist.

$C: \lim\limits_{x\to 0^+} \frac{1}{2-2^{1/x}} = 0, \qquad \lim\limits_{x\to 0^-} \frac{1}{2-2^{1/x}} = \frac{1}{2}$

Approx f(x) exists.

Hence, C is correct.

110. $\lim\limits_{x\to 0^+} \frac{x\left(e^{[x]+|x|}-2\right)}{[x]+|x|} = -1$

$$\lim_{x\to 0^-} \frac{x\left[e^{[x]+|x|}-2\right)}{[x]+|x|} = \lim_{x\to 0^-} \frac{x\left(e^{-1-x}-2\right)}{-1-x} = 0$$

Approx $f(x) = -1$

Hence, A is correct.

111. $\lim\limits_{x\to 0^-} f(x) = 0, \qquad \lim\limits_{x\to 0^+} f(x) = a$

For approx f(x) to exist $[|-a|] \le 1$

$\Rightarrow 0 \le |-a| < 2$

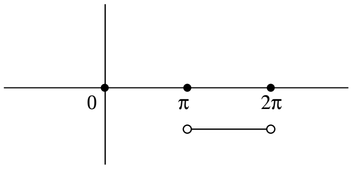
$\Rightarrow -2 < a < 2$

$\Rightarrow a \in (-2, 2)$

Hence, B is correct.

Comprehension-3:

112. $f(x) = \left[\frac{\sin x}{x}\right]$



At $x = 0^+, \quad f(x) = 0$

In I^{st} quad, $f(x) = 0$

In II^{nd} quad, $f(x) = 0$

at $x = n\pi, \; f(x) = 0$

In III^{rd} & IV^{th} quad, $f(x) = -1$

Hence, D is correct.

113. As $f \circ f(x) = 0$

⇒ Hence, C is correct.

114. $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \operatorname{sgn} f(x) = 0$

Hence, A is correct.

Comprehension-4:

115. $$\lim_{x\to 0} \frac{\displaystyle\int_0^x \frac{t^2\,dt}{\left(a+t^r\right)^{1/p}}}{bx-\sin x}$$

$$\Rightarrow \lim_{x\to 0} \frac{x^2}{\left(a+x^r\right)^{1/p}} \times \frac{1}{b-\cos x}$$

For limit to exist and non zero $\Rightarrow b = 1$

Hence, D is correct.

116. $\lim\limits_{x\to 0} \frac{x^2}{(1-\cos x)} \times \frac{1}{\left(a+x^r\right)^{1/3}} = 1$

$\Rightarrow a^{1/3} = 2$

$\Rightarrow a = 8$

Hence, A is correct.

117. $\lim\limits_{x\to 0} \left(\frac{x^2}{1-\cos x}\right)\left(\frac{1}{9+x^r}\right)^{1/2} = \frac{2}{3}$

Hence, B is correct.

Comprehension-5:

118. $$\lim_{x\to 0^+} F(x) = \frac{\lim}{x\to 0^+} e^{\frac{\left(p_1a_1^x+p_2a_2^x+\cdots+p_na_n^x-1\right)}{x}}$$

$$= \frac{\lim}{x\to 0^+} e^{\left(\frac{p_1\left(a_1^x-1\right)}{x}+\frac{p_2\left(a_2^x-1\right)}{x}+\cdots+p_n\frac{\left(a_n^x-1\right)}{x}\right)}$$

$$= \lim_{x\to 0^+} e^{\left(p_1 \ln a_1+p_2 \ln a_2+\cdots p_n \ln a_n\right)}$$

$$= a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n}$$

Hence, C is correct.

119. $F(x) = \left(p_1a_1^x + p_2a_2^x + \ldots\right)^{1/x}$

$$\log F(x) = \frac{1}{x} \log\left(p_1a_1^x + p_2a_2^x + \ldots\right)$$

$$\lim_{x\to\infty} \log F(x) = \lim_{x\to\infty} \frac{\log\left(p_1a_1^x + p_2a_2^x + \ldots\right)}{x}$$

$$= \lim_{x\to\infty} \frac{p_1 a_1^x \ln a_1 + P_2 a_2^x \ln a_2 + \cdots}{p_1 a_1^x + p_2 a_2^x + \cdots}$$

$$= \lim_{x\to\infty} \frac{p_1 \ln a_1 + p_2 (a_2/a_1)^x \ln a_2 + \cdots}{p_1 + p_2 (a_2/a_1)^x + \cdots}$$

$$= \ln a_1$$

$$\Rightarrow \lim_{x\to\infty} F(x) = a_1$$

Hence, C is correct.

120. Put, $x = -1/y$

$$\lim_{y\to 0} \log F(-1/y) = \lim_{y\to 0} \frac{\log\left(p_1 a_1^{-1/y} + p_2 a_2^{-1/y} + \ldots\right)}{-1/y}$$

$$\frac{p_1 a_1^{-1/y} \ln a_1 + p_2 a_2^{-1/y} \ln a_2 + \ldots + p_n a_n^{-1/y} \ln a_n}{p_1 a_1^{-1/y} + p_2 a_2^{-1/y} + \cdots p_n a_n^{-1/y}}$$

$$\Rightarrow \lim_{x\to -\infty} F(x) = a_n$$

Hence, D is correct.

MATCH THE COLUMN FOR JEE ADVANCED

121. A) $\lim_{n\to\infty} \cos^2\left(n\pi\left(1+\frac{1}{n}+\frac{2}{n^2}\right)^{1/3}\right)$

$$\lim_{n\to\infty} \cos^2\left(n\pi\left(1+\frac{1}{3n}+\frac{2}{3n^2}+\cdots\right)\right)$$

$$\lim_{n\to\infty} \cos^2\left(n\pi+\frac{\pi}{3}+\frac{2\pi}{3n}+\cdots\right)$$

$$\cos^2\frac{\pi}{3}=\frac{1}{4}$$

B) $\lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2}\right)$

$$= \lim_{n\to\infty} n\sin\left(2\pi n\left(1+\frac{1}{n^2}\right)^{1/2}\right)$$

$$= \lim_{n\to\infty} n\sin\left(2\pi n\left(1+\frac{1}{2n^2}+\cdots\right)\right)$$

$$= \lim_{n\to\infty} n\sin\frac{\pi}{n} = \pi$$

C) $\lim_{n\to\infty} (-1)^n \sin\left(\pi\sqrt{n^2+0\cdot 5n+1}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)$

$$\lim_{n\to\infty} (-1)^n \sin n\pi\left(1+\frac{1}{4n}+\frac{1}{2n^2}+\ldots\right)\cos\left(\frac{\pi}{2n}\right)$$

$$\lim_{n\to\infty} (-1)^n \sin\left(n\pi+\frac{\pi}{4}+\frac{\pi}{2n}+\ldots\right)\cos\left(\frac{\pi}{2n}\right)$$

$$= \frac{1}{\sqrt{2}}$$

D) $\lim_{x\to\infty} e^{\left(\frac{x+a}{x-a}-1\right)x}$

$$\lim_{x\to\infty} e^{\frac{2ax}{x-a}} = e^{2a} = e$$

$$\Rightarrow a = \frac{1}{2}$$

$A \to Q, B \to R, C \to P, D \to P$.

122. A) $\lim_{x\to\infty} \sqrt{x^2-x-1}-ax-b$

$$\lim_{x\to\infty} \frac{x^2-x-1-(ax+b)^2}{\sqrt{x^2-x-1}+ax+b}$$

$$\lim_{x\to\infty} \frac{\left(1-a^2\right)x^2-(1+2ab)x-\left(1+b^2\right)}{\sqrt{x^2-x-1}+ax+b}$$

For limit to be zero; $a^2=1,\ 2ab+1=0$

$\Rightarrow a=+1,\ b=-\frac{1}{2}$ or $a=-1,\ b=1/2$

$(a, 2b) \Rightarrow (1,-1)$ which lies on $3x-2y-5=0$

B) $\lim_{x\to\infty} \frac{\left(1+a^3\right)+8e^{1/x}}{1+\left(1-b^3\right)e^{1/x}}$

$$\frac{9+a^3}{2-b^3}=2$$

$\Rightarrow 5+a^3+2b^3=0$ ....(I)

$\Rightarrow \left(a,b^3\right)$ can be $(1,-3)$, $\left(0,-\frac{5}{2}\right)$

For $15x-2y-11=0$

$\Rightarrow 15a-2b^3-11=0$ .... (II)

Solving (I) and (II) $\Rightarrow a^3+15a-6=0$

Which will have a real solution

$\& \ b^3 \neq 2$

C) $\lim_{x\to\infty}\sqrt{x^4-x^2+1}-\left(ax^2+b\right)$

$$\lim_{x\to\infty}\frac{x^4-x^2+1-a^2x^4-b^2-2abx^2}{\sqrt{x^4-x^2+1}+\left(ax^2+b\right)}$$

$$\lim_{x\to\infty}\frac{\left(1-a^2\right)x^4-(1+2ab)x^2+\left(1-b^2\right)}{\sqrt{x^4-x^2+1}+\left(ax^2+b\right)}=0$$

$\Rightarrow a^2=1\Rightarrow a=+1, b=-\frac{1}{2}$

$\&$ $(1+2ab=0)$

$\Rightarrow (a,-2b)$

$=(1,1)$

D) $\lim_{x\to -a}\frac{x^7-(-a)^7}{x-(-a)}=7(-a)^6=7$

$a=\pm 1\Rightarrow a=-1$ (as $a<0$)

so, $(-1,2)$ lies only on $y=2$

123. A) $f(x)=|x-a|+|x-10|+|x-a-10|$

At $x=a$, $f(x)=|a-10|+|-10|=10$ (Min value)

At $x=10$, $f(x)=|10-a|+|-a|=10$ (Min value)

At $x=a+10$, $f(x)=|10|+|a|=10$ (Min value)

B) $\lim_{x\to 0}\frac{x(1-\cos 2x)^2-a(\sin x-\tan x)^2}{\tan^5 x+a\sin^8 x}$

$$=\lim_{x\to 0}\frac{\sin^5 x}{\sin^5 x}\frac{\left(\frac{x}{\sin x}\times 4-a\frac{(2\sin^2 x/2)^2}{\cos^2 x\sin^3 x}\right)}{\sin^5 x\left(\sec^5 x+a\sin^3 x\right)}$$

$$=\lim_{x\to 0}\left[4\left(\frac{x}{\sin x}\right)-\frac{4a\sin^4 x/2}{\cos^2 x\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^3}\right]\Bigg/\left(\sec^5 x+a\sin^3 x\right)$$

$=4$

C) $\lim_{n\to\infty}\frac{n^a\sin^2\angle n}{n+1}==\lim_{n\to\infty}\frac{\left(\sin^2\angle n\right)n^{a-1}}{1+1/n}=0$

D) $\log_a(2a-x)=\left(\log_a x\right)^3$ has only one solution at $x=a$

A → S, B → R, C → P, D → Q

124. A) $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)$

$$=\lim_{x\to\infty}\left(\frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{2}{\sqrt{1+1/\sqrt{x}}+\sqrt{1-1/\sqrt{x}}}\right)=1$$

B) $\lim_{x\to 0}\frac{\sin 2x-2\tan x}{\ln\left(1+x^3\right)}$

$$=\lim_{x\to 0}\frac{-2\sin x}{\cos x}\frac{\left(\sin^2 x\right)}{x^3}\times\frac{x^3}{\ln\left(1+x^3\right)}$$

$=-2$

C) $\lim_{x\to 0^+}\ln\sin^3 x-\ln\left(x^3(x+e)\right)$

$$=\lim_{x\to 0^+}\ln\left(\frac{\sin^3 x}{x^3(x+e)}\right)=\ln\left(\frac{1}{e}\right)=-1$$

D) $2\pi|\sin\theta|+2\pi|\cos\theta|=\frac{5\pi}{2}$

$\Rightarrow |\sin\theta|+|\cos\theta|=\frac{5}{4}$

$f(x)=\left(\frac{5}{4}\right)^x$

$$\lim_{x\to\infty}\left[2\times\left(\frac{4}{5}\right)^x\right]=0$$

A → S, B → P, C → Q, D → R

125. A) $\lim_{x\to 0}\frac{1-\cos 2x}{e^{x^2}-e^x+x}$

$$\lim_{x\to 0}\frac{2\sin 2x}{2xe^{x^2}-e^x+1}=\lim_{x\to 0}\frac{4\cos 2x}{4x^2e^{x^2}+2e^{x^2}-e^x}=4$$

B) $\lim_{x\to 0^+}\left(\frac{3+x}{3-x}\right)^{1/x}=\lim_{x\to 0^+}e^{\left(\frac{3+x}{3-x}-1\right)\times\frac{1}{x}}$

$$= \lim_{x\to 0^+} e^{\frac{2}{3-x}} = e^{2/3}$$

$$\Rightarrow p+q=5$$

C) $\lim\limits_{x\to 0} \dfrac{\tan^3 x - \tan x^3}{x^5}$

$$= \lim_{x\to 0} \frac{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^3 - \left(x^3+\frac{x^9}{3}+\frac{2}{15}x^{15}+\cdots\right)}{x^5}$$

$$=1 \quad (\text{As coeff of } x^5 \text{ in} \left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^3$$

D) $\lim\limits_{x\to 0}\left(\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x - x+1}}\right)$

$$\lim_{x\to 0} \frac{(x+2\sin x)\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1}\right)}{\left(x^2+x+2\sin x-\sin^2 x\right)}$$

$$\lim_{x\to 0} \frac{(1+\frac{2\sin x}{x})\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1}\right)}{\left(1+x+\frac{2\sin x}{x}-\frac{\sin^2 x}{x}\right)}$$

$$=2$$

$A \to R,\ B \to S,\ C \to P,\ D \to Q$

Review Exercises for JEE Advanced

1. (i) $\lim\limits_{h\to 0} \dfrac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$

$$\lim_{h\to 0} \frac{\tan(h)(1+\tan(a+2h)\tan(a+h)) - \tan(h)(1+\tan a\tan(a+h))}{h^2}$$

$$\lim_{h\to 0} \frac{\tan h}{h}[\tan(a+h)]\frac{\tan(2h)}{h}[1+\tan a\tan(a+2h)]$$

$$\Rightarrow 2\tan a\sec^2 a = \frac{2\sin a}{\cos^3 a}$$

(ii) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$

$$\lim_{x\to 0} -\left(\frac{\tan^{-1}3x+\sin^{-1}3x}{\tan^{-1}2x+\sin^{-1}2x}\right)$$

$$\left(\frac{\sqrt{1-\sin^{-1}2x}+\sqrt{1+\tan^{-1}2x}}{\left(1+\tan^{-1}3x\right)^{2/3}+\left(1-\sin^{-1}3x\right)^{2/3}+\left(1+\tan^{-1}3x\right)^{1/3}\left(1-\sin^{-1}3x\right)^{1/3}}\right)$$

$$= -\frac{3}{2}\times\frac{2}{3} = -1$$

2. (i) $\lim\limits_{x\to 0} \dfrac{1-\cos x\cos 2x\cos 3x}{\sin^2 2x}$

$$= \lim_{x\to 0} \frac{\sin x\cos 2x\cos 3x + 2\sin 2x\cos x\cos 3x + 3\sin 3x\cos x\cos 2x}{2\cdot 2\sin 2x\cos 2x}$$

$$= \frac{1+4+9}{8} = \frac{7}{4}$$

(ii) $\lim\limits_{x\to 0} \dfrac{(\sin x-\tan x)^2 - 4x\sin^4 x}{\sin^5 x\left(5+8\sin^3 x\cos^8 x\right)}$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)^2}{\cos^2 x\sin^3 x} - \frac{4x}{\sin x}\right]\times\frac{1}{\left(5+8\sin^3 x\cos^8 x\right)}$$

$$= -\frac{4}{5}$$

3. (i) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1+x^2}\cos x}{\tan^4 x}$

$$= \lim_{x\to 0} \frac{1-\sqrt{1+x^2}\left(1-\frac{x^2}{\angle 2}+\frac{x^4}{\angle 4}\cdots\right)}{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^4}$$

$$= \lim_{x\to 0} \frac{1-\left(1+\frac{x^2}{2}-\frac{x^4}{8}\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{\angle 4}+\cdots\right)}{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^4} \qquad \text{(I)}$$

Coeff of x^4 in Numerator $=\dfrac{1}{3}$

Hence, from (I),

$$\lim_{x\to 0} \frac{\frac{1}{3}x^4+ax^6+bx^8+\cdots}{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^4}$$

$$= \frac{1}{3}$$

(ii) $$\lim_{x\to0}\frac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2 x/2}$$

$$=\lim_{x\to0}\frac{(1-\cos 2x+x\sin x)}{\tan^2 x/2}\times\frac{1}{\sqrt{1+x\sin x}+\sqrt{\cos 2x}}$$

$$=\lim_{x\to0}\frac{\sin x(x+2\sin x)}{\tan^2\frac{x}{2}}\times\frac{1}{(\sqrt{1+x\sin x}+\sqrt{\cos 2x})}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\times\left(1+2\frac{\sin x}{x}\right)\times\frac{x^2}{\tan^2 x/2}\times\frac{1}{(\sqrt{1+x\sin x}+\sqrt{\cos 2x})}$$

$$=6$$

4. (i) $$\lim_{x\to0}\frac{1}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$$

$$=\lim_{x\to0}\frac{\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)}{x^8}$$

$$=\lim_{x\to0}\frac{2\sin^2 x^2/4}{x^4}\times\frac{2\sin^2 x^2/8}{x^4}$$

$$=4\times\frac{1}{16}\times\frac{1}{6^4}=\frac{1}{2^8}$$

(ii) $$\lim_{x\to0}\frac{\sin\sin\tan\left(x^2/2\right)}{\ln\cos 3x}$$

$$=\lim_{x\to0}\frac{\sin\sin\tan x^2/2}{\sin\tan x^2/2}\times\frac{\sin\tan x^2/2}{\tan x^2/2}\times\frac{\tan x^2/2}{x^2/2}\times\frac{x^2\times(\cos 3x-1)}{2\ln(1+\cos 3x-1)(\cos 3x-1)}$$

$$=-\frac{1}{9}$$

5. $$\lim_{x\to0}\frac{\sin(3x+a)-\sin a-3(\sin(2x+a)-\sin(x+a))}{x^3}$$

$$=\lim_{x\to0}\frac{2\cos\left(\frac{3x+2a}{2}\right)\sin 3x/2-6\cos\left(\frac{3x+2a}{2}\right)\sin(x/2)}{x^3}$$

$$=\lim_{x\to0}\frac{2\cos\left(\frac{3x+2a}{2}\right)\left[\frac{3\sin x}{2}-4\sin^3\frac{x}{2}-3\sin\frac{x}{2}\right]}{x^3}$$

$$=\lim_{x\to0}-8\cos\left(\frac{3x+2a}{2}\right)\frac{\sin^3 x/2}{x^3}=-\cos a$$

6. $$\lim_{x\to1}\frac{(1-x)\left(1-x^2\right)\cdots\left(1-x^{2n}\right)}{\left((1-x)\left(1-x^2\right)\cdots\left(1-x^n\right)\right]^2}$$

Divide Nr &Dr by $(1-x)^{2n}$

$$\Rightarrow\lim_{x\to1}\frac{\left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\cdots\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\cdots\left(\frac{1-x^n}{1-x}\right)\right]^2}$$

$$\Rightarrow\frac{2\cdot3\cdot4\cdots2n}{(2\cdot3\cdots\cdots n)^2}=\frac{\angle 2n}{(\angle n)^2}$$

7. $$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{2^2 n}\right)\cdots\left(1+\frac{1}{2^{n-1}\cdot n}\right)\right]^n$$

$$=\lim_{n\to\infty}e^{\left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{2^2 n}\right)\cdots\left(1+\frac{1}{2^{n-1}n}\right)-1\right]^n}$$

$$=\lim_{n\to\infty}e^{\left[\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\frac{1}{2^{n-1}}\right)+\text{higher power of n}\right]}$$

$$=e^2$$

8. $$y=\lim_{x\to0^+}\left(\log_{\sin x}\sin 2x\right)^{\log\sin 2x}=1^\infty\text{ form}$$

$$y=e^{\lim\limits_{x\to0^+}\left(\frac{(\log\sin 2x-\log\sin x)}{\log\sin x}\right)\log\sin 2x}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

Now, $$\lim_{x\to0^+}\frac{(\log\sin 2x-\log\sin x)}{\log\sin x}\times\log\sin 2x$$

$$=\lim_{x\to0}\left(\log 2+\log\cos x\right)\times\frac{\log\sin 2x}{\log\sin x}$$

$$=\log 2\quad\left(\text{As}\lim_{x\to0}\frac{\log\sin 2x}{\log\sin x}=1\right)$$

From (I), $y=e^{\log 2}=2$ Ans

9. $$\lim_{x\to0}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{\tan x-\sin x+\cdots\sqrt{\infty}}}}{-1+\sqrt{x^3+\sqrt{x^3+\cdots\sqrt{\infty}}}}$$

Let $y^2=t+\sqrt{t+\sqrt{t}+\cdots}$

$\Rightarrow y^2-y-t=0$

$\Rightarrow \quad y=\dfrac{1\pm\sqrt{1+4t}}{2}\Rightarrow y=\dfrac{1+\sqrt{1+4t}}{2}$

Put $t=\tan x-\sin x$ $\&\,x^3$ & use eq (1)

$$\lim_{x\to0}\frac{-1+\sqrt{1+4(\tan x-\sin x)}}{-1+\sqrt{1+4x^3}}$$

$$=\lim_{x\to0}\frac{4(\tan x-\sin x)}{4x^3}\times\frac{\left(\sqrt{1+4x^3}+1\right)}{\left(\sqrt{1+4(\tan x-\sin x)}+1\right)}$$

$$=\frac{1}{2}\qquad\left(\text{As }\lim_{x\to0}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}\right)$$

10. $\displaystyle\lim_{n\to\infty}\frac{\sqrt{1-x_0^2}}{x_1x_2\cdots x_n}$ where $x_{r+1}=\sqrt{\dfrac{1+x_r}{2}}$

Let $x_0=\cos\theta$

$\Rightarrow \quad x_1=\cos\theta/2$

$$\Rightarrow\lim_{n\to\infty}\frac{\sqrt{1-x_0^2}}{x_1x_2\cdots x_n}=\lim_{n\to\infty}\frac{\sin\theta}{\left(\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^n}\right)}$$

$$=\lim_{n\to\infty}2^n\frac{\sin\theta\sin\theta/2n}{\sin\theta}$$

$$=\lim_{n\to\infty}\frac{\sin\theta/2^n}{\left(\theta/2^n\right)}\times\theta=\theta=\cos^{-1}x_0$$

11. $$T_r=\frac{r(n-(r-1))(n-(r-2))}{2}$$

$$=\frac{r((n+1)-r)((n+2)-r)}{2}$$

$$=\frac{r\left((n+1)(n+2)-r(2n+3)+r^2\right)}{2}$$

$$=\frac{1}{2}\left[r^3-r^2(2n+3)+r(n+1)(n+2)\right]$$

$$S_n=\sum_{r=1}^{n}T_r=\frac{1}{2}\left[\frac{n^2(n+1)^2}{4}-\frac{(2n+3)n(n+1)(2n+1)}{6}+\frac{n(n+1)^2(n+2)}{2}\right]$$

Now $$\lim_{n\to\infty}\frac{1\sum_{r=1}^{n}r+2\sum_{r=1}^{n-1}r+\cdots+n\cdot1}{n^4}$$

$$=\lim_{n\to\infty}\frac{1}{24}\left[\frac{3n^2(n+1)^2-2n(n+1)(2n+1)(2n+3)+6(n)(n+1)^2(n+2)}{n^4}\right]$$

$$=\frac{1}{24}[3-8+6]=\frac{1}{24}$$

12. $$T_r=\frac{r(r+1)(2r+1)\times4}{6\cdot r^2(r+1)^2}=\frac{2(2r+1)}{3r(r+1)}=\frac{2}{3}\left[\frac{1}{r}+\frac{1}{r+1}\right]$$

$$\lim_{x\to\infty}s_n=-T_1+T_2-T_3+T_4+\cdots\infty$$

$$=\frac{2}{3}\left[\left(-1-\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}\right)+\cdots\infty\right]$$

$$=-\frac{2}{3}$$

13. $$\lim_{x\to\infty}\ln_{\frac{x^2+1}{x+1}}\sqrt{x^2+1}-x$$

$$=-\lim_{x\to\infty}\frac{\ln\left(\sqrt{x^2+1}+x\right)}{\ln\left(x^2+1\right)-\ln(x+1)}$$

$$=-\lim_{x\to\infty}\frac{\left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\times\dfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{\dfrac{2x}{x^2+1}-\dfrac{1}{x+1}}$$

$$=-\lim_{x\to\infty}\frac{(x+1)\left(x^2+1\right)}{\left(\sqrt{x^2+1}\right)\left(x^2+2x-1\right)}$$

$$=-\lim_{x\to\infty}\frac{(1+1/x)\left(1+1/x^2\right)}{\sqrt{1+1/x^2}\left(1+2/x-1/x^2\right)}=-1$$

14. $$\lim_{x\to\pm\infty}\left(\frac{x^3+1}{x}-x^2\right)$$

$$= \lim_{x\to\pm\infty}\left(\frac{1}{x}\right) = 0$$

which means at *x* tends to infinity, value of both the functions are tending towards same finite value. Hence, they are asymptotic to each other.

15. $$\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{x+1}{x+2}\right)-\tan^{-1}\left(\frac{x}{x+2}\right)}{1/x}$$

$$\lim_{x\to\infty}\left(\frac{1}{1+\frac{(x+1)^2}{(x+2)^2}}\times\frac{1}{(x+2)^2}-\frac{1}{1+\frac{x^2}{(x+2)^2}}\times\frac{2}{(x+2)^2}\right)\times -x^2$$

$$\lim_{x\to\infty}\frac{\left(-((x+2)^2+x^2\right)+2\left((x+2)^2+(x+1)^2\right)\right)}{((x+2)^2+(x+1)^2)((x+2)^2+x^2)}x^2$$

$$\lim_{x\to\infty}\left(\frac{(x+2)^2+2(x+1)^2-x^2)x^2}{\left((x+2)^2+(x+1)^2\right)((x+2)^2+x^2)}\right)$$

$$=\frac{2}{4}=\frac{1}{2}$$

16. $$\lim_{x\to 0}\frac{(\sin x-\tan x)^2+(1-\cos 2x)^4+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin^2 x}{\cos^2 x}\left(2\sin^2\frac{x}{2}\right)^2+2^4\sin^8 x+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$$

$$=\lim_{x\to 0}\frac{\frac{4\sec^2 x\sin^4 x/2}{\sin^3 x}+2^4\sin^3 x+(x/\sin x)^5}{7\sin^2 x\cos^7 x+\sin x+2}=\frac{1}{2}$$

17. A) $$\lim_{x\to 1/2}\frac{\frac{|x-1|}{2}+x|x-1|+2-\frac{2}{x}}{\sqrt{x-2+\frac{1}{x}}}$$

$$=\left(\frac{1}{4}+\frac{1}{4}+2-4\right)/\sqrt{1/2}$$

$$=-\frac{3}{\sqrt{2}}$$

B) $$\lim_{x\to 3/2}\frac{\frac{|x-1|}{2}+x|x-1|+2-\frac{2}{x}}{\sqrt{x-2+\frac{1}{x}}}$$

$$=\left(\frac{1}{4}+\frac{3}{4}+2-\frac{4}{3}\right)\Big/\sqrt{\frac{2}{3}-\frac{1}{2}}$$

$$=\frac{5}{3}\times\sqrt{6}=\frac{5\sqrt{2}}{\sqrt{3}}$$

18. $$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}=\lim_{x\to 0}\frac{x+x^2}{x\left(\sqrt{1+x+x^2}+1\right)}=\frac{1}{2}$$

$$\&\ \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-2\cos x}{\sin\left(x-\frac{\pi}{4}\right)}=\lim_{h\to 0}\frac{\sqrt{2}-2\cos(\pi/4+h)}{\sin h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2}-\sqrt{2}\cos h+\sqrt{2}\sinh}{\sin h}$$

$$=2\sqrt{2}\lim_{h\to 0}\frac{\sin h/2\left[\sin h/2+\cos h/2\right]}{2\sin h/2\cos h/2}=\sqrt{2}$$

Now LHS $=\frac{1}{2}+\sqrt{2}=1.914$

RHS $=\log 0.005=-2.301$

$\Rightarrow$ LHS $>$ RHS

19. $$\lim_{x\to -0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}-\frac{11}{24}ex^2}{x^3}$$

$$=\lim_{x\to 0}\frac{\left(e-\frac{ex}{2}+\frac{11}{24}ex^2-\frac{7x^3e}{16}+\cdots\right)-e+\frac{ex}{2}-\frac{11}{24}ex^2}{x^3}$$

$$=-\frac{7}{16}e$$

20. For $r=1$,

$$\lim_{x\to 4}\frac{d_1-2}{x-4}=\lim_{x\to 4}\frac{\sqrt{2+\sqrt{x}}-2}{x-4}$$

$$=\lim_{x\to 4}\frac{\sqrt{x}-2}{(x-4)\left(\sqrt{2+\sqrt{x}}+2\right)}$$

$$=\lim_{x\to 4}\frac{1}{\left(\sqrt{x}+2\right)\left(\sqrt{2+\sqrt{x}}+2\right)}=\frac{1}{16}$$

For $r=3$

$$d_3=\sqrt{2+d_2}=\sqrt{2+\sqrt{2+d_1}}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{x}}}}$$

$$\lim_{x\to 4}\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{x}}}}-2}{x-4}$$

Apply L-Hospital Rule,

$$\lim_{x\to4}\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}}\times\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}}\times\frac{1}{2\sqrt{2+\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{4^4}=2^{-8}$$

21. $\phi_n(x)=\dfrac{x^n f(x)+g(x)}{x^n+1}$

Case 1: $|x|<1$

$$\lim_{n\to\infty}\frac{x^n f(x)+g(x)}{x^n+1}=g(x)$$

Case 2: $|x|>1$

$$\lim_{n\to\infty}\frac{f(x)+x^{-n}g(x)}{1+x^{-n}}=f(x)$$

Case 3: x = 1

$$\lim_{n\to\infty}\frac{x^n f(x)+g(x)}{x^n+1}=\frac{f(x)+g(x)}{2}$$

Case 4: $x=-1$

$$\lim_{n\to\infty}\frac{x^n f(x)+g(x)}{x^n+1}=\text{ Not defined}$$

22. $\lim\limits_{x\to0}\left[\dfrac{a\sin x}{x}\right]+\left[\dfrac{b\tan x}{x}\right]$

As $\dfrac{\sin x}{x}<1$ & $\dfrac{\tan x}{x}>1$ when $x\to0$

$$\Rightarrow\frac{a\sin x}{x}<a\quad\&\quad\frac{b\tan x}{x}>b$$

$$\Rightarrow\lim_{x\to0}\left[\frac{a\sin x}{x}\right]+\left[\frac{b\tan x}{x}\right]=a-1+b$$

23. As $f_1(x)=\dfrac{x}{2}+10=\dfrac{x+20}{2}$

$$f_n(x)=\cfrac{\cfrac{\cfrac{\cfrac{x+20}{2}+20}{2}+20}{2}+20}{+}\cdots\infty$$

$$f_n(x)=\frac{x+20+40+80+160+\cdots}{2^n}$$

$$f_n(x)=\frac{x}{2^n}+\frac{20\left(2^n-1\right)}{2^n}$$

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\left(\frac{x}{2^n}+\frac{20\left(1-2^{-n}\right)}{1}\right)$$

$= 20$

24. If $0<a<1$,

$$\lim_{x\to\infty}\left(\frac{a^x-1}{x(a-1)}\right)^{1/x}=(0)^0=1$$

If $a>1$

$$y=\lim_{x\to\infty}\left(\frac{a^x-1}{x(a-1)}\right)^{1/x}$$

$$\ln y=\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{a^x-1}{x(a-1)}\right)$$

$$=\lim_{x\to\infty}\frac{x(a-1)}{a^x-1}\times\frac{x(a-1)a^x\ln a-\left(a^x-1\right)(a-1)}{x^2(a-1)^2}$$

$$=\lim_{x\to\infty}\frac{xa^x\ln a-a^x+1}{x\left(a^x-1\right)}$$

$$=\lim_{x\to\infty}\left(\frac{a^x\ln a}{\left(a^x-1\right)}-\frac{1}{x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\ln a}{1-a^{-x}}-\frac{1}{x}\right)=\ln a$$

$\Rightarrow y=a$

25. $y=1-\lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty}\left(\cos(m!\pi x)^{2n}\right)\right)$

When x is rational, $m!\pi x$ → Integral multiple of π

$\Rightarrow\cos m!\pi x=\pm1$

$$\Rightarrow\lim_{n\to\infty}(\cos m!\pi x)^{2n}=1$$

$\Rightarrow y=0$

When x is irrational, $m!\pi x$ → Irrational multiple of π

$\cos m!\pi x=(-1,1)$

$$\Rightarrow\lim_{m\to\infty}\lim_{n\to\infty}\cos(m!\pi x)^{2n}=0$$

$\Rightarrow y=1$

26. $\lim\limits_{x\to\infty}\dfrac{x^{1/3}(1-x)^{2/3}}{x}=\lim\limits_{x\to\infty}\left(\dfrac{1}{x}-1\right)^{\frac{2}{3}}=+1=m$

$$b = \lim_{x\to\infty} x^{1/3}(1-x)^{2/3} - x$$

$$= \lim_{x\to\infty} \frac{x(1-x)^2 - x^3}{x^{2/3}(1-x)^{4/3} + x^2 + x\cdot x^{1/3}(1-x)^{2/3}}$$

$$= \lim_{x\to\infty} \frac{x(1-2x)}{x^2\left[\left(\frac{1}{x}-1\right)^{4/3} + 1 + \left(\frac{1}{x}-1\right)^{2/3}\right]} = \frac{-2}{3}$$

Now as $b = \lim_{x\to\infty}(f(x) - mx)$

$\Rightarrow \lim_{x\to\infty}(f(x) - mx - b) = 0$ As b is independent of x.

27. $$y = \lim_{n\to\infty} \frac{1}{1 + n\sin^2 \pi x}$$

when x is not an integer

$$y = \lim_{n\to\infty} \frac{1/n}{1/n + \sin^2 \pi x} = 0$$

when x is an integer

$\sin^2 \pi x$ will be equal to zero

$\Rightarrow n\sin^2 \pi x = 0$

$$y = \lim_{n\to\infty} \frac{1}{1 + n\sin^2 \pi x} = 1$$

28. $$y = \lim_{n\to\infty} \frac{\varphi(x) + n\phi(x)\sin^2 \pi x}{1 + n\sin^2 \pi x}$$

When x is an integer

$$y = \lim_{n\to\infty} \frac{\varphi(x) + n\phi(x)\sin^2 \pi x}{1 + n\sin^2 \pi x}$$

$= \varphi(x)$

When x is not an integer

$$y = \lim_{n\to\infty} \frac{\frac{\varphi(x)}{n} + \phi(x)\sin^2 \pi x}{1/n + \sin^2 \pi x} = \phi(x)$$

29. $$\lim_{x\to\infty} \frac{x\tan x + 2x - 1}{x + 1}$$

As $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

$\Rightarrow$ Power of x in numerator is larger than the denominator, then limit is not defined.

Now, if x tends to $n\pi + \frac{\pi}{4}$

$$\lim_{n\to\infty} \lim_{x\to n\pi+\frac{\pi}{4}} \left(\frac{\left(n\pi + \frac{\pi}{4}\right) + 2\left(n\pi + \frac{\pi}{4}\right) - 1}{nx + \frac{\pi}{4} + 1} \right) =$$

$$\lim_{n\to\infty} \frac{3\left(n\pi + \frac{\pi}{4}\right) - 1}{n\pi + \frac{\pi}{4} + 1} = 3$$

30. $$\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\tan x}$$

$$= \lim_{x\to 0} \frac{x\sin(1/x)}{\left(\frac{\tan x}{x}\right)} = 0$$

Whereas,

$f'(x) = -\cos(1/x) + 2x\sin(1/x)$

$\phi'(x) = \sec^2 x$

$$\lim_{x\to 0} \frac{f'(x)}{\phi'(x)} = \lim_{x\to 0} \frac{2x\sin 1/x - \cos 1/x}{\sec^2 x}$$

$$= \lim_{x\to 0}\left(2x\cos^2 x \sin\frac{1}{x} - \cos^2 x \cos\frac{1}{x}\right)$$

= Not defined

31. $$y = \lim_{n\to\infty} \frac{f(x) + n\phi(x)\sin^2 \pi x}{1 + n\sin^2 \pi x}$$

Similar to Q.28

32. $$y = \lim_{n\to\infty} \left[\frac{\left(n^3+1^3\right)\left(n^3+2^3\right)\cdots\left(n^3+n^3\right)}{\left(n^3\right)^n}\right]^{1/n}$$

$$\ln y = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \ln\left(1 + \left(\frac{r}{n}\right)^3\right)$$

$$= \int_0^1 \ln\left(1 + x^3\right)dx$$

$$= x\ln\left(1 + x^3\right) - \int \frac{3x^3}{1 + x^3}dx$$

$$= x\ln\left(1 + x^3\right) - 3x + 3\int \frac{dx}{1 + x^3}$$

$$= x\ln\left(1 + x^3\right) - 3x + 3\left[\begin{array}{l} \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln\left(x^2 - x + 1\right) \\ + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) \end{array}\right]_0^1$$

$$= \left(\ln 2 - 3 + \ln 2 + \sqrt{3}\frac{\pi}{6}\right) - \left(-\sqrt{3}\frac{\pi}{6}\right)$$

$$= 2\ln 2 - 3 + \frac{\pi}{\sqrt{3}} = \ln 4 - 3 + \frac{\pi}{\sqrt{3}}$$

$$\Rightarrow y = 4e^{\frac{\pi}{\sqrt{3}} - 3}$$

33. $$\lim_{x\to\infty}\left(\frac{3x^2+1}{3x^2-1}\right)^{\frac{x^3}{1+x}}$$

$$= \lim_{x\to\infty} e^{\left(\frac{3x^2+1}{3x^2-1}-1\right)\frac{x^3}{(1+x)}}$$

$$= \lim_{x\to\infty}\frac{2x^3}{(1+x)\left(3x^2-1\right)} = e^{2/3} = k$$

$$\Rightarrow 6\ln k = 6\ln e^{2/3} = 4\ln e = 4$$

$$\Rightarrow [6\ln k] = 4$$

34. $$\lim_{x\to 2}\frac{\sqrt{x^2+5}-\sqrt[3]{2x^3+3x+5}}{\sqrt[3]{x+6}-\sqrt[5]{4x^2+6x+4}}$$

$$\Rightarrow \lim_{x\to 2}\frac{\dfrac{\left(x^2+5\right)^{1/2}-(9)^{1/2}}{x-2}-\left(\dfrac{\left(2x^3+3x+5\right)^{1/3}-(27)^{1/3}}{x-2}\right)}{\dfrac{(x+6)^{1/3}-(8)^{1/3}}{x-2}-\left(\dfrac{\left(4x^2+6x+4\right)^{1/5}-(32)^{1/5}}{x-2}\right)}$$

$$= \lim_{x\to 2}\frac{\left[(x+2)\dfrac{((x^2+5)^{1/2}-(9)^{1/2})}{\left(x^2+5-9\right)}-\left(2x^2+4x+11\right)\dfrac{\left(\left(2x^3+3x+5\right)^{1/3}-(27)^{1/3}\right)}{\left(2x^3+3x+5-27\right)}\right]}{\left[\dfrac{(x+6)^{1/3}-8^{1/3}}{(x+6-8)}-2(2x+7)\left[\dfrac{\left(4x^2+6x+4\right)^{1/5}-(32)^{1/5}}{4x^2+6x+4-32}\right]\right]}$$

$$= \frac{4\cdot\frac{1}{2}(9)^{-1/2}-27\cdot\frac{1}{3}\cdot(27)^{-2/3}}{\frac{1}{3}(8)^{-2/3}-22\cdot\frac{1}{5}(32)^{-4/5}} = \frac{\frac{2}{3}-1}{\frac{1}{12}-\frac{22}{80}} = \frac{\frac{-1}{3}}{\frac{80-22\times 12}{12\times 80}}$$

$$= \frac{+1}{3}\times\frac{12\times 80}{184} = \frac{+40}{23}$$

$$\Rightarrow 230b = 400$$

35. $$\lim_{x\to 0}\frac{\cos 4x + a\cos 2x + b}{x^4} = \text{finite quantity}$$

$a + b + 1 = 0$ .... (1)

$$\lim_{x\to 0}\frac{-4\sin 4x - 2a\sin 2x}{4x^3}$$

$$\lim_{x\to 0}\frac{-2\sin 2x(4\cos 2x + a)}{4x^3}$$

$\Rightarrow$ For limit to exist, $a = -4$

$\Rightarrow b = 3$

$\Rightarrow a^{b+1} = (-4)^4 = 4^4 = 256$

Target Exercises for JEE Advanced

1. $$\lim_{x\to\infty}\left(\frac{x^{x+1}+x+1}{x^{x+1}}\right)^x = 1^\infty \text{ form.}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{x^{x+1}+x+1-x^{x+1}}{x^{x+1}}\right)x}$$

$$= \lim_{e^{x\to\infty}}\frac{(x+1)}{x^x} = 1$$

2. $$\lim_{x\to a}\left[\frac{(x^2-2ax^2-a^2x+2a^3)(x-2a)^{-1}+2a(x+a)}{x\left(x^2-a^2\right)}-\frac{2}{x-a}\right]$$

$$\Rightarrow \lim_{x\to a}\left[\frac{\left(x^2-2ax^2-a^2x+2a^3\right)+2a(x+a)(x-2a)-2x(x+a)(x-2a)}{x\left(x^2-a^2\right)(x-2a)}\right]$$

$$\Rightarrow \lim_{x\to a}\left[\frac{x^2-2ax^2-a^2x+2a^3+2ax^2-4a^3-2a^2x-2x^3+4a^2x+2ax}{x(x-a)(x+a)(x-2a)}\right]$$

$$\Rightarrow \lim_{x\to a}\frac{-2x^3+x^2+a^2x+2ax-2a^3}{x(x-a)(x+a)(x-2a)}$$

As Lim x→a, Numerator tends to finite value & denominator tends to zero. Hence, Limit does not exist.

3. $$\lim_{x\to a}\left(\frac{\left(a^{x-a}-1\right)\left(x^p-a^p\right)}{(x-a)^2\left(\ln a^p\right)\left(a^{p-1}\right)}\right)^{\frac{1}{x-a}}=1^\infty \text{ form}$$

$$\Rightarrow e^{\lim\limits_{x\to a}\left(\frac{\left(a^{x-a}-1\right)\left(x^p-a^p\right)-(x-a)^2\ln a^p\left(a^{p-1}\right)}{(x-a)^2\ln a^p\left(a^{p-1}\right)}\right)\times\frac{1}{x-a}}$$

$$\Rightarrow e^{\lim\limits_{x\to a}\frac{\left(a^{x-a}-1\right)px^{p-1}+\left(x^p-a^p\right)\left(a^{x-a}\ln a\right)-2(x-a)\ln a^p\left(a^{p-1}\right)}{3(x-a)^2\left(\ln a^p\right)\left(a^{p-1}\right)}}$$

$$\Rightarrow e^{\lim\limits_{x\to a}\frac{\left(a^{x-a}-1\right)p(p-1)x^{p-2}+\left(a^{x-a}px^{p-1}\ln a\right)+p\left(x^{p-1}\right)\left(a^{x-a}\ln a\right)+\left(x^p-a^p\right)a^{x-a}(\ln a)^2-2\ln a^p\left(a^{p-1}\right)}{6(x-a)\ln a^p\left(a^{p-1}\right)}}$$

$$\Rightarrow e^{\lim\limits_{x\to 0}\frac{p\ln a(p-1)\left(a^{p-2}\right)+p\cdot a^{p-1}(\ln a)^2+2\left(\ln a^p\right)\left(a^{(p-1)}\ln a+(p-1)a^{p-2}\right)}{6p\ln a a^{p-1}}}$$

$$\Rightarrow e^{\frac{(p-1)a^{p-2}+a^{p-1}\ln a+2\left(a^{(p-1)}\ln a+(p-1)a^{p-2}\right)}{6a^{p-1}}}$$

$$\Rightarrow e^{\frac{(p-1)+a\ln a}{2a}}$$

4. $$\lim_{x\to\infty}x\ln\left(e\left(1+\frac{1}{x}\right)^{1-x}\right)$$

Put $x=\frac{1}{y}$

$$\lim_{y\to 0}\frac{1}{y}\ln\left(e(1+y)^{1-\frac{1}{y}}\right)$$

$$\lim_{y\to 0}\frac{1+\left(1-\frac{1}{y}\right)\ln(1+y)}{y}$$

$$\Rightarrow \lim_{y\to 0}\frac{y+(y-1)\ln(1+y)}{y^2}$$

$$\Rightarrow \lim_{y\to 0}\frac{1+\frac{y-1}{y+1}+\ln(1+y)}{2y}$$

$$\Rightarrow \lim_{y\to 0}\frac{2y+\ln(1+y)}{2y(y+1)}=\lim_{y\to 0}\frac{1+\frac{\ln(1+y)}{2y}}{y+1}$$

$$\Rightarrow 1+\frac{1}{2}=\frac{3}{2}$$

5. $$\lim_{x\to 1^-}\frac{(3+ax)^{5/2}-b\ln x-c(\sin(x-1)}{(x-1)^2}$$

For limit to exist, $a=-3$

Now,

$$\lim_{x\to 1^-}\frac{-3^{5/2}\frac{5}{2}(1-x)^{3/2}-\frac{b}{x}-c\cos(x-1)}{2(x-1)}$$

For limit to exist ⇒ b+c = 0

Now,

$$\lim_{x\to 1^-}\frac{+3^{5/2}\frac{15}{4}(1-x)^{1/2}+\frac{b}{x^2}+c\sin(x-1)}{2}$$

$\Rightarrow\ b=4$

$\Rightarrow c=-4$

$a^2+b^2+c^2=41$

6. $$\lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{(1+bx)-(1+ax)\sqrt{1+x}}{x^3(1+bx)\sqrt{1+x}}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\left(1+bx-(1+ax)\left(1+\frac{1}{2}x-\frac{\frac{1}{2}\times\frac{1}{2}x^2}{2}+\frac{\frac{1}{2}x+\frac{1}{2}\times\frac{3}{2}}{6}x^3+....\right)\right)}{x^3(1+bx)\sqrt{1+x}}$$

$$\Rightarrow \lim_{x\to 0}\frac{\left(\left(b-a-\frac{1}{2}\right)x+\left(\frac{1}{8}-\frac{a}{2}\right)x^2+...\right)}{x^3(1+bx)\sqrt{1+x}}$$

For limit to exist, $b-a-\frac{1}{2}=0$ &

$\frac{1}{8}-\frac{a}{2}=0$

$a=\frac{1}{4},b=\frac{3}{4}$

$l=\frac{-1}{16}+\frac{1}{8}\times\frac{1}{4}=\frac{+1}{16}$

$\Rightarrow \frac{1}{a}-\frac{2}{l}+\frac{3}{b}=4-32+4=-24$

7. $$\lim_{x\to\infty} x^2\left(\sin x\sin^3\frac{1}{x}-\left(x\tan^3\frac{1}{x}\right)\operatorname{sgn}(x)\right)$$

Put $x=\frac{1}{y}$

$$\lim_{y\to0}\frac{1}{y^2}\left[\sin\frac{1}{y}\sin^3 y-\frac{1}{y}\tan^3 y\right]$$

$$\lim_{y\to0}\frac{\sin^2 y}{y^2}\left[\sin y\cdot\sin\frac{1}{y}-\frac{1}{y}\frac{\sin y}{\cos^3 y}\right]$$

$= -1$

8. $$\lim_{y=n\to\infty}\frac{n\cos^2\frac{n\pi}{2}+\sin^2\frac{n\pi}{2}}{n\left(\cos^2\frac{n\pi}{2}+n\sin^2\frac{n\pi}{2}\right)}$$

If n is even integer, Then

$$y=\lim_{n\to\infty}\frac{n}{n}=1$$

If n is odd integer, then

$$y=\lim_{n\to\infty}\frac{1}{n^2}=0$$

If n is Real no, then

$$y=\lim_{n\to\infty}\frac{\cos^2\frac{n\pi}{2}+\frac{\sin^2 n\pi/2}{n}}{\cos^2\frac{n\pi}{2}+n\sin^2\frac{n\pi}{2}}$$

= Does not exist

9. $$a_{n+1}=\left(a_n+x\right)^{-1}=\frac{1}{a_n+x}$$

$$b_{n+1}=\frac{1}{b_n+y}$$

$$a_{n+1}-b_{n+1}=\frac{b_n-a_n}{\left(a_n+x\right)\left(b_n+y\right)}+(y-x)$$

Put n = 0

$$a_1-b_1=\frac{y-x}{xy}\,\text{.............................}\,(I)$$

For n = 2

$$\lim_{x\to y}\frac{a_2-b_2}{(x-y)(x+y)}$$

$$=\lim_{x\to y}\frac{b_1-a_1+y-x}{\left(a_1+x\right)\left(b_1+y\right)(x-y)(x+y)}$$

$$\Rightarrow\lim_{x\to y}\frac{-\frac{1}{xy}+1}{\left(a_1+x\right)\left(b_1+y\right)(x+y)}$$

$$\Rightarrow\lim_{x\to y}\frac{xy-1}{xy\left(\frac{1}{x}+x\right)\left(\frac{1}{y}+y\right)(x+y)}$$

$$\Rightarrow\frac{x^2-1}{\left(x^2+1\right)^2(2x)}$$

10. Let $x^4e^{\frac{1}{x^2}}=y$

As $x\to0,\ y\to\infty$

If n is odd, $\int_0^{\frac{\pi}{4}+y}\cos^n t\,dt$ is finite

Hence the limit is zero.

11. $$\lim_{x\to0^+}\frac{\frac{\ln x}{(1+x)^2}-\ln\left(\frac{x}{1+x}\right)}{\ln\left(1-x^2\right)+2x}$$

$$\lim_{x\to0^+}\frac{\ln x-(\ln x)(1+x)^2+(\ln(1+x))(1+x)^2}{(1+x)^2\cdot x\cdot\left(\frac{x\ln\left(1-x^2\right)}{x^2}+2\right)}$$

$$\lim_{x\to0^+}\frac{(\ln x)(-x-2)+\frac{\ln(1+x)(1+x)^2}{x}}{(1+x)^2\left(x\frac{\ln\left(1-x^2\right)}{x^2}+2\right)}$$

$=\infty$

12. $$f(x-1)=\begin{cases}1-|x-1|, & |x-1|\le1\\ |x-1|, & |x-1|>1\end{cases}$$

$$=\begin{cases}x, & 0\le x\le1\\ 2-x, & 1<x\le2\\ x-1, & x>2\\ 1-x, & x<0\end{cases}$$

$$f(x+1)=\begin{cases}x+2, & -2\le x\le-1\\ -x, & -1<x\le0\\ x+1, & x>0\\ -x-1, & x<-2\end{cases}$$

$$g(x)=\begin{cases} 2x+1, & 0<x\le 1 \\ 0 & x=0 \\ 3, & 1\le x\le 2 \\ 2x & x>2 \\ 1-2x & -1<x<0 \\ 3 & -2\le x\le -1 \\ -2x & x<-2 \end{cases}$$

Now, $\lim\limits_{x\to\frac{\pi}{2}} g(\sin x)=3$

13. $\cot^{-1}\left(\dfrac{r^4-r^2+1}{2r}\right)=\tan^{-1}\left(\dfrac{2r}{r^4-r^2+1}\right)$

$$=\tan^{-1}\left(\frac{2r}{1+r^2\left(r^2-1\right)}\right)$$

$$=\tan^{-1}\left(\frac{2r}{1+\left(r^2+r\right)\left(r^2-r\right)}\right)$$

$$=\tan^{-1}\left(r^2+r\right)-\tan^{-1}\left(r^2-r\right)$$

Now, $\sum\limits_{r=1}^{\infty}\cot^{-1}\left(\dfrac{r^4-r^2+1}{2r}\right)$

$$=\tan^{-1}(2)-\tan^{-1}0+\tan^{-1}6-\tan^{-1}2+\cdots$$

$$=\tan^{-1}\infty-\tan^{-1}0$$

$$=\frac{\pi}{2}$$

14. $\lim\limits_{x\to 0}\dfrac{(1-\cos x)\left(e^x-\cos x\right)}{x^n}$

case - 1 $n=3$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\left(\frac{e^x-\cos x}{x}\right)$$

$$=\frac{1}{2}$$

case - 2 $n>3$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\left(\frac{e^x-\cos x}{x}\right)\frac{1}{x^{n-3}}=\infty$$

case - 3 $n<3$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\left(\frac{e^x-\cos x}{x}\right)x^{3-n}=0$$

15. α,β are roots of $ax^3+x-1-a=0$

$$\alpha+\beta+1=0$$
$$\alpha\beta+\alpha+\beta=1/a$$
$$\alpha\beta=\frac{1+a}{a}$$

$\Rightarrow \dfrac{1}{\alpha}$ is a root of $(a+1)x^3-x^2-a=0$

$$\&\,(a+1)x^3-x^2-a=(x-1/\alpha)\left(x-1/\beta\right)(x-1)(a+1)$$

$$\lim_{x\to\frac{1}{\alpha}}\frac{\ln\left(1+(a+1)x^3-x^2-a\right)}{((a+1)x^3-x^2-a)}\times\frac{((a+1)x^3-x^2-a)}{\dfrac{\left(e^{1-\alpha x}-1\right)}{1-\alpha x}\times(1-\alpha x)(x-1)}$$

$$=\frac{1}{-\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)(a+1)$$

$$=-\frac{a}{\alpha}(\beta-\alpha)=a\left(1-\frac{\beta}{\alpha}\right)$$

Hence Proved

16. $\lim\limits_{x\to 1}\left[\left(\dfrac{4}{x^2-x^{-1}}-\dfrac{1-3x+x^2}{1-x^3}\right)^{-1}+3\left(\dfrac{x^4-1}{x^3-x^{-1}}\right)\right]$

$$=\lim_{x\to 1}\left[\left(\frac{4x+1-3x+x^2}{x^3-1}\right)^{-1}+3x\right]$$

$$\lim_{x\to 1}(x-1+3x)=3$$

17. $2f(\sin x)+\sqrt{2}f(\cos x)=\tan x$

Put $\quad x=\dfrac{\pi}{2}-x$

$$2f(\cos x)+\sqrt{2}f(\sin x)=\cot x$$

$$\Rightarrow 2f(\sin x)=2\tan x-\sqrt{2}\cot x$$

$$\Rightarrow f(\sin x)=\tan x-\frac{\cot x}{\sqrt{2}}$$

$$\Rightarrow f(x)=\frac{x}{\sqrt{1-x^2}}-\frac{\sqrt{1-x^2}}{\sqrt{2}x}$$

Now, $\quad \lim\limits_{x\to 1^-}\sqrt{1-x}\,f(x)$

$$=\lim_{x\to 1^-}\left(\frac{x}{\sqrt{1+x}}-\frac{(1-x)\sqrt{1+x}}{\sqrt{2}x}\right)=\frac{1}{\sqrt{2}}$$

18. $$\lim_{x\to a}\left\{\left[\frac{\sqrt[4]{a}-\sqrt[4]{x}}{\sqrt{a}+\sqrt{x}}-\frac{2\sqrt[4]{ax}}{\left(x^{1/4}-a^{1/4}\right)\left(x^{1/2}+a^{1/2}\right)}\right]^{-1}-2\log_2 a^{1/4}\right\}^{-8}$$

$$\lim_{x\to a}\left\{\left[\frac{\left(x^{1/4}-a^{1/4}\right)\left(x^{1/2}+a^{1/2}\right)}{-\left(x^{1/4}-a^{1/4}\right)^2-2(ax)^{1/4}}\right]-a^{1/4}\right\}^{8}$$

$$\lim_{x\to a}\left\{-\left[\frac{\left(x^{1/4}-a^{1/4}\right)\left(x^{1/2}+a^{1/2}\right)}{x^{1/2}+a^{1/2}}\right]-a^{1/4}\right\}^{8}$$

$$=\left\{-a^{1/4}\right\}^8=a^2$$

19. $$\lim_{x\to\infty}\sqrt{ax^2+2bx+c}-(\alpha x+\beta)=0$$

$$\lim_{x\to\infty}\frac{\left(ax^2+2bx+c\right)-(\alpha x+\beta)^2}{\sqrt{ax^2+2bx+c}+\alpha x+\beta}$$

$$\lim_{x\to\infty}\frac{\left(a-\alpha^2\right)x^2+(2b-2\alpha\beta)x+\left(c-\beta^2\right)}{\sqrt{ax^2+2bx+c}+\alpha x+\beta}=0$$

$$\Rightarrow a=\alpha^2 \quad \& \quad 2b=2\alpha\beta$$

$$\Rightarrow \alpha=\pm\sqrt{a}, \qquad \beta=\frac{b}{\pm\sqrt{a}}$$

$$\Rightarrow \alpha=\sqrt{a}, \qquad \beta=\frac{b}{\sqrt{a}}$$

Now, $$\lim_{x\to\infty}\frac{x\left(c-\beta^2\right)}{\sqrt{ax^2+2bx+c}+\alpha x+\beta}$$

$$=\frac{c-\beta^2}{\sqrt{a}+\alpha}$$

$$=\frac{\left(c-\frac{b^2}{a}\right)}{2\sqrt{a}}=\frac{-\left(b^2-ac\right)}{2a\sqrt{a}}$$

Hence Proved.

20. Case I → If $x\neq I$

$$\Rightarrow \quad y=1-x-[x]-[1-x]$$

$$y=(1-x)-[1-x]-[x]$$

$$y=\{1-x\}-[x]=1-\{x\}-[x]$$

$$y=1+x$$

and

$$y=1-x-\lim_{x\to\infty}\left(\cos^{2n+1}2\pi x\right)=1-x$$

Case II ⇒ If $x=I$

$$y=1-x-x-1+x=-x$$

and

$$y=1-x-\lim_{n\to\infty}\cos^{2n+1}2\pi x=-x$$

Hence both functions are identical.

21. $f(x+y)=f(x)+f(y) \quad \forall x,y\in R, f(x)=1.$

Solving functional eqn

$f(x) = x$

Then, $$\lim_{x\to 0}\frac{2^{\tan x}-2^{\sin x}}{x^2\sin x}=\lim_{x\to 0}\frac{2^{\sin x}\left[2^{\tan x-\cos x}-1\right]}{x^2\frac{\sin x}{x}\cdot x}$$

$$=\frac{1}{2}\ln 2$$

22. $$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^{n}c_{k}}{n^k(k+3)}$$

Now

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots c_nx^n$$

Multiply by x^2 & integrate

$$\int_0^{1/n}x^2(1+x)^n\,dx=\frac{c_0x^3}{3}+\frac{c_1x^4}{4}+\ldots$$

$$\Rightarrow \frac{1}{n^3}\sum_{k=0}^{n}\frac{{}^{n}c_{k}}{n^k(k+3)}=\int_0^{1/n}x^2(1+x)^n\,dx$$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^{n}c_{k}}{n^k(k+3)}=\lim_{n\to\infty}n^3\int_0^{1/n}x^2(1+x)^n\,dx \quad \ldots\ldots \text{(I)}$$

$$I=\int_0^{1/n}x^2(1+x)^n\,dx=\frac{x^2(1+x)^{n+1}}{n+1}-\int\frac{2x(1+x)^{n+1}}{n+1}dx$$

$$=\frac{x^2(1+x)^{n+1}}{n+1}-\frac{2x(1+x)^{n+2}}{(n+1)(n+2)}+\frac{2(1+x)^{n+3}}{(n+1)(n+2)(n+3)}\Bigg|_0^{1/n}$$

$$=\frac{(1+n)^n}{n^{n+3}}-\frac{2(n+1)^{n+1}}{n^{n+3}(n+2)}+\frac{2(n+1)^{n+2}}{\left(n^{n+3}\right)(n+2)(n+3)}$$

$$-\frac{2}{(n+1)(n+2)(n+3)}$$

From (I)

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n-2\left(1+\frac{1}{n}\right)^{n+1}\cdot\frac{n}{n+2}\right.$$

$$\left.+\frac{2(1+1/n)^{n+2}n^2}{(n+2)(n+3)}\right.$$

$$\left.-\frac{2n^3}{(n+1)(n+2)(n+3)}\right]$$

$$= e - 2e + 2e - 2 = e - 2$$

23. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

As $n \to \infty, x_{n+1} \to x_n$

$$\Rightarrow x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$\Rightarrow \frac{x_n}{2} = \frac{a}{2x_n}$$

$$\Rightarrow x_n^2 = a \Rightarrow x_n = \pm\sqrt{a}$$

As x, is a negative ? $x_n = -\sqrt{a}$

as $n \to \infty$

24. $U_n = {}^m c_n x^n$

Now, if m is smaller than n, then one of the bracket among $m(m-1)(m-2)\cdots(m-(n-1))$

becomes zero

$\Rightarrow U_n = 0$

Else,

$$U_{n+1} = {}^m c_{n+1} x^{n+1} \quad \& \quad U_n = {}^m c_n x^n$$

For $n \to \infty$, $\lim\limits_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{\left({}^m c_{n+1}\right) x^{n+1}}{{}^m c_n x^n}$

$$= \lim_{x\to\infty} \frac{!m}{!n+1\ !m-n-1} \times \frac{!n\,!m-n}{!m} x$$

$$= \lim_{n\to\infty} \frac{m-n}{n+1} \cdot x = \lim_{n\to\infty}\left(\frac{m/n-1}{1+1/n}\right) x = -x$$

Hence, $\lim\limits_{n\to\infty} {}^m c_n x^n = 0$ for all values of m

25. Let $d_n = \cos\theta + \cos 2\theta + \cdots + \cos n\theta$

$$= \frac{\sin n\theta/2}{\sin\theta/2} \cos\left(\frac{\theta + n\theta}{2}\right)$$

$$\lim_{n\to\infty} \frac{d_1 + \cdots + d_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\sin r\theta/2 \cos\left(\frac{\theta + r\theta}{2}\right)}{\sin\theta/2}$$

$$= \lim_{n\to\infty} \frac{1}{2n} \sin\theta/2 \sum_{r=1}^{n}\left(\sin\left(r\theta + \frac{\theta}{2}\right) + \sin(-\theta/2)\right)$$

$$= \frac{1}{2\sin\theta/2} \lim_{n\to\infty}\left(\frac{1}{n}\sum(\sin-\theta/2) + \frac{1}{n}\sum \sin(r\theta + \theta/2)\right)$$

$$= \frac{1}{2\sin\theta/2}\left[\begin{matrix} -\sin\frac{\theta}{2} + \lim\limits_{n\to\infty} \frac{1}{n} \frac{\sin(n\theta/2)}{\sin\theta/2} \\ \sin\left(\frac{\theta + \frac{\theta}{2} + n\theta + \theta/2}{2}\right) \end{matrix}\right] = \frac{-1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{c_1 + c_2 + \ldots + c_n}{n} =$$

$$\lim_{n\to\infty} \frac{n \cdot 1/2 + d_1 + d_2 + \ldots + d_n}{n} = 0$$

Similarly, $s_n = \sin\theta + \sin 2\theta + \ldots + \sin n\theta$

$$= \frac{\sin\frac{n\theta}{2}}{\sin\theta/2} \sin\left(\frac{\theta + n\theta}{2}\right)$$

$$\lim_{n\to\infty} \frac{S_1 + S_2 + \ldots + S_n}{n} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\sin r\theta/2}{\sin\theta/2} \sin\left(\frac{\theta + r\theta}{2}\right)$$

$$\frac{1}{2\sin\frac{\theta}{2}} \lim_{n\to\infty} \frac{1}{n} \sum \cos(\theta/2) - \cos(r\theta + \theta/2)$$

$$\frac{1}{2\sin\theta/2} \lim_{n\to\infty}\left(\cos\frac{\theta}{2} - \frac{1}{n}\sum\cos(r\theta + \theta/2)\right)$$

$$= \frac{\cot\theta/2}{2}$$

Hence Proved.

26. $y = \lim\limits_{n\to\infty} \frac{x^{2n} \sin\frac{\pi x}{2} + x^2}{x^{2n} + 1}$

If $x = 1$, or -1

$$y = \lim_{n\to\infty} \frac{x^{2n} \sin\frac{\pi x}{2} + x^2}{x^{2n} + 1} = 1$$

If $|x| < 1$

$$y=\lim_{n\to\infty}\frac{x^{2n}\sin\frac{\pi x}{2}+x^2}{x^{2n}+1}=x^2$$

If $|x|>1$

$$y=\lim_{n\to\infty}\frac{\sin\pi x/2+x^{2-2n}}{1+x^{-2n}}=\sin\frac{\pi x}{2}$$

$$\Rightarrow y=\begin{cases}1, & x=1\text{ or }-1\\ x^2, & -1<x<1\\ \sin\frac{\pi x}{2}, & x<-1\text{ or }x>1\end{cases}$$

27. $y=\lim\limits_{n\to\infty}\dfrac{x^n\phi(x)+x^{-n}\varphi(x)}{x^n+x^{-n}}$

case-1: $x=1$ of $x=-1$

$$y=\frac{\phi(x)+\varphi(x)}{2}$$

case-2: when $|x|<1$

$$y=\lim_{n\to\infty}\frac{(x)^{2n}\phi(x)+\varphi(x)}{x^{2n}+1}=\varphi(x)$$

case-3: when $|x|>1$

$$y=\lim_{n\to\infty}\frac{\phi(x)+x^{-2n}\varphi(x)}{1+x^{-2n}}=\phi(x)$$

Hence, y is composed of two parts of graphs $\varphi(x)$ & $\phi(x)$, together with two isolated points at $x=\pm 1$.

28. $y=\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}\dfrac{2}{n}\arctan\left(n\sin^2 m!\pi x\right)$

Case 1: When x is rational, $\sin^2 m!\pi x$ be equal to integral multiple of π & is equal to zero.

$$\Rightarrow \tan^{-1}\left(n\sin^2 m!\pi x\right)=0$$

$$\Rightarrow \quad y=0$$

Case 2: When x is irrational, $\sin^2 m!\pi x$ is always positive & a fraction, then, $\tan^{-1}n\sin^2 m!\pi x$ tends to $\dfrac{\pi}{2}$

$$\Rightarrow y=1$$

29. $\lim\limits_{x\to 0}\dfrac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(\cdots\cos^2 x\right)\right)\right)}{\left(\pi\cdot\dfrac{\sqrt{x+4}-2}{x}\right)}$

Now, as

$$\frac{\sqrt{x+4}-2}{x}=\frac{(x+4)-4}{x(\sqrt{x+4}+2)}=\frac{1}{\sqrt{x+4}+2}$$

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\frac{1}{4}$$

$$\&\ \lim_{x\to 0}\cos^2\left(1-\cos^2\left(1-\cos^2\left(\cdots\cos^2 x\right)\right)\right)=1$$

$$\Rightarrow \lim_{x\to 0}\frac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(\cdots\cos^2 x\right)\right)\right)}{\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)}=\frac{4}{\pi}$$

30. As $x_n=\sqrt{a+\sqrt{a+\sqrt{a+}}}\cdots\infty$

$$\Rightarrow x_n=\sqrt{a+x_n}$$

$$\Rightarrow x_n^2-x_n-a=0$$

$$\Rightarrow x_n=\frac{1\pm\sqrt{1+4a}}{2}$$

As x_n is positive $\Rightarrow x_n=1+\sqrt{1+4a}$

31. $[x]+\dfrac{1}{2}[2x]+\dfrac{1}{3}[3x]+\ldots+\dfrac{1}{n}[nx]$

$$=nx-\left(\{x\}+\frac{1}{2}\{2x\}+\cdots\frac{1}{n}\{nx\}\right)$$

Now,

$$\Rightarrow \lim_{n\to\infty}\frac{[x]+\frac{1}{2}[2x]+\frac{1}{3}[3x]+\cdots+\frac{1}{n}[nx]}{\frac{n(n+1)(2n+1)}{6}}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{6nx}{n(n+1)(2n+1)}\right)-\lim_{n\to\infty}\frac{6\left(\{x\}+\frac{1}{2}\{2x\}+\cdots+\frac{1}{n}\{nx\}\right)}{n(n+1)(2n+1)}$$

$$=0$$

32. let $y=\sin x+2\sin^2 x+3\sin^3 x+\cdots\infty$

$$xy=\sin^2 x+2\sin^3 x+\cdots\infty$$

$$(1-\sin x)y=\sin x+\sin^2 x+\sin^3 x+\cdots\infty$$

$$y=\frac{\sin x}{(1-\sin x)^2}$$

Now, $\lim\limits_{x\to\frac{\pi}{2}}\left((1-\sin x)^2 f(x)\right)^{\frac{1}{\sin x-1}}$

$= \lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{1/\sin x-1}$

$= \lim\limits_{x\to\frac{\pi}{2}} e^{(\sin x-1)\times\frac{1}{(\sin x-1)}} = e$

33. $v_1 = \dfrac{1}{2} < 2$

$v_2 = \dfrac{1}{2}+\dfrac{1}{2\cdot 4} < \dfrac{1}{2}+\dfrac{1}{2^2} < 2$

Now, $U_n = \dfrac{1}{2}+\dfrac{1}{2\cdot 4}+\dfrac{1}{2\cdot 4\cdot 6}+\cdots$

$< \dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots$

As $n\to\infty$

$\lim\limits_{n\to\infty} U_n < \dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots\infty$

$\Rightarrow \lim\limits_{n\to\infty} U_n < \dfrac{1/2}{1-1/2}$

$\Rightarrow \lim\limits_{n\to\infty} U_n < 1 < 2$

Hence proved.

34. $\lim\limits_{n\to\infty}\dfrac{1^k+2^k+\cdots+n^k}{n\cdot n^k}$

$= \lim\limits_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{n}\left(\dfrac{r}{n}\right)^k = \int_0^1 x^k dx = \left.\dfrac{x^{k+1}}{k+1}\right|_0^1$

$= \dfrac{1}{k+1}$

Now, coefficient of x^{n-2} in $(x-1^k)\left(x-2^k\right)\cdots\left(x-n^k\right)$ is F(n)

$\Rightarrow F(n) = 1^k2^k+1^k3^k+\cdots+2^k3^k+2^k4^k+\cdots+(n-1)^k m^k$

$= \sum\limits_{1\le i<j\le n}(ij)^k$

$= \dfrac{\sum\limits_{1\le i\le n}\sum\limits_{i\le j\le n} i^k j^k - \sum\limits_{i=j}(ij)^k}{2}$

Previous Year's Questions (JEE Advanced)

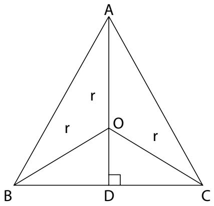
1. $\lim\limits_{x\to 0^+} g(f(x)) = gf\left(0^+\right) = g\left(\sin 0^+\right) = g\left(0^+\right) = 1$

$\lim\limits_{x\to 0^-} g(f(x)) = g(f(0^-)) = g\left(\sin 0^-\right) = g\left(0^-\right) = 1$

$\Rightarrow$ Hence, $\lim\limits_{x\to 0} gf(x) = 1$

2. $\lim\limits_{x\to-\infty}\left[\dfrac{x^4\sin\frac{1}{x}+x^2}{\left(1+|x|^3\right)}\right] = \lim\limits_{x\to-\infty}\left[\dfrac{\frac{\sin(1/x)}{1/x}+1/x}{\frac{1}{x^3}-1}\right] = -1$

3.



$BD = \sqrt{r^2-(h-r)^2} = \sqrt{2rh-h^2}$

$BC = 2\sqrt{2rh-h^2}$

Area of $\Delta ABC = h\sqrt{2rh-h^2}$

Also, $\lim\limits_{h\to 0}\dfrac{A}{P^3} = \lim\limits_{h\to 0}\dfrac{h\sqrt{2rh-h^3}}{8\left(\sqrt{2rh-h^2}+\sqrt{2hr}\right)^3}$

$= \lim\limits_{h\to 0}\dfrac{\sqrt{2r-h}}{8\left(\sqrt{2r-h}+\sqrt{2r}\right)^3}$

$= \dfrac{1}{128r}$

4. $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

$\lim\limits_{x\to\infty} e^{\left(\frac{x+6}{x+1}-x\right)^{x+4}} = e^5$

5. $\lim\limits_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{1/x^2}$

$= \lim\limits_{x\to 0} e^{\left(\frac{1+5x^2}{1+3x^2}-1\right)\frac{1}{x^2}}$

$= e^2$

6. $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos t^2 dt}{x\sin x}$

$= \lim\limits_{x\to 0}\dfrac{(\cos x^4)2x}{x\cos x+\sin x}$

$= \lim\limits_{x\to 0}\dfrac{2\cos x^4}{\cos x+\dfrac{\sin x}{x}} = 1$

7. $$\lim_{h\to 0}\frac{\ln(1+2h)-2\ln(1+h)}{h^2}$$

$$\lim_{h\to 0}\frac{\ln\left[\frac{1+2h}{(1+h)^2}\right]}{h^2}$$

$$\lim_{h\to 0}\frac{\ln\left[1+\frac{-h^2}{1+h^2+2h}\right]}{\frac{-h^2}{1+h^2+2h}}\times-\frac{1}{1+2h+h^2}$$

$= -1$

8. Consider $f(x)=\frac{|x-a|}{x-a}, g(x)=\frac{x-a}{|x-a|}$

Then, $\lim\limits_{x\to a} f(x)\times g(x)$ exist but $\lim\limits_{x\to a} f(x)$ or $\lim\limits_{x\to a} g(x)$ does not exist.

Hence, it is a false statement.

9. $$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\sqrt{\frac{1-\sin x/x}{1+\cos^2 x/x}}=1$$

Hence, C is correct.

10. $$\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+.....+\frac{n}{1-n^2}\right)$$

$$\lim_{n\to\infty}\frac{(1+2+3+.....+n)}{\left(1-n^2\right)}$$

$$\lim_{n\to\infty}\frac{n(n+1)}{2\left(1-n^2\right)}=\frac{-1}{2}$$

Hence, B is correct.

11. $$f(x)=\begin{cases}\frac{\sin[x]}{[x]} & ,x\in(-\infty,0)\cup[1,\infty)\\ 0 & ,x\in[0,1)\end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0}\frac{\sin[-h]}{[-h]}=\sin 1$$

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} 0=0$$

$\therefore$ Limit does not exist

Hence, D is correct.

12. $$\lim_{x\to\infty}\frac{x^n}{e^x}=0$$

$\Rightarrow$For n is any whole no, it will hold true

Hence, B is correct.

13. $$\lim_{x\to 0}\left(\frac{x}{\tan^{-1}2x}\right)=\lim_{x\to 0}\frac{2x}{2\left(\tan^{-1}2x\right)}=\frac{1}{2}$$

Hence, B is correct.

14. $$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x}$$

$$\lim_{x\to 0} e^{\left(\frac{1+\tan x}{1-\tan x}-1\right)\times\frac{1}{x}}=\lim_{x\to 0} e^{\frac{2\tan x}{x}\times\frac{1}{1-\tan x}}=e^2$$

15. $$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+\left(r/n\right)^2}}$$

$$=\int_0^2\frac{x}{\sqrt{1+x^2}}dx$$

$$=\frac{1}{2}\times 2\sqrt{1+x^2}\Big|_0^2=\sqrt{5}-1$$

Hence, B is correct.

16. $$\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{4\sin^4 x}$$

$$\lim_{x\to 0}\frac{x}{4\sin^4 x}\left[\frac{2\tan x}{1-\tan^2 x}-2\tan x\right]$$

$$\frac{1}{2}\lim_{x\to\infty}\frac{x}{\sin x}\cdot\frac{1}{\cos^3 x}\cdot\frac{1}{1-\tan^2 x}=\frac{1}{2}$$

Hence, C is correct.

17. $$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=\lim_{x\to\infty} e^{\left(\frac{x-3}{x+2}-1\right)^x}=e^{-5}$$

Hence, C is correct.

18. $$\lim_{x\to 0}\frac{\sin\left(\pi\cos^2 x\right)}{x^2}=\lim_{x\to 0}\frac{\sin\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}=x$$

Hence, B is correct.

19. $$L=\lim_{x\to 0}\frac{(\cos x-1)\left(\cos x-e^x\right)}{x^n}$$

$$=\lim_{n\to 0}\frac{(\cos x-1)}{x^2}\cdot\frac{\left(\cos x-e^x\right)}{x}\times\frac{1}{x^{n-3}}$$

For L to be finite non zero, n–3 = 0

$\Rightarrow$n = 3

Hence, C is correct.

20. $$\lim_{x\to 0}\frac{((a-n)nx-\tan x)\sin nx}{x^2}=0$$

$$\Rightarrow n\lim_{x\to 0}\frac{\sin nx}{nx}\left[(a-n)n-\frac{\tan x}{x}\right]=0$$

$$\Rightarrow (a-n)n-1=0$$

$$\Rightarrow a=n+\frac{1}{n}$$

Hence, D is correct.

21. $$\lim_{x\to 0}\left[(\sin x)^{1/x}+\left(\frac{1}{x}\right)^{\sin x}\right]$$

$$=\lim_{x\to 0}(\sin x)^{1/x}+\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$$

$$= 0 + e^{\lim\limits_{x\to 0}\sin x\ln(1/x)} = e^{\lim\limits_{x\to 0} -\frac{\ln x}{\operatorname{cosec} x}}$$

$$= e^{\lim\limits_{x\to 0}\frac{\sin x}{x}\tan x} = e^0 = 1$$

Hence, C is correct.

22. $$\lim_{x\to\pi/4}\frac{\int_2^{\sec^2 x} f(t)dt}{x^2-\frac{\pi^2}{16}}$$

$$\lim_{x\to\frac{\pi}{4}}\frac{f\left(\sec^2 x\right)2\sec^2 x\tan x}{2x} = \frac{2f(2)}{\pi/4} = \frac{8f(2)}{\pi}$$

Hence, A is correct.

23. $$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$$

$$= \lim_{x\to 0}\frac{x\ln(1+x)}{\left(x^4+4\right)3x^2} = \frac{1}{12}$$

Hence, B is correct.

24. $$e^{\ln\left(1+b^2\right)} = 2b\sin^2\theta$$

$$\text{or } \sin^2\theta = \frac{1+b^2}{2b} = 1 \text{ as } \left(\frac{1+b^2}{2b}\geq 1\right)$$

$$\Rightarrow \theta = \pm\frac{\pi}{2}$$

Hence, D is correct.

25. $$\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right) = 4$$

$$\lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{x+1} = 4$$

$$\Rightarrow 1-a=0 \quad \& \quad 1-a-b=4$$

$$\Rightarrow a=1 \quad \& \quad b=-4$$

Hence, B is correct.

26. Here, $\lim\limits_{x\to 0}\dfrac{x^2\sin(\beta x)}{\alpha x-\sin x} = 1$

$$\Rightarrow \lim_{x\to 0}\frac{x^2\left(\beta x-\frac{(\beta x)^3}{3!}+\frac{(\beta x)^5}{5!}-\cdots\right)}{\alpha x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\left(\beta-\frac{\beta^3x^2}{3!}+\frac{\beta^5x^4}{5!}\cdots\right)}{(\alpha-1)x+\frac{x^3}{3!}+\frac{x^5}{5!}\cdots} = 1$$

Limit exists only, when $\alpha - 1 = 0$

$$\alpha = 1$$

$$\therefore \lim_{x\to 0}\frac{x^3\left(\beta-\frac{\beta^3x^2}{3!}+\frac{\beta^5x^4}{5!}\cdots\right)}{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}\cdots\right)} = 1$$

$$\Rightarrow \quad 6\beta = 1$$

$$6(\alpha+\beta) = 6\alpha+6\beta$$

$$= 6+1 \Rightarrow 7$$

Ans is 7

27. Given, $\lim\limits_{\alpha\to 0}\left[\dfrac{e^{\cos\left(a^n\right)}-e}{\alpha^m}\right] = \dfrac{-e}{2}$

$$\Rightarrow \lim_{\alpha\to 0}\frac{e\left\{e^{\cos\left(a^n\right)-1}-1\right\}}{\cos\left(\alpha^n\right)-1}\cdot\frac{\cos\left(\alpha^n\right)-1}{\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow \lim_{\alpha\to 0} e\left\{\frac{e^{\cos\left(\alpha^n\right)-1}-1}{\cos\left(\alpha^n\right)-1}\right\}\cdot\lim_{\alpha\to 0}\frac{-2\sin^2\alpha^n/2}{\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow e\times 1\times(-2)\lim_{\alpha\to 0}\frac{\sin^2\left(\alpha^n/2\right)}{\frac{\alpha^{2n}}{4}}\cdot\frac{\alpha^{2n}}{4a^m} = \frac{-e}{2}$$

$$\Rightarrow e\times 1\times(-2)\times 1\times\lim_{\alpha\to 0}\frac{\alpha^{2n-m}}{4} = \frac{-e}{2}$$

For this to exists, $2n-m=0$

$$\Rightarrow \frac{m}{2} = 2$$

Hence ans is 2

28. $$\lim_{x\to 0^-}\frac{\sqrt{1/2(1-\cos 2x)}}{x} = \lim_{x\to 0^-}\frac{-\sin x}{x} = -1$$

Hence, B is correct.

29. $$\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1} = \sqrt{2}\lim_{x\to 1}\frac{|\sin(x-1)|}{x-1}$$

⇒Limit does not exist

Hence, C is correct.

30. $$L = \lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 0}\frac{1}{x^2\left(a+\sqrt{a^2-x^2}\right)}-\frac{1}{4x^2}$$

$$= \lim_{x\to 0}\frac{(4-a)-\sqrt{a^2-x^2}}{4x^2\left(a+\sqrt{a^2-x^2}\right)}$$

For Numerator $\to 0$, if $a = 2$

Then, $L = \dfrac{1}{64}$

Hence, A, C is correct.

31. $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$

Now, $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\frac{1-x(1+1-x)}{1-x}\cos\left(\frac{1}{1-x}\right)$

$=\lim\limits_{x\to 1^-}(1-x)\cos\left(\frac{1}{1-x}\right)=0$

and $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}\frac{1-x(1-1+x)}{x+1}\cos\left(\frac{1}{1-x}\right)$

$=\lim\limits_{x\to 1^+}(x+1)\cos\left(\frac{1}{x+1}\right)$, which doesn't exists.

B, D are correct options.

32. $\lim\limits_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Rationalising both Numerator & Denominator

$\lim\limits_{x\to a}\frac{a+2x-3x}{3a+x-4x}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$=\frac{1}{3}\cdot\frac{4\sqrt{9}}{2\sqrt{3}\sqrt{a}}=\frac{2}{3\sqrt{3}}$

33. $\lim\limits_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$\lim\limits_{h\to 0}\frac{a^2\big(\sin(a+b-\sin a)+2ah\sin(a+h)+h^2\sin(a+h)}{h}$

$\lim\limits_{h\to 0}\frac{a^2\left(2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}\right)}{2\frac{h}{2}}+\lim\limits_{h\to 0}2a\sin(a+h)$

$=a^2\cos a+2a\sin a$

34. $\lim\limits_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$

$\lim\limits_{x\to 0}\frac{2^x-1}{x}\times\frac{x}{(1+x)^{1/2}-1}$

$=\frac{\ln 2}{1/2}=2\ln 2$

35. $\lim\limits_{n\to\infty}\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n$

$\lim\limits_{n\to\infty}\left(\frac{2}{\pi}n\cos^{-1}\left(\frac{1}{n}\right)-n\right)+\lim\limits_{n\to\infty}\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)$

$\lim\limits_{n\to\infty}\frac{2}{\pi}n\left(\cos^{-1}\left(\frac{1}{n}\right)-\frac{\pi}{2}\right)+1$

$-\lim\limits_{n\to\infty}\left(\frac{\sin^{-1}\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)\frac{2}{\pi}+1$

$1-\frac{2}{\pi}$

Previous Year's Questions (AIEEE/JEE MAINS)

1. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$$= \lim_{x\to 0}\frac{2\sin^2 x(3+\cos x)}{\dfrac{x\times\tan 4x}{4x}\times 4x}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{(3+\cos x)}{4}\times\frac{1}{\lim\limits_{x\to 0}\dfrac{\tan 4x}{4x}}$$

$$= 2\times\frac{4}{4}\times 1$$

$$\left\{\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ and } \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right\}$$

$$= 2$$

Hence, the correct option is C.

2. $\lim\limits_{x\to 0}\dfrac{\sin\left(\pi\cos^2 x\right)}{x^2}$

$$= \lim_{x\to 0}\frac{\sin\pi\left(1-\sin^2 x\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin\left(\pi-\pi\sin^2 x\right)}{x^2} \Rightarrow \lim_{x\to 0}\frac{\sin\left(\pi\sin^2 x\right)}{x^2}$$

$$\{\because \sin(\pi-\theta)=\sin\theta\}$$

$$= \lim_{x\to 0}\frac{\sin\pi\sin^2 x}{\pi\sin^2 x}\times(\pi)\left(\frac{\sin^2 x}{x^2}\right)$$

$$= \pi \qquad \left\{\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right.$$

Hence, D is correct.

3. $\lim\limits_{x\to 2}\dfrac{\tan(x-2)\left[x^2+(k-2)x-2k\right]}{x^2-4x+4}=5$

$$= \lim_{x\to 2}\frac{\tan(x-2)}{(x-2)}\times\frac{\left(x^2+(k-2)x-2k\right)}{x-2}$$

$$= \lim_{x\to 2}\frac{\tan(x-2)}{(x-2)}\times\frac{(x-2)(x+k)}{(x-2)}=2+k=5$$

$$\Rightarrow k=3$$

Hence, D is correct.

4. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{\sin^2 x}$

Apply L-Hospital Rule,

$$\lim_{x\to 0}\frac{e^{x^2}\cdot 2x+\sin x}{2\sin x\cos x}$$

$$= \lim_{x\to 0}\left(2e^{x^2}+\frac{\sin x}{x}\right)\Big/ 2\frac{\sin x}{x}\cdot\cos x=\frac{3}{2}$$

Hence, B is correct.

5. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$$= \lim_{x\to 0}\frac{2\sin^2 x(3+\cos x)}{\dfrac{x\times\tan 4x}{4x}\times 4x}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{(3+\cos x)}{4}\times\frac{1}{\lim\limits_{x\to 0}\dfrac{\tan 4x}{4x}}$$

$$= 2\times\frac{4}{4}\times 1\left\{\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ and } \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right\}$$

$$= 2$$

Hence, the correct option is C

6. $P=\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{1/2x} \quad \left[1^{\infty}\text{ form }\right]$

$$\Rightarrow e^{\lim\limits_{x\to 0^+}\left(\tan^2\sqrt{x}+1-1\right)\frac{1}{2x}}$$

$$\Rightarrow e^{\lim\limits_{x\to 0^+}\frac{\tan^2\sqrt{x}}{2x}}$$

$$\Rightarrow e^{\frac{1}{2}\lim\limits_{x\to 0^+}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2} \Rightarrow e^{\frac{1}{2}\lim\limits_{x\to 0^+}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2}$$

$$\Rightarrow e^{1/2}$$

$$\therefore \log P=\log e^{1/2}$$

$$\therefore P=1/2$$

Hence, option C is correct.

7. $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}-\dfrac{4}{x^2}\right)^{2x}=e^3$

$$\Rightarrow e^{\lim\limits_{x\to\infty}\left(\frac{a}{x}-\frac{4}{x^2}\right)\times 2x}$$

$$= e^{\lim\limits_{x\to\infty}\left(a-\frac{4}{x}\right)\times 2} \Rightarrow e^{2a}=e^3$$

$$\Rightarrow a=\frac{3}{2}$$

Hence, B is correct.

8. $$L=\lim_{x\to 0}\frac{(1-\cos 2x)^2}{2x\tan x-x\tan 2x}$$

$$=\lim_{x\to 0}\frac{4\sin^4 x}{2x\tan x\left(1-\frac{1}{1-\tan^2 x}\right)}$$

$$=\lim_{x\to 0}\frac{2\sin^4 x}{x\tan^3 x}\left(\tan^2 x-1\right)$$

$$=-2$$

Hence, C is correct

9. $$\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$$

$$\lim_{x\to 3}\frac{(\sqrt{3x}-3)(\sqrt{3x}+3)(\sqrt{2x-4}+\sqrt{2})}{(\sqrt{2x-4}-\sqrt{2})(\sqrt{2x-4}+\sqrt{2})(\sqrt{3x}+3)}$$

$$\lim_{x\to 3}\frac{3(x-3)}{2(x-3)}\frac{\left(\sqrt{2x-4}+\sqrt{2}\right)}{\left(\sqrt{3x}+3\right)}$$

$$\Rightarrow \frac{1}{\sqrt{2}}$$

Hence, B is correct

10. $$\lim_{n\to\infty}\frac{1^a+2^a+\cdots+n^a}{[n+1]^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]}=\frac{1}{60}$$

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}(r/n)^a}{\left(1+\frac{1}{n}\right)^{a-1}\left(\frac{1}{n}\sum_{r=1}^{n}\left(a+\frac{r}{n}\right)\right)}$$

$$\Rightarrow \frac{\int_0^1 x^a dx}{\int_0^1 (a+x)dx}\Rightarrow \frac{\frac{1}{a+1}-0}{a+\frac{1}{2}-0}=\frac{2}{(a+1)(2a+1)}=\frac{1}{60}$$

$$\Rightarrow (a+1)(2a+1)=120$$

$$\Rightarrow a=7$$

Hence, A is correct

11. $$\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$$

$$=\lim_{x\to\pi/2}\frac{1}{8}\frac{\cos x(1-\sin x)}{\sin x\left(\frac{\pi}{2}-x\right)^3}$$

$$=\lim_{h\to 0}\frac{1}{8}\cdot\frac{\cos(\pi/2-h)[1-\sin(\pi/2-h)]}{\sin\left(\frac{\pi}{2}-h\right)\left(\frac{\pi}{2}-\frac{\pi}{2}+h\right)^3}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\sin h(1-\cos h)}{\cos h\cdot h^3}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\sin h\left(2\sin^2 h/2\right)}{\cos h\cdot h^3}$$

$$=\frac{1}{4}\lim_{h\to 0}\frac{\sin h\cdot\sin^2(h/2)}{h^3\cos h}$$

$$=\frac{1}{4}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\left(\frac{\sin h/2}{h/2}\right)^2\cdot\frac{1}{\cos h}\cdot\frac{1}{4}\Rightarrow\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$$

Hence, C is correct

12. $$\lim_{x\to 0^+}x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\cdots+\left[\frac{15}{x}\right]\right)$$

$$\lim_{x\to 0^+}x\left(\frac{15\times 8}{x}-\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}+\left\{\frac{3}{x}\right\}\ldots\left\{\frac{15}{x}\right\}\right)\right)$$

$$\Rightarrow 120$$

Hence, C is correct

13. $$\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$$

$$\lim_{x\to 0}\frac{2x\tan x\left(\frac{1}{1-\tan^2 x}-1\right)}{4\sin^4 x}$$

$$\lim_{x\to 0}\frac{2x\tan^3 x}{4\sin^4 x\left(1-\tan^2 x\right)}=\frac{1}{2}$$

Hence, C is correct

14. $$\lim_{x\to 0}\frac{(27+x)^{1/3}-3}{9-(27+x)^{2/3}}$$

$$\lim_{x\to 0}\frac{(27+x)^{1/3}-3}{\left(3-(27+x)^{1/3}\right)\left(3+(27+x)^{1/3}\right)}$$

$$\Rightarrow\frac{-1}{6}$$

Hence, C is correct

15. $$\lim_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$$

$$=\lim_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}\times\frac{\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}}{\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}}$$

$$=\lim_{y\to 0}\frac{1+\sqrt{1+y^4}-2}{y^4\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)}$$

$$= \lim_{y\to 0} \frac{\sqrt{1+y^4}-1}{y^4\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)} \times \frac{\sqrt{1+y^4}+1}{\sqrt{1+y^4}+1}$$

$$= \lim_{y\to 0}\frac{1+y^4-1}{y^4\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)\left(\sqrt{1+y^4}+1\right)}$$

$$= \lim_{y\to 0}\frac{1}{\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)\left(\sqrt{1+y^4}+1\right)}$$

$$= \frac{1}{\left(\sqrt{1+\sqrt{1+0}}+\sqrt{2}\right)\left(\sqrt{1+0}+1\right)} = \frac{1}{4\sqrt{2}}$$

Hence, A is correct

16. $\lim\limits_{x\to 0^-}\frac{x([x]+|x|)\sin[x]}{|x|}$

$$= \lim_{x\to 0^-}\frac{x(-1-x)\sin(-1)}{-x}$$

$$\left[\because \lim_{x\to 0^-}[x]=-1 \text{ and } \lim_{x\to 0^-}|x|=-x\right]$$

$$= -\sin 1$$

Therefore the correct option is A

17. $\lim\limits_{x\to 1^+}\frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|\cdot[1-x]}$

$$= \lim_{x\to 1^+}\frac{(1-x)+\sin(x-1)}{(x-1)(-1)}\times\sin\left(\frac{\pi}{2}(-1)\right)$$

$$= \lim_{x\to 1^+}\left(1-\frac{\sin(x-1)}{(x-1)}\right)(-1)$$

$$= (1-1)(-1)$$

$$= 0$$

Hence, D is correct

18. $\lim\limits_{x\to 0}\frac{\tan\left(\pi\sin^2 x\right)+\left(|x|-\sin(x[x])\right)^2}{x^2}$

$$\text{RHL} = \lim_{x\to 0^+}\frac{\tan\left(\pi\sin^2 x\right)+x^2}{x^2} \qquad \left(\because x\to 0^+ \Rightarrow [x]=0\right)$$

$$= \lim_{x\to 0^+}\frac{\tan\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}+1 \quad \Rightarrow \pi+1$$

Now,

$$\text{LHL} = \lim_{x\to 0^-}\frac{\tan\left(\pi\sin^2 x\right)+(-x+\sin x)^2}{x^2}$$

$$\left(\because x\to 0^- \Rightarrow [x]=-1\right)$$

$$= \lim_{x\to 0^-}\frac{\tan\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}+\left(-1+\frac{\sin x}{x}\right)^2 = \pi$$

∴ LHL≠RHL

So, Limit does not exist

Hence, D is correct

19. $\lim\limits_{x\to 0}\frac{x\cot(4x)}{\sin^2 x\cot^2(2x)}$

$$\lim_{x\to 0}\frac{x\tan^2 2x}{\tan 4x\sin^2 x}$$

$$\lim_{x\to 0}\frac{4x}{4\tan 4x}\times\frac{x^2}{\sin^2 x}\times\frac{\tan^2 2x}{4x^2}.4 = 1$$

Hence, D is correct

20. $\lim\limits_{x\to \pi/4}\frac{\cot^3 x-\tan x}{\cos(x+\pi/4)}$

Applying L-Hospital,

$$\lim_{x\to\frac{\pi}{4}}\frac{\left(-3\cot^2 x\,\mathrm{cosec}^2 x-\sec^2 x\right)}{-\sin\left(x+\frac{\pi}{4}\right)}$$

Hence, C is correct

21. $\lim\limits_{x\to 1^-}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$

$$= \lim_{x\to 1^-}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}\times\frac{\sqrt{\pi}+\sqrt{2\sin^{-1}x}}{\sqrt{\pi}+\sqrt{2\sin^{-1}x}}$$

$$= \lim_{x\to 1^-}\frac{2\left(\frac{\pi}{2}-\sin^{-1}x\right)}{\sqrt{1-x}\left(\sqrt{\pi}+\sqrt{2\sin^{-1}x}\right)}$$

$$= \lim_{x\to 1^-}\frac{2\cos^{-1}x}{\sqrt{1-x}}\times\frac{1}{2\sqrt{\pi}}$$

$$= \lim_{\theta\to 0^+}\frac{2\theta}{\sqrt{2}\sin(\theta/2)}\times\frac{1}{2\sqrt{\pi}} \quad \{\text{Putting } x=\cos\theta\}$$

$$= \frac{4}{2\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

Hence, C is correct.

2

CONTINUITY OF FUNCTIONS

2.1 Definition of Continuity

Earlier we came across continuous functions and widely used their properties when constructing the graphs of simple functions, though the term "continuous function" was not used that time since the definition of this notion was not given then.

In the first stage the graphs of functions, say, $y = ax + b$, $y = ax^2$ or $y = ax^3$ were plotted point by point. The procedure was as follows: we first tabulated the values of a given function for certain values of the argument and then we constructed the points whose coordinates were put down in the table and then joined the plotted points with a "continuous curve". Thus, we obtained the graph of the given functions. We did not notice then that the graph of such functions were continuous.

The geometrical concept of continuity for a function which possesses a graph is that the function is continuous if its graph is an unbroken curve. A point at which there is a sudden break in the curve is thus a point of discontinuity.

The notion of continuity is a direct consequence of the concept of limit. The special class of functions known as continuous functions possesses many important properties which will be investigated in this chapter.

Continuity at a Point

The question for our consideration is as follows: given any function f(x) defined in the neighbourhood of a, is the function f(x) continuous or not at x = a?

The graph of y = f(x) is said to be continuous at x = a if it can be traced with a continuous motion – without any jump – of the pen from left to right at x = a.

Definition A function f is continuous at x = a if the following three conditions are met:

(i) f(x) is defined at x = a.
(ii) $\lim\limits_{x\to a} f(x)$ exists.
(iii) $\lim\limits_{x\to a} f(x) = f(a)$.

In other words, function f(x) is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x) = f(a)$.

In terms of one-sided limits, f(x) is continuous at x = a if L.H.L. = f(a) = R.H.L.

i.e. $\lim\limits_{x\to a^-} f(x) = f(a) = \lim\limits_{x\to a^+} f(x)$

i.e. $\lim\limits_{h\to 0} f(a-h) = f(a) = \lim\limits_{h\to 0} f(a+h)$.

There is another way to discuss about continuity. Let there be a function $y = f(x)$, $x \in (a, b)$, and let x_0 be a certain value of the argument from the interval (a, b). Then, if $x \in (a, b)$ is another value of the argument, the difference $x - x_0$ is called the increment of the argument and is denoted by Δx, and the difference

$$f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

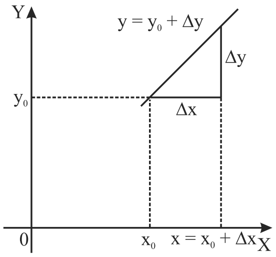
is termed as the increment of the function f at the point x_0 and is denoted as Δf or Δy.

If the function f is continuous at the point x_0, then, by definition, $\lim\limits_{x\to x_0} f(x) = f(x_0)$ and

consequently, $\lim\limits_{x\to x_0} \left(f(x) - f(x_0)\right) = 0$,

which means $\lim\limits_{\Delta x\to 0} \Delta f = 0$.

It follows from the last relation that if f(x) is continuous at the point x_0, then to a small increment of the argument there corresponds a small increment of the function or, the increment of the function f is an infinitely small quantity as $\Delta x \to 0$.



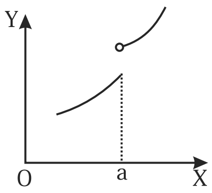
Formally, the function f(x) is continuous when x = a, if given ε, a number δ can be found, such that, whenever $|x - a| \le \delta$, we have $|f(x) - f(a)| < \varepsilon$.

Points of Discontinuity

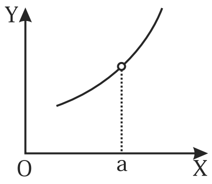
If a function f(x) is continuous at a point x = a, then the point a is called the point of continuity of the function f(x). Otherwise, when the limit of the function f(x) at the point a does not exist, or exists but is not equal to f(a), the function is said to be discontinuous at the point x = a, the latter being called the point of discontinuity of the function f(x).

In particular, if f(x) is defined for all points of the interval $(a - \delta, a + \delta)$ except for the point a, then x = a is also a point of discontinuity of the function f(x) in the interval. It follows from the aforegoing that a function is discontinuous at a given point if either (i) the given point fails to be in the domain of the function, or (ii) the function fails to be have a limit at the given point, or (iii) the limit of the function is unequal to the function's value at the given point.

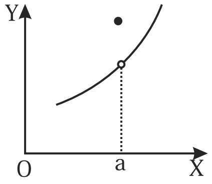
(i)



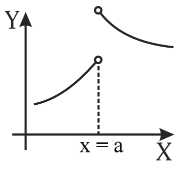
(ii)



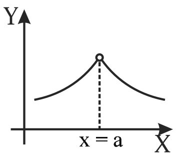
(iii)



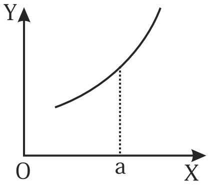
(iv)



(v)



(vi)



The function shown in figures (i) to (v) are discontinuous at x = a while that in (vi) is continuous at x = a. Thus, a function f can be discontinuous due to any of the following three reasons:

(i) $\lim\limits_{x \to a} f(x)$ does not exist

i.e. $\lim\limits_{x \to a^-} f(x) \ne \lim\limits_{x \to a^+} f(x)$ [figures (i) and (iv)]

(ii) f (x) is not defined at x = c [figures (ii) and (v)]

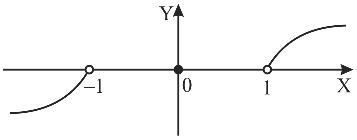
(iii) $\lim\limits_{x \to a} f(x) \ne f(c)$ [figure (iii)]

Geometrically, the graph of the function will exhibit a break at x= c .

Note: It should be noted that continuity of a function is the property of interval and is meaningful at x = a only if the function has a graph in the immediate neighbourhood of x = a. For example, the discussion of continuity of

$f(x) = \dfrac{1}{x - 1}$ at x = 1 is meaningful, but continuity of f(x) = ln x at x = –2 is meaningless.

Similarly, if f(x) has a graph as shown in the figure below, then continuity at x = 0 is meaningless since we can't approach 0 from either side of the point. Such a point is called an isolated point.



Consider the following examples:

(i) The function $f(x) = \dfrac{1}{(1 - x)^2}$ is discontinuous at x = 1.

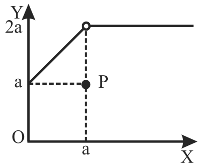
This function is not defined at the point x = 1.

(ii) The function $f(x) = \dfrac{\sin x}{|x|}$ has a discontinuity at

$x = 0$, since $f(0^+) = \lim\limits_{x \to 0^+} \dfrac{\sin x}{x} = 1$ and

$f(0^-) = \lim\limits_{x \to 0^-} \dfrac{\sin x}{-x} = -1$.

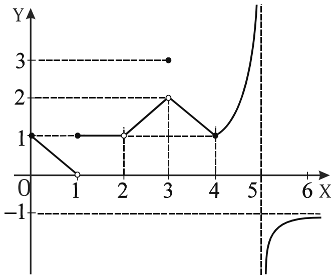
(iii) Let $f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a} & \text{when } 0\le x<a,\\ a & \text{when } x=a,\\ 2a & \text{when } x>a.\end{cases}$



Here $\lim\limits_{x\to a^-} f(x)=2a$, $f(a)=a$, $\lim\limits_{x\to a^+} f(x)=2a$,

and there is a discontinuity at x = a due to the isolated point P.

To understand explicitly the reasons of discontinuity, consider the graph of the following function y = f(x).



Let us comment on the continuity of the function.

(i) f is continuous at x = 0 and x = 4

(ii) f is discontinuous at x = 1 as limit does not exist

(iii) f is discontinuous at x = 2 as f (2) is not defined although the limit exists.

(iv) f is discontinuous at x = 3 as $\lim\limits_{x\to 3} f(x) \ne f(3)$

(v) f is discontinuous at x = 5 as neither the limit exist nor f is defined at x = 5.

Example 1: Discuss the continuity of the function [cos x] at $x=\dfrac{\pi}{2}$, where [·] denotes the greatest integer function.

Solution: $\text{L.H.L}=\lim\limits_{x\to\frac{\pi}{2}^-}\left[\cos x\right]=0.$

$$\text{R.H.L}=\lim_{x\to\frac{\pi}{2}^+}\left[\cos x\right]=-1.$$

$$f\left(\frac{\pi}{2}\right)=\left[\cos\frac{\pi}{2}\right]=0.$$

Since, L.H.L ≠ R.H.L the limit does not exist.

So, the function is discontinuous at $x=\dfrac{\pi}{2}$.

Example 2: Test the continuity of the function f(x) at x = 0, where $f(x)=\dfrac{e^{1/x}}{1+e^{1/x}}$, when x ≠ 0 and f(0) = 0.

Solution: Given, f(0) = 0

$$\text{L.H.L.} = \lim_{x\to0^-} f(x)=\lim_{x\to0^-}\frac{e^{1/x}}{1+e^{1/x}}=\frac{0}{1+0}=0$$

$$\text{since}\ \lim_{x\to0^-}\left(\frac{1}{x}\right)=-\infty$$

$$\text{R.H.L.} = \lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{e^{1/x}}{1+e^{1/x}}$$

$$\lim_{x\to0^+}\frac{1}{\frac{1}{e^{1/x}}+1}=\frac{1}{0+1}=1$$

$$\text{since}\ \lim_{x\to0^+}\left(\frac{1}{x}\right)=\infty$$

Since $\lim\limits_{x\to0^-} f(x)\ne\lim\limits_{x\to0^+} f(x)$,

f(x) is discontinuous at x = 0.

Example 3: Find whether f(x) is continuous or not at x = 1, if $f(x)=\begin{cases}\sin\dfrac{\pi x}{2}, & x<1\\ [x], & x\ge1\end{cases}$,

where [·] denotes the greatest integer function.

Solution: For continuity at x = 1, we determine, f(1), $\lim\limits_{x\to1^-} f(x)$ and $\lim\limits_{x\to1^+} f(x)$.

Now, f(1) = [1] = 1.

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\sin\frac{\pi x}{2}=\sin\frac{\pi}{2}=1.$$

and $\lim\limits_{x\to1^+} f(x)=\lim\limits_{x\to1^+}[x]=1.$

So $f(1)=\lim\limits_{x\to1^-} f(x)=\lim\limits_{x\to1^+} f(x).$

∴ f(x) is continuous at x = 1.

One-Sided Continuity

A function f defined in some neighbourhood of a point a for x ≤ a is said to be continuous at a from the left if $\lim\limits_{x\to a^-} f(x)=f(a)$.

A function f defined in some neighbourhood of a point a for x ≥ a is said to be continuous at a from the right if $\lim\limits_{x\to a^+} f(x)=f(a)$.

One-sided continuity is a collective term for functions continuous from the left or from the right.

If the function f is continuous at a, then it is continuous at a from the left and from the right . Conversely, if the function f is continuous at a from the left and from the right, then $\lim\limits_{x\to a}$ f(x) exists, and $\lim\limits_{x\to a} f(x)=f(a)$.

The function y = sgn x is neither left continuous nor right continuous at x = 0.

The function $y = \sin^{-1} x$ is left continuous at $x = 1$. We cannot discuss the right continuity here as the function is not defined in the right neighbourhood of $x = 1$.

The function $y = \{x\}$ is right continuous at $x = 0$ but left discontinuous there.

At each integer n, the function $f(x) = [x]$ is continuous from the right but discontinuous from the left because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [x] = n = f(n)$$

but $\displaystyle\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [x] = n-1 \neq f(n)$.

Example 4: Let $f(x) = \begin{cases} x^2 & \text{if } \; x > 2 \\ x+1 & \text{if } \; x \le 2 \end{cases}$

Show that f is continuous from the left at 2, but not from the right.

Solution: $f(2) = 2 + 1 = 3$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+1) = 3 \quad \text{and}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4$$

Since $\displaystyle\lim_{x\to 2^-} f(x) = f(2)$, f is continuous from the left at 2 and $\displaystyle\lim_{x\to 2^+} f(x) \neq f(2)$, f is not continuous from the right at 2.

Continuity at End Points

Let a function $y = f(x)$ be defined on $[a, b]$.

Then the function f(x) is said to be continuous at the left end point $x = a$ if, $f(a) = \displaystyle\lim_{x\to a^+} f(x)$,

and f(x) is said to be continuous at the right end point $x = b$ if, $f(b) = \displaystyle\lim_{x\to b^-} f(x)$.

For example, consider the function $f(x) = \{x\}$, $0 \le x \le 1$. It is continuous at $x = 0$ and discontinuous at $x = 1$.

Example 5: Discuss the continuity of $f(x) = \sqrt{\dfrac{x-a}{b-x}}$ where $a < b$, at $x = a$ and $x = b$.

Solution: We notice that the domain of the function is (a, b). At the left end point $x = a$, we have $\displaystyle\lim_{x\to a^+} f(x) = 0 = f(a)$.

Hence f is continuous at $x = a$.

At the right end point $x = b$, f(b) is undefined and $\displaystyle\lim_{x\to b^-} f(x) = \infty$.

Hence f is discontinuous at $x = b$.

Continuous Extension to a Point

A function may have a limit at a point where it is undefined. Then we can extend the definition of the function at that point to make it continuous there. Suppose f(a) is not defined, but $\displaystyle\lim_{x\to a} f(x) = L$ exists. If the point a is added to the domain of definition of the function f(x) and if at that new point the value of the function is put equal to the common value of the left hand and right hand limits, the (new) function F(x) thus obtained, is continuous at the point a. Then, we can define the new function F(x) by the formula

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = a \end{cases}$$

The function F is continuous at $x = a$. It is called the continuous extension of f to $x = a$.

For example, the function $y = \dfrac{\sin x}{x}$ is not defined at the point $x = 0$. But since $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ we can introduce a new function, defined for all the values of x and coinciding with the old one for $x \neq 0$, which is everywhere continuous:

$$F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } \; x \neq 0 \\ 1 & \text{for } \; x = 0 \end{cases}.$$

Example 6: Let $f(x) = \dfrac{\ln(1+ax) - \ln(1-bx)}{x}$. Find the value which should be assigned to f at $x = 0$, so that it is continuous at $x = 0$.

Solution: $\displaystyle\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} a\left[\frac{\ln\ (1+ax)}{ax}\right] + b\left[\frac{\ln\ (1-bx)}{(-bx)}\right]$$

$$= a.\,1 + b.\,1 = a + b.$$

$$\left[\because \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right]$$

So, $f(0) = a + b$, if f is continuous at $x = 0$.

Example 7: A function $y = f(x)$ is defined as

$$f(x) = \begin{cases} k\sin\dfrac{(x+3)\pi}{6} & \text{for } \; x \le 2 \\ \dfrac{3-\sqrt{11-x}}{x-2} & \text{for } \; x > 2 \end{cases}$$

If f (x) is continuous at $x = 2$, then find the value of k.

Solution: Since f (x) is continuous at $x = 2$

$$\lim_{x\to 2} f(x) = f(2) \Rightarrow \lim_{x\to 2} f(x) = \frac{k}{2}.$$

Now, $f(2^-) = k\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{k}{2}$ and

$$f(2^+) = \lim_{x\to 2} \frac{3-\sqrt{11-x}}{x-2}$$

$$= \lim_{x\to 2} \frac{(x-2)}{(x-2)\left(3+\sqrt{11-x}\right)} = \frac{1}{6}.$$

$$\therefore \quad \frac{k}{2} = \frac{1}{6} \quad \Rightarrow \quad k = \frac{1}{3}.$$

Example 8: Let $f(x) = \dfrac{\ln(1+x)^{1+x} - x}{x^2}$, then find the value of f(0) so that the function f is continuous at x = 0.

Solution: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{(1+x)\ln(1+x) - x}{x^2} \left(\dfrac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0} \frac{\ln(1+x)+1-1}{2x} \quad \text{(by L'Hospital's Rule)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\ln(1+x)}{x} = \frac{1}{2}.$$

For continuity, we must have f(0) = $\lim\limits_{x\to 0} f(x)$.

Hence, f(0) = $\dfrac{1}{2}$.

Example 9: Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^2-2|x-1|-1}, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}, \text{ at } x = 1.$$

Solution: R.H.L = f(1⁺) = $\lim\limits_{x\to 1^+} \dfrac{x^2-1}{x^2-2|x-1|-1}$

$$= \lim_{x\to 1^+} \frac{x^2-1}{x^2-2(x-1)-1}$$

$$= \lim_{x\to 1^+} \frac{(x+1)}{(x+1)-2} = \infty$$

There is no need to find the L.H.L. at x = 1 since R.H.L. is non-existent.

Thus, f(x) is discontinuous at x = 1.

Example 10: Check the continuity of the function $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0.

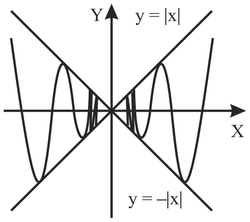
Solution: Consider the limit $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

If $x > 0$, $-x \le x\sin(1/x) \le x$,

and if $x < 0$, $x \le x\sin(1/x) \le -x$.

Thus, for $x \neq 0$, $-|x| \le x\sin(1/x) \le |x|$.

Since both $|x| \to 0$ and $-|x| \to 0$ as $x \to 0$, the Sandwich theorem applies and we can conclude that $x\sin(1/x) \to 0$ as $x \to 0$. This is illustrated in the following figure.



$y = x\sin\left(\frac{1}{x}\right)$

It follows that the function is continuous at x = 0, since the value of the function and the value of the limit are the same at 0. This shows that the behaviour of a function can be very complex in the vincinity of x = a, even though the function is continuous at a.

Example 11: Test the continuity of f(x) at x = 0 if

$$f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Solution: L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)}$$

$$= \lim_{h\to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)}$$

$$= \lim_{h\to 0} (1-h)^2 = (1-0)^2 = 1.$$

R.H.L = $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} (h+1)^{2\left(\frac{1}{|h|}+\frac{1}{h}\right)}$$

$$= \lim_{h\to 0} (h+1)^{2-\frac{2}{h}}$$

$$= \lim_{h\to 0} (h+1)^{-\frac{2}{h}} = e^{-2}$$

f(0) = 0.

∴ L.H.L. ≠ R.H.L.

Hence, f(x) is discontinuous at x = 0.

Example 12: Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{a^{2[x]+\{x\}}-1}{2[x]+\{x\}}, & x \neq 0 \\ \log_e a, & x = 0 \end{cases} \quad (a \neq 1)$$

at x = 0, where [x] and {x} are the greatest integer part and fractional part of x respectively.

Solution: $f(0) = \log_e a$

L.H.L. $= \lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{a^{2[0-h]+\{0-h\}}-1}{2[0-h]+\{0-h\}}$$

$$= \lim_{h\to 0} \frac{a^{2[0-h]+\{-1+(1-h)\}}-1}{2[0-h]+\{-1+(1-h)\}}$$

$$= \lim_{h\to 0} \frac{a^{-2+(1-h])}-1}{-2(1-h)} = \frac{a^{-1}-1}{-1} = 1-\frac{1}{a}.$$

R.H.L. $= \lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{a^{2[-)]+\{0+h\}-1}}{2[0+h]+\{0+h\}}$$

$$= \lim_{h\to 0} \frac{a^{0+h}-1}{0+h}$$

$$= \lim_{h\to 0} \frac{a^{h}-1}{h} = \log_e a$$

We find that $1 - 1/a = \log_e a$ only when $a = 1$, which is not acceptable. Since L.H.L. $\neq$ R.H.L. $= f(0)$, $f(x)$ is discontinuous at $x = 0$.

Example 13: If the function $f(x) = \dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x - \sin x}$ $(x \neq 0)$ is continuous at $x = 0$, then find the value of $f(0)$.

Solution: $f(0) = \lim\limits_{x\to 0} \dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$

$$= \lim_{x\to 0} \frac{\tan(\tan x)-\sin(\sin x)}{\dfrac{\tan x}{x}\left(\dfrac{1-\cos x}{x^2}\right)x^3}$$

$$= 2\lim_{x\to 0} \frac{\tan(\tan x)-\sin(\sin x)}{x^3}$$

$$= 2\lim_{x\to 0} \frac{1}{x^3}\left(\left(\tan x + \frac{\tan^3 x}{3} + \frac{2}{15}\tan^5 x + \ldots\ldots\right)\right.$$

$$\left.-\left(\sin x - \frac{\sin^3 x}{3!} + \frac{\sin^5 x}{5!} - \ldots\ldots\ldots\right)\right)$$

$$= 2\lim_{x\to 0}\left(\left(\frac{\tan x - \sin x}{x^3}\right) + \frac{\left(\dfrac{\tan^3 x}{3} + \dfrac{\sin^3 x}{3!}\right)}{x^3} + \ldots.\right)$$

$$= 2\lim_{x\to 0}\left(\left(\frac{\tan x}{x}\right)\left(\frac{1-\cos x}{x^2}\right) + \frac{1}{3} + \frac{1}{6}\right)$$

$$= 2\left[\frac{1}{2}+\frac{1}{2}\right] = 2.$$

Example 14: Let $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$ $\forall x \in \left(0, \dfrac{\pi}{2}\right)$ except at $x = \dfrac{\pi}{4}$. Define $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ may be continuous at $x = \dfrac{\pi}{4}$.

Solution: $f(x)$ will be continuous at $x = \dfrac{\pi}{4}$, if $\lim\limits_{x\to\pi/4} f(x) = f\left(\dfrac{\pi}{4}\right)$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \lim_{x\to\pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x\to\pi/4} \frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$$

$$= \lim_{x\to\pi/4} \frac{(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1)(\cos x + \sin x)\sin x}{(\sqrt{2}\cos x + 1)(\cos x - \sin x)(\cos x + \sin x)}$$

$$= \lim_{x\to\pi/4} \frac{\left(2\cos^2 x - 1\right)}{(\cos^2 x - \sin^2 x)} \frac{\left(\cos x + \sin x\right)\sin x}{\left(\sqrt{2}\cos x + 1\right)}$$

$$= \lim_{x\to\pi/4} \frac{\sin x(\cos x + \sin x)}{(\sqrt{2}\cos x + 1)}$$

$$= \frac{\dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right)}{\sqrt{2}.\dfrac{1}{\sqrt{2}} + 1} = \frac{1}{2}.$$

Example 15: Let $f(x) = \dfrac{e^{\tan x} - e^{x} + \ln(\sec x + \tan x) - x}{\tan x - x}$ be a continuous function at $x = 0$. Find the value of $f(0)$.

Solution: For continuity of f at $x = 0$, we have $f(0) = \lim\limits_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{e^{\tan x} - e^{x}}{\tan x - x} + \lim_{x\to 0} \frac{\ln(\sec x + \tan x) - x}{\left(\dfrac{\tan x - x}{x^3}\right)x^3}$$

$$= \lim_{x\to 0} \frac{e^{x}(e^{\tan x - x} - 1)}{\tan x - x} + 3\lim_{x\to 0} \frac{\ln(\sec x - \tan x) - x}{x^3}$$

$$= 1 + 3\lim_{x\to 0} \frac{\sec x - 1}{3x^2} \quad \text{(by L'Hospital's Rule)}$$

$$= 1 + \frac{1}{2} = \frac{3}{2}.$$

Example 16: Let a function f(x) be defined in the neighborhood of as

$$f(x) = \begin{cases} \dfrac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)} & \text{for } x<0 \\[2ex] \dfrac{e^{\sin 2x}-1}{\ln(1+\tan 9x)} & \text{for } x>0 \end{cases}$$

Find whether it is possible to define $f(0)$ so that f may be continuous at $x = 0$.

Solution: $\lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{e^{\sin 2h}-1 \quad \sin 2h.(2h)}{\ln(1+\tan 9h)\ \sin 2h.(2h)}$$

$$= \lim_{h\to 0} \frac{e^{\sin 2h}-1}{\sin 2h} . \lim_{h\to 0} \frac{\sin 2h}{2h} \lim_{h\to 0} \frac{2h}{\ln(1+\tan 9h)}$$

$$= 1.1. \lim_{h\to 0} \frac{2h}{\tan 9h\ \ln(1+\tan 9h)\cot 9h}$$

$$= \lim_{h\to 0} \frac{2h}{\tan 9h} = \frac{2}{9} \quad . \quad \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{\ln(1+2\sin^2 h)}{2\sin^2 h} . \lim_{h\to 0} \frac{2\sin^2 h}{\ln^2(1-\sin 3h)}$$

$$= \lim_{h\to 0} \frac{2\sin^2 h}{h^2} . \lim_{h\to 0} \frac{h^2}{\ln^2(1-\sin 3h)}$$

$$= \lim_{h\to 0} \frac{2\cdot h^2(-\sin 3h)^2}{\sin^3 3h\cdot \ln^2(1-\sin 3h)} = \frac{2}{9}$$

Since $f(0^+) = f(0^-) = \dfrac{2}{9}$ the limit exists.

Therefore, it is possible to define f(0) such that f is continuous at $x = 0$.

$\therefore \quad f(0) = \dfrac{2}{9}$.

Example 17: Determine the value of p, so that the

$$\text{function } f(x) = \begin{cases} \dfrac{x^2+2\cos x-2}{x^4} & \text{for } x<0 \\[1ex] p & \text{for } x=0 \\[1ex] \dfrac{\sin x-\ell n(e^x\cos x)}{6x^2} & \text{for } x>0 \end{cases}$$

is continuous at $x = 0$.

Solution: $\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} \dfrac{h^2-2(1-\cos h)}{h^4}$

$$= \lim_{h\to 0} \frac{h^2-4\sin^2\frac{h}{2}}{h^4}$$

$$= \lim_{h\to 0} \frac{h+2\sin\frac{h}{2}}{h} . \lim_{h\to 0} \frac{h-2\sin\frac{h}{2}}{h^3}$$

$$= 2\cdot\lim_{t\to 0} \frac{2t-2\sin t}{8t^3} \quad \text{where } h = 2t$$

$$= \lim_{t\to 0} \frac{t-\sin t}{2t^3} = \frac{1}{12}.$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{\sin h-\ln(e^h\cos h)}{6h^2}$$

$$= \lim_{h\to 0} \frac{\sin h-h-\ln\cos h}{6h^2}$$

$$= \frac{\sin h-h}{6h^2} - \frac{1}{6}\ln(\cos h)^{\frac{1}{h^2}}$$

$$= \lim_{h\to 0} \frac{\sin h-h}{h^3}\cdot h + \lim_{h\to 0} -\frac{1}{6}\ln(\cos h)^{\frac{1}{h^2}}$$

$$= 0 - \frac{1}{6}\ln e^{\lim\limits_{h\to 0}\frac{1}{h^2}(\cos h-1)} = \frac{1}{12}$$

Hence, $\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(0+h) = f(0) = \dfrac{1}{12} = p$.

Example 18: If $f(x) = \dfrac{\sin 2x + A\sin x + B\cos x}{x^3}$ is continuous at $x = 0$, find the values of A and B. Also find f(0).

Solution: As f(x) is continuous at $x = 0$,

$$f(0) = \lim_{x\to 0} \frac{\sin 2x + A\sin x + B\cos x}{x^3}.$$

As denominator → 0, when x → 0, numerator should also approach 0, which is possible only if

$\sin 2(0) + A\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$

$$\therefore \quad f(0) = \lim_{x\to 0} \frac{\sin 2x + A\sin x}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{2\cos x + A}{x^2}\right)$$

$$= \lim_{x\to 0} \left(\frac{2\cos x + A}{x^2}\right)$$

Again we can see that denominator → 0 as x → 0.

∴ Numerator should also approach 0 as x → 0

$\Rightarrow \quad 2 + A = 0 \quad \Rightarrow \quad A = -2.$

$$\Rightarrow \quad f(0)=\lim_{x\to 0}\left(\frac{2\cos x-2}{x^2}\right)=\lim_{x\to 0}\left(\frac{-4\sin^2 x/2}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{-\sin^2 x/2}{x^2/4}\right)=-1$$

So, we get A = – 2, B = 0 and f(0) = – 1.

Example 19: Let

$$f(x)=\begin{cases}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & ,\ x<0\\ 3 & ,\ x=0\\ \left\{1+\left(\dfrac{cx+dx^3}{x^2}\right)\right\}^{1/x} & ,\ x>0\end{cases}$$

If f is continuous at x = 0, then find the values of a, b, c and d.

Solution: f(0) = 3

$$\text{R.H.L.}=\lim_{x\to 0^+}f(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)$$

$$=\lim_{h\to 0}\left\{1+\left(\frac{ch+dh^3}{h^2}\right)\right\}^{1/h}$$

Since f is continuous at x = 0, R.H.L. exists.
For existence of R.H.L., c must be 0.

$\therefore$ $\text{R.H.L.}=\lim_{h\to 0}\ \{1+dh)^{1/h}$ (form 1^∞)

$$=e^{\lim\limits_{h\to 0}(1+dh-1)}=e^d$$

$$\text{L.H.L.}=\lim_{x\to 0^-}f(x)=\lim_{x\to 0}f(0-h)$$

$$=\lim_{h\to 0}\frac{a(1-(-h)\sin(-h))+b\cos(-h)+5}{(-h)^2}$$

$$=\lim_{h\to 0}\frac{a(1-h\sin h)+b\cos h+5}{h^2}$$

For finite value of L.H.L. the numerator must tend to 0 as h → 0.

$\therefore$ a + b + 5 = 0.

$$\text{L.H.L.}=\lim_{h\to 0}\frac{a-ah\sin h-(5+a)\cos h+5}{h^2}$$

$$=\lim_{h\to 0}\frac{(5+a)(1-\cos h)-a\,h\sin h}{h^2}$$

$$=\lim_{h\to 0}\left\{\frac{(a+5)(1-\cos h)(1+\cos h)}{h^2(1+\cos h)}-\frac{a\,h\sin h}{h^2}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{(a+5)\sin^2 h}{h^2(1+\cos h)}-\frac{a\sin h}{h}\right\}$$

$$=\frac{(a+5)(1)^2}{1+1}-a\,.\,1$$

$$=\frac{a+5}{2}-a=\left(\frac{5-a}{2}\right).$$

Now, f(x) is continuous at x = 0.

$\therefore$ L.H.L. = R.H.L. = f(0)

$$\frac{5-a}{2}=e^d=3$$

$\therefore$ a = 1, d = ln 3, c = 0, b = –6.

Example 20: Let f (x)

$$=\begin{cases}\dfrac{1+a\cos 2x+b\cos 4x}{x^2\sin^2 x} & \text{if } x\neq 0\\ c & \text{if } x=0\end{cases},$$

be continuous at x = 0, then find the values of a, b and c.

Solution: $\lim\limits_{x\to 0}\dfrac{1+a\cos 2x+b\cos 4x}{x^4}$

As $x\to 0$, Denominator $\to 0$ and
Numerator $\to$ 1 + a + b
For existence of limit, a + b + 1 = 0 ...(1)

$$\therefore \quad c=\lim_{x\to 0}\frac{a\cos 2x+b\cos 4x-(a+b)}{x^4} \quad ...(2)$$

$$=-\lim_{x\to 0}\frac{\dfrac{a(1-\cos 2x)}{x^2}+\dfrac{b(1-\cos 4x)}{x^2}}{x^2}$$

The limit of numerator $=4a\left(\dfrac{1}{2}\right)+16b\left(\dfrac{1}{2}\right)$

$\Rightarrow$ 2a + 8b = 0 $\Rightarrow$ a = – 4b ...(3)

From (1) and (2) – 4b + b = – 1

$$\Rightarrow \quad b=\frac{1}{3} \quad \text{and} \quad a=-\frac{4}{3}$$

$$\text{From (2), } c=\lim_{x\to 0}\frac{4(1-\cos 2x)-(1-\cos 4x)}{3x^2}$$

$$=\lim_{x\to 0}\frac{8\sin^2 x-2\sin^2 2x}{3x^4}$$

$$=\lim_{x\to 0}\frac{8\sin^2 x-8\sin^2 x\cos^2 x}{3x^4}$$

$$=\lim_{x\to 0}\frac{8}{3}\cdot\frac{\sin^2 x}{x^2}\cdot\frac{\sin^2 x}{x^2}=\frac{8}{3}.$$

Example 21: Discuss the continuity of the function,

$$f(x)=\lim_{n\to\infty}\frac{\ln(2+x)-x^{2n}\sin x}{1+x^{2n}} \text{ at } x=1.$$

Solution: We have f(1) = (ln3 – sin1)/2

$$\text{and } \lim_{n\to\infty}x^{2n}=\begin{cases}0, & \text{if } x^2<1\\ \infty, & \text{if } x^2>1\end{cases}$$

$\therefore$ For $x^2<1$, we have

$$f(x)=\lim_{n\to\infty}\frac{\ln(2+x)-x^{2n}\sin x}{1+x^{2n}}=-\ln(2+x)$$

Again for $x^2 > 1$, we have

$$f(x) = \lim_{n\to\infty}\frac{\frac{1}{x^{2n}}\ln(2+x)-\sin x}{1+\frac{1}{x^{2n}}} = -\sin(x)$$

Here, as $x \to 1$

$$\lim_{x\to1^-} f(x) = \ln 3 \text{ and } \lim_{x\to1^+} f(x) = -\sin 1$$

So, $\lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$.

Therefore, f(x) is discontinuous at x = 1.

Example 22: Find the value of f(1) if the function

$$f(x) = \lim_{n\to\infty}\frac{x^{m+1}-(m+1)x+m}{(x-1)^2},\ x\neq 1$$

is continuous at x = 1.

Solution: We have

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{x(x^m-1)-m(x-1)}{(x-1)^2},$$

$$= \lim_{x\to0}\frac{x(x-1)(1+x+x^2+....+x^{m-1})-m(x-1)}{(x-1)^2}$$

$$= \lim_{x\to0}\frac{(x+x^2+x^3+....+x^m)-(1+1+...m\text{ times})}{x-1}$$

$$= \lim_{x\to0}\frac{(x-1)+(x^2-1)+(x^3-1)+.....+(x^m-1)}{x-1}$$

$$= \lim_{x\to0}\left[1+\frac{x^2-1}{x-1}+\frac{x^3-1}{x-1}+.....+\frac{x^m-1}{x-1}\right]$$

$$= 1+2+3+....+m = \frac{m(m+1)}{2}$$

Hence, for f to be continuous at x = 1, we should have $f(1) = \frac{m(m+1)}{2}$.

Example 23: A function f(x) satisfies f(x + y) = f(x). f(y) for all x and y ∈ R. Show that the function is continuous for all values of x if it is continuous at x = 1.

Solution: We have f(x + y) = f(x) . f(y) ...(1)

Putting x = 0, y = 0 in (1)

we have f(0) = f(0). f(0) ⇒ f(0) = 0 or 1.

If f(0) = 0 then putting y = 0 in (1)

we get f(x) = f(x) . f(0)

⇒ f(x) = 0 for all x.

Hence f is continuous for all x. Here the continuity of f at x = 1 is not required as a condition.

If f(0) = 1 then we have the following:

Putting x = 1, y = –1 in (1)

we have f(0) = f(1). f(–1) ⇒ f(1). f(–1) = 1

Hence f(1) is non-zero. ...(2)

As the function is continuous at x = 1,

we have $\lim\limits_{h\to0} f(1+h) = f(1)$

(using f(x + y) = f(x) . f(y))

$\Rightarrow \lim\limits_{h\to0} f(1) . f(h) = f(1)$

$\Rightarrow \lim\limits_{h\to0} f(h) = 1$ using (2) ...(3)

Now, we consider any arbitrary point x.

$$\lim_{h\to0} f(x+h) = \lim_{h\to0} f(x) . f(h)$$

$$= f(x)\lim_{h\to0} f(h)$$

$= f(x)$ using (3)

Hence, at any arbitrary point x, limit = function's value. Therefore, the function is continuous for all values of x.

Example 24: Let f be a function satisfying

$$f(x+y)+\sqrt{6-f(y)} = f(x)f(y) \text{ and } \lim_{x\to0} f(x) = 6.$$

Discuss the continuity of f.

Solution: $\lim\limits_{h\to0} f(x+h) = \lim\limits_{h\to0}\left[f(x)f(h)-\sqrt{6-f(h)}\right]$

$$= f(x)\lim_{h\to0} f(h) - \lim_{h\to0}\sqrt{6-f(h)}$$

$= f(x).6 - 0 = 6f(x) \neq f(x)$

Putting x = 0, y = 0 in the given relation, we get

$f(0)+\sqrt{6-f(0)} = f^2(0)$. $\Rightarrow f(0) \neq 0$

∴ f(x) = 0 for all x is not possible.

Thus, $\lim\limits_{h\to0} f(x+h) \neq f(x)$.

Hence, f is discontinuous at all x.

Concept Problems A

1. Check the continuity of

$$f(t) = \begin{cases}\dfrac{\tan(\sin t)}{\sin t}, & \sin t\neq 0\\ 1, & \sin t = 0\end{cases} \text{ at } t = 0.$$

2. If $f(x) = \begin{cases}x+\lambda, & x<3\\ 4, & x=3\\ 3x-5, & x>3\end{cases}$ is continuous at x = 3 then find the value of λ.

3. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin(2x)}$, $x\neq 0$ is continuous function at $x=0$, then find the value of $f(0)$.

4. If the function $f(x)=\begin{cases}\dfrac{x^2-(a+2)x+2a}{x-2}, & x\neq 2\\ 2, & x=2\end{cases}$ is continuous at $x=2$ then find the value of a.

5. If f and g are continuous functions with $f(3)=5$ and $\lim\limits_{x\to 3}[2f(x)-g(x)]=4$, find $g(3)$.

6. Use continuity to evaluate the limits :

(i) $\lim\limits_{x\to 4}\dfrac{5+\sqrt{x}}{\sqrt{5+x}}$

(ii) $\lim\limits_{x\to 1}\tan^{-1}\left(\dfrac{x^2-4}{3x^2-6x}\right)$

7. For the function $\phi(x)=x.l\text{n}\sin^2x$ when $x>0$, and $\phi(0)=0$, discuss the right continuity at $x=0$.

8. The function $f(x)=(x+1)^{\cot x}$ is undefined at $x=0$. Find the value which should be assigned to f at $x=0$, so that it is continuous at $x=0$.

Practice Problems A

9. If possible find value of λ for which f(x) is continuous at $x=\dfrac{\pi}{2}$.

$$f(x)=\begin{cases}\dfrac{1-\sin x}{1+\cos 2x}, & x<\dfrac{\pi}{2}\\ \lambda, & x=\dfrac{\pi}{2}\\ \dfrac{\sqrt{2x-\pi}}{\sqrt{4+\sqrt{2x-\pi}}-2}, & x>\dfrac{\pi}{2}\end{cases}$$

10. Find the value of f(0) so that the function

$f(x)=\dfrac{\sqrt{1-x}-\sqrt[3]{1-x}}{x}$ is continuous at $x=0$.

11. Show that the function f defined on R by setting $f(x)=|x|^m\sin\dfrac{1}{x}$, when $x\neq 0$, $f(0)=0$, is continuous at $x=0$ whenever $m>0$.

12. The function $f(x)=1-x\sin\dfrac{1}{x}$ is meaningless for $x=0$. How should one choose the value f(0) so that f(x) is continuous for $x=0$?

13. Find the values of a and b such that the function

$$f(x)=\begin{cases}x+a\sqrt{2}\sin x, & 0\le x<\dfrac{\pi}{4}\\ 2x\cot x+b, & \dfrac{\pi}{4}\le x\le\dfrac{\pi}{2}\\ a\cos 2x-b\sin x, & \dfrac{\pi}{2}<x\le\pi\end{cases}$$

is continuous at $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$.

14. If $f(x)=(1+ax)^{\frac{1}{x}}$ $\quad x<0$

$=b$ $\quad x=0$

$=\dfrac{(x+c)^{\frac{1}{3}}-1}{x}$ $\quad x>0$

find the values of a, b, c, f(x) is continuous at $x=0$.

15. If $f(x)=\cos\left(x\cos\dfrac{1}{x}\right)$ and $g(x)=\dfrac{l\text{n}(\sec^2 x)}{x\sin x}$ for $x\neq 0$ and they are both continuous at $x=0$ then show that $f(0)=g(0)=1$.

16. The function $f(x)=a[x+1]+b[x-1]$, where [.] is the greatest integer function then find the condition for which f(x) is continuous at $x=1$.

17. Let f (x)

$$=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & \text{for } -\dfrac{\pi}{6}<x<0\\ b & \text{for } x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0<x<\dfrac{\pi}{6}\end{cases}$$

Find ‘a’ and ‘b’ if f is continuous at $x=0$.

18. If f(x) is continuous in [0, 1] and $f\left(\dfrac{1}{2}\right)=1$ then find $\lim\limits_{n\to\infty}f\left(\dfrac{\sqrt{n}}{2\sqrt{n}+1}\right)$.

19. Let $f(x)=\dfrac{\sin(e^{x-2}-1)}{\log(x-1)}$, $x\neq 2$.

If f(x) is continuous at $x=2$ find f(2).

2.2 Continuity In An Interval

We can extend the concept of continuity and say that a function f(x) is continuous in an interval if it is continuous at every point in the interval.

Roughly speaking, a function is said to be continuous on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, function with this property have many other desirable properties.

Continuity in an Open Interval

A function f is said to be continuous in an open interval (a , b) if f is continuous at each and every point lying in the interval (a , b) .

Continuity in a Closed Interval

A function f is said to be continuous in a closed interval [a, b] if :

(i) f is continuous in the open interval (a , b)

(ii) f is right continuous at ‘a’

i.e. $\lim_{x \to a^+} f(x) = f(a)$.

(iii) f is left continuous at ‘b’

i.e. $\lim_{x \to b^-} f(x) = f(b)$.

For instance, $f(x) = \sqrt{-x^2 + 3x - 2}$, where $1 \le x \le 2$, is a function continuous on this interval, since it is continuous at every point of the interval (1, 2), continuous on the right at the point x = 1, and continuous on the left at the point x = 2.

The function $y = \dfrac{1}{\sqrt{1 - x^2}}$ is continuous in the open interval (–1, 1). It is discontinuous at the points x = – 1 and x = 1.

Similarly, f is continuous on the half-open interval (a, b] if it is continuous at each number between a and b and is continuous from the left at the endpoint b.

In the case where f is continuous on (–∞, ∞), we will say that f is continuous everywhere. The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an arbitrary real number.

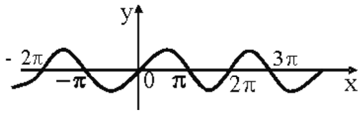
Continuity of Elementary Functions

All basic elementary functions are continuous in the intervals where they are defined.

The constant function and the identity function are continuous over R.

We know that y = sin x and y = cos x are continuous for every value of x.

From the graph we see that f(x) = sin x is continuous in its entire domain.



We see that the functions

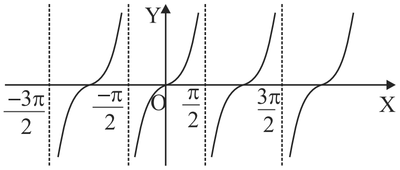
$$y = \tan x = \frac{\sin x}{\cos x},\ y = \cot x = \frac{\cos x}{\sin x}$$

$$y = \sec x = \frac{1}{\cos x},\ y = \text{cosec}\ x = \frac{1}{\sin x}$$

are continuous for all those values of x for which they are defined. The discontinuities of these four functions arise only when the denominators become zero and for such values of x, these functions themselves cease to be defined.

f(x) = tan x is continuous at all points except

$x = (2n + 1)\dfrac{\pi}{2}$, $n \in I$.



Thus, the domain of continuity of each of the functions sin x, cos x, tan x, cot x, sec x, and cosec x coincides with the corresponding domain of definition.

f(x) = tan x is continuous at all points except $x = (2n + 1)\dfrac{\pi}{2}$, $n \in I$.

Note: f is a continuous function if it is continuous at each number a in its domain.

The examination of the graph of the function y = 1/x in the vicinity of the point x = 0 clearly shows that it "splits" into two separate curves at that point.

However, the function f(x) = 1/x, whose domain consists of the intervals (–∞, 0) and (0, ∞) is continuous. Although this function explodes at 0, this does not prevent it from being a continuous function. The key to being continuous is that the function is continuous at each number in its domain. The number 0 is not in the domain of f(x) = 1/x.

However, f(x)=1/x is discontinuous in the interval (–∞, ∞)

Theorem Every elementary function is contin-uous at every point in its domain.

The polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \ldots.. + a_{n-1}x + a_n$ is continuous over R .

It follows from the preceding theorem and the fact that constant function and the identity function are continuous over R.

If f(x) is continuous for any particular value of x, then any polynomial in f(x), such as

$a_0\{f(x)\}^n + a_1\{f(x)\}^{n-1} + \ldots.. + a_{n-1}\{f(x)\} + a_n$ is also continuous

The rational function $f(x) = \dfrac{a_0x^n + a_1x^{n-1} + \ldots.. + a_n}{b_0x^m + b_1x^{m-1} + \ldots.. + b_m}$

is continuous at every value of x except at those points where the denominator becomes zero.

For instance, the function $f(x) = \dfrac{3 - x}{4x + 7}$ is continuous throughout the entire number line except for the point $x = -\dfrac{7}{4}$, at which the denominator of the fraction vanishes. And the function $f(x) = -\dfrac{x^3 + 4x^2 + x + 1}{x^2 + x + 1}$ is continuous everywhere on R, since the denominator never vanishes.

Now, each of the irrational functions $\sqrt{(x - a)(b - x)}$, $\sqrt[3]{(x - a)(b - x)}$ and $\sqrt[3]{\left(\dfrac{x - a}{b - x}\right)}$ are continuous at each point in their domain, since these functions are elementary.

All trigonometric functions, exponential and logarithmic functions are continuous in their domain.

Example 1: Test the following functions for continuity

(a) $f(x) = \dfrac{2x^5 - 8x^2 + 11}{x^4 + 4x^3 + 8x^2 + 8x + 4}$

(b) $f(x) = \dfrac{3\sin^3 x + \cos^2 x + 1}{4\cos x - 2}$

Solution:

(a) A function representing a ratio of two continuous functions (polynomials in this case) is discontinuous only at points for which the denominator is zero.
But in this case $(x^4 + 4x^3 + 8x^2 + 8x + 4)$
$= (x^2 + 2x + 2)^2 = [(x + 1)^2 + 1]^2 > 0$ (non-zero).
Hence f(x) is continuous everywhere.

(b) The function f(x) suffers discontinuities only at points for which the denominator becomes zero i.e. at the roots of the equation

$4\cos x - 2 = 0 \Rightarrow \cos x = 1/2.$

$\Rightarrow \quad x = 2n\pi \pm \pi/3, n \in I.$

Thus the function f(x) is continuous for all real x, except at the points $2n\pi \pm \pi/3$, $n \in I$.

Example 2: Let $f(x) = \dfrac{\sqrt{x^2 + kx + 1}}{x^2 - k}$. Find all possible values of k for which f is continuous for every $x \in R$.

Solution: $f(x) = \dfrac{\sqrt{x^2 + kx + 1}}{x^2 - k}$

For f to be continuous $\forall\, x \in R$

$x^2 + kx + 1 \geq 0$ and $x^2 - k$ must not have any real root.

$\therefore \quad k^2 - 4 \leq 0$ and $k < 0$

$\Rightarrow \quad k \in [-2, 2]$ and $k < 0$

From above, $k \in [-2, 0)$.

Example 3: Where is the function $F(x) = \ln(1 + \cos x)$ continuous?

Solution: We know that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, $F(x) = f(g(x))$ is continuous wherever it is defined.

Now $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$. So it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi$, $\pm 3\pi$, ... Thus, F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values.

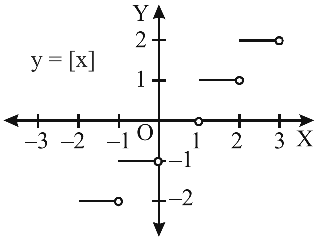
Example 4: Where is the function

$f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Solution: We know that the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on R. Thus, $y = \ln x + \tan^{-1} x$ is continuous on $(0, \infty)$. The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, f is continuous at all positive numbers x except where $x^2 - 1 = 0$. So f is continuous on the intervals (0, 1) and $(1, \infty)$.

We need to examine the continuity of non-elementary functions carefully.

For instance, the greatest integer function $f(x) = [x]$, is continuous everywhere except for the integral values of x.



$y = [x]$ is discontinuous at $x \in I$.

Also, the fractional part function $f(x) = \{x\}$, is continuous everywhere except for the integral values of x.

Hence, the continuity of functions $\{f(x)\}$ and $[f(x)]$ should be checked at all points where f(x) becomes an integer.

The function $y = \text{sgn } x$ (or $y = \dfrac{|x|}{x}$) is discontinuous at the point $x = 0$.

Hence, the continuity of the function sgn (f(x)) should be checked at the points where $f(x) = 0$. (Note that if f(x) is constanly equal to 0 when $x \longrightarrow a$ then $x = a$ may not be a point of discontinuity).

Usually there are only a few points in the domain of a given function f where a discontinuity can occur.

To find the points of discontinuity, we collect all the doubtful points and examine them for continuity.

We use the term **suspicious point** for a number a where

(i) The definition of the function changes or domain of f splits, or

(ii) substitution of $x = a$ causes division by 0 in the function.

A function may be discontinuous at a suspicious point which can be found using the test of continuity.

For example, $y = \dfrac{x^2 - 1}{x^2 - 4}$ has two suspicious points $x = \pm 2$ (where the denominator becomes 0).

$y = x \sin \dfrac{1}{x}$ has one suspicious point $x = 0$.

For the function $y = |x^2 - 4|$ we have

$y = x^2 - 4$ when $x^2 - 4 \geq 0$
and $y = 4 - x^2$ when $x^2 - 4 < 0$.

This means the definition of the function changes when $x^2 - 4 = 0$, i.e. $x = \pm 2$. Thus, the function has two suspicious points $x = \pm 2$.

Note: There is no chance of continuity at points where the function is not defined. We know that the function cannot be continuous at such points.

Example 5: If $f(x) = [\sin \pi x], \quad 0 \le x < 1$

$$= \left\{x - \frac{2}{3}\right\} \operatorname{sgn}\left(x - \frac{5}{4}\right), \ 1 \le x \le 2,$$

where {.} represents the fractional function then find the suspicious points for continuity of function in the interval [0, 2].

Solution:

(i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2.

(ii) For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$ i.e. x = 0, 1/2

(iii) For $\left\{x - \frac{2}{3}\right\} \operatorname{sgn}\left(x - \frac{5}{4}\right)$, continuity should be checked when $x - 5/4 = 0$ i.e. $x = 5/4$ and when $x - 2/3 \in I$ i.e. $x = 5/3$.

Finally, continuity should be checked at the suspicious points x = 0, 1/2, 1, 5/4, 5/3 and 2 .

Example 6: Show that the function

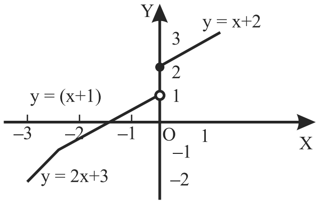
$$f(x) = \begin{cases} 2x+3, & -3 \le x < -2 \\ x+1, & -2 \le x < 0 \\ x+2, & 0 \le x \le 1 \end{cases}$$

is discontinuous at x = 0 and continuous at other points in the interval [–3, 1].

Solution: The graph of

$$f(x) = \begin{cases} 2x+3, & -3 \le x < -2 \\ x+1, & -2 \le x < 0 \\ x+2, & 0 \le x \le 1 \end{cases}$$

is plotted below :



Here, we observe from the graph that at x = 0, $\lim_{x \to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 2$, which shows that the function is discontinuous at x = 0 and continuous at every other point in [–3, 1].

Example 7: Let

$$f(x) = \begin{cases} -2\sin x & \text{if} \quad x \le -\pi/2 \\ A\sin x + B & \text{if} \quad -\dfrac{\pi}{2} < x < \pi/2 \\ \cos x & \text{if} \quad x \ge \pi/2 \end{cases}.$$

Find A and B so as to make the function continuous.

Solution: At x = – π/2

$$\text{L.H.L.} = \lim_{x \to \frac{\pi^-}{2}} (-2\sin x) = 2$$

$$\text{R.H.L.} = \lim_{x \to -\frac{\pi^+}{2}} A\sin x + B = -A + B$$

So $B - A = 2$ ...(1)

At x = π/2

$$\text{L.H.L.} = \lim_{x \to \frac{\pi^-}{2}} A\sin x + B = A + B$$

$$\text{R.H.L.} = \lim_{x \to \frac{\pi^+}{2}} \cos x = 0$$

So $A + B = 0$ ...(2)

Solving (1) and (2) we get A = – 1 and B = 1.

Example 8: Find the intervals on which f and g are continuous where $f(x) = \begin{cases} 3-x & \text{if} \quad -5 \le x < 2 \\ x-2 & \text{if} \quad 2 \le x < 5 \end{cases}$

and $g(x) = \begin{cases} 2-x & \text{if} \quad -5 \le x < 2 \\ x-2 & \text{if} \quad 2 \le x < 5 \end{cases}$.

Solution: The domain for both function is [–5, 5). Both functions are continuous except possibly at the suspicious point x= 2. Examining f, we see

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1$$

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2) = 0$

so $\lim_{x \to 2} f(x)$ does not exist and f is discontinuous at x = 2. Thus, f is continuous for $-5 \le x < 2$ and for $2 < x < 5$.

For g, we have g(2) = 0

and $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x) = 0$

and $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x - 2) = 0.$

Therefore, $\lim_{x \to 2} g(x) = 0 = g(2)$, and g is continuous at x = 2.

Hence, g is continuous throughout the interval [–5, 5).

Example 9: Find the points of discontinuity of the following functions for x ∈ R.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$

(ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1}$

(iv) $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

(v) f(x) = [[x]] – [x – 2], where [.] represents the greatest integer function.

Solution: A function is discontinuous at all such points where it is undefined.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$

f(x) is discontinuous when 2 sinx – 1 = 0

i.e. $\sin x = \dfrac{1}{2} \Rightarrow x = 2n\pi + \dfrac{\pi}{6}, 2n\pi + \dfrac{5\pi}{6}, n \in I$

(ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

f(x) is discontinuous when $x^2 - 3|x| + 2 = 0$

$\Rightarrow \quad |x|^2 - 3|x| + 2 = 0$

$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$

$\Rightarrow \quad |x| = 1, 2$

$\Rightarrow \quad x = \pm 1, \pm 2$

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1} = \dfrac{1}{\left(x^2 + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}}$

Now, $x^4 + x^2 + 1 = \left(x^2 + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \geq 1 \; \forall \, x \in R$

$\Rightarrow$ f(x) is continuous $\forall \, x \in R$

(iv) $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

f(x) is discontinuous when x – 2 = 0 and also

when $1 - e^{\frac{x-1}{x-2}} = 0 \Rightarrow e^{\frac{x-1}{x-2}} = 1 \Rightarrow \dfrac{x-1}{x-2} = 0 \Rightarrow x = 1.$

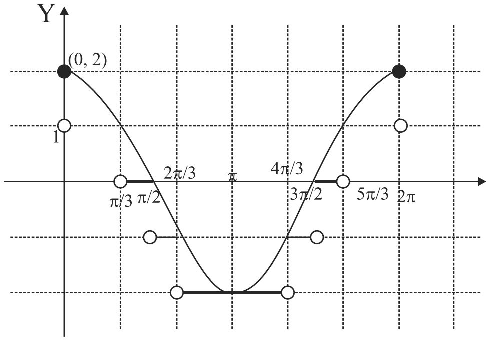
Thus f(x) is discontinuous at x = 1, 2.

(v) $f(x) = [[x]] - [x - 2] = [x] - ([x] - 2) = 2$

$\Rightarrow \quad$ f(x) is continuous $\forall \, x \in R$

Example 10: Draw the graph and find the points of discontinuity for f(x) = [2cos x], x ∈ [0, 2π], (where [.] represents the greatest integer function)

Solution:



The suspicious points are:

when $2\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$;

when $2\cos x = \pm 1 \Rightarrow x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$;

when $2\cos x = \pm 2 \Rightarrow x = 0, \pi, 2\pi$.

Clearly from the graph given above, f(x) is discontinuous at

$x = 0, \dfrac{\pi}{3}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, 2\pi.$

Example 11: Draw the graph and discuss the continuity of f(x) = [sinx + cosx], x ∈ [0, 2π], where [.] represents the greatest integer function.

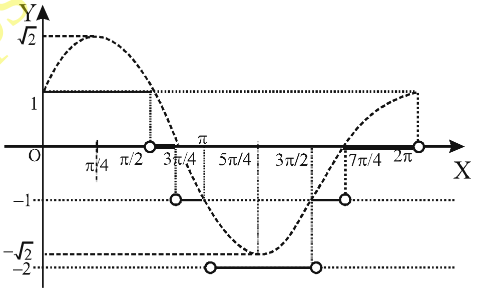
Solution: f(x) = [sinx + cosx]

Let $g(x) = \sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$

The range of g(x) is $[-\sqrt{2}, \sqrt{2}]$.

The suspicious points occur when g(x) = 0 , ± 1

$\Rightarrow \quad x = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, 2\pi$



Clearly from the graph given above f(x) is discontinuous at

$x = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, 2\pi.$

Example 12: Discuss the continuity of f(x) in [0,2] where

$$f(x) = \begin{cases} [\cos \pi x], & x \leq 1 \\ |2x - 3|[x - 2], & x > 1 \end{cases}$$

and [.] denotes the greatest integer function.

Solution: For x ∈ [0, 1], f(x) = [cos πx]

Since [x] is discontinuous at x ∈ I, we must check continuity of [cos πx] at points where cos π x is an integer. This happens at $x = 0, \dfrac{1}{2}, 1$.

When x ∈ [0, 1], the suspicious points are $x = 0, \dfrac{1}{2}, 1$.

Now for x ∈ (1, 2) , $\quad -1 < x - 2 < 0$

$\therefore \quad [x - 2] = -1$ Thus, $f(x) = -|2x - 3|$

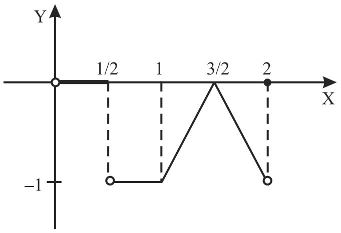
Since | x | is a continuous function, f(x) is continuous in (1, 2).

The right end point x = 2 is a suspicious point.

Hence, the suspicious points are $0, \dfrac{1}{2}, 1, \dfrac{3}{2}, 2$.

Now $$f(x) = \begin{cases} 1 & , \quad x = 0 \\ 0 & , \quad 0 < x \le \frac{1}{2} \quad (0 \le \cos \pi x < 1) \\ -1 & , \quad \frac{1}{2} < x \le 1 \quad (-1 \le \cos \pi x < 0) \\ 2x - 3 & , \quad 1 < x \le \frac{3}{2} \quad (|2x-3| = 3 - 2x) \\ 3 - 2x & , \quad \frac{3}{2} < x < 2 \quad (|2x-3| = 2x - 3) \\ 0 & , \quad x = 2 \quad ([x-2] = 0) \end{cases}$$

Graph of f(x)



It is clear from the graph that f(x) is discontinuous at x = 0, $\frac{1}{2}$, 2 and continuous at all other points in [0, 2].

Example 13: If $f(x) = \begin{cases} \text{sgn}(x-2) \times [\ln x], & 1 \le x \le 3 \\ \{x^2\} & 3 < x \le 3.5 \end{cases}$

where [.] denotes the greatest integer function and {.} represents the fractional part function, find the point where the continuity of f(x) should be checked. Hence, find the points of discontinuity.

Solution: Continuity should be checked at the end-points of intervals of each definition, i.e., x = 1, 3, 3.5.
$\{x^2\}$ is discontinuous for those values of x where x^2 is an integer.
Hence, continuity should be checked when x^2 = 10, 11, 12 or x = $\sqrt{10}, \sqrt{11}, \sqrt{12}$.
sgn(x–2) should be checked when x –2 = 0 or x = 2.
[lnx] should be checked when lnx = 1 or x = e.
Hence, continuity must be checked at x =1, 2, e, 3, $\sqrt{10}, \sqrt{11}, \sqrt{12}$, 3.5.
Now, f(1) = 0 and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \text{sgn}(x-2) \times [\ln x] = 0$$

Hence f(x) is continuous at x = 1.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \text{sgn}(x-2) \times [\ln x] = (-1) \times 0 = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \text{sgn}(x-2) \times [\ln x] = (1) \times 0 = 0$$

Hence, f(x) is continuous at x = 2.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \text{sgn}(x-2) \times [\ln x] = 1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \{x^2\} = 0$$

Hence, f(x) is discontinuous at x = 3.

Also f(x) is discontinuous at

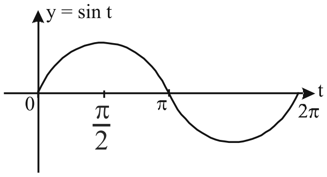
x = $\sqrt{10}, \sqrt{11}, \sqrt{12}$ because of $\{x^2\}$.

$$\lim_{x \to 3.5^-} f(x) = \lim_{x \to 3.5^-} \{x^2\} = 0.25 = f(3.5).$$

Hence, f(x) is discontinuous at x = 3, $\sqrt{10}, \sqrt{11}, \sqrt{12}$.

Example 14: Let f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π. Discuss the continuity of this function at x = $\frac{\pi}{2}$.

Solution: f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π



If $x \in \left[0, \frac{\pi}{2}\right]$, sin t is an increasing function

Hence if t ∈ [0, x], sin t will attain its maximum value at t = x.

∴ f(x) = sin x if $x \in \left[0, \frac{\pi}{2}\right]$

If $x \in \left(\frac{\pi}{2}, 2\pi\right]$ and t ∈ [0, x]

then sin t will attain its maximum value when t = $\frac{\pi}{2}$.

⇒ f(x) = sin $\frac{\pi}{2}$ = 1 if $x \in \left(\frac{\pi}{2}, 2\pi\right]$

$$\therefore \quad f(x) = \begin{cases} \sin x & , \quad \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 1 & , \quad \text{if } x \in \left(\frac{\pi}{2}, 2\pi\right] \end{cases}$$

Now, $f\left(\frac{\pi}{2}\right) = 1$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \sin x = 1$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} 1 = 1.$$

Since $f(\frac{\pi}{2})$ = L.H.L.= R.H.L., f(x) is continuous at x = $\frac{\pi}{2}$.

Example 15: If the function

$$f(x) = \frac{2 - (256 - 7x)^{1/8}}{(5x + 32)^{1/5} - 2}, \ x \ne 0 \text{ is continuous}$$

everywhere then find the value of f(0).

Solution: f(x) is continuous at all points except at the point where $(5x + 32)^{1/5} = 2$ i.e. x = 0.
For continuity at x = 0,

$$f(0) = \lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 + h)$$

[note that here h assumes both positive and negative values close to 0]

$$= -\lim_{h\to 0}\frac{(256-7h)^{1/8}-(256)^{1/8}}{(5h+32)^{1/5}-(32)^{1/5}}$$

$$= -\lim_{h\to 0}\frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}.(-7h)}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}.(5h)}$$

$$= \frac{7}{5}\lim_{h\to 0}\frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}}$$

$$= \frac{7}{5}.\frac{\frac{1}{8}.(256)^{1/8-1}}{\frac{1}{5}.(32)^{1/5-1}} \quad \left\{\because \lim_{x\to a}\frac{x^n-a^n}{x-a}-na^{n-1}\right\}$$

$$= \frac{7}{8}\cdot\frac{(2)^{-7}}{(2)^{-4}} = \frac{7}{8}\cdot\frac{1}{2^3} = \frac{7}{64}.$$

Hence, $f(0) = \frac{7}{64}$.

Example 16: If f(x) is a continuous function for all real values of x and satisfies $x^2 + (f(x) - 2)\,x + 2\sqrt{3} - 3 - \sqrt{3}\,.\,f(x) = 0,\ \forall\ x \in R$, then find the value of $f(\sqrt{3})$.

Solution: As f(x) is continuous for all $x \in R$.

$$\lim_{x\to\sqrt{3}} f(x) = f(\sqrt{3}) \text{ where}$$

$$f(x) = \frac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x},\ x \neq \sqrt{3}$$

$$\lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}}\frac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}$$

$$= \lim_{x\to\sqrt{3}}\frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)} = 2(1-\sqrt{3})$$

$\therefore\quad f(\sqrt{3}) = 2(1 - \sqrt{3})$.

Example 17: Let y = f(x) be defined parametrically as $y = t^2 + t\,|\,t\,|$, $x = 2t - |\,t\,|$, $t \in R$. Then examine the continuity of f(x) at $x = 0$.

Solution: $y = t^2 + t\,|\,t\,|$ and $x = 2t - |\,t\,|$

When $t \geq 0$,

$x = 2t - t = t$, $y = t^2 + t^2 = 2t^2$

$\Rightarrow\quad x = t$ and $y = 2t^2$

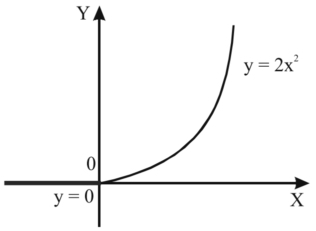
$\Rightarrow\quad y = 2x^2\ \forall\ x \geq 0$

When $t < 0$,

$\Rightarrow\quad x = 2t + t = 3t$ and $y = t^2 - t^2 = 0$.

$\Rightarrow\quad y = 0$ for all $x < 0$.

Hence, $f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ which is clearly continuous for all x as shown graphically below.



Example 18: Let $f(x) = x^3 - 3x^2 + 6\ \forall\ x \in R$, and

$$g(x) = \begin{cases} \max f(t), x+1 \leq t \leq x+2, & -3 \leq x < 0 \\ 1-x, \quad \text{for} & x \geq 0 \end{cases}$$

Test continuity of g (x) for $x \in [-3, 1]$.

Solution: Since $f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

For maxima and minima, $f'(x) = 0$

$\Rightarrow\quad x = 0, 2$

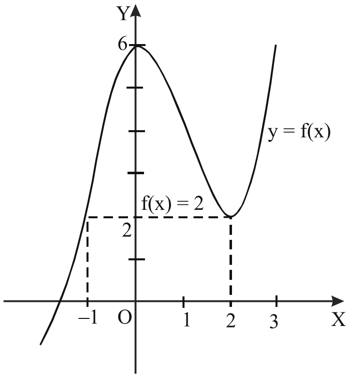
$f''(x) = 6x - 6$

$f''(0) = -6 < 0$ (local maxima at x = 0)

$f''(2) = 6 > 0$ (local minima at x = 2)

Also, $f(0) = 6$.

Now graph of f(x) is :



Clearly f(x) is increasing in $(-\infty, 0)$ and $(2, \infty)$ and decreasing in (0, 2).

Consider $x + 2 < 0 \Rightarrow x < -2$

For $-3 \leq x < -2$,

$-2 \leq x + 1 < -1$ and $-1 \leq x + 2 < 0$

Since f(x) increases, the maximum value of g(x) is f(x +2).

$\therefore\quad g(x) = f(x + 2)$ if $-3 \leq x < -2$.

Now consider $x + 1 < 0$ and $0 \leq x + 2 < 2$

For $-2 \le x < -1$, $g(x) = f(0)$.

Now for $x + 1 \ge 0$ and $x + 2 < 2$

i.e. $-1 \le x < 0$, $g(x) = f(x + 1)$.

$$\text{Hence, } g(x) = \begin{cases} f(x+2) & , \quad -3 \le x < -2 \\ f(0) & , \quad -2 \le x < -1 \\ f(x+1) & , \quad -1 \le x < 0 \\ 1 - x & , \quad x \ge 0 \end{cases}$$

Now, we can check that g(x) is continuous in the interval [–3, 1].

Example 19: Consider the function

$$f(x) = x\left[\frac{1}{(1+x)} + \frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \ldots.\infty\right] \text{ for } x \ne 0.$$

Find $f(0)$ if $f(x)$ is continuous at $x = 0$.

Solution: We have the sum upto n terms as

$$\frac{1}{1+x} + \frac{(1+2x)-(1+x)}{(1+x)(1+2x)} + \frac{(1+3x)-(1+2x)}{(1+2x)(1+3x)} + \ldots\ldots + \frac{(1+nx)-\left(1+\overline{n-1}\,x\right)}{\left(1+\overline{n-1}\,x\right)(1+nx)}$$

$$= \frac{2}{1+x} - \frac{1}{1+nx} \quad \text{upto } n \text{ terms when } x^{1}\ 0.$$

Now, $f(x) = \dfrac{2}{1+x}$ if $x \ne 0$ and $n \to \infty$.

Thus, for continuity of f at x = 0, f(0) should be equal to the limiting value of f which is 2.

Example 20: Examine the continuity of the function

$$f(x) = \lim_{n\to\infty} \frac{\sin x}{1 + (2\sin x)^{2n}} \text{ in } (0, \pi).$$

Solution: $f(x) = \lim\limits_{n\to\infty} \dfrac{\sin x}{1 + (2\sin x)^{2n}}$

$$= \begin{cases} \sin x, & 0 < x < \frac{\pi}{6} \text{ or } \frac{5\pi}{6} < x < \pi \\ \frac{1}{4}, & x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \\ 0, & \frac{\pi}{6} < x < \frac{5\pi}{6} \end{cases}$$

We can see from the simplified definition of f that it is discontinuous at $x = \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$.

Example 21: Given $f(x) = \sum\limits_{r=1}^{n}(x^r + x^{-r})^2$, $x \ne \pm 1$ and

$$g(x) = \begin{cases} \lim\limits_{n\to\infty}\left(\left(f(x)-2n\right)x^{-2n-2}(1-x^2)\right) & \text{for } x \ne \pm 1 \\ -1, \quad \text{for } x = \pm 1 \end{cases}$$

show that g(x) is continuous for all x in the domain.

Solution: $f(x) = \sum\limits_{r=1}^{n}\left(x^{2r} + \dfrac{1}{x^{2r}} + 2\right)$

$$= \sum_{r=1}^{n} x^{2r} + \sum_{r=1}^{n}\frac{1}{x^{2r}} + 2n$$

$$= (x^2+x^4+\ldots+x^{2n}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \ldots.. + \frac{1}{x^{2n}}\right) + 2n$$

$$= \frac{x^2(1-x^{2n})}{(1-x^2)} + \frac{1}{x^2}\,\frac{\left(1-\frac{1}{x^{2n}}\right)}{1-\frac{1}{x^2}} + 2n$$

$$\text{Thus, } f(x) = \frac{x^2(1-x^{2n})}{1-x^2} + \frac{\left(1-x^{2n}\right)}{(1-x^2)x^{2n}} + 2n$$

$$f(x) = \frac{(1-x^{2n})}{1-x^2}\left(x^2 + \frac{1}{x^{2n}}\right) + 2n, \quad x \ne \pm 1$$

$$\therefore \quad (f(x) - 2n)(1 - x^2) = (1-x^{2n})\left(x^2 + \frac{1}{x^{2n}}\right)$$

Now consider

$$g(x) = \lim_{n\to\infty}\left(\left(f(x)-2n\right)x^{-2n-2}(1-x^2)\right) \text{ for } x \ne \pm 1$$

$$= \lim_{n\to\infty}(1-x^{2n})\left(x^2 + \frac{1}{x^{2n}}\right)\cdot\frac{1}{x^{2n+2}}, \quad x \ne \pm 1$$

$$= \lim_{n\to\infty}\frac{(1-x^{2n})(x^{2n+2}+1)}{(x^{2n})(x^{2n+2})}$$

$$= \lim_{n\to\infty}\left(-1-\frac{1}{x^{2n}}\right)\left(1+\frac{1}{x^{2n+2}}\right)$$

$$\text{Now } g(x) = \begin{cases} -1 & \text{if } |x| > 1 \\ \text{undefined} & \text{if } |x| < 1 \\ -1 & \text{if } x = \pm 1 \end{cases}$$

The domain of g(x)is $x \in (-\infty, -1] \cup [1, \infty)$.

Now clearly g is continuous for all x in the domain.

Example 22: Let $f : R \to R$ be a function which satisfies $f(x + y^3) = f(x) . (f(y))^3 \ \forall \ x, y \in R$. If f is continuous at x = 0, prove that f is continuous everywhere.

Solution: To prove $\lim\limits_{h\to 0} f(x + h) = f(x)$.

Put x = y = 0 in the given relation

$$f(0) = f(0) + (f(0))^3 \Rightarrow f(0) = 0$$

Since f is continuous at x = 0, $\lim\limits_{h\to 0} f(h) = f(0) = 0$.

Now, $\lim\limits_{h\to 0} f(x + h) = \lim\limits_{h\to 0} f(x) + (f(h))^3$

$$= f(x) + \lim_{h\to 0} (f(h))^3 = f(x) + 0 = f(x).$$

Hence f is continuous for all $x \in R$.

Example 23: If $f(x + y) = f(x) . f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x) . G(x)$ where $\lim\limits_{x\to 0} g(x) = 0$ and $\lim\limits_{x\to 0} G(x)$ exists, prove that f(x) is continuous at all $x \in R$.

Solution: $\lim\limits_{x\to 0} g(x) = 0 \quad \Rightarrow \quad \lim\limits_{h\to 0} g(h) = 0 \quad ...(1)$

$\lim\limits_{x\to 0} G(x)$ exists $\Rightarrow \lim\limits_{h\to 0} G(0 + h) =$ finite ...(2)

Now, $\lim\limits_{h\to 0} f(x + h) = \lim\limits_{h\to 0} f(x) . f(h) = f(x) \lim\limits_{h\to 0} f(h)$

$[\because f(x + y) = f(x) . f(y)]$

$= f(x) . \lim\limits_{h\to 0} \{1 + g(h) G(h)\}$,

[using the given relation]

$= f(x) . \{1 + \lim\limits_{h\to 0} g(h) . \lim\limits_{h\to 0} G(h)\}$

$= f(x) . \{1 + 0 .$ finite$\}$, using (1) and (2)

$= f(x)$

$\therefore \quad \lim\limits_{h\to 0} f(x + h) = f(x)$.

$\therefore \quad$ f(x) is continuous everywhere.

Single Point Continuity

There are some functions which are continuous only at one point in an interval, though they are defined everywhere in the interval.

For example, $f(x) = \begin{cases} x & \text{if} \quad x \in Q \\ -x & \text{if} \quad x \notin Q \end{cases}$ is continuous only at x = 0. The limit of the function does not exist anywhere except at the point x = 0.

$f(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$ is continuous only at x = 0.

$f(x) = \begin{cases} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{cases}$ is continuous only at x = 1/2.

$f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$ is continuous only at x = 1 or – 1.

Example 24: Find the values of a and b if the function

$f(x) = x^2 + ax + 1$, if x is rational

$= ax^2 + bx + 1$, if x is irrational

is continuous at x = 1 and 2.

Solution: Continuity at x = 1 implies

$$a + 2 = a + b + 1$$

which gives b = 1.

Continuity at x = 2 implies

$$2a + 5 = 4a + 2b + 1$$

which gives a = 1.

Note: Point functions are treated as continuous. If the domain of the function contains a countable number of points then it is continuous at all of these points.

For example, $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is continuous at x = 1. Here the function is defined only at x = 1.

Also, $f(x) = \dfrac{1}{\{x\} + \{-x\} - 1}$ is defined only for integers. Hence, it is continuous at all points in the domain.

Concept Problems B

1. At what points are the tangent and cotangent functions continuous?
2. Where are the following functions discontinuous?

   (i) sec x (ii) $\dfrac{1 + x^2}{1 - x^2}$

3. Prove that the function

   $f(x) = \begin{cases} x + 1, & -1 \le x \le 0, \\ -x, & 0 < x \le 1 \end{cases}$ is discontinuous.

4. For what value of 'k' is the function,

   $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & x \ne 0 \\ k, & x = 0 \end{cases}$ continuous ?

5. Is the function

   $f(x) = \begin{cases} 2x + 1 & \text{if} \quad -3 < x < -2 \\ x - 1 & \text{if} \quad -2 \le x < 0 \\ x + 2 & \text{if} \quad 0 \le x < 1 \end{cases}$

   continuous everywhere in (–3, 1) ?

6. If b and c are given, find all values of a for which

$$f(x) = \begin{cases} 2\cos x & \text{if } x \le c, \\ ax^2 + b & \text{if } x > c \end{cases} \quad \text{is continuous.}$$

7. Find constants a and b so that the given function will be continuous for all x in the domain.

(i) $$f(x) = \begin{cases} \dfrac{\tan ax}{bx} & \text{if } x < 0 \\ 4 & \text{if } x = 0 \\ ax + b & \text{if } x > 0 \end{cases}$$

(ii) $$f(x) = \begin{cases} 2\sin(a\cos^{-1} x) & \text{if } -1 \le x < 0 \\ \sqrt{3} & \text{if } x = 0 \\ ax + b & \text{if } x > 0 \end{cases}$$

8. For what (if any) values of x are the following functions discontinous:

(i) $y = [x^2]$, (ii) $y = [\sqrt{x}\,]$,

(iii) $y = \sqrt{(x - [x])}$, (iv) $y = [2x]$,

(v) $y = [x] + [-x]$?

9. (i) For $x \neq 0$, let $f(x) = [1/x]$, where [.] denotes the greatest integer function. Sketch the graph of f over the intervals $[-2, -\frac{1}{5}]$ and $[\frac{1}{5}, 2]$.

What happens to f(x) as $x \to 0$ through positive values? and through negative values?

Can you define f(0) so that f becomes continuous at 0?

(ii) Answer the same when $f(x) = (-1)^{[1/x]}$ for $x \neq 0$.

(iii) Answer the same when $f(x) = x(-1)^{[1/x]}$ for $x \neq 0$.

10. If f(x) is continuous on $(-\infty, \infty)$, then it has no vertical asymptotes. True or false.

11. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither ?

(i) $$f(x) = \begin{cases} 1 + x^2 & \text{if } x \le 0 \\ 2 - x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

(ii) $$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

12. Choose parameters, entering into the definition of the function such that the function f(x) becomes continuous:

(i) $$f(x) = \begin{cases} 2^{1/(x-1)}, & x < 1 \\ ax^2 + bx + 1, & x \ge 1 \end{cases}$$

(ii) $$f(x) = \begin{cases} x^2 + x + 1, & x \ge -1 \\ \sin \pi(x + a)), & x < -1. \end{cases}$$

13. Locate the discontinuities of the following functions:

(i) $y = \dfrac{1}{1 + e^{1/x}}$ (ii) $y = \ln(\tan^2 x)$

14. Let $f(x) = 2^{-1/x^2}$ for $x \neq 0$.

(i) Find $\lim\limits_{x \to \infty} f(x)$ (ii) Find $\lim\limits_{x \to 0} f(x)$

(iii) Is it possible to define f(0) in such a way that f is continuous for all real x ?

15. Determine the set of all points where the function $f(x) = \dfrac{x^3}{4 + |x|}$ is continuous.

16. Examine the continuity of the function over $x \in R$.

(i) $f(x) = \tan \dfrac{1}{x}$ (ii) $f(x) = \dfrac{x - 1}{x(x+1)(x^2 - 4)}$

(iii) $f(x) = [x] + [-x]$.

Practice Problems B

17. Let $$f(x) = \begin{cases} \dfrac{1 - \cos(4x)}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{(16 + \sqrt{x})^{1/2} - 4}, & x > 0 \end{cases}$$

then find the value of 'a' for which f(x) is continuous at x = 0.

18. If $f(x) = \text{sgn}\left(\left\{x - \dfrac{1}{2}\right\}\right)[\ln x], 1 < x \le 3$

$= \{x^2\}, \quad 3 < x \le 3.5$

Find the point where the continuity of f(x) should be checked.

19. The function f(x) is defined as

$$f(x) = \begin{cases} x^2 \cos e^{1/x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Show that f(x) is discontinuous at x = 0.

20. If

$$f(x) = \begin{cases} x & \text{for rational values of } x \text{ in } [0, 1] \\ 1 - 2x & \text{for irrational values of } x \text{ in } [0, 1] \end{cases}$$

Show that f(x) is continuous only at $x = 1/3$.

21. Find the points of continuity of $f(x) = \sqrt{\dfrac{x^2 - 2x}{x - 1}}$.

22. If the function f given by

$$f(x) = \begin{cases} \dfrac{\tan 2ax}{3bx} & \text{if } x < 0 \\ c & \text{if } x = 0 \\ \dfrac{|x|}{x} + \dfrac{x^3 - 1}{x - 1} & \text{if } 0 < x < 1 \\ b & \text{if } x \ge 1 \end{cases}$$

is everywhere continuous, find a, b, and c.

23. Let $f : [0, \infty) \to [0, \infty)$,

$f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots..}}}$ Is f right continuous at 0?

24. What are the points of discontinuity of the function

$$f(x) = \begin{cases} \cos x & \text{if } x \in Q \\ \sin x & \text{if } x \notin Q \end{cases} ?$$

25. Let f be the "nearest integer, with rounding down" function. That is,

$$f(x) = \begin{cases} \text{the integer nearest to } x \text{ if } x \text{ is not midway between two consecutive integers} \\ x - \frac{1}{2} \text{ if } x \text{ is midway between two consecutive integers} \end{cases}$$

(i) Does $\lim\limits_{x \to 3.5^-} f(x)$ exist ? If so, evaluate it.

(ii) Does $\lim\limits_{x \to 3.5} f(x)$ exist ? If so, evaluate it.

(iii) Is f continuous at 3.5 ?

(iv) Where is f not continuous ?

26. Test the continuity of the function

$$y = \lim_{n \to \infty} \frac{x}{1 + (2\cos x)^{2n}}.$$

27. If $f(x) = x + \{-x\} + [x]$,where [·] denotes the greatest integer function. and {.} denotes the fractional part function. Discuss the continuity of f in [–2 , 2] .

28. Discuss the continuity of the function $f(x) = [[x]] - [x - 1]$, where [.] denotes the greatest integer function.

29. Find all possible values of a and b so that f (x) is continuous for all $x \in R$ if

$$f(x) = \begin{cases} |ax + 3| \ , & \text{if } x \le -1 \\ |3x + a| \ , & \text{if } -1 < x \le 0 \\ \dfrac{b \sin 2x}{x} - 2b, & \text{if } 0 < x < \pi \\ \cos^2 x - 3 \ , & \text{if } x \ge \pi \end{cases}.$$

30. Examine the continuity at x= 0 of the function

$$f(x) = \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots.\infty.$$

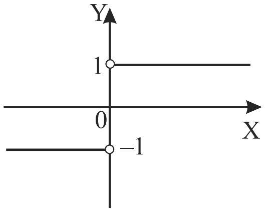
2.3 Classification of Discontinuity

Discontinuity of First and Second Kind

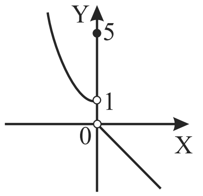
Let a function f be defined in the neighbourhood of a point a, except perhaps at a itself. Also let both the one-sided limits $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist, but the conditions of continuity are not satisfied. Then the function f(x) is said to have a **discontinuity of the first kind** at the point a.

For instance, the function $f(x) = \begin{cases} \dfrac{x}{1 + 2^{\frac{1}{x}}}, & x \ne 0 \\ 1, & x = 0 \end{cases}$ has a discontinuity of the first kind at x = 0, since both the one-sided limits are 0 but this is not equal to f(0) which is 1.

The function $f(x) = \dfrac{x}{|x|}$ has a discontinuity of the first kind at x = 0 since the left hand limit is $\lim\limits_{x \to 0^-} f(x) = -1$, while the right hand limit is $\lim\limits_{x \to 0^+} f(x) = 1$ i.e. the one sided limits exists.



The function $y = \begin{cases} x^2 + 1 & \text{for} & x < 0, \\ 5 & \text{for} & x = 0, \\ -x & \text{for} & x > 0, \end{cases}$



has a discontinuity of the first kind at x = 0 because the one-sided limits exist.

A function f(x) having a finite number of discontinuities of first kind in a given interval is called sectionally or **piecewise continuous** function.

Sometimes, a function f is discontinuous at x = a because $\lim_{x\to a} f(x)$ does not exist as one or both of the one-sided limits do not exist.

The function f(x) is said to have **discontinuity of the second kind** at x = a, if atleast one of the one-sided limits (L.H.L. or R.H.L.) at the point x = a does not exist or equals to infinity.

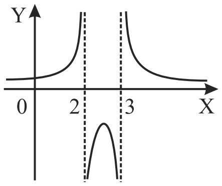
Let us show that for the function $f(x) = \frac{1}{3-x}$, the point x = 3 is a point of discontinuity of the second kind.

Consider the limits on the left and on the right at the point x = 3:

$$\lim_{x\to 3^-} \frac{1}{3-x} = \infty, \quad \lim_{x\to 3^+} \frac{1}{3-x} = -\infty$$

Thus, the one-sided limits of the function f(x) at the point x = 3 are infinite, and according to the definition, this means that the function f(x) has a point of discontinuity of the second kind at this point.

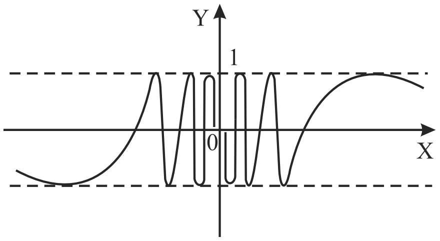
The function $y = \frac{1}{(x-2)(x-3)}$ has no one-sided limits at x = 2 and x = 3. Therefore x = 2 and x = 3 are discontinuities of the second kind. We can see the graph of the function below.



The function y = ln |x| at the point x = 0 has the limit $\lim_{x\to 0} \ln|x| = -\infty$. Consequently, $\lim_{x\to 0} f(x)$ (and also the one-sided limits) does not exist. Hence, x = 0 is a discontinuity of the second kind.

It is not true that discontinuities of the second kind only arise when $\lim_{x\to a} f(x) = \infty$. The situation can be different. The function y = sin (1/x), does not have the one-sided limits as $x \to 0$ since the values of the function sin (1/x) do not approach a certain number, but oscillate an infinite number of times within the interval from –1 to 1 as $x \to 0$.

It has a discontinuity of the second kind at x = 0. The graph is shown below.



Removable Discontinuity

A function f is said to have a removable discontinuity at x = a, if $\lim_{x\to a} f(x)$ exists but is not equal to f(a). In this case we can redefine the function such that $\lim_{x\to a} f(x) = f(a)$ and make it continuous at x = a.

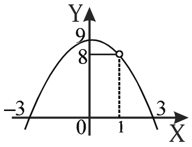
For example, f(x) = [x] + [–x] has a removable discontinuity at x = 1 since $\lim_{x\to 1} f(x) = -1$, but it is not equal to f(1) which is 0.

Removable discontinuity can be further classified as:

(i) Missing Point Discontinuity

A function f is said to have a missing point discontinuity at x = a if $\lim_{x\to a} f(x)$ exists and the function is undefined at x = a.

For example, $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 since $\lim_{x\to 1} f(x) = 8$ but the function is undefined at x = 1.

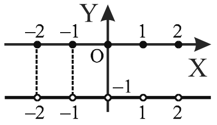


(ii) Isolated Point Discontinuity

A function f is said to have an isolated point discontinuity at x = a if $\lim_{x\to a} f(x)$ exists and the function is defined at x = a, but they are unequal.

For example, $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \in R-\{2\} \\ 1, & x = 2 \end{cases}$ has an isolated point discontinuity at x = 2 since $\lim_{x\to 2} f(x)$ exists and is equal to 4, but it does not match with the function's value at x = 2, which is 1.

Also the function $f(x) = [x]+[-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has isolated point discontinuities at all integral x as shown in the figure

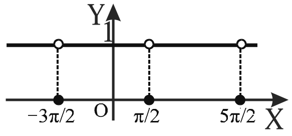


The function f(x) = sgn(cos2x–2sinx+3)

= sgn(2(2 + sin x)(1 – sinx))

$$= \begin{cases} 0 & \text{if } x = 2n\pi + \dfrac{\pi}{2} \\ 1 & \text{if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$

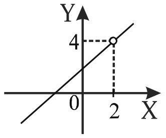
has an isolated point discontinuity as $x = 2n\pi + \dfrac{\pi}{2}$.



Continuous Extension of a Function

A function F is called a continuous extension of a given function f, at a removable discontinuity a of f, if the values of F and f agree at every point of the domain of f and the value of F at a is the limit of f at a.

For example, let $f(x) = (x^4 - 4)/(x - 2)$, $x \in R - \{2\}$. Then x = 2 is a missing point discontinuity.



But since $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$,

we see that x = 2 is a removable discontinuity.

Here F is defined as

$$F(x) = \begin{cases} \dfrac{x^2-4}{x-2} & , x \in R - \{2\} \\ 4 & , x = 2 \end{cases}$$

It is the continuous extension of f at x = 2.

The function $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & , x \in R - \{3\} \\ 4 & , x = 2 \end{cases}$ is defined for all $x \in R$. At x = 3 it has the following left hand and right hand limits:

$$\lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3-0}(x+3) = 6,$$

$$\lim_{x\to 3^+} \frac{x^2-9}{x-3} = \lim_{x\to 3-0}(x+3) = 6,$$

that is, the one-sided limits are equal. Hence, $\lim\limits_{x\to 3} f(x) = 6$.

However, the point x=3 is a discontinuity in the function since the limit is not equal to the function's value.

Let us see how we can make the function continuous at x = 3. Consider the function F(x) defined as

$$F(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

The values of F(x) coincide with those of the function $f(x) = (x^2 - 9) / (x - 3)$ everywhere except at x = 3, where the "old" function was defined as 4, while at x = 3 the "new" function has a value of 6.

The function F(x) fulfills the equality

$$\lim_{x\to 3} F(x) = F(3) = 6,$$

which means that F(x) is continuous at x = 3. Thus, the discontinuity of f(x) has been removed.

Note that we were able to get a continuous function F(x) from the discontinuous function f(x) because the one-sided limits were equal.

Irremovable Discontinuity

A function f is said to have an irremovable discontinuity at x = a, if $\lim\limits_{x\to a} f(x)$ does not exist. In this case we cannot redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ and make it continuous at x = a.

Irremovable discontinuities can be further classified as :

(i) Finite Discontinuity

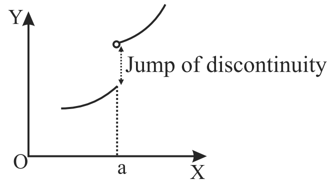
A function f is said to have a finite or jump discontinuity at x = a if $\lim\limits_{x\to a} f(x)$ does not exist since the left hand limit and the right hand limit are unequal, but the one-sided limits do exist.

For instance, f(x) = [x] has a finite or jump discontinuity at all integral x.

Jump of a Function at a Point

If x = a is a point of finite discontinuity of the function f(x), then the graph of this function undergoes a jump at x = a.

The difference R.H.L. – L.H.L. i.e. $f(a^+) - f(a^-)$ is called the jump in the function at x = a.



The graph of the function f(x) = sgn(x) makes a jump of 2 units at the point x = 0 since

$$f(0^+) - f(0^-) = 1 - (-1) = 2.$$

The difference between the greatest and least of the three numbers $f(a^+)$, $f(a^-)$, $f(a)$ is the **saltus** or measure of discontinuity of the function at the point a.

(i) $$\lim_{x\to 0}\tan^{-1}\left(\frac{1}{x}\right)\left<\begin{array}{l} f(0^+) = \frac{\pi}{2} \\ \\ f(0^-) = -\frac{\pi}{2}\end{array}\right. ;\ \text{jump} = \pi$$

(ii) $$\lim_{x\to\pi}\frac{|\sin x|}{x-\pi}\left<\begin{array}{l} f(\pi^+) = -1 \\ \\ f(\pi^-) = 0\end{array}\right. ;\ \text{jump} = -1$$

(iii) $$\lim_{x\to 2}\frac{[x]}{x}\left<\begin{array}{l} f(2^+) = 1 \\ \\ f(2^-) = \frac{1}{2}\end{array}\right. ;\ \text{jump} = \frac{1}{2}$$

(iv) $f(x) = \dfrac{1}{1 - e^{-1/x}}$ and $f(0) = e$.

$$f(0^+) = \lim_{h\to 0}\frac{1}{1 - e^{-1/(0+h)}} = \lim_{h\to 0}\frac{1}{1 - e^{-1/h}}$$

$$= \frac{1}{1 - e^{-\infty}} = \frac{1}{1-0} = 1$$

and $$f(0^-) = \lim_{h\to 0}\frac{1}{1 - e^{-1/(h-h)}} = \lim_{h\to 0}\frac{1}{1 - e^{1/h}}$$

$$= \frac{1}{1 - e^{\infty}} = \frac{1}{1-\infty} = 0.$$

Hence f is discontinuous at x = 0, the discontinuity being of the first kind and irremovable. This function has a jump of one unit at 0 since $f(0^+) - f(0^-) = 1$. The saltus is e.

(ii) Infinite Discontinuity

A function is said to have an infinite discontinuity at x = a if atleast one of the one-sided limits is infinite. For instance, the following functions have infinite discontinuity:

(i) $$f(x) = \frac{x}{1-x} \text{ at } x = 1\left<\begin{array}{l} f(1^+) = -\infty \\ \\ f(1^-) = \infty\end{array}\right.$$

(ii) $$f(x) = 2^{\tan x} \text{ at } x = \frac{\pi}{2}\left<\begin{array}{l} f\left(\frac{\pi^+}{2}\right) = 0 \\ \\ f\left(\frac{\pi^-}{2}\right) = \infty\end{array}\right.$$

(iii) $$f(x) = \frac{1}{x^2} \text{ at } x = 0\left<\begin{array}{l} f(0^+) = \infty \\ \\ f(0^-) = \infty\end{array}\right.$$

(iv) $$f(x) = \begin{cases} x^2, & x \le 0 \\ \ln x, & x > 0\end{cases}$$

has an infinite discontinuity at x = 0 since the right hand limit is infinite (note that the left hand limit is 0.

Pole Discontinuities

The concept of pole discontinuity is related with infinite limit. For a point x = a to qualify as a pole of a function f, we must have $\lim_{x\to a}\dfrac{1}{f(x)} = 0$.

What this means roughly is that "f(x) becomes numerically big and stays big as x gets close to a".

The function $y = \dfrac{1}{(x-2)(x-3)}$ has pole at x = 2 and x = 3.

Note that $f(x) = \begin{cases} x^2, & x \le 0 \\ \ln x, & x > 0\end{cases}$

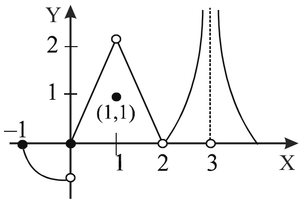
does not have a pole discontinuity at x = 0, even if it has an infinite discontinuity at x = 0 since the left hand limit is finite.

Again, the reciprocal of a polynomial of degree n has atmost n poles : in fact, the real zeros of a polynomial are the poles of its reciprocal.

In particular, the reciprocal of the quadratic polynomial with negative discriminant has no poles. The poles of a rational function in "lowest terms" are the zeros of the denominator.

The zeros of sin x and cos x are respectively the poles of cosec x and sec x.

Consider the following graph to understand the nature of discontinuity.



From the above graph note that

(i) f is continuous at $x = -1$

(ii) f has non removable (finite type) discontinuity at $x=0$

(iii) f has isolated discontinuity at $x = 1$

(iv) f has missing point discontinuity at $x = 2$

(v) at $x = 3$ non removable infinite type discontinuity.

(iii) Oscillatory Discontinuity

A function is said to have an oscillatory discontinuity at $x = a$ if atleast one of the one-sided limits does not exist because of too much oscillation in the values of the function.

For example, $f(x) = \sin\frac{1}{x}$ has an oscillatory discontinuity at $x = 0$.

The function $f(x) = \left[1+\frac{1}{3}\sin(\ln|x|)\right]$ oscillates between 0 and 1 at $x = 0$ and hence has a oscillatory discontinuity at $x = 0$.

Note: In all the cases of irremovable discontinuity the value of the function at $x = a$ (point of discontinuity) i.e. $f(a)$ may or maynot be defined.

Example 1: If $f(x) = \begin{cases} x, & x<1 \\ x^2, & x>1 \end{cases}$

then find the type of discontinuity at $x = 1$.

Solution: $f(x) = \begin{cases} x, & x<1 \\ x^2, & x>1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1$

$$\therefore \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$$

but $f(1)$ is not defined.

So, $f(x)$ has a missing point removable discontinuity at $x = 1$. The discontinuity is of first kind.

Example 2: Let $f(x) = \cos^{-1}\{\cot x\}, \ x < \frac{\pi}{2}$

$$= \pi[x]-1, \qquad x \ge \frac{\pi}{2}$$

Find jump of discontinuity at $x = \frac{\pi}{2}$.

Solution: $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & \text{if} \quad x < \dfrac{\pi}{2} \\ \pi[x]-1 & \text{if} \quad x \ge \dfrac{\pi}{2} \end{cases}$

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} \cos^{-1}\{\cot x\}$$

$$= \cos^{-1} 0 = \frac{\pi}{2}.$$

$$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} \pi[x]-1 = \pi - 1.$$

$\therefore$ The jump of discontinuity = R.H.L. – L.H.L.

$$= \pi - 1 - \frac{\pi}{2} = \frac{\pi}{2} - 1.$$

Example 3: Let $f(x) = \dfrac{2\cos x - \sin 2x}{(\pi-2x)^2}$,

$g(x) = \dfrac{e^{-\cos x}-1}{8x-4\pi}$, and $h(x) = \begin{cases} f(x), & x < \pi/2 \\ g(x), & x > \pi/2 \end{cases}$

then show that h has an irremovable discontinuity at $x = \pi/2$.

Solution: $h(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi-2x)^2}, & x < \dfrac{\pi}{2} \\ \dfrac{e^{-\cos x}-1}{8x-4\pi}, & x > \dfrac{\pi}{2} \end{cases}$

L.H.L. at $x = \pi/2$

$$= \lim_{h\to 0} \frac{2\sin h - \sin 2h}{4h^2}$$

$$= \lim_{h\to 0} \frac{2\sin h(1-\cos h)}{4h^2} = 0.$$

$$\text{R.H.L.} = \lim_{h\to 0} \frac{e^{\sin h}-1}{\theta\left((\pi/2)+h\right)-4\pi}$$

$$= \lim_{h\to 0} \frac{e^{\sin h}-1}{8h}\cdot\frac{\sin h}{\sin h} = \frac{1}{8}.$$

Thus, $h\left(\dfrac{\pi}{2}^+\right) = 0$ and $h\left(\dfrac{\pi}{2}^-\right) = \dfrac{1}{8}$

We have $h\left(\dfrac{\pi}{2}^+\right) \ne h\left(\dfrac{\pi}{2}^-\right)$.

Since L.H.L. ≠ R.H.L., $h(x)$ has an irremovable discontinuity at $x = \pi/2$.

Example 4: State the number of point of discontinuity and discuss the nature of discontinuity

for the function $f(x) = \dfrac{1}{\ln|x|}$ and also sketch its graph.

Solution: $f(x) = \begin{cases} \dfrac{1}{\ln x} & \text{if } x>0,\ x\ne 1 \\ \dfrac{1}{\ln(-x)} & \text{if } x<0,\ x\ne -1 \end{cases}$

The function is obviously discontinuous at $x = 0, 1, -1$. as it is not defined.

$$\left.\begin{aligned} \lim_{x\to 0^+} f(x) = 0 \\ \lim_{x\to 0^-} f(x) = 0 \end{aligned}\right\} \text{Limit exists at } x = 0.$$

Hence there is a removable discontinuity (missing point) at x = 0.

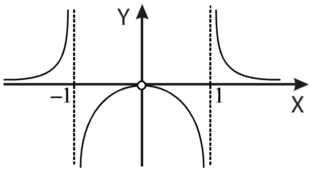
$$\left.\begin{array}{l}\lim\limits_{x\to1^+} f(x)=\infty\\ \lim\limits_{x\to1^-} f(x)=-\infty\end{array}\right\}\text{Limit dne.}$$

Hence there is a nonremovable discontinuity (infinite type) at x = 1.

$$\left.\begin{array}{l}\lim\limits_{x\to-1^+} f(x)=-\infty\\ \lim\limits_{x\to-1^-} f(x)=\infty\end{array}\right\}\text{Limit dne.}$$

Hence there is a non removable discontinuity (infinite type) at x = –1.

Note that f (x) is even ⇒ the graph is symmetric about y axis. The graph of f (x) is as follows.



Example 5: Let $f(x) = \lim\limits_{n\to\infty}\frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0.

Solution: Let $f(x) = \lim\limits_{n\to\infty}\frac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right)=\lim_{n\to\infty}\frac{1}{1+n\,.\,\sin^2\frac{\pi}{4}}$$

$$=\lim_{n\to\infty}\frac{1}{1+n\left(\frac{1}{2}\right)}=0$$

Now,

$$f(0)=\lim_{n\to\infty}\frac{1}{n\,.\,\sin^2(0)+1}=\frac{1}{1+0}=1$$

$$\lim_{x\to0}f(x)=\lim_{x\to0}\left[\lim_{n\to\infty}\frac{1}{1+n\sin^2 x}\right]\{\text{form}\left[\frac{1}{1+\infty}\right]\}=0$$

Here $\sin^2x$ is a very small quantity but not zero.

∴ Thus f(x) is discontinuous at x = 0. The type of discontinuity is isolated point removable discontinuity.

Example 6: Examine the function $f(x)=\lim\limits_{n\to\infty}\frac{x}{1+(4\sin^2 x)^n}$ for continuity in (0, π). Plot its graph and state the nature of discontinuity.

Solution: For $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, $4\sin^2x = 1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+1^n}=\frac{x}{2}\,.$$

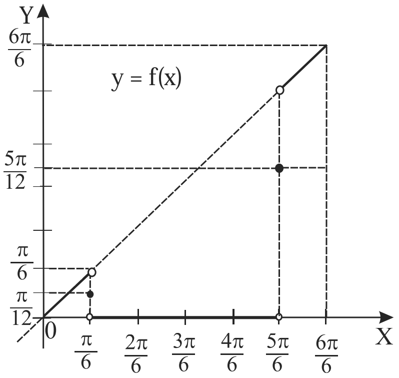
For $\frac{\pi}{6} < x < \frac{5\pi}{6}$, $4\sin^2x > 1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+(\text{greater than }1)^n}=0\,.$$

For $0\le x<\frac{\pi}{6}$ or $\frac{5\pi}{6}<x<\pi$, $4\sin^2x < 1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+(\text{less than }1)^n}=x\,.$$

$$\text{Hence } f(x)=\begin{cases}x & \text{for } 0\le x<\frac{\pi}{6}\text{ or }\frac{5\pi}{6}<x\le\pi\\[4pt] \frac{x}{2} & \text{for } x=\frac{\pi}{6}\text{ or }\frac{5\pi}{6}\\[4pt] 0 & \text{for } \frac{\pi}{6}<x<\frac{5\pi}{6}\end{cases}$$



The graph of f(x) is as shown above.

From the graph it is clear that f (x) is continuous everywhere in (0,π) except at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ and where it has removable discontinuity of finite type.

Example 7: Consider f(x)

$$=\frac{1}{\sqrt{b-a}}\,\frac{\sqrt{\frac{b-a}{a}}\sin 2x}{\sqrt{1+\left(\sqrt{\frac{b-a}{a}}\sin x\right)^2}}\sqrt{a+b\tan^2 x}\,,$$

for b > a > 0 and g (x) = sgn (f(x)). Find whether g (x) is continuous at x = 0 or not and state the nature of discontinuity, if discontinuous.

Solution: $f(x)=\frac{1}{\sqrt{b-a}}\cdot\frac{\sqrt{b-a}\cdot\sin 2x\cdot\sqrt{a+b\tan^2 x}}{\sqrt{a}\cdot\sqrt{\frac{a+(b-a)\sin^2 x}{a}}}$

$$= \frac{\sin 2x \cdot \sqrt{a + b\tan^2 x}}{\sqrt{a\cos^2 x + b\sin^2 x}}$$

$$= \frac{\sin 2x \cdot \sqrt{a\cos^2 x + b\sin^2 x}}{|\cos x|\sqrt{a\cos^2 x + b\sin^2 x}}$$

Hence, $f(x) = \dfrac{\sin 2x}{|\cos x|}$

Now, $g(x) = \text{sgn}\left(\dfrac{\sin 2x}{|\cos x|}\right)$

We have $g(0) = 0$

$$g(0^+) = \lim_{h\to 0}\left(\frac{\sin 2h}{\cos h}\right) = \lim_{h\to 0}\text{sgn}\left(2\sin h\right) = 1.$$

$$g(0^-) = \lim_{h\to 0}\text{sgn}\left(\frac{-2\sin 2h}{\cos h}\right) = \lim_{h\to 0}\text{sgn}\left(-2\sin h\right) = -1$$

Hence, $g(x) = \text{sgn}(f(x))$ is discontinuous at $x = 0$. The nature of discontinuity is irremovable discontinuity of finite type.

Example 8: Let

$$f(x) = \begin{cases} \dfrac{e^x - 1}{\sqrt{(1+x^2)} - \sqrt{1-x^2}} & \text{if } x \neq 0 \\ \sqrt{\dfrac{2}{3}} & \text{if } x = 0 \end{cases},$$

then find whether $f(x)$ is continuous at $x = 0$ or not and state the nature of discontinuity, if discontinuous.

Solution:

$$\text{Limit} = \lim_{x\to 0}\frac{\left(\dfrac{e^x - 1}{x}\right)\cdot x\cdot\sqrt{(1+x^2)} + \sqrt{1-x^2}}{\sqrt{(1+x^2)} - \sqrt{1-x^2}\cdot\sqrt{(1+x^2)} + \sqrt{1-x^2}}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}\cdot x}{\sqrt{(1+x^2)^2 - (1-x^2)}} = \lim_{x\to 0}\frac{\sqrt{2}\,x}{\sqrt{x^4 + 3x^2}}$$

Now R.H.L. $= \dfrac{\sqrt{2}}{\sqrt{3}}$ and L.H.L. $= -\dfrac{\sqrt{2}}{\sqrt{3}}$

Hence, $f(x)$ is discontinuous at $x = 0$. The nature of discontinuity is irremovable discontinuity of finite type.

Concept Problems

C

1. Let $f(x) = x,\ x < 1$
$= x^2,\ x > 1$
$= 2,\ x = 1$
Find the type of discontinuity at $x = 1$.

2. Let $f(x) = x,\ x < 1$
$= 2x,\ x \geq 1$
Find the type of discontinuity at $x = 1$.

3. Prove that the function $h(x) = 2^{\tan x}$ has infinite discontinuity at $x = \pi/2$.

4. If $f(x) = x + \dfrac{x+2}{|x+2|}$ find the points of discontinuity and determine the jumps of the function at these points.

5. If $f(x) = \dfrac{1}{x-a}\sin\dfrac{1}{x-a}$ find $f(a^+)$ and $f(a^-)$. Is the function continuous at $x = a$?

6. Test the discontinuity of the following function at $x = a$ and specify the type of discontinuity.

$$f(x) = \frac{1}{x-a}\,\text{cosec}\,\frac{1}{x-a},\quad x \neq a \text{ and } f(a) = 0.$$

7. The function $f(x) = \tan^{-1}\dfrac{1}{x-2}$ is meaningless for $x = 2$. Is it possible to define the value of $f(x)$ in such a way that the redefined function becomes continuous at $x = 2$?

8. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on R.

(i) $f(x) = \dfrac{x^2 - 2x - 8}{x+2},\ a = -2$

(ii) $f(x) = \dfrac{x-7}{|x-7|},\ a = 7$

Practice Problems

C

9. Test the function $f(x)$ for continuity and indicate the kind of discontinuity, if any.

(i) $f(x) = x\sin(1/x)$

(ii) $f(x) = \sin(1/x)$,

(iii) $f(x) = \{x\}$,

(iv) $f(x) = \begin{cases} x^2 & \text{if } x \text{ is an irrational number} \\ 1 & \text{if } x \text{ is a rational number} \end{cases}$

(v) $f(x) = \dfrac{x^2+2}{x^3+1}$,

(vi) $f(x) = \tan^{-1}(1/x)$,

(vii) $f(x) = \dfrac{1}{1-e^{x/(1-x)}}$,

(viii) $f(x)=\ln\dfrac{x^2}{(x+1)(x-3)}$

(ix) $f(x) = \begin{cases} x^2 & \text{for } 0 \le x < 1, \\ 2-x & \text{for } 1 < x \le 2, \end{cases}$

(x) $f(x) = \begin{cases} \cos(\pi x/2), & |x| \le 1, \\ |x-1|, & |x| > 1 \end{cases}$

10. What is the nature of the discontinuities at x = 0 of the functions:

(i) $y = \dfrac{\sin x}{x}$ (ii) $y = [x] + [-x]$

(iii) $y = \operatorname{cosec} x$ (iv) $y = \sqrt{\left(\dfrac{1}{x}\right)}$,

(v) $y = \sqrt[3]{\left(\dfrac{1}{x}\right)}$ (vi) $y = \operatorname{cosec}\dfrac{1}{x}$

(vii) $y = \dfrac{\sin(1/x)}{\sin(1/x)}$?

11. Can $f(x) = x(x^2-1)/|x^2-1|$ be extended to be continuous at x = 1 or –1?

12. Show that the only discontinuities a rational function can have are either removable or infinite. That is, if r(x) is a rational function that is not continuous at a, show that either a is a removable discontinuity or $\lim\limits_{x\to a} |r(x)| = \infty$.

13. Let $f(x) = \lim\limits_{n\to\infty} (1+x)^n$. Comment on the continuity of f(x) at 0.

14. Investigate the following functions for continuity:

(i) $y = \dfrac{\sqrt{7+x}-3}{x^2-4}$

(ii) $y = (1+x)\tan^{-1}\dfrac{1}{1-x^2}$

(iii) $y = \dfrac{1}{1+e^{\frac{1}{1-x}}}$

15. Investigate the following functions for continuity:

(i) $y = \lim\limits_{n\to\infty}\dfrac{1}{1+x^n}$ $(x \ge 0)$

(ii) $y = \lim\limits_{n\to\infty}(x\tan^{-1} nx)$

2.4 Algebra of Continuous Functions

It is easily deduced from the theorems on limits that the sum, product, difference or quotient of two functions which are continuous at a certain point are themselves continuous at that point (except that, in the case of the quotient, the denominator must not vanish at the point in question).

Further it is true that composition of a continuous function with a continuous funcion is a continuous function.

1. If f(x) and g(x) are continuous at x = a, then the following functions are also continuous at x = a.
  (i) cf(x) is continuous at x = a, where c is any constant.
  (ii) f(x) ± g(x) is continuous at x = a.
  (iii) f(x). g(x) is continuous at x = a.
  (iv) f(x)/g(x) is continuous at x = a, provided $g(a) \neq 0$.

Theorem If the functions f(x) and g(x) are continuous at a point x = a, then the sum h(x) = f(x) + g(x) is also continuous at the point x = a.

Proof Since f(x) and g(x) are continuous, we can write

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

By theorem on limits, we can write

$$\lim_{x\to a} h(x) = \lim_{x\to a}\,[f(x) + g(x)]$$

$$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a) + g(a) = h(a).$$

Thus, the sum h(x) = f(x) + g(x) is a continuous function.

Note, as a corollary, that the theorem holds true for any finite number of terms.

Theorem The sum of a finite number of functions continuous at a point is a continuous function at the point.

Proof Suppose we are given a definite number of functions $f_1, f_2, \ldots, f_n$ continuous at a point x = a. We have to prove that their sum $g = f_1 + f_2 + \ldots + f_n$ is a continuous function at that point. The functions $f_1, f_2, \ldots, f_n$ being continuous, we have

$$\lim_{x\to a} f_1(x) = f_1(a),\ \lim_{x\to a} f_2(x) = f_2(a), \ldots\ldots$$

$$\lim_{x\to a} f_n(x) = f_n(a),$$

By the theorem on the limit of a sum we write

$$\lim_{x\to a} g(x) = \lim_{x\to a}(f_1 + f_2 + \ldots + f_n)$$

$$= \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \ldots\ldots + \lim_{x\to a} f_n(x)$$

$$= f_1(a) + f_2(a) + \ldots. + f_n(a) = g(a).$$

Thus, $\lim\limits_{x\to a} g(x) = g(a)$ which is what we wished to prove.

By the above theorem we at once recognize that the functions $y = x^2 + 3x + \sqrt{x}$,

$y = \sin^3 x - x \sin x - (x^4 - 1) \cos x$, and

$y = \sin x - 2x/(x^2 + 1)$ are continuous at every point in a domain common to all functions involved.

We can show that a polynomial is a continuous function. The polynomial function $P : R \to R$ is given by $P(x) = a_0 + a_1x + ... + a_nx^n$, where $a_0, a_1, ..., a_n \in R$. The functions $f_1, f_2, ..., f_n : R \to R$ defined by

$f_1(x) = x$, $f_2(x) = x^2$, ..., $f_n(x) = x^n$ are continuous.

Hence, the functions a_1f_1, a_2f_2, ..., a_nf_n are also continuous functions. Therefore, the function

$$P(x) = a_0 + a_1f_1 + ... + a_nf_n$$

is also continuous as the sum of continuous functions is a continuous function.

Theorem The product of a finite number of functions continuous at a point is a continuous function at that point.

Proof Let $g = f_1 . f_2 ..... f_n$ retaining the notation of the above theorem and using the theorem on the limit of a product, we get

$$\lim_{x\to a} g(x) = \lim_{x\to a} (f_1.f_2....f_n)$$

$$= \lim_{x\to a} f_1(x). \lim_{x\to a} f_2(x). ..... . \lim_{x\to a} f_n(x)$$

$$= f_1(a) . f_2(a) ...... f_n(a) = g(a)$$

which is what we set out to prove.

Theorem The quotient of two functions continuous at a point $x = a$ is a continuous function at the point $x = a$ provided that the denominator does not turn into zero at the point.

Proof If $g = \frac{f_1}{f_2}$, then by the theorem on the limit of a quotient, we have

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)} = \frac{f_1(a)}{f_2(a)} = g(a)$$

if $\lim_{x\to a} f_2(x) = f_2(a) \neq 0$. Therefore $g = \frac{f_1}{f_2}$ is a continuous function at the point $x = a$.

2. If $f(x)$ is continuous at $x = a$ and $g(x)$ is discontinuous at $x = a$, then we have the following results.

(i) Both the functions $f(x) + g(x)$ and $f(x) - g(x)$ are discontinuous at $x = a$.

For example, consider, $f(x) = x$ and $g(x) = \{x\}$.

Here $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$. Both the sum function $x + \{x\}$ and the difference function $x - \{x\}$ are discontinuous at $x = 0$.

(ii) $f(x). g(x)$ is not necessarily discontinuous at $x = a$. We need to find the result by getting the limit of the product $f(x). g(x)$ and comparing it with $f(a). g(a)$.

For example, consider, $f(x) = x^3$ and $g(x) = \text{sgn}(x)$.

Here $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$. But the product function

$$f(x)g(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases} \text{ is continuous at } x = 0$$

As another example, the product of the functions

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$ even when $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$.

However, the product of the functions $f(x) = x$ and $g(x) = [x]$ is discontinuous at $x = 1$.

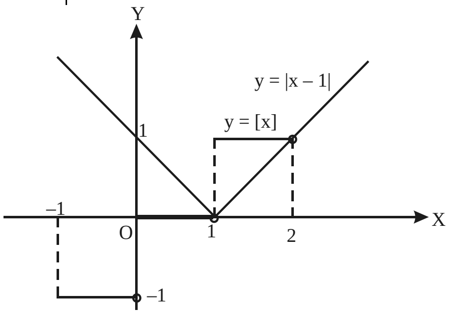
(iii) $f(x)/g(x)$ is not necessarily discontinuous at $x = a$. Here also we need to work on the function $f(x)/g(x)$ to get the result.

Let $f(x) = x(x^2 - 1)$ and $g(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$.

Here $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$. We can check that $f(x)/g(x)$ is continuous at $x = 0$.

Example 1: Discuss the continuity of $f(x) = [x] + |x - 1|$.

Solution: Let us draw the graphs of the functions $y = [x]$ and $y = |x - 1|$



It is a clear from the figure that $f(x) = [x]$ is discontinuous at all integral points and $g(x) = |x - 1|$ is continuous for all $x \in R$. The sum of a discontinuous and a continuous function is discontinuous. Hence $f(x) + g(x)$ is discontinuous at all integral points.

3. If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then we have the following results.

(i) The functions $f(x) + g(x)$ and $f(x) - g(x)$ are not necessarily discontinuous at $x = a$. However, atmost one of $f(x) + g(x)$ or $f(x) - g(x)$ can be continuous at $x = a$. That is, both of them cannot be continuous simultaneously at $x = a$. We have the following reason.

Let us assume that both $f(x) + g(x)$ and $f(x) - g(x)$ are continuous. Then the sum of functions $(f(x) + g(x)) + (f(x) - g(x)) = 2f(x)$ must be continuous at $x = a$, which is wrong as it is given that $f(x)$ is discontinuous at $x = a$. Hence our assumption is wrong. So, both the functions cannot be continuous simultaneously at $x = a$.

For example, consider, $f(x) = [x]$ and $g(x) = \{x\}$.

Here both $f(x)$ and $g(x)$ are discontinuous at $x=0$

The sum function $[x] + \{x\}$ being equal to x is continuous at $x = 0$. The difference function $[x] – \{x\}$ however is discontinuous at $x = 0$.

But this does not mean that one of the functions $f(x) + g(x)$ or $f(x) – g(x)$ must be continuous. We can have both the functions discontinuous. For example,

if $f(x) = 2[x]$ and $g(x) = \{x\}$, then both the functions $f(x) + g(x)$ or $f(x) – g(x)$ are discontinuous simultaneously at $x = 0$.

(ii) $f(x) \cdot g(x)$ is not necessarily discontinuous at $x = a$. We need to find the result by getting the limit of the product $f(x). g(x)$ and comparing it with $f(a). g(a)$.

For example, consider, $f(x) = [x]$ and $g(x) = [–x]$.

Here both $f(x)$ and $g(x)$ are discontinuous at $x = 0$ but, the product function $[x]\cdot[–x]$ is continuous at $x = 0$.

Further, $f(x) = [x]$ and $g(x) = \{x\}$ are both discontinuous at $x = 0$ and, the product function $[x]\cdot\{x\}$ is discontinuous at $x = 0$.

Hence, we cannot comment in advance, when both the functions are discontinuous at $x = a$.

(iii) $f(x)/g(x)$ is not necessarily discontinuous at $x = a$. Here also we need to work on the function $f(x)/g(x)$ to get the result.

Let $f(x) = \begin{cases} x^2 - 1, & x \ge 0 \\ x + 1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} x + 1, & x \ge 0 \\ x - 1, & x < 0 \end{cases}$.

Here both $f(x)$ and $g(x)$ are discontinuous at $x = 0$. But we find that $f(x)/g(x)$ is continuous at $x = 0$.

Example 2: If $f(x) = [\sin(x–1)] – \{\sin(x–1)\}$ then comment on continuity of $f(x)$ at $x = \frac{\pi}{2} + 1$.

Solution: $f(x) = [\sin (x – 1)] – \{\sin (x – 1)\}$Let $g(x) = [\sin (x – 1)] + \{\sin (x – 1\} = \sin (x – 1)$

which is obviously continuous at $x = \frac{\pi}{2} + 1$

Here $[\sin (x – 1)]$ and $\{ \sin (x – 1)\}$ are both discontinuous at $x = \frac{\pi}{2} + 1$.

$\therefore$ Atmost one of $f(x)$ or $g(x)$ can be continuous at $x = \frac{\pi}{2} + 1$.

As $g(x)$ is continuous at $x = \frac{\pi}{2} + 1$, therefore, $f(x)$ must be discontinuous.

Continuity of Composite Functions

Theorem A function composed of a finite number of continuous functions is a continuous function.

It is sufficient to prove this assertion for a composite function formed by two continuous functions because after that it can be extended, consecutively, to an arbitary number of constituent functions.

Theorem If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$ then the composite function $(gof)(x)$ is continuous at $x = a$.

Proof Let $y = g(u)$, $u = f(x)$ and $y = g(f(x)) = F(x)$

where $f(x)$ is continuous for $x = a$ and $g(u)$ is continuous for $u = b = f(a)$. We have to prove that

$y = F(x))$ is continuous at the point a.

Indeed, let $x \to a$. The continuity of the function

$u = f(x)$ implies that $\lim\limits_{x\to a} f(x) = f(a) = b$,

that is $u \to b$. The function $g(u)$ being continuous at the point b. we have $\lim\limits_{u\to b} g(u) = g(b)$.

Now, since $u = f(x)$, we can rewrite the last relation in the form

$\lim\limits_{x\to a} g(f(x)) = g(f(b))$ or, equivalently,

$\lim\limits_{x\to a} F(x) = F(a)$ which is what we wished to prove.

This theorem is also named as the chain rule for continuity.

For example, $f(x) = \sin x$ is continuous at $x = \frac{\pi}{2}$ and

$g(x) = \begin{cases} x^2 - 1, & x \le 1 \\ x - 1, & x > 1 \end{cases}$ is continuous at $x = f(\frac{\pi}{2}) = 1$. Hence

the composite function $(gof)(x)$ is continuous at $x = \frac{\pi}{2}$.

Note:

1. Let a function $f(x)$ be continuous at all points in the interval $[a, b]$, and let its range be the interval $[A, B]$ and further the function $g(x)$ be continuous in the interval $[A, B]$, then the composite function $(gof)(x)$ is continuous in the interval $[a, b]$.
2. If the function f is continuous everywhere and the function g is continuous everywhere, then the composition gof is continuous everywhere.
3. All polynomials, trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions are continuous at all points in their domains.
4. If $f(x)$ is continuous, then $|f(x)|$ is also continuous.

For example, $f(x) = \frac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous for all x. Hence, the composite function $(gof)(x) = \left|\frac{x \sin x}{x^2 + 2}\right|$ is also continuous for all x.

Example 3: Check the continuity of

$$f(x) = \frac{x^3\cos x + x^2\sin x}{\cos(1/\sin x)}.$$

Solution: The numerator is continuous for all x. As far as the denominator is concerned, according to the theorem on continuity of a composite function, it is continuous at points where the function $u = 1/\sin x$ is continuous, since the function $\cos u$ is continuous everywhere.

Hence the denominator is continuous everywhere, except at the point $x = k\pi$ (k an integer). Besides, we must exclude the points at which $\cos(1/\sin x) = 0$. i.e. the points at which $1/\sin x = (2p + 1)\pi/2$ ($p \in I$), or $\sin x = 2/[2p + 1)\pi]$. Thus, the function f(x) is continuous everywhere except at the points

$$x = k\pi \text{ and } x = (-1)^n \sin^{-1}\frac{2}{(2p+1)\pi} + n\pi,$$

where $k, p, n \in I$.

Example 4: What can you say about the continuity of the function $f(x) = \sqrt{9-x^2}$?

Solution: Because the natural domain of this function is the closed interval [–3, 3], we will need to investigate the continuity of f on the open interval (–3, 3) and at the two endpoints. If c is any number in the interval (–3, 3), then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}\sqrt{9-x^2}$

$$= \sqrt{\lim_{x\to c}(9-x^2)} = \sqrt{9-c^2} = f(c)$$

which proves f is continuous at each number in the interval (–3, 3). The function f is also continuous at the endpoints since

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\sqrt{9-x^2} = \sqrt{\lim_{x\to 3^-}(9-x^2)} = 0 = f(3);$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}\sqrt{9-x^2}$$

$$= \sqrt{\lim_{x\to -3^+}(9-x^2)} = 0 = f(-3).$$

Thus, f is continuous on the closed interval [–3, 3].

Note: The nth-root function $f(x) = \sqrt[n]{x}$ is continuous everywhere if n is odd and it is continuous for $x \ge 0$ if n is even.

We may combine this result with the previous theorem. Then we see that a root of a continuous function is continuous, wherever it is defined. That is, the composition

$$h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$$

of $f(x) = \sqrt[n]{x}$ and the function g(x) is continuous at a assuming that $g(a) \ge 0$ if n is even (so that $\sqrt[n]{g(a)}$ is defined).

For example, the above theorem tells us that composite functions like $\sqrt{\sin x}$, $\sqrt{x}$, $(\sqrt{x} + 1)^3$, $\sqrt{\cos^2 x + 3}$, are continuous at all points at which the functions are defined, because polynomials and trigonometric functions like sine and cosine are continuous at every point, while root functions are continuous on their proper domains.

Example 5: Show that the function

$$f(x) = \left(\frac{x-7}{x^2+2x+2}\right)^{2/3} \text{ is continuous everywhere.}$$

Solution: Note first that the denominator

$x^2 + 2x + 2 = (x + 1)^2 + 1$is never zero.

Hence the rational function

$$r(x) = \frac{x-7}{x^2+2x+2}$$

is defined and continuous everywhere. It then follows from the continuity of the cube root function that $f(x) = [r(x)]^{2/3} = \sqrt[3]{[r(x)]^2}$ is continuous everywhere.

As was mentioned earlier, all the basic elementary functions are continuous in the intervals where they are defined and therefore the theorems proved here imply that every elementary function is continuous in those intervals where it is defined.

An elementary function can only be discontinuous at those points where some of the constituent functions it is formed of are not defined or where the denominators of some fractions involved vanish.

For instance, the function $y = \dfrac{x}{x^2-4}$ is discontinuous at the points $x = \pm 2$ and continuous at all the other points.

The function $y = x^2\tan x$ is discontinuous at the points $x = (2k + 1)\dfrac{\pi}{2}$, $k \in I$.

Example 6: Find the points of discontinuity of

$$y = \frac{1}{u^2+u-2}, \text{ where } u = \frac{1}{x-1}.$$

Solution: The function $u = f(x) = \dfrac{1}{x-1}$ is discontinuous at the point $x = 1$. ...(1)

The function $y = g(u) = \dfrac{1}{u^2+u-2} = \dfrac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$.

When $u = -2$, $\dfrac{1}{x-1} = -2$

$\Rightarrow\quad x - 1 = -\dfrac{1}{2} \quad\Rightarrow\quad x = 1/2$ ...(2)

When $u = 1$, $\dfrac{1}{x-1} = 1$

$\Rightarrow\quad x - 1 = 1 \quad\Rightarrow\quad x = 2$ ...(3)

Hence, the composite function $y = g(f(x))$ is discontinuous at three points $x = \dfrac{1}{2}, 1, 2$.

Example 7: If $f(x)=\frac{x+1}{x-1}$ and $g(x)=\frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and (fog)(x).

Solution: $f(x)=\frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain.
f is not defined at x = 1
∴ f is discontinuous at x = 1

$$g(x)=\frac{1}{x-2}$$

g(x) is also a rational function. It must be continuous in its domain and it is not defined at x = 2
∴ g is discontinuous at x = 2
Now fog(x) may be discontinuous at
(i) x = 2 (point of discontinuity of g(x))
(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x)).

If g(x) = 1 then $\frac{1}{x-2}=1 \Rightarrow \quad x=3.$

∴ The discontinuity of fog(x) should be checked at x = 2 and x = 3

At x = 2, $\quad fog(x)=\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$

We see that fog (2) is not defined.

$$\lim_{x\to 2} fog(x)=\lim_{x\to 2}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}=\lim_{x\to 2}\frac{1+x-2}{1-x+2}=1.$$

∴ fog (x) is discontinuous at x = 2 and it has a removable discontinuity at x = 2.
At x = 3, fog (3) is not defined.

$$\lim_{x\to 3^+} fog(x)=\lim_{x\to 3^+}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}=\infty$$

$$\lim_{x\to 3^-} fog(x)=\lim_{x\to 3^-}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}=-\infty$$

∴ fog (x) is discontinuous at x = 3 and it has an infinite irremovable discontinuity there.

Example 8: Given the function f(x) = 1/(1 – x). Find the points of discontinuity of the composite function y = f(f(f(x))).

Solution: The point x = 1 is a discontinuity of the function $v=f(x)=\frac{1}{1-x}$. If x ≠ 1, then

$$u=f(f(x))=\frac{1}{1-1/(1-x)}=\frac{x-1}{x}.$$

Hence, the point x = 0 is a discontinuity of the function u = f(f(x)). If x ≠ 0, x ≠ 1, then

$$y=f(f(f(x)))=\frac{1}{1-(x-1)/x}=x \text{ is continuous everywhere.}$$

Thus, the points of discontinuity of this composite function are x = 0, x = 1, both of them being removable.

Example 9: If f(x) = sgn(2sinx + a) is continuous for all x, then find the possible values of a.

Solution: Since f(x) = sgn(2sinx + a) is continuous for all x, we should have 2sinx + a ≠ 0 for any real x.
⇒ sin x ≠ –a/2
⇒ |a/2| > 1
⇒ a < –2 or a > 2.

Continuity of the Inverse of a Continuous Invertible Function

Theorem If the function y = f(x) is defined, continuous and strictly monotonic on the interval I, then there exist the inverse function $y=f^{-1}(x)$ defined, continuous and also strictly monotonic in the range of the function y = f(x).

Theorem Assume f is strictly increasing and continuous on an interval [a, b]. Let c = f(a) and d = f(b) and let g be the inverse of f. Then
(i) g is strictly increasing on [c, d], and
(ii) g is continuous on [c, d]

There is a coresponding theorem for decreasing functions. That is, the inverse of a strictly decreasing continuous function f is strictly decreasing and continuous.

For example, we prove that the function $y=\sqrt[3]{x}$ is continuous ∀ x ∈ R, considering it as the inverse of $y=x^3$. The function $y=x^3$ is continuous ∀ x ∈ R, and its range is y ∈ R. Also, it is strictly increasing and hence invertible. Hence, its inverse function,

$$y=\sqrt[3]{x},\ x\in R, \text{ is continuous } \forall\ x\in R.$$

Continuity of Integrals

Assume f is integrable on [a, x] for every x in [a, b] and let $A(x)=\int_a^x f(t)dt$. Then the integral A is continuous at each point of [a, b]. (At each endpoint we have one-sided continuity)

Proof We choose a point c in [a, b]. Now we prove that A(x) → A(c) as x → c. We have

$$A(x)-A(c)=\int_c^x f(t)dt$$

We estimate the size of this integral. Since f is bounded on [a, b], there is a constant M > 0 such that $-M(x - c) \le A(x) - A(c) \le M(x - c)$.

If x < c, we obtain the same inequalities with x – c replaced by c – x. Therefore, in either case we can let $x \to c$ and apply the Sandwitch theorem to find that $A(x) \to A(c)$. This proves the theorem. If c is an endpoint of [a, b], we must let $x \to c$ from inside the interval, so the limits are one-sided.

Note: If f is continuous on [a, b], then it is integrable on the interval. Hence, the integral of a continuous function is continuous. For example, $\int_1^x \frac{\sin t}{t}dt$ is continuous in the interval $[1, \infty)$.

Concept Problems D

1. Let $f(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$ and

$g(x) = \begin{cases} -1 & x \ge 0 \\ 1 & x < 0 \end{cases}$

Show that f + g is continuous at x = 0 even though f and g are both discontinuous there.

2. Will the sum of two functions f(x) + g(x) be necessarily discontinuous at a given point x_0 if :

(i) The function f(x) is continuous and the function g(x) is discontinuous at $x = x_0$,

(ii) Both functions are discontinuous and the function at $x = x_0$?

3. Is the product of the two functions f(x) g(x) necessarily discontinuous at a given point x_0 if :

(i) The function f(x) is continuous and the function g(x) is discontinuous at this point;

(ii) Both functions f(x) and g(x) are discontinuous at $x = x_0$?

4. Prove that if the function f(x) is continuous and non-negative in the interval (a, b), then the function $F(x) = \sqrt{f(x)}$ is likewise continuous in this interval.

5. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere.

6. Suppose that the function f is continuous everywhere and that the composition f(g(x)) is continuous at x = a. Does it follow that g(x) is continuous at a ?

7. Can one assert that the square of a discontinuous function is also a discontinuous function? Give an example of a function discontinuous everywhere whose square is a continuous function.

8. Give an example of a function f such that f is not continuous but |f| is continuous. Show that f^2 can be continuous when f is not.

Practice Problems D

9. Let $f(x) = \begin{cases} x & \text{if} \quad x \ne 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$ and

$g(x) = \begin{cases} 3x & \text{if} \quad x \ne 0 \\ -2 & \text{if} \quad x = 0 \end{cases}$

Show that f + g is continuous at x = 0 even though f and g are both disontinuous there.

10. Let $f(x) = x^2$ and

$g(x) = \left[\begin{array}{ll} \cos\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{array}\right.$

Show that f.g is continuous at x = 0.

11. Let f and g be two functions defined as follows:

$f(x) = \frac{x + |x|}{2}$ for all x, $g(x) = \begin{cases} x & \text{for} \quad x < 0 \\ x^2 & \text{for} \quad x \ge 0 \end{cases}$

Find a formula for the composite function h(x) = f[g(x)]. For what values of x is h continuous?

12. Let f and g be two functions defined as follows:

$$f(x) = \begin{cases} 1 + x^3 & , \quad x < 0 \\ x^2 - 1 & , \quad x \ge 0 \end{cases};$$

$$g(x) = \begin{cases} (x-1)^{\frac{1}{3}} & , \quad x < 0 \\ (x+1)^{\frac{1}{2}} & , \quad x \ge 0 \end{cases}$$

Comment the continuity of gof(x)

13. Let f and g be two functions defined as follows :

$f(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$, $g(x) = \begin{cases} 2 - x^2 & \text{if} \quad |x| \le 2 \\ 2 & \text{if} \quad |x| > 2 \end{cases}$

Find a formula for the composite function h(x) = f[g(x)]. For what values of x is h continuous?

14. Discuss the continuity of the composite function h(x) = f(g(x)) where $f(x) = \frac{1}{(x-6)}$, $g(x) = x^2 + 5$.

15. If $f(x) = \dfrac{1}{1-x}$. Find the points of discontinuity of the function $y = f(f(f(x)))$.

16. Let $f(x) = \begin{cases} 1+x & , \ 0 \le x \le 2 \\ 3-x & , \ 2 < x \le 3 \end{cases}$.

Determine the form of $g(x) = f(f(x))$ and hence find the point of discontinuity of g, if any.

17. If $f(x) = -1 + |x-1|$, $-1 \le x \le 3$ and $g(x) = 2 - |x+1|$, $-2 \le x \le 2$, then discuss the continuity of $f(g(x))$.

18. Let $f(x) = \begin{cases} x-1, & -1 \le x < 0 \\ x^2, & 0 \le x \le 1 \end{cases}$ and $g(x) = \sin x$.

Further let $h(x) = f(|g(x)|) + |f(g(x))|$. Discuss the continuity of $h(x)$ in $[-1, 1]$.

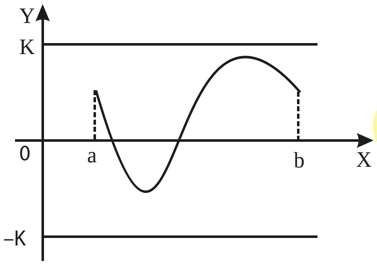
2.5 Properties of Functions Continuous on a Closed Interval

Functions continuous on a closed interval possess d in the following theorems.

Boundedness Theorem

If a function f(x) is continuous at every point of a closed interval [a, b], then the function f(x) is bounded on this interval.

Note that the boundedness of a function on the interval [a, b] means that there is a number $K > 0$ such that $|f(x)| < K$ for all $x \in [a, b]$.



Represented in the figure is the graph of a continuous function f on a closed interval [a, b]. Obviously, there exists a number $K > 0$ such that the graph is located between the straight lines $y = K$ and $y = -K$.

Note that if a function is continuous on an open interval (a, b) or on a half-open interval [a, b) or (a, b], then it is not necessarily bounded on such an interval. For instance, the function $y = 1/x$ is continuous but not bounded in the interval (0, 1]. There is no contradiction with the theorem since the function in question is not continuous on a closed interval but only in a half-open one.

If the continuity condition is not take into account, then the assertion that the function f(x) is bounded may not be true. For instance, the function

$$y = \begin{cases} 1/x & \text{for} \quad 0 < x \le 1 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

is defined on the closed interval [0, 1], but is not bounded on this interval since it is discontinuous at $x = 0$.

Weierstrass Theorem (Extreme Value Theorem)

If a function is continuous in a closed interval there exists atleast one point at which the function assumes the greatest value and atleast one point at which it assumes the least value on that interval.

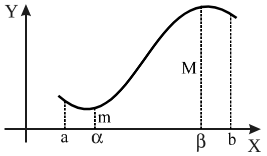
If a function f is continuous on a closed interval [a, b], then there exist its minimum and maximum values on [a, b], i.e. there exist points α, β, ∈ [a, b] such that $f(\alpha) \le f(x) \le f(\beta)$ for all $x \in [a, b]$. In other words,

$$\min_{x\in[a,b]} f(x) = f(\alpha), \qquad \max_{x\in[a,b]} f(x) = f(\beta).$$

In other words, if $m = \min\{f(x) : a \le x \le b\}$ and $M = \max\{f(x) : a \le x \le b\}$, then there are two points $\alpha, \beta \in [a, b]$ such that $f(\alpha) = m$ and $f(\beta) = M$. Note that these points α and β need not be unique.

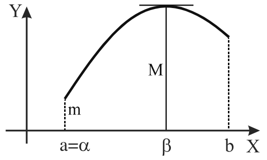
The continuous function $y = f(x)$ represented in Figure 1 attains its minimum on [a, b] at the point $x = \alpha$ and maximum at the point $x = \beta$. In this case both points α and β belong to the interval (a, b).

The continuous function $y = f(x)$ represented in Figure 2 reaches its minimum on [a, b] at its left end point and maximum at a certain interior point β of this interval.



Minimum value 'm' occurs at $x = \alpha$ and maximum value M occurs at $x = \beta$, $\alpha, \beta \in (a, b)$

Figure 1



Minimum value 'm' occurs at the end point $x = a$ and the maximum value M occurs inside the interval

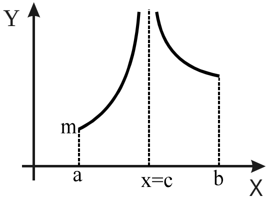
Figure 2

The value (α) and f(β), whose existence is affirmed by the theorem, are respectively the absolute minimum and absolute maximum of the function on the interval [a, b].

The extreme value theorem guarantees that the maximum and the minimum exists, but does not tell us how to find them. The problem of finding them is discussed in the chapter of maxima – minima.

If the function f(x) is continuous in the open interval (a, b) or in the half-open interval (a, b], then the function may not attain its least or greatest value. Further, if f(x) is a discontinuous function, then the theorem may not hold true.

To see that continuity is necessary for the extreme value theorem to be true refer the graph shown below.



There is a discontinuity at x = c in the interval. The function has a minimum value at the left end point x = a and f has no maximum value.

Also, the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1, \\ 0 & \text{for } \quad x = 1 \end{cases}$$

which is continuous on an open interval [0, 1) has no absolute maximum. As $x \to 1$ from the left, $f(x) \to 1$, but f(x) does not attain the value 1. Incidentally, f takes on its minimum value 0 on the interval [0,1] at two places, at x = 0 and at x = 1.

If we consider the function y = x in the interval 0 < x < 1, there is no least and no greatest values among them. There is no extreme left point, since no matter what point $x = x_1$ we take there will be a point to the left of it, for instance, the point $\frac{x_1}{2}$. Likewise there is no extreme right point; consequently, there is no least and no greatest value of the function y = x, 0 < x < 1.

Consider the function $f(x) = \tan^{-1}x$ for $x \ge 0$.

It is obvious that $\lim\limits_{x\to\infty} \tan^{-1} x = \pi/2$. But there is no x for which the function $\tan^{-1}x$ takes on the value $\pi/2$, and it does not attain maximum on $x \ge 0$. In this case the conditions of the theorem are not fulfilled : here the domain of the function $[0, \infty)$ is unbounded.

If a function f is discontinuous, then it may have both a maximum and a minimum value, but this is not always true.

For example, the function $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 4 \end{cases}$

has a maximum at x = 2 and a minimum at x = 4, even if it is discontinuous of x = 2. This can be concluded from the graph of y = f(x).

Example 1: Let $f(x) = 1/(1–x^2)$ in the open interval (–1, 1). Show that f does not have a maximum value in this interval.

Solution: For x near 1, f(x) gets arbitrarily large since the denominator $1–x^2$ is close to 0. The graph of f, for $x \in (–1,1)$, shows that the function is continuous throughout the open interval (–1, 1), but there is no number c in (–1,1) at which f has a maximum value. However, f has a minimum value, f(0) = 1.

Sign Preserving Property of Continuous Functions

If f is continuous at c and $f(c) \ne 0$, then there exists an interval $(c – \delta, c + \delta)$ around c such that f(x) has the sign of f(c) for every value of x in this interval.

Its truth is obvious if we understand that a continuous function does not undergo sudden changes so that if f(c) is positive for the value c of x and also f is continuous at c, it cannot suddenly become negative or zero and must, therefore, remain positive for values of x in a certain neighbourhood of c.

If $\lim\limits_{x\to c} f(x) = b$ and $b > 0$, then there exists a deleted neighbourhood D of c such that f(x) > 0 for every x in D. Similarly, if b < 0 there exists a D such that f(x) < 0 for every x in D.

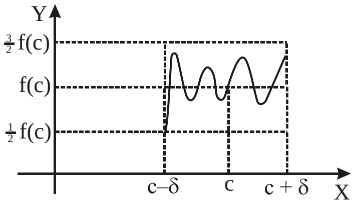
Proof Suppose f(c) > 0. By continuity, for every $\in > 0$ there is a $\delta > 0$ such that

$$f(c) – \in < f(x) < f(c) + \in \qquad ...(1)$$

whenever $c – \delta < x < c + \delta$.

If we take the δ corresponding to $\in = f(c)/2$ (this $\in$ is positive), then (1) becomes

$$\frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c),\ c – \delta < x < c + \delta.$$



Here f(x) > 0 for x near c because f(c) > 0.

Therefore f(x) > 0 in this interval , and hence f(x) and f(c) have the same sign. If f(c) < 0, we take the δ corresponding to $\in = \frac{1}{2}f(c)$ and arrive at the same conclusion.

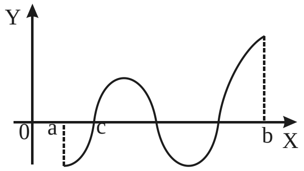
Corollary. If f(x) is continuous at x = c, and f(x) vanishes for values of x as near as we please, or assumes, for values of x as near to c as we please, both positive and negative values, then f(c) = 0.

This is an obvious corollary to sign preserving property. If f(c) is not zero, it must be positive or negative; and if it were, for example, positive, f(x) would be positive for all values of x sufficiently near to c, which contradicts the hypotheses of the theorem.

Note: If there is one-sided continuity at c, then there is a corresponding one-sided interval $[c, c + \delta)$ or $(c – \delta, c]$ in which f has the same sign as f(c).

Bolzano's Theorem

If a function f is continuous on a closed interval [a, b] and the numbers f(a) and f(b) are different from zero and have opposite signs, then there is atleast one point c on the open interval (a, b) such that f(c) = 0.



The function whose graph is depicted in the above figure satisfies the conditions of Bolzano's theorem. It is continuous on [a, b] and $f(a) < 0$, $f(b) > 0$. Geometrically, it is obvious that the graph must intersect the x-axis atleast at one point $c \in (a, b)$. This is just what is stated by the theorem.

In other words, if f is continuous in [a, b] and f(a) and f(b) have opposite signs, then there is atleast one value of x for which f(x) vanishes in the interval (a, b).

Proof To fix the ideas, suppose that $f(a) < 0$ and $f(b) > 0$. Since f(x) is continuous, it will be negative in the neighbourhood of a and positive in the neighbourhood of b. The set of values of x between a and b which make f(x) positive is bounded below by a, and hence possesses an exact lower bound k: clearly $a < k < b$. From the definition of the lower bound, the values of f(x) must be negative or zero in $a \le x < k$. Since f(x) is continuous when x = k,

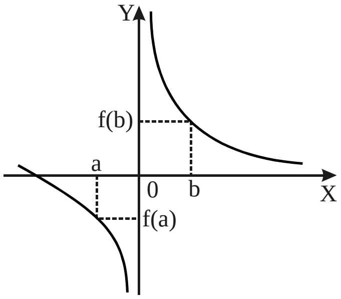
$$\lim_{x\to k^-} f(x) = f(k).$$

Hence f(k) is also negative or zero. We shall show that f(k) cannot be negative; for if $f(k) = -c$, where c is positive, then there exists a positive number δ such that

$$|f(x) - f(k)| < c \text{ when } |x - k| \le \delta,$$

since f(x) is continuous when x = k. The function f(x) would when be negative for whose values of x in (a, b) which lie between k and $k + \delta$, which contradicts the fact that k is the lower bound of the set of values of x between a and b which make f(x) positive. It follows that f(k) = 0, and the theorem is therefore proved.

In the following figure, f (a) and f (b) are of opposite signs but f (x) has no root in (a, b) as f is discontinuous.



Hence, if a function is discontinuous in the interval [a, b] and f(a) and f(b) have opposite signs then f(x) may or maynot have a root in (a, b).

Example 2: The equation $x^3 - x - 2 = 0$ may have a solution somewhere between x = 1 and x = 2. Apply the Bolzano's theorem to show that this is true.

Solution: The function $f(x) = x^3 - x - 2$ is continuous on [1, 2] because it is a polynomial. $f(1) = -2$ and $f(2) = 4$. Since, they have opposite signs, Bolzano's theorem implies that the function vanishes at some point in (1, 2). In particular, there exists atleast one c in (1, 2) such that $f(c) = c^3 - c - 2 = 0$.

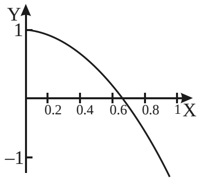
So x = c is a solution of the equation $x^3 - x - 2 = 0$ which lies in (1, 2).

Example 3: What can be said about the roots of $f(x) = \cos x - 2x^2$ in the interval $0 \le x \le 1$?

Solution: $f(x) = \cos x - 2x^2$ in the interval $0 \le x \le 1$. Since f(x) is a difference of the two continuous functions, it is continuous. We find values of f at different values of x as shown in the table.

TABLE

x	f(x)
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	–0.58
1.0	–1.46



Roots occur where the graph of a function crosses the x-axis

We conclude that f(x) has atleast one root in the interval $0.6 < x < 0.8$, since f(x) changes sign from positive to negative on that interval. The graph of f(x) suggests that there is only one root in the interval $0 \le x \le 1$, but we cannot find the root from the graph or the table of values.

Example 4: Given that $a > b > c > d$ then prove that the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ will have two real and distinct roots.

Solution: $(x - a)(x - c) + 2(x - b)(x - d) = 0$

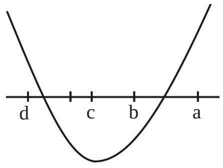
Let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$f(a) = (a - a)(a - c) + 2(a - b)(a - d) = +\text{ve}$

$f(b) = (b - a)(b - c) + 0 = -\text{ve}$

$f(c) = 0 + 2(c - b)(c - d) = -\text{ve}$

$f(d) = (d - a)(d - c) + 0 = +\text{ve}$



Hence $(x - a)(x - c) + 2(x - b)(x - d) = 0$ has atleast one real root in (d, c) and atleast one real root in (b, a). Since, a quadratic equation can have atmost two real roots, exactly one real root lies in each of the two intervals. Thus the roots are real and distinct.

Example 5: Show that $e^{-x} \sin x = \ln x$ has atleast one solution on the interval [1, 2].

Solution: Notice that the function $f(x) = e^{-x} \sin x - \ln x$ is continuous on [1, 2].

We find that

$$f(1) = e^{-1} \sin 1 - \ln 1 \approx 0.31 > 0 \text{ and}$$

$$f(2) = e^{-2} \sin 2 - \ln 2 \approx -0.57 < 0$$

Therefore, by Bolzano theorem there is atleast one number c on (1, 2) for which $f(c) = 0$, and it follows that $e^{-c} \sin c = \ln c$.

Example 6: Given a function on the interval [–2, 2]

$$f(x) = \begin{cases} x^2 + 2 & \text{if } -2 \le x < 0, \\ -(x^2 + 2) & \text{if } 0 \le x \le 2 \end{cases}$$

is there a point on this closed interval at which $f(x) = 0$?

Solution: At the endpoints of the interval [–2, 2] the given function has different signs :

$$f(-2) = 6;\ f(2) = -6.$$

But it is easy to notice that it does not become zero at any point of the interval [–2, 2]. Indeed, $x^2 + 2 > 0$ and $-(x^2 + 2) < 0$ at any x; this is due to the fact that f(x) has a discontinuity at the point $x = 0$.

Example 7: Let f be a continuous function defined from [0, 1] to [0, 1] with range [0, 1]. Show that is some 'c' in [0, 1] such that $f(c) = 1 - c$.

Solution: Consider $g(x) = f(x) - 1 + x$

$$g(0) = f(0) - 1 \le 0 \quad (\text{as } f(0) \le 1)$$

$$g(1) = f(1) \ge 0 \quad (\text{as } f(1) \in [0, 1])$$

There are three cases:

Case I: $g(0) = 0$

This happens when $f(0) = 1$. In such a case $c = 0$, which lies in [0, 1].

Case II: $g(1) = 0$

This happens when $f(1) = 0$. In such a case $c = 1$, which lies in [0, 1].

Case III: g (0) and g (1) are of opposite signs

By Bolzano's theorem there exists atleast one $c \in (0, 1)$ such that $g(c) = 0$.

$\Rightarrow g(c) = f(c) - 1 + c = 0 \Rightarrow f(c) = 1 - c.$

Combining the three cases, there exists some c in [0, 1] such that $f(c) = 1 - c$.

Example 8: Let $f : [0, 2] \to R$ be continuous and $f(0) = f(2)$. Prove that there exists x_1 and x_2 in (0, 2) such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$

Solution: Consider the continuous function g on [0, 2] defined as $g(x) = f(x + 1) - f(x) \quad (x_2 = x_1 + 1)$

Now, $g(0) = f(1) - f(0) = f(1) - f(2)$ ...(1)

$g(1) = f(2) - f(1) = f(2) - f(1)$ ...(2)

Thus, g (0) and g (1) are of opposite signs.

Hence by Bolzano's theorem there exists some $c \in (0, 1)$ where $g(c) = 0$,

i.e. $f(c + 1) = f(c) \quad \{c + 1 \in (1, 2) \text{ as } c \in (0, 1)\}$

Putting $c = x_1$; $c + 1 = x_2$, obviously $x_1, x_2 \in (0, 2)$ we have $f(x_2) = f(x_1)$ where $x_2 - x_1 = 1$.

Example 9: Prove that the function

$$f(x) = a\sqrt{x-1} + b\sqrt{2x-1} - \sqrt{2x^2 - 3x + 1}$$

where $a + 2b = 2$ and $a, b \in R$ always has a root in (1, 5) $\forall\, b \in R$.

Solution: There are three cases:

Case I: Let $b > 0$, then $f(1) = b > 0$

and $f(5) = 2a + 3b - 6 = 2(a + 2b) - b - 6$

$$= 4 - b - 6 = -(2 + b) < 0$$

Hence by Bolzano's theorem, there exists some $c \in (1, 5)$ such that $f(c) = 0$.

Case II: If $b = 0$ then $a = 2$.

$$f(x) = 2\sqrt{x-1} - \sqrt{2x^2 - 3x + 1} = 0$$

$\Rightarrow \quad 4(x - 1) = 2x^2 - 3x + 1 = (2x - 1)(x - 1)$

$\Rightarrow \quad (x - 1)(2x - 5) = 0 \quad \Rightarrow x = \dfrac{5}{2}$

Hence $f(x) = 0$ if $x = \dfrac{5}{2}$ which lies in (1, 5).

Case III: If $b < 0$, $f(1) = b < 0$ and

$$f(2) = a + b\sqrt{3} - \sqrt{3}$$

$$= (a + 2b) + \left(\sqrt{3} - 2\right) b - \sqrt{3}$$

$$= (2 - \sqrt{3}) - (2 - \sqrt{3})\, b$$

$$= (2 - \sqrt{3})(1 - b) > 0 \quad (\text{as } b < 0)$$

Thus, f(1) as f(2) have opposite signs

Hence there exists some $c \in (1, 2) \subset (1, 5)$ for which $f(c) = 0$.

Isolation of Roots

A real number x_1 is said to be a real root of the equation $f(x) = 0$ if $f(x_1) = 0$. We say that a real root of an equation has been isolated if we exhibit an interval [a, b] containing this root.

If f is a continuous function in an interval I and $f(a)\, f(b) < 0$ for some $a, b \in I$, then by Bolzano’s theorem there is a point c between a and b for which $f(c) = 0$. This is often used to locate the roots of equations of the form $f(x) = 0$.

For example, consider the equation $x^3 + x - 1 = 0$.

Let $f(x) = x^3 + x - 1$.

Note that $f(0) = -1$ whereas $f(1) = 1$. This shows that the above equation has a root between 0 and 1.

Now we try with 0.5, $f(0.5) = -0.375$. So there must be a root of the equation between 1 and 0.5. We again try 0.75, $f(0.75) > 0$, which means that the root is between 0.5 and 0.75.

So we may try 0.625, $f(0.625) < 0$. So the root is between 0.75 and 0.625.

Now if we take the approximate root to be .6875, then we are away from the exact root by atmost a distance of .0625. If we continue this process further, we shall get better and better approximations to the root of the equation.

Concept Problems E

1. The boundedness theorem tells us that every continuous real-function on the closed interval [0, 1] is bounded and attains its bounds. For each of the following intervals I give an example of a continuous function f from I to R which is unbounded, and an example of a function g from I to R which is continuous and bounded on I, but such that g does not have a maximum value on I.

(i) $I = [0, 1)$; (ii) $I = (1, \infty)$; (iii) $I = [0, \infty)$.

2. Prove that the equations have a solution in the given intervals

(i) $x^4 + 2x - 1 = 0$ on [0, 1] ;

(ii) $x^5 - 5x^3 + 3 = 0$ on [–3, –2]

3. Prove that the equation $\sin x - x\cos x = 0$ has a root between π and $\frac{3}{2}\pi$.

4. Let $f(x) = \tan x$. Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in the interval $(\pi/4, 3\pi/4)$ such that $f(x) = 0$. Explain why this does not contradict Bolzano's theorem.

5. With the aid of Bolzano's theorem, isolate the real roots of each of the following equations (each has four real roots).

(i) $2x^4 - 14x^2 + 14x - 1 = 0$

(ii) $x^4 + 4x^3 + x^2 - 6x + 2 = 0$

6. Suppose that f is continuous on the interval [0, 1], that $f(0) = 2$, and that f has no zeros in the interval. Prove that $f(x) > 0$ for all x in [0, 1].

Practice Problems E

7. A function f from R to R is said to be periodic, and a real number $T > 0$ is said to be a period of f it, for all $x \in R$, $f(x + T) = f(x)$. Suppose that f is a continuous function from R to R and that f is periodic. Prove that f is bounded.

8. Does the function $2^x - x^3 + x^5$ have (a) a maximum value for x in [–3, 10] ? (b) a minimum value for x in [–3, 10] ?

9. Prove that the function

$$f(x) = \begin{cases} x+1, & -1 \le x \le 0, \\ -x, & 0 < x \le 1 \end{cases}$$

is discontinuous at $x = 0$ and still has the maximum and minimum value on [–1, 1].

10. Show that there is a number x between $\pi/2$ and π such that $\tan x = -x$.

11. Show that the given equation has atleast one solution on the indicated interval.

(i) $\sqrt[3]{x} = x^2 + 2x - 1$ on [0, 1]

(ii) $\frac{1}{x+1} = x^2 - x - 1$ on [1, 2]

(iii) $\sqrt[3]{x-8} + 9x^{2/3} = 29$ on [0, 8]

(iv) $\cos x = x^2 - 1$ on $[0, \pi]$

12. Prove that the equation $\tan x = x$ has infinite number of real roots.

13. Let $\varphi(x) = \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$, show that $\phi(0)$ and $\phi(2)$ differ in sign but $\phi(x)$ does not vanish in [0, 2].

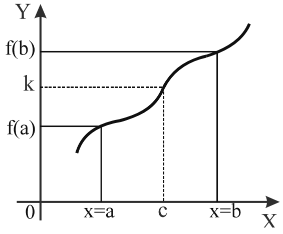
14. (i) Prove that the only polynomial function (with real coefficient) p such that, for all $x > 0$, $p(x)/x^2 \in [2, 3]$ are those of the form $p(x) = Ax^2$ for some constant $A \in [2, 3]$.

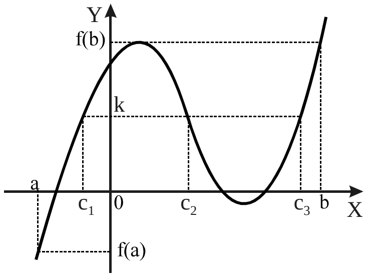
(ii) Give an example of a continuous function f from $(0, \infty)$ to R which is not a polynomial but such that, for all $x > 0$, $2 \le \frac{f(x)}{x^2} \le 3$.

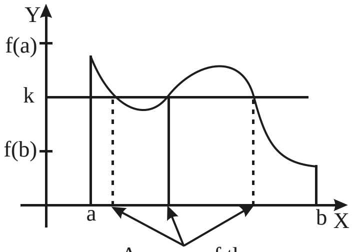
2.6 Intermediate Value Theorem (I.V.T.)

Theorem If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is atleast one number c in (a, b) such that $f(c) = k$.

In other words, a continuous function defined on [a, b] takes on all values between f(a) and f(b). Pictorially, it asserts that a horizontal line of height k must meet the graph of f atleast once if k is between f(a) and f(b), as shown in the figure. That is, when you move a pencil along the graph of a continuous function from one height to another, the pencil passes through all intermediate heights. This is a way of saying that the graph has no gaps or jumps, suggesting that the idea of being able to trace such a graph without lifting the pen from the paper is accurate. The following graphs of functions continuous in a closed interval illustrate the theorem.

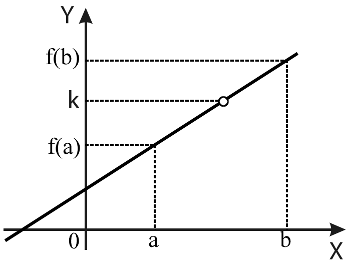






Any one of these three numbers serves as c.

Note that continuity through the interval [a, b] is essential for the validity of this theorem. This is illustrated by the following figure



Even though the theorem guarantees the existence of c, it does not tell how to find it. Such theorems are called existence theorems. To find c, we must solve the equation, namely f(c) = k.

Proof Let us define a new function g(x) = f(x) – k, where k is a constant number lying between f(a) and f(b). Since f is a continuous function on [a, b], so is the function g.

Without loss of generality let us assume that f(a) < k < f(b). Now g(a) = f(a) – k < 0 and g(b) = f(b) – k > 0. The function g, obviously, attains values of opposite signs at the end points of the interval [a, b]. Therefore, by Bolzano theorem , there is a point c lying inside (a, b) such that g(c) = 0 or f(c) – k = 0, that is f(c) = k. This is what we wished to prove.

Since k is any value between f(a) and f(b), it follows that f takes all values between f(a) and f(b) atleast once. In other words, a continuous function cannot pass from one value to another without assuming once (atleast) every intermediate value.

Alternative proof: For definiteness, let f(a) < 0 and f(b) > 0. Divide the interval [a, b] into n equal parts and consider the values of the function with their proper sign at the points of division, say $x_1, x_2, \ldots, x_{n-1}$. Let x_{p+1} be the first of these points for which f(x) is positive. Then at $x = x_p$, the function is either negative or zero. If it is zero, the theorem is at once established. If not, we rename the interval $[x_p, x_{p+1}]$ as $[a_1, b_1]$.

Divide it into n equal parts and consider the values of f(x) at the points of division again. Suppose $[a_2, b_2]$ is the first of the new sub-intervals such that $f(b_2) > 0$ and $f(a_2) \le 0$.

We take $f(a_2) \ne 0$, for otherwise the theorem is established. Continuing this process indefinitely, we shall get a sequence of intervals

$[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots$

such that $[a_1, b_1) \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$

Also $b_k - a_k = (b - a) / n^k \to 0$ as $k \to \infty$.

This sequence of intervals will therefore define a limiting point x_0. We will now show that $f(x_0) = 0$. Since f is continuous at $x = x_0$, we have $|f(x) - f(x_0)| < \in$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$. Now we can choose m so large that the interval $[a_m, b_m]$ lies within $(x_0 - \delta, x_0 + \delta)$, such that $f(a_m) < 0$ and $f(b_m) > 0$. Also by the sign preserving property if $f(x_0) \ne 0$, f(x) must have the same sign as $f(x_0)$ in the interval $(x_0 - \delta, x_0 + \delta)$. Hence $f(a_m)$ and $f(b_m)$ cannot differ in sign unless $f(x_0)$ is zero, thus establishing the theorem.

As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

Example 1: Show that the function

$f(x) = x - 1 - \cos \pi x, x \in [1, 2]$ takes on the value 1/2.

Solution: The function $f(x) = x - 1 - \cos \pi x, x \in [1, 2]$ is continuous everywhere in the given interval.

Also, we have $f(1) = -1$ and $f(2) = 1$.

Since $-1 < 1/2 < 1$, hence by I.V.T. there exists atleast one point x = c in [1, 2] such that f(c) = 1/2.

Example 2: Does the function $f(x) = x^3/4 - \sin \pi x + 3$ take on the value $2\frac{1}{3}$ within the interval [–2, 2]?

Solution: The function $f(x) = x^3/4 - \sin \pi x + 3$ is continuous within the interval $[-2, 2]$. Furthermore, at the end points of this interval it attains the values $f(-2) = 1$; $f(2) = 5$. Since $1 < 2\frac{1}{3} < 5$, then, by I.V.T. within the interval $[-2, 2]$ there exists at least one point x such that $f(x) = 2\frac{1}{3}$.

Example 3: Show that the function $f(x) = (x-a)^2(x-b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Solution: f is continuous in [a, b] and f(a) = a , f(b) = b. We know that the average value of a and b

i.e. $\frac{a+b}{2} \in [a, b]$.

By Intermediate Value Theorem, there exists atleast one $c \in (a, b)$ such that $f(c) = \frac{a+b}{2}$.

Example 4: Use Intermediate Value Theorem to show that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution in the interval [1, 2].

Solution: Let $P(x) = 2x^3 + x^2 - x + 1$. Then

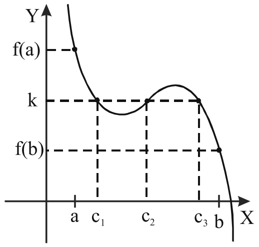
$P(1) = 2 . 1^3 + 1^2 - 1 + 1 = 3$

$P(2) = 2 . 2^3 + 2^2 - 2 + 1 = 19$

Since P is continuous and 5 is between P(1) = 3 and P(2) = 19, we may apply the Intermediate Value Theorem to P in the case a = 1, b = 2, and k = 5. Thus there is atleast one number c between 1 and 2 such that P(c) = 6. This completes the answer.

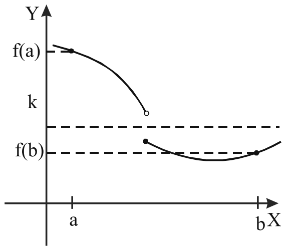
To get a more accurate estimate for a number c such that P(c) = 5, we can find a shorter interval for which the Intermediate Value Theorem can be applied. For instance, $P(1.2) \approx 4.7$ and $P(1.3) \approx 5.8$. By the intermediate value theorem, there is a number c in [1.2, 1.3] such that P(c) = 5.

Note: The Intermediate Value Theorem guarantees the existence of atleast one number c in the closed interval [a, b]. There may, of course, be more than one number c such that f(c) = k, as shown in Figure 1. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 2 jumps over the horizontal line given by y = k and for this function there is no value of c in [a, b] such that f(c) = k.



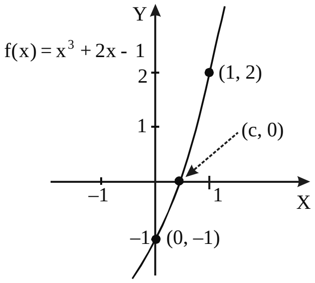
(Fig. 1)

f is continuous on [a, b].
(For k, there exist 3 c's.)



(Fig. 2)

f is not continuous on [a, b].
(For k, there are no c's.)



(Fig. 3)

f is continuous on [0, 1] with f(0) < 0 and f(1) > 0.

The converse of I.V.T. is however not true. Consider, for example, the function f defined as follows :

$$f(x) = \sin(1/x),\ x \neq 0 \text{ and } f(0) = 0.$$

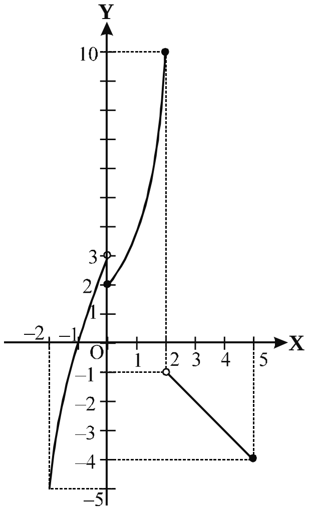
In the interval $[-2/\pi, 2/\pi]$, this function takes all values between $f(-2/\pi)$ and $f(2/\pi)$, that is, between –1 and 1 and infinite number of times as x varies from $-2/\pi$ to $2/\pi$ but the function is not continuous in this interval, being discontinuous at x = 0.

Example 5: Let

$$f(x) = \begin{cases} 3 - x^2 + 2x & -2 \le x < 0 \\ x^3 + 2 & 0 \le x \le 2 \\ 1 - x & 2 < x \le 5 \end{cases}$$

Show that f(x) attains all intermediate values between f(–2) and f(5) even if f is discontinuous in the interval.

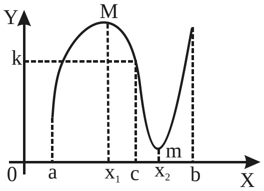
Solution: This is self evident from the graph of the function shown below. Howevever, this could not be concluded from I.V.T. since f(x) is discontinuous at x = 0 and x = 2.



Corollary. The I.V.T can also be reformulated in the following way : a function continuous on a closed interval [a, b] assumes all the intermediate values lying between its least and greatest values on [a, b] (which exist by virtue of Extreme Value Theorem).

In other words, let f be continuous on [a, b] an let $k \in [m, M]$ where m = absolute minimum value of f and M = absolute maximum value of f on [a, b]. Then there exists $c \in [a, b]$ such that f(c) = k.

Proof By extreme value theorem, there exist $x_1, x_2 \in [a, b]$ such that $m = f(x_1)$ and $M = f(x_2)$. If $x_1 = x_2$, then f is constant on [a, b] and the result follows. Let $x_1 < x_2$. Since, f is continuous on $[x_1, x_2] \subset [a, b]$, by intermediate value theorem there exists $c \in [x_1, x_2] \subset [a, b]$ such that f(c) = k.



Similarly the result can be proved when $x_1 > x_2$.

Note: A continuous function whose domain is closed must have a range in a closed interval but it is not necessary that if the domain is open then range is also open (range can be closed).

Theorem Let f be a continuous strictly increasing function on a closed interval [a, b] and let $\alpha = f(a)$, $\beta = f(b)$. Then (i) the range of f for the closed interval [a, b] is the closed interval $[\alpha, \beta]$, (ii) there exists a function x = g(y), the inverse of f, which is one-valued, strictly increasing and continuous on $[\alpha, \beta]$.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function y = sin x is continuous and strictly increasing; it has a continuous inverse which is, as we know, designated as

$$x = \sin^{-1}y \ (-1 \le y \le 1).$$

A strictly decreasing function f(x), continuous on [a, b] has an inverse, which is a strictly decreasing continuous function on $[\beta, \alpha]$ where $\alpha = f(a)$, $\beta = f(b)$.

Theorem Let a strictly increasing function y = f(x) map the closed interval [a, b] onto the closed interval $[\alpha, \beta]$, i.e. $f([a, b]) = [\alpha, \beta]$. Then f is continuous on [a, b].

Proof Let us be given an arbitrary point x_0 belonging for the time being to the open interval (a, b). By virtue of the fact that f is strictly increasing, the corresponding point $y_0 = f(x_0)$ will belong to the interval (α, β) $(\alpha < y_0 < \beta)$.

Let us take $\varepsilon > 0$ so small that $\alpha < y_0 - \varepsilon < y_0 < y_0 + \varepsilon < \beta$. By the hypothesis, there exist points $x_1, x_2 \in (a, b)$, $(x_1 < x_0 < x_2)$ such that $y_0 - \varepsilon = f(x_1)$, $y_0 + \varepsilon = f(x_2)$.

The interval (x_1, x_2) can be regarded as a neighbourhood of the point x_0 $(x_0 \in (x_1, x_2))$.

Since the function f increases, for $x \in (x_1, x_2)$ we shall have $y_0 - \varepsilon < f(x) < y_0 + \varepsilon$ or $|f(x) - y_0| < \varepsilon$, i.e. $|f(x) - f(x_0)| < \varepsilon$, and we have proved the continuity of the function f at the point x_0.

If $x_0 = a$ or $x_0 = b$, then we prove the one-sided continuity of the function f in a similar way.

Let f(x) be a function defined and bounded in the interval [a, b], then, if M and m are the bounds of f(x), the number M – m is called the **span** or **oscillation** of the function f(x) in the interval.

Theorem A continuous function attains its bounds. If f(x) is continuous in [a, b] and M and m are its maximum and minimum values, there are atleast two points x_1 and x_2 in (a, b) such that

$$f(x_1) = M, \ f(x_2) = m.$$

Proof Suppose that M is not attained; then M – f(x) does not vanish at any point of [a, b]. Hence

$\dfrac{1}{M - f(x)}$ is a continuous function, and therefore bounded. If G > 0 be its upper bound, we have

$$\frac{1}{M - f(x)} \le G,$$

So that $\quad M - f(x) \ge \dfrac{1}{G},$

that is, $\quad f(x) \le M - \dfrac{1}{G}$

but this contradicts the fact that M is the maximum value of f(x) in (a, b). Hence M must be attained. Similarly it may be proved that m is attained.

Consider the following examples:

(i) If f(x) = 1/x except when x = 0 and f(0) = 0 , then f(x) has neither an upper nor a lower bound in any interval which includes x = 0 in its interior, such as the interval (–1, 1).

(ii) If $f(x) = 1/x^2$ except when x = 0, and f(0) = 0, then f(x) has the lower bound 0, but no upper bound, in the interval (–1, 1).

(iii) Let f(x) = sin(1/x) except when x = 0 and f(0) = 0, then f(x) is discontinuous for x = 0. In any interval $(-\delta, \delta)$ the lower bound is –1 and the upper bound 1 and each of these values is assumed by f(x) an infinity of times.

(iv) Let f(x) = x – [x]. This function is discontinous for all integral values of x. In the interval (0, 1) its lower bound in 0 and its upper bound 1. It is equal to 0. when x = 0 or x = 1, but it is never equal to 1. Thus f(x) never assumes a value equal to its upper bound.

(v) Let f(x) = 0 when x is irrational, and f(x) = q when x is a rational fraction p/q. Then f(x) has the lower bound 0, but no upper bound, in any interval (a, b).

But if $f(x) = (-1)^p\, q$ when x = p/q then f(x) has neither an upper nor a lower bound in any interval.

Corollary. If f(x) is continuous in the interval [a, b], then (a, b) can be subdivided into a finite number of sub-intervals in each of which the span or oscillation of f(x) is less than any given ε.

Existence of Solutions of Equations

The Intermediate Value Theorem can be often used to locate the zeros of a function (i.e.solutions of equations written in the form $f(x) = 0$) that is continuous on a closed interval. Specifically, if f is continuous on [a, b] and f(a) and f(b) differ in sign, then the Intermediate Value Theorem guarantees the existence of atleast one zero of f in the open interval (a, b).

Let $f(x) = x^2 - 2 = 0$

$f(1) = -1 < 0$, whereas $f(2) = 2 > 0$

We note that the function f is continuous on [1, 2] and that $k = 0$ is an intermediate value of f on the interval [1, 2]. Therefore, it follows from I.V.T. that $f(c) = c^2 - 2 = 0$ for some number c in (1, 2).

i.e. $c^2 = 2$

This number c is the desired square root of 2. Thus it is the intermediate value property of continuous functions that guarantees the existence of the number $\sqrt{2}$.

Consider another example. Prove that the equation $x - \cos x = 0$ possesses a root in the interval $(0, \pi)$.

The function $f(x) = x - \cos x$ is continuous on the closed interval $[0, \pi]$ and assumes $f(0) = -1$, and $f(\pi) = \pi + 1$ values having opposite signs at its end points. Since $-1 < 0 < \pi + 1$, by I.V.T. f(x) assumes the value 0 at some x between 0 and π. Thus, the equation $x - \cos x = 0$ possesses a root in the interval $(0, \pi)$.

Example 6: Show that the equation $x^5 - 2x^2 + x + 11 = 0$ has atleast one real root.

Solution: For x large and positive the polynomial $P(x) = x^5 - 2x^2 + x + 11$ is positive

(since $\lim_{x\to\infty} P(x) = \infty$).

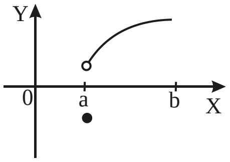
Thus, there is a number b such that $P(b) > 0$. Similarly, for x negative and of large absolute value P(x) is negative (since $\lim_{x\to-\infty} P(x) = -\infty$). Let us now select a number a such that

$P(a) < 0$.

The number 0 is between P(a) and P(b). Since P is continuous on the interval [a, b]. There is a number c in [a, b] such that $P(c) = 0$. This number c is a real solution to the equation $x^5 - 2x^2 + x + 11 = 0$.

Note that the argument in this example applies to any polynomial of odd degree.

Note: In both Bolzano's theorem and the intermediate value theorem, it is assumed that f is continuous at each point of [a, b], including the endpoints a and b. To understand why continuity at both endpoints is necessary, we refer to the curve in the figure below.



Here f is continuous everywhere in (a, b], excluding the end point a. Although f(a) is negative and f(b) is positive, there is no x in (a, b) for which $f(x) = 0$.

Example 7: If n is a positive integer and if $a > 0$, then prove that there is exactly one positive b such that $b^n = a$.

Solution: Choose $c > 1$ such that $0 < a < c$, and consider the function f defined on the interval [0, c] by the equation $f(x) = x^n$. This function is continuous on [0, c], and at the end points we have $f(0) = 0$, $f(c) = c^n$.

Since $0 < a < c < c^n$, the given number a lies between the function values f(0) and f(c). Therefore, by the Intermediate Value Theorem, we have $f(x) = a$ for some x in (0, c) say for $x = b$. This proves the existence of atleast one positive b such that $b^n = a$. There cannot be more than one such b because f is strictly increasing on [0, c]. This completes the proof.

Example 8: Prove that the equation

$$\sqrt{x-5} = \frac{1}{x+3} \text{ has atleast one real root.}$$

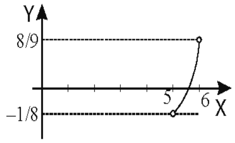
Solution: Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$

The function is continuous on [5, 6].

$$\text{Also } f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0$$

$$f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$$

Hence by intermediate value theorem there exists atleast one value of $c \in (5, 6)$ for which $f(c) = 0$



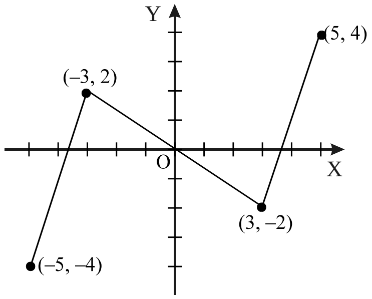
$$\therefore \quad \sqrt{c-5} - \frac{1}{c+3} = 0$$

Thus, c is root of the equation $\sqrt{x-5} = \dfrac{1}{x+3}$ where $c \in (5, 6)$.

Example 9: Let $f : R \to R$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall\ x \in R$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then prove that the equation $f(x) = 0$ has atleast three real roots.

Solution: $f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function. Since points $(-3, 2)$ and $(5, 4)$ lie on the curve, points $(3, -2)$ and $(-5, -4)$ will also lie on the curve.

For minimum number of roots, the graph of a continuous function f (x) can be drawn as follows.



From the above graph of f(x), it is clear that equation f(x) = 0 has atleast three real roots.

Example 10: Using Intermediate Value Theorem, prove that there exists a number x such that

$$x^{200} + \frac{1}{1+\sin^2 x} = 200.$$

Solution: Let $f(x) = x^{200} + (1 + \sin^2 x)^{-1}$.

f is continuous and f(0) = 1 < 200 and $f(2) > 2^{200}$, which is much greater than 200. Hence, from the Intermediate Value Theorem there exists a number c in (0, 2) such that f(c) = 200.

Example 11: Let f : R → R satisfy

$$f(x) - f(y) = e^{x-y} - 1 \ \forall \ x, y \in R.$$

Prove that f is a continuous function. Also prove that the function f(x) has atleast one zero if f(0)< 1.

Solution: $\lim\limits_{h\to 0} f(x + h) - f(x) = \lim\limits_{h\to 0} (e^{x+h-x} - 1) = 0$

Hence f is continuous everywhere.

Putting y = 0, we have $f(x) = f(0) + e^x - 1$.

Also $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = f(0) - 1 < 0$

Since f(x) is positive for large positive x and negative of large negative x, by Intermediate Value Theorem f(x) = 0 has atleast one root.

Example 12: If f(x) be a continuous function in [0, 2π] and f(0) = f(2π) then prove that there exists point c ∈ [0, π] such that f(c) = f(c + π).

Solution: Let $g(x) = f(x) - f(x + \pi)$ ....(1)

at $x = \pi$; $g(\pi) = f(\pi) - f(2\pi)$ ....(2)

at $x = 0$, $g(0) = f(0) - f(\pi)$ ...(3)

Adding (2) and (3), $g(0) + g(\pi) = f(0) - f(2\pi)$

⇒ $g(0) + g(\pi) = 0$ [given $f(0) = f(2\pi)$]

⇒ $g(0) = -g(\pi)$

⇒ g(0) and g(π) are opposite in sign.

⇒ There exists a point c between 0 and π such that g(c) = 0.

From (1) putting x = c, $g(c) = f(c) - f(c + \pi) = 0$

Hence, $f(c) = f(c + \pi)$.

Example 13: Suppose that f is a continuous function of [0, 1] and that f(0) = f(1). Let n be a positive integer. Prove that there is some number x ∈ [0, 1] such that f(x) = f(x + 1/n). [Universal Chord Theorem]

Solution: Consider g(x) = f(x) – f(x + 1/n), which is clearly continuous. If g is never zero in [0, 1] then g must be either strictly positive or strictly negative. But then

$$0 = f(0) - f(1) = \left(f(0) - f\left(\frac{1}{n}\right)\right) + \left(f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right)$$

$$+ \left(f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right)\right) + \ldots + \left(f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)\right).$$

The sum of each parenthesis on the right is strictly positive or strictly negative and hence never 0, a contradiction.

Example 14: A function f is continuous in the interval [0,1] and assumes only rational values in the entire interval. If $f(\frac{1}{2}) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ everywhere.

Solution: Suppose, if possible, that there exists a point c ∈ [0,1] such that $f(c) \neq \frac{1}{2}$. It is obvious that $c \neq \frac{1}{2}$. By the law of trichotomy, either $c < \frac{1}{2}$ or $\frac{1}{2} < c$. Without loss of generality, let us assume that $c < \frac{1}{2}$. The function f is continuous in $[c, \frac{1}{2}]$. and therefore by I.V.T. it must every value lying between f(c) and $f\left(\frac{1}{2}\right)$. But this is not possible, because f(c) and $f\left(\frac{1}{2}\right)$ are two distinct rational numbers between which there lie infinitely many irrational numbers and f(x) does not take any irrational value. The contradiction shows that there does not exist any c ∈ [0, 1] such that f(c) is different from $\frac{1}{2}$. Hence $f(x) = \frac{1}{2}$ everywhere.

Example 15: Let f and g be continuous functions on an interval I, let f(x) ≠ 0 for any x ∈ I and let $(f(x))^2 = (g(x))^2$ for all x ∈ I. Prove that either f(x) = g(x) for all x ∈ I or f(x) = –g(x) for all x ∈ I.

Solution: Suppose, if possible, that there exist $x_1 \in I$ and $x_2 \in I$, such that $f(x_1) = g(x_1)$ and $f(x_2) = -g(x_2)$.

Since f is continuous on I and f(x) ≠ 0 anywhere on I, therefore $f(x_1)$ and $f(x_2)$ must be of the same sign. Consequently, $g(x_1)$ and $g(x_2)$ must be of opposite signs.

Now g is continuous on I and $g(x_1)$, $g(x_2)$ are of opposite signs. Therefore, by Intermediate Value Theorem, there exists x_0 lying between x_1 and x_2, such that $g(x_0) = 0$. Combining this with $(f(x))^2 = (g(x))^2$ for all $x \in I$, we have $f(x_0) = 0$, which is not possible.

Therefore either $f(x) = g(x)$ for all $x \in I$ or $f(x) = -g(x)$ for all $x \in I$.

Example 16: Let f be a continuous function defined over the real numbers that satisfies the Cauchy functional equation

$f(x + y) = f(x) + f(y), \forall\, x, y \in R$,

Then prove that f is linear, that is, there is a constant c such that $f(x) = cx$.

Solution: We first prove the assertion for positive integers n using induction. We then extend our result to negative integers. Then we extend the result to reciprocals of integers and after that to rational numbers. Finally we extend the result to all real numbers.

We prove by induction that for integer $n \geq 0$, $f(nx) = nf(x)$. Using the functional equation,

$$f(0 \cdot x) = f(0 \cdot x + 0 \cdot x) = f(0 \cdot x) + f(0 \cdot x)$$

$$\Rightarrow \quad f(0 \cdot x) = 0\, f(x),$$

and the assertion follows for $n = 0$. Assume $n \geq 1$ is an integer and that $f((n - 1)x) = (n - 1)f(x)$.

Then

$$f(nx) = f((n - 1)x + x) = f((n - 1)x) + f(x)$$
$$= (n - 1)\, f(x) + f(x) = nf(x),$$

proving the assertion for all strictly positive integers.

Let $m < 0$ be an integer. Then $-m > 0$ is a strictly positive integer, for which the result proved above holds, and thus, $f(-mx) = -mf(x)$. Now.

$$0 = f(0) \Rightarrow 0 = f(mx + (-mx)) = f(mx) + f(-mx)$$

$$\Rightarrow \quad f(mx) = -f(-mx) = -(mf(x)) = mf(x),$$

and the assertion follows for negative integers.

We have thus proved the theorem for all integers.

Assume now that $x = \dfrac{a}{b}$, with $a \in I$ and $b \in I-\{0\}$.

Then $f(a) = f(a \cdot 1) = af(1)$ and

$f(a) = f\left(b\dfrac{a}{b}\right) = bf\left(\dfrac{a}{b}\right)$ by the result we proved for integers

and hence

$af(1) = bf\left(\dfrac{a}{b}\right) \Rightarrow f\left(\dfrac{a}{b}\right) = f(1)\left(\dfrac{a}{b}\right)$.

We have established that for all rational numbers $x \in Q$, $f(x) = xf(1)$.

We have not used the fact that the function is continuous so far. Since the rational are dense in the reals the extension of the result to all real numbers now follows.

Example 17: f is a continuous real-valued function such that $f(x + y) = f(x)\, f(y)$ for all real x, y. If $f(2) = 5$, find $f(5)$.

Solution: More generally, since $f(nx) = f(x)^n$ for all integers n, $f(1) = c = f(1/n)^n$ for some constant c and all integers n. Thus $f(k/n) = f(1/n)^k = f(1)^{k/n} = c^{k/n}$ for all rational numbers k/n. By continuity, it follows that $f(x) = c^x$ for all real numbers x. Since $f(2) = 5$, $c = \sqrt{5}$, so $f(5) = 25\sqrt{5}$.

Example 18: Let f(x) be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

Solution: Let $g(x) = f(x) - x$. Then, g(x) is a polynomial that never vanishes. We argue that it must always have the same sign. Suppose if possible that $g(a) < 0 < g(b)$ for some reals a and b. Since g(x), being a polynomial, is continuous, the Intermediate Value Theorem applies and there must be a number c between a and b for which $g(c) = 0$, yielding a contradiction.

Thus, either $g(x) > 0$ for all x or else $g(x) < 0$ for all x. Then,

$$f(f(x)) - x = f(f(x)) - f(x) + f(x) - x$$
$$= g(f(x)) + g(x)$$

for all real x. Since g never changes sign, both g(x) and g(f(x)) have the same sign (either positive or negative) and so their sum cannot vanish. Hence $f(f(x)) \neq x$ for any real x.

Concept Problems F

1. Use the Intermediate Value Theorem to show that there is a number c in (1, 2] such that $4 - c = 2^c$.
2. Verify that the Intermediate Value Theorem applies in the indicated interval and find the value of c guaranteed by the theorem.
   (i) $f(x) = x^2 + x - 1$, [0, 5], $f(c) = 11$
   (ii) $f(x) = x^3 - x^2 + x - 2$, [0, 3], $f(c) = 4$
   (iii) $f(x) = \dfrac{x^2 + x}{x - 1}$, $\left[\dfrac{5}{2}, 4\right]$, $f(c) = 6$
3. Use the Intermediate Value Theorem to show that the equation has atleast one root :
   (i) $2x^3 + x^2 - x = 4$ (ii) $x^5 - 2x^2 + x + 11 = 0$
4. Prove that the equation $2x^3 + 5x^2 - 5x - 3 = 0$ has a root between $-\infty$ and -1, another between -1 and 0 and a third between 1 and 2.
5. Show that the equation $x^3 - 3x + 1 = 0$ has a real root in the interval (1, 2). Approximate this root.

6. Show that the equation $x^3 - 3x^2 + 1 = 0$ has three distinct roots by calculating the values of the function at $x = -3, -2, -1, 0, 1, 2$ and 3 and then applying the Intermediate Value Theorem of continuous functions on appropriate closed intervals.

7. Prove that $x^3 + x^2 - 3x - 3 = 0$ has a root between 1 and 2, between 1.5 and 1.75, between 1.625 and 1.75, etc. Show that if we continue this procedure, we can approximate the root of the equation as closely as we want.

8. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

9. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specific interval :

   (i) $\sqrt[3]{x} = -1 - x$, (0, 1) (ii) $\ln x = e^{-x}$, (1, 2)

10. Is there a number that is exactly 1 more than its cube?

11. Apply the Intermediate Value Theorem to prove that every real number has a cube root.

12. Apply the Intermediate Value Theorem to prove that the equation $x^5 + x = 1$ has a solution.

13. Apply the Intermediate Value Theorem to prove that the equation $x^3 - 4x^2 + 1 = 0$ has three solutions.

14. Let f be continuous on [a, b] and let f(x) be always rational. What can we say about f ?

15. Suppose that f and g are two functions both continuous on the interval [a, b], and such that $f(a) = g(b) = p$ and $f(b) = g(a) = q$ where $p \neq q$. Apply Intermediate Value Theorem to the function $h(x) = f(x) - g(x)$ to show that $f(c) = g(c)$ at some point c of (a, b).

16. Suppose that today you leave your home at 1 P.M. and drive to school, arriving at 2 P.M. Tomorrow you leave your school at 1 P.M. and retrace the same route, arriving home at 2 P.M. Show that at some instant between 1 and 2 P.M. you are at precisely the same point on the road both days.

17. Let $f : [1, e] \rightarrow [0,1]$ be continuous then prove that $f(x) = \ln x$ has atleast one solution in [1, e].

18. Let $f : (1, 10) \rightarrow [2,11]$ be a continuous function, then prove that it cannot be an invertible function.

19. Let f be continuous on [a, b] and suppose that $f(x) = 0$ for every rational x in [a, b]. Prove that $f(x) = 0$ for all x in [a, b].

20. Let f, g be continuous functions from R to R. Suppose that $f(x) = g(x)$ for all $x \in Q$. Prove that $f(x) = g(x)$ for all $x \in R$, i.e. f = g.

Practice Problems F

21. Show that the equation

$$x + \sin x = \frac{1}{\sqrt{x}+3}$$

has atleast one solution on the interval $[0, \pi]$.

22. Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has atleast one root in the interval [0, 1].

23. Show that the equation $x^5 - 2x^3 + x^2 - 3x + 1 = 0$ has atleast one root in the interval [1, 2].

24. Find roughly the situations of the roots of $2x^3 - 3x^2 - 36x + 10 = 0$.

25. Apply the Intermediate Value Theorem to show that every positive number a has a square root. That is, given $a > 0$, prove that there exists a number r such that $r^2 = a$.

26. Suppose that $a < b < c$. If the function f is continuous on the closed interval [a, b] and on the closed interval [b, c], does it follow that f is continuous on [a, c] ? If f is continuous on the closed interval [n, n + 1] for every integer n, does it follow that f is continuous on the entire real line?

27. Let f be a continuous function on R and periodic with fundamental period 1 i.e. $f(x + 1) = f(x)$, then prove that there will be a real number x_0, such that $f(x_0 + \pi) = f(x_0)$.

28. Use the Intermediate Value Theorem to show that there is a square with a diagonal length that is between r and 2r and an area that is half the area of a circle of radius r.

29. Let $f : [0, 1] \rightarrow [0, 1]$ and $g : [0, 1] \rightarrow [0, 1]$ be continuous fractions. Given that $f(0) < g(0)$ and $(f(1))^3 > g(1)$ prove that there is a number c in (0, 1) such that $(f(c))^3 = g(c)$

30. Prove that $\sqrt{x+3} = e^x$ has atleast one real root.

31. Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} c_k x^k$, such that the first and last coefficients c_0 and c_n have opposite signs. Prove that $f(x) = 0$ for atleast one positive x.

32. Given a real valued function f which is continuous on the closed interval [a, b]. Assume that $f(a) \leq a$ and that $f(b) \geq b$. Prove that f has a fixed point in [a, b].

33. Prove that if f is continuous and has no zeros in [a, b], then either $f(x) > 0$ for all x in [a, b] or $f(x) < 0$ for all x in [a, b].

34. A rational function can have infinitely many x-values at which it is discontinuous. True or false.

35. Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval [a,b]. If $f_1(a) < f_2(a)$ and $f_2(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

36. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has atleast two solutions in the interval [–6, 2].

37. If f(x) is a continuous function in [2, 3] which takes only irrational values for all $x \in [2, 3]$ and $f(2.5) = \sqrt{5}$ then find f(2.8).

38. Find an interval in which the given equation has atleast one solution. Note that the interval is not unique.

(i) $\ln x = (x-2)^2$ (ii) $e^{-x} = x^3$

(iii) $\cos x - \sin x = x$ (iv) $\tan x = 2x^2 - 1$

39. Let f be a continuous function from R to R. Suppose that (i) f(1) = 1 and that (ii) f(x + y) = f(x) + f(y) for all real x and y.

(a) Prove that f(x)= x for every positive rational x.

(b) Prove that f(0)=0 and that f(–x)=–f(x) for all real x.

(c) Prove that f(x) = x for all real x.

40. Let f be a continuous function such that f(x) f(f(x)) = 1 and f(2006) = 2005, then consider the following assertion : $f(1000) = \frac{1}{1000}$. If the reason is : as $f(f(x)) = \frac{1}{f(x)}$ we have $f(t) = \frac{1}{t}$

$$\Rightarrow \quad f(1000) = \frac{1}{1000}$$

Prove that the assersion is correct but the reason is false.

41. Find all functions f : R → R, continuous at x = 1 such that $\forall\, x \in R,\ f(x) = -f(x^2)$.

42. Let f and g be continuous functions defined for all x. Assume that f(x) = g(x) for all rational x. Deduce that f(x) = g(x) for all real numbers x.

43. Let us determine all continuous functions f such that f(x + y) = f(x)f(y) for all real numbers x and y and such that the values of f are always positive.

(a) Let b be a fixed positive number. Let $f(x) = b^x$. Check that f(x + y) = f(x)f(y). [Part(b) will show that there are no other functions that satisfy the stated conditions]

(b) Assume that f is a continuous function such that f(x) > 0 for all x and f(x + y) = f(x) f(y) for all x and y. Let f(1) = c.

(i) Show that $f(n) = c^n$ for any positive integer n.

(ii) Show that f(0) = 1.

(iii) Show that $f(n) = c^n$ for any negative integer n.

(iv) Show that $f(1/n) = \sqrt[n]{c}$ for any positive integer n.

(v) Show that $f(m/n) = (\sqrt[n]{c})^m$ for any integer m and positive integer n.

(vi) By (v), $f(x) = c^x$ for any rational number x. Assuming that f is continuous and that the exponential function c^x is continuous, deduce that $f(x) = c^x$ for all real numbers x.

44. Let f be a continuous function whose domain is the x axis and which has the property that f(x + y) = f(x) + f(y) for all numbers x and y. This question shows that f must be of the form f(x) = cx for some constant c.

(a) Let f(1) = c. Show that f(2) = 2c.

(b) Show that f(0) = 0.

(c) Show that f(–1) = –c.

(d) Show that for any positive integer n, f(n) = cn.

(e) Show that for any negative integer n, f(n) = cn.

(f) Show that $f\left(\frac{1}{2}\right) = c/2$.

(g) Show that for any nonzero integer n, f(1/n) = c/n.

(h) Show that for nay integer m and positive integer n, f(m/n) = c(m/n).

(i) Show that for any irrational number x, f(x) = cx. (This is where the continuity of f enters).

Target Problems for JEE Advanced

Problem 1: Find the values of a and b if f is continuous at x = π/2, where

$$f(x) = \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}, \qquad 0 < x < \frac{\pi}{2}$$

$$= a + 2, \qquad x = \frac{\pi}{2}$$

$$= (1 + |\cot x|)^{\frac{b|\tan x|}{a}}, \qquad \frac{\pi}{2} < x < \pi$$

Solution: We have $f\left(\frac{\pi}{2}\right) = a + 2$

$$\lim_{h\to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan 6(\pi/2 - h)}{\tan(5\pi/2 - 5h)}}$$

$$= \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan(5\pi/2 - 5h)}} = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{-\tan 6h}{\cot 5\pi}}$$

$$= \lim_{h\to 0}\left(\frac{6}{5}\right)^{-\tan 6h \tan 5h} = \lim_{h\to 0}\left(\frac{6}{5}\right)^{0} = 1$$

$$\lim_{h\to 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h\to 0}\left(1 + \left|\cot\left(\frac{\pi}{2} + h\right)\right|\right)^{\frac{b|\tan(\pi/2 + h)|}{a}}$$

$$= \lim_{h\to 0}(1 + \tan h)^{\frac{b\coth}{a}} = e^{b/a}.$$

Since f is continuous at x = π/2, we have

$$a + 2 = 1 = e^{b/a}$$

which gives a = –1, b = 0

Problem 2: Obtain a relation in a and b, if possible, so that the function

$$f(x) = \lim_{n\to\infty} \frac{x^n(a+\sin(x^n)) + (b-\sin(x^n))}{(1+x^n)\sec(\tan^{-1}(x^n+x^{-n}))}$$

is continuous at x = 1.

Solution: $f(x) = \lim_{n\to\infty} \frac{x^n(a+\sin(x^n)) + (b-\sin(x^n))}{(1+x^n)\sec(\tan^{-1}(x^n+x^{-n}))}$

For continuity at x = 1

$\lim_{x\to 1} f(x)$ must exist and equals $f(1)$

$$f(1) = \lim_{n\to\infty} \frac{1^n(a+\sin 1^n) + b - \sin(1^n)}{(1+1^n)\cdot\sec\left(\tan^{-1}(1^n+1^{-n})\right)}$$

$$= \frac{a+\sin 1 + b - \sin 1}{\sec(\tan^{-1} 2)} = \frac{a+b}{2\sqrt{5}}$$

Now for x > 1 in the immediate neighbourhood

$$f(x) = \lim_{n\to\infty} \frac{a+\sin(x^n)+\dfrac{b-\sin x^n}{x^n}}{\left(1+\dfrac{1}{x^n}\right)\sec(\tan^{-1}(x^n+x^{-n}))}$$

$$= \frac{a + (\text{some quantity between 1 and} -1) + 0}{1\cdot\sec(\tan^{-1}\infty)} = 0$$

Similarly for x < 1 in the immediate neighbourhood of 1

$$f(x) = \frac{b}{1\cdot\sec(\tan^{-1}\infty)} = 0$$

Hence $f(x) = 0$ for $x \neq 1$

$\therefore \lim_{x\to 1} f(x) = 0 = a + b.$

Problem 3: If $f(x) = \dfrac{A\cos x + Bx\sin x - 5}{x^4}$, $(x \neq 0)$ is continuous at x = 0, then find the value of A and B. Also find $f(0)$.

Solution: For continuity $\lim_{x\to 0} f(x) = f(0)$

Now for $\lim_{x\to 0} \dfrac{A\cos x + Bx\sin x - 5}{x^4}$ to exist

as $x \to 0$, Numerator $\to A-5$ and Denominator $\to 0$. Hence $A - 5 = 0 \Rightarrow A = 5$

Hence $\lim_{x\to 0} \dfrac{Bx\sin x - 5(1-\cos x)}{x^4}$

$$= \lim_{x\to 0} \frac{B\cdot\dfrac{\sin x}{x} - \dfrac{5}{1+\cos x}\,\dfrac{\sin^2 x}{x^2}}{x^2}$$

as $x \to 0$, Numerator $\to B - \dfrac{5}{2}$ and

Denominator $\to 0 \Rightarrow B = \dfrac{5}{2}$

$$\lim_{x\to 0} \frac{x\sin x - 2(1-\cos x)}{x^4}$$

$$= \frac{5}{2}\lim_{x\to 0} \frac{2x\sin\frac{x}{2}\cos\frac{x}{2} - 4\sin^2\frac{x}{2}}{x^4}$$

$$= \frac{5}{2}\lim_{x\to 0} \frac{2\sin\frac{x}{2}}{x}\lim_{x\to 0} \frac{x\cos\frac{x}{2} - 2\cos\frac{x}{2}}{x^3}$$

Let $x = 2\theta$

$$= \frac{5}{16}\lim_{\theta\to 0} \frac{2\theta\cos\theta - 2\sin\theta}{\theta^3}$$

$$= \frac{5}{16}\lim_{\theta\to 0} 2\cos\theta\,\frac{(\theta-\tan\theta)}{\theta^3}$$

$$= \frac{5}{8}\lim_{\theta\to 0} \frac{\theta-\tan\theta}{\theta^3} = \frac{5}{8}\left(-\frac{1}{3}\right) = -\frac{5}{24}.$$

Since f is continuous $f(0) = -\dfrac{5}{24}$.

Problem 4: Let S denotes the sum of an infinite geometric progression whose first term is the value of the function $f(x) = \dfrac{\sin(x-(\pi/6))}{\sqrt{3}-2\cos x}$ at $x = \pi/6$, if $f(x)$ is continuous at $x = \pi/6$ and whose common ratio is the limiting value of the function $g(x) = \dfrac{\sin(x)^{1/3}\ln(1+3x)}{(\tan^{-1}\sqrt{x})^2(e^{5x^{1/3}}-1)}$ as $x \to 0$. Find the value of S.

Solution: $a = f\left(\dfrac{\pi}{6}\right) = \lim_{x\to\pi/6} \dfrac{\sin\left(x-(\pi/6)\right)}{\sqrt{3}-2\cos x}$

$$= \lim_{x\to\pi/6} \frac{\sin\left(x-(\pi/6)\right)}{2\left(\cos(\pi/6)-\cos x\right)}$$

$$= \lim_{x\to\pi/6} \frac{2\sin((x/2)-(\pi/12))\cos((x/2)-(\pi/12))}{4\sin((\pi/12)+(x/2))\sin((\pi/12)+(x/2))}$$

$$= \frac{2}{2} = 1.$$

Hence, a = 1

$$r = \lim_{x\to 0} \frac{\sin(x)^{1/3}.\ln(1+3x)}{\left(\dfrac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\right)^2\left(e^{5.x^{1/3}}-1\right)}\,\frac{\left(5x^{1/3}\right)}{\left(5x^{1/3}\right)}$$

$$= \lim_{x\to 0} \frac{3\ln(1+3x)^{1/3x}}{5} = \frac{3}{5}.$$

$$\therefore \text{ Sum } S = \frac{a}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$$

Problem 5: Let f (x)

$$= \begin{cases} \dfrac{a^x - 1}{x^n \sin x}\left(\dfrac{b\sin x - \sin bx}{\cos x - \cos bx}\right)^n & x > 0 \\ \\ \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax} & x < 0 \end{cases}$$

be continuous at x = 0 (a, b > 0 ,b ≠ 1, a ≠ b). Obtain f(0) and a relation between a, b and n.

Solution:

$$f(0^+)\ \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{a^h - 1}{\sinh . h^n}\left[\frac{b\sin h - \sin b h}{\cos h - \cos b h}\right]^n$$

$$= = \lim_{h\to 0} \frac{a^h - 1}{h}\ \frac{h}{\sin h}$$

$$\left[\left(\frac{b\sin(h) - \sin(bh)}{h.h^2}\right)\frac{h^2}{\cos h - \cos bh}\right]^n$$

= *l*n a · *l*^{n} (say)

$$\text{where } l = \lim_{h\to 0} \frac{b\sin h - \sin bh}{h^3}\cdot\frac{h^2}{\cos h - \cos bh}$$

$$= \lim_{h\to 0}\left[\frac{b(\sin h - h) - (\sin bh - bh)}{h^3}\right]$$

$$\cdot\left[b^2\left(\frac{1-\cos bh}{b^2 h^2}\right) - \left(\frac{1-\cos h}{h^2}\right)\right]^{-1}$$

$$= \lim_{h\to 0}\left[b\left(\frac{\sin h - h}{h^3}\right) - \left(\frac{\sin bh - bh)}{b^3h^3}\right)b^3\right]$$

$$\cdot\left[\left(\frac{b^2}{2} - \frac{1}{2}\right)^{-1}\right]$$

$$= \left[b\left(-\frac{1}{6}\right) - b^3\left(-\frac{1}{6}\right)\right]\cdot\frac{2}{(b^2-1)}$$

$$[\text{using } \lim_{h\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}\ ]$$

$$= \frac{b(b^2-1)}{6}\cdot\frac{2}{(b^2-1)} = \left(\frac{b}{3}\right)$$

Hence f (0$^+$) = *l*n a · $\left(\frac{b}{3}\right)^n$

Now f (0$^-$) = $\lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{a^{-h}\sin(-bh) - b^{-h}\sin(-ah)}{\tan(-bh) - \tan(-ah)}$$

(multiply D^r & N^r by a^h · b^h)

$$= \lim_{h\to 0}\frac{a^h \sin ah - b^h \sin bh}{a^h . b^h [\tan ah - \tan bh]}$$

$$= \lim_{h\to 0}\frac{(1 + h\ln a.....)(ah....) - (1 + h\ln b)(bh...)}{(ah...) - (bh....)}$$

$$= \lim_{h\to 0}\frac{(a-b)h + ...}{(a-h)h + ...} = 1$$

∴ If f (x) is continuous at x = 0, then f (0) = 1.

Also the continuity relationship between a, b and n should be *l*n a · $\left(\frac{b}{3}\right)^n = 1$.

Problem 6: If f (x)

$$= \frac{\sin 3x + A\ \sin 2x + B\ \sin x}{x^5}\ (x \neq 0)$$

is continuous at x = 0, then find f (0).

Solution: We have

$$f(0) = \lim_{x\to 0}\frac{1}{x^5}\left(\left[3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - ......\right]\right.$$

$$+A\left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - .....\right]$$

$$\left.+B\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ......\right]\right) \quad ...(1)$$

Now f (x) is continuous at x = 0, so we must have

$$2A + 3 + B = 0 \quad ...(2)$$

$$\text{and} \quad \frac{27}{6} + \frac{8A}{6} + \frac{B}{6} = 0$$

$$\Rightarrow \quad 8A + B = -27 \quad ...(3)$$

∴ On solving (2) and (3), we get

A = – 4, B = 5

$$\text{Hence, } f(0) = \frac{3^5}{5!} + \frac{A.2^5}{5!} + \frac{B}{5!} = 1.$$

Alternative:

We have f (0)

$$= \lim_{x\to 0}\frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x + B\sin x}{x^5}$$

For limit to exist $3 + 2A + B = 0$ ...(4)

$$= \lim_{x\to 0}\frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x - (3+2A)\sin x}{x^5}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\left[\frac{3 - 4\sin^2 x + 2A\cos x - 3 - 2A}{x^4}\right]$$

$$= \lim_{x\to 0}\frac{-2A(1-\cos x) - 4\sin^2 x}{x^4}$$

$$= \lim_{x\to 0}\frac{-4A\sin^2\dfrac{x}{2} - 4\sin^2 x}{x^4}$$

$$= \lim_{x\to 0} \frac{-4A\sin^2\frac{x}{2} - 16\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{x^4}$$

$$= -\lim_{x\to 0} \frac{\sin^2\frac{x}{2}}{\frac{x^4}{4}} \left[\frac{A + 4\cos^2\frac{x}{2}}{x^2}\right]$$

$\therefore$ $A + 4 = 0 \Rightarrow A = -4 \Rightarrow B = 5$

Also $f(0) = 1$.

Problem 7: The function

$f(x) = \dfrac{e^{2x} - 1 - x\left(e^{2x} + 1\right)}{x^3}$ is not defined at $x = 0$. What should be the value of $f(x)$ so that $f(x)$ is continuous at $x = 0$?

Solution: $\ell = \lim\limits_{x\to 0} \dfrac{e^{2x} - 1 - x(e^{2x} + 1)}{x^3}$

Put $x = 3t = \lim\limits_{t\to 0} \dfrac{e^{6t} - 1 - 3t\,(e^{6t} + 1)}{27\,t^3}$

$$= \lim_{t\to 0} \frac{(e^{2t} - 1)^3 + 3e^{2t}\,(e^{2t} - 1) - 3t\,(e^{6t} + 1)}{27\,t^3}$$

$$= \lim_{t\to 0} \frac{(e^{2t}-1)^3 + 3e^{2t}[e^{2t}-1-t(e^{2t}+1)] + 3t[e^{2t}(e^{2t}+1) - e^{6t} - 1]}{27\,t^3}$$

$$= \lim_{t\to 0} \frac{(e^{2t} - 1)^3}{27\,t^3} + \frac{1}{9}\lim_{t\to 0} e^{2t} \times \ell$$

$$- \lim_{t\to 0} \frac{1}{9\,t^2}\,(e^{2t} - 1)\,(e^{4t} - 1)$$

$$\Rightarrow \frac{8\,\ell}{9} = \frac{8}{27} - \frac{8}{9}\lim_{t\to 0}\left(\frac{e^{2t} - 1}{2\,t}\right) \times \left(\frac{e^{4t} - 1}{4\,t}\right)$$

$$\Rightarrow \ell = \frac{1}{3} - 1 \Rightarrow \ell = -\frac{2}{3}.$$

The value of $f(0) = \ell = -\dfrac{2}{3}$

Problem 8: Let $f(x) = \operatorname{cosec} 2x + \operatorname{cosec} 2^2 x + \operatorname{cosec} 2^3 x + \ldots\ldots\ldots\ \operatorname{cosec} 2^n x$, $x \in (0, \pi/2)$

and $g(x) = f(x) + \cot 2^n x$

If

$$H(x) = \begin{cases} (\cos x)^{g(x)} + (\sec x)^{\operatorname{cosec} x} & \text{if } x > 0 \\ p & \text{if } x = 0 \\ \dfrac{e^x + e^{-x} - 2\cos x}{x\sin x} & \text{if } x < 0 \end{cases}$$

find the value of p, if possible to make the function H (x) continuous at x = 0.

Solution: $f(x) = \operatorname{cosec} 2x + \operatorname{cosec} 2^2x + \ldots. + \operatorname{cosec} 2^n x$

Now $\operatorname{cosec} 2x = \dfrac{1}{\sin 2x} = \dfrac{\sin(2x - x)}{\sin x \sin 2x} = \cot x - \cot 2x$

Similarly $\operatorname{cosec} 2^2 x = \cot 2x - \cot 2^2 x$

$\operatorname{cosec} 2^3 x = \cot 2^2 x - \cot 2^3 x$

$\vdots$

$\operatorname{cosec} 2^n x = \cot 2^{n-1} x - \cot 2^n x$

$\Rightarrow f(x) = \cot x - \cot 2^n x$

$\therefore$ $g(x) = f(x) + \cot 2^n x = \cot x$

Now $\lim\limits_{h\to 0} H(0 + h) = \lim\limits_{h\to 0} ((\cos h)^{\cot h} + (\sec h)^{\operatorname{cosech}})$

$$= e^{\lim\limits_{h\to 0}\coth(\cos h - 1)} + e^{\lim\limits_{h\to 0}\operatorname{cosech}(\sec h - 1)}$$

$= 1 + 1 = 2$ ...(1)

$$H(0 - h) = \lim_{h\to 0} \frac{e^{-h} + e^{h} - 2\cos h}{h\sec h}$$

$$= \lim_{h\to 0}\left\{\frac{e^{h} + e^{-h} - 2}{h^2} + \frac{2(1 - \cos h)}{h^2}\right\} = 2 \quad \ldots(2)$$

From (1) and (2) H (x) will be continuous if p = 2.

Problem 9: Check continuity of the function

$$f(x) = \lim_{n\to\infty} \frac{\cos \pi x - x^{2n}\sin(x - 1)}{1 + x^{2n+1} - x^{2n}}$$

Solution: We have

$$f(x) = \begin{cases} \dfrac{\cos\pi x - 0}{1 + x - 0}, & |x| < 1 \quad \left[\because \lim\limits_{n\to\infty} x^{2n} = 0 \text{ if } |x| < 1\right] \\ \dfrac{\cos\pi x - \sin(x - 1)}{1 + 1 + 1}, & |x| = 1 \\ \lim\limits_{n\to\infty} \dfrac{\dfrac{\cos\pi x}{x^{2n}} - \sin(x - 1)}{\dfrac{1}{x^{2n}} + x - 1} = \dfrac{-\sin(x - 1)}{x - 1}, & |x| > 1 \end{cases}$$

i.e. $$f(x) = \begin{cases} \cos\pi x, & |x| < 1 \\ -1 + \sin 2, & x = -1 \\ -1, & x = 1 \\ \dfrac{-\sin(x - 1)}{x - 1}, & |x| > 1 \end{cases}$$

The above function may have discontinuities only at $x = \pm 1$.

At $x = -1$, we have

$$\lim_{h\to 0} f(-1 + h) = \lim_{h\to 0} \cos\pi(-1 + h) = -1.$$

$f(-1) = -1 + \sin 2$.

This implies that f(x) is discontinuous at $x = -1$.

At $x = 1$, we have

$$\lim_{h\to 0} f(1 + h) = \lim_{h\to 0} \frac{-\sin(1 + h - 1)}{1 + h - 1} = \lim_{h\to 0} \frac{-\sinh}{h} = -1.$$

$$\lim_{h\to0} f(1-h) = \lim_{h\to0} \cos\pi(1-h) = -1.$$

$f(1) = -1$

Thus, f(x) is discontinuous at x = 1.

Hence, f(x) is continuous on $x \in R - \{-1\}$.

Problem 10: If $g(x) = \lim_{m\to\infty} \dfrac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3}$ is continuous at x = 1 and $g(1) = \lim_{x\to1}\{\ln(ex)\}^{2/\ln x}$, then find the value of $2g(1) + 2f(1) - h(1)$; assume that f(x) and h(x) are continuous at x = 1.

Solution: Here,

$$g(1) = \lim_{x\to1} \{\ln e + \ln x\}^{2/\ln x}$$

$$= \lim_{x\to1} \{1 + \ln x\}^{2/\ln x}$$

$$= e^{\lim_{x\to1} \ln x \frac{2}{\ln x}}$$

$$g(1) = e^2 \qquad \text{...(1)}$$

Now, $$\lim_{x\to1^-} g(x) = \lim_{x\to1^-}\lim_{m\to\infty}\left\{\frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \frac{h(1)+1}{3+3} \qquad \{\text{since } x < 1 \Rightarrow \lim_{m\to\infty} x^m = 0\}$$

$$\therefore \quad \lim_{x\to1^-} g(x) = \frac{h(1)+1}{6} \qquad \text{...(2)}$$

And, $$\lim_{x\to1^+} g(x) = \lim_{x\to1^+}\lim_{m\to\infty}\left\{\frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \lim_{x\to1^+}\lim_{m\to\infty} \frac{f(1) + h(x)/x^m + 1/x^m}{2 + 3/x^{m-1} + 3/x^m} = \frac{f(1)}{2}$$

$$\therefore \quad \lim_{x\to1^+} g(x) = \frac{f(1)}{2} \qquad \text{...(3)}$$

As g(x) is continuous at x = 1, from (1), (2) and (3)

$$e^2 = \frac{h(1)+1}{6} = \frac{f(1)}{2}$$

$\Rightarrow \quad h(1) = 6e^2 - 1 \qquad$ and $\qquad f(1) = 2e^2$

$\Rightarrow \quad 2g(1) + 2f(1) - h(1) = 2e^2 + 4e^2 - 6e^2 + 1 = 1$

$\therefore \quad 2g(1) + 2f(1) - h(1) = 1.$

Problem 11: Prove that $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$. (where [.] denotes greatest integer function) is continuous in $\left[0, \dfrac{\pi}{2}\right)$.

Solution: $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$

Let $\quad g(u) = [u] + \sqrt{u - [u]}$,

Then f(x) = g(u(x)) where $u(x) = \tan x \geq 0$

We discuss continuity of g(u) for $u = a \in N$.

LHL at u = a:

$$\lim_{u\to a^-} g(u) = \lim_{h\to0}\, [a-h] + \sqrt{a - h - [a-h]}$$

$$= \lim_{h\to0}\, (a-1) + \sqrt{a - h - (a-1)}$$

$$= a - 1 + 1 = a$$

Now RHL at u = a :

$$\lim_{u\to a^+} g(u) = \lim_{h\to0}\, [a+h] + \sqrt{a + h - [a+h]}$$

$$= \lim_{h\to0}\, a + \sqrt{a + h - a}$$

$$= a,$$

and $\quad g(a) = [a] + \sqrt{a - [a]} = a \qquad$ as $a \in N$.

So, g(u) is continuous ∀ $a \in N$, now g(u) is clearly continuous in (a – 1, a) ∀ $a \in N$.

Hence g(u) is continuous in $[0, \infty)$.

Now u(x) = tan x is continuous in $[0, \pi/2)$.

So, f(x) = g{u(x)} is continuous in $[0, \pi/2)$.

Problem 12: If $f(x) = \dfrac{x}{(1+x)} + \dfrac{x}{(1+x)(1+2x)} + \dfrac{x}{(1+2x)(1+3x)} + \ldots\ldots$ to infinity, then examine the continuity of f at x = 0.

Solution: For $x \neq 0$, sum of n terms of the series

$$\left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{1+2x}\right) + \left(\frac{1}{1+2x} - \frac{1}{1+3x}\right) + \ldots + \left(\frac{1}{1+(n-1)x} - \frac{1}{1+nx}\right)$$

$$= 1 - \frac{1}{1+nx}.$$

$$\therefore \quad f(x) = \lim_{n\to\infty}\left(1 - \frac{1}{1+nx}\right) = 1 - 0 = 1.$$

and f(0) = 0

$$\therefore \quad f(x) = \begin{cases} 1, & x \neq 0 \\ x, & x = 0 \end{cases}$$

Clearly f(x) is discontinuous at x = 0.

Problem 13: Show that the function $f : R \to R$ defined by $f(x) = \dfrac{1}{1 + e^{1/(\sin n!\, \pi x)}}$ can be made discontinuous at any rational point in the interval [0, 1] by a proper choice of n.

Solution: Let x be rational say p/q, where p and q are integers prime to each other. Then taking n = q, we see that

$$n!\, \pi x = q!\, .\, \pi\, .\, (p/q)$$

is an integral multiple of π and therefore sin n! πx = 0 but cos n! πx = 1 or –1 according as n! x is an even or an odd integer. Now

$\sin n!\,\pi(x \pm h) = \sin n!\,\pi x \cos n!\,\pi h \pm \cos(n!\,\pi x)\sin(n!\,\pi h)$

$$= \pm \sin n!\,\pi h \text{ or } \mp \sin(n!\,\pi h) \qquad ...(1)$$

according as n! x is even or odd.

(i) Let n! x be an even integer.

$$\text{Then } f(x^+) = \lim_{h\to 0} \frac{1}{1+e^{1/\sin\{n!\pi(x+h)\}}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{1/\sin(n!\,\pi h)}} = \frac{1}{1+e^{\infty}} = 0 \quad [\text{using (1)}]$$

$$\text{and } f(x^-) = \lim_{h\to 0} \frac{1}{1+e^{1/\sin\{n!\pi(x-h)\}}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{-1/\sin(n!\pi h)}} \quad [\text{using (1)}]$$

$$= \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

(ii) Let n! x be an odd integer. Then

$$f(x^+) = \lim_{h\to 0} \frac{1}{1+e^{-1/\sin(n!\pi h)}} = 1$$

$$\text{and } f(x^-) = \lim_{h\to 0} \frac{1}{1+e^{1/\sin(n!\pi h)}} = 0.$$

Hence $f(x^+) \neq f(x^-)$ at any rational point x.

Problem 14: Show that the function $f : R \to R$ defined by

$$f(x) = \lim_{t\to\infty} \frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}$$

has discontinuity of first kind at the points

x = 0, 1, 2, ..., n .... .

Solution: At x = 0, 1, 2, 3, ... , n .... , we have

$$\sin \pi x = 0, \text{ so that } f(x) = \lim_{t\to\infty} \frac{(1+0)^t - 1}{(1+0)^t + 1} = 0$$

at these values.

Now, if 2m < x < 2m + 1 (m being an integer), then sin πx is positive. Hence for such values of x, we have

$$f(x) = \lim_{t\to\infty} \frac{1 - 1/(1+\sin \pi x)^t}{1 + 1/(1+\sin \pi x)^t} = \frac{1-(1/\infty)}{1+(1/\infty)} = 1$$

And if 2m + 1 < x < 2m + 2. sin πx is negative and so

$$\lim_{t\to\infty} (1+\sin \pi x)^t = 0.$$

$$\therefore \quad f(x) = \frac{0-1}{0+1} = -1 \text{ for these values of x.}$$

Hence if x an even integer, then

$f(x) = 0,\ f(x^+) = 1$ and $f(x^-) = -1$,

and if x is an odd integer, then

$f(x) = 0,\ f(x^+) = -1$ and $f(x^-) = 1$.

Hence f has discontinuities of the first kind at

x = 0, 1, 2, ... , n , ... ,

Problem 15: Discuss the nature of the discontinuity of the function f defined by

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}} \text{ at } x = 1.$$

Show that f(0) and f(π/2) differ in sign and explain still why f does not vanish in [0, π/2].

Solution: We shall first of all obtain an expression for f in [0, π/2] in a form free from limits.

If $0 \le x < 1$, then $x^{2n} \to 0$ as $n \to \infty$, and if $x > 1$, then $x^{-2n} \to 0$ as $n \to \infty$, therefore we have:

$$\text{If} \quad 0 \le x < 1, \text{ then } f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

$$= \log(2+x) \qquad ...(1)$$

$$\text{If} \quad x = 1, \text{ then } f(x) = \lim_{n\to\infty} \frac{\log(3) - 1^{2n}\sin 1}{+1^{2n}}$$

$$= \frac{1}{2}(\log 3 - \sin 1) \qquad ...(2)$$

$$\text{If} \quad x > 1, \text{ then } f(x) = \lim_{n\to\infty} \frac{x^{-2n}\log(2+x) - \sin x}{x^{-2n}+1}$$

$$= -\sin x \qquad ...(3)$$

From (1), (2) and (3), we have

$$f(x) = \begin{cases} \log(2+x) & \text{if } 0 \le x < 1 \\ \frac{1}{2}(\log 3 - \sin 1), & \text{if } x = 1 \\ -\sin x & \text{if } x > 1 \end{cases} \qquad ...(4)$$

$$f(1^-) = \lim_{h\to 0^+} f(1-h) = \lim_{h\to 0^+} \log(3-h) = \log 3$$

$$f(1^+) = \lim_{h\to 0^+} f(1+h) = \lim_{h\to 0^+} -\sin(1+h) = -\sin 1.$$

We find that $f(1^-)$ and $f(1^+)$ both exist but are unequal; also neither of them is equal to f(1). Therefore f has a discontinuity at x = 1.

From (4), we find that

f(0) = log 2 > 0, f(π/2) = –sin π/2 = –1,

so that f(0) and f(π/2) are of opposite signs.

Again, from (4) it is clear that f does not vanish anywhere in [0, π/2].

The function f is not continuous in [0, π/2], the point x = 1 being a point of discontinuity. This explains as to why f does not vanish anywhere in [0, π/2] even though f(0) and f(π/2) are of opposite signs.

The hypothesis as well as the conclusion of the Intermediate Value Theorem are not satisfied for the function f in [0, π/2].

Things to Remember

1. A function f(x) is said to be continuous at $x = a$, if $\lim\limits_{x\to a} f(x) = f(a)$.
2. A function f(x) is said to be continuous at the left end point $x = a$ if, $f(a) = \lim\limits_{x\to a^+} f(x)$ and f(x) is said to be continuous at the right end point $x = b$ if, $f(b) = \lim\limits_{x\to b^-} f(x)$
3. $$F(x)=\begin{cases} f(x) & \text{if } x \text{ is in the domain of } f\\ L & \text{if } x=a\end{cases}$$
   The function F is continuous is called the continuous extension of f to $x = a$, provided $\lim\limits_{x\to a} f(x) = L$ exists.
4. A function f is said to be continuous in an open interval (a , b) if f is continuous at each and every point lying in the interval (a, b).
5. A function f is said to be continuous in a closed interval [a, b] if :
   (i) f is continuous in the open interval (a , b)
   (ii) f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a)$.
   (iii) f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b)$.
6. We use the term suspicious point for a number c where
   (i) The definition of the function changes or domain of f splits,
   (ii) Substitution of $x = c$ causes division by 0 in the function.
7. If both the one-sided limits $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist, but the conditions of continuity are not satisfied. Then the function f(x) is said to have a discontinuity of the first kind at the point a.
8. A function f(x) having a finite number of discontinuities of first kind in a given interval is called sectionally or piecewise continuous function.
9. The function f(x) is said to have discontinuity of the second kind at $x = a$, if atleast one of the one-sided limits (L.H.L. or R.H.L.) at the point $x = a$ does not exist or equals to infinity.
10. A function is said to have a removable discontinuity at $x = a$, if $\lim\limits_{x\to a} f(x)$ exists but is not equal to f(a). In this case we can redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ and make it continuous at $x = a$.
11. A function is said to have a missing point discontinuity at $x = a$ if $\lim\limits_{x\to a} f(x)$ exists while the function is undefined at $x = a$.
12. A function is said to have an isolated point discontinuity at $x = a$ if $\lim\limits_{x\to a} f(x)$ exists and the function is defined at $x = a$, but they are unequal.
13. A function is said to have an irremovable discontinuity at $x = a$, if $\lim\limits_{x\to a} f(x)$ does not exist. In this case we cannot redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ and make it continuous at $x = a$.
14. A function is said to have a finite or jump discontinuity at $x = a$ if $\lim\limits_{x\to a} f(x)$ does not exist since the left hand limit and the right hand limit are unequal, but the one-sided limits do exist.
15. If $x = a$ is a point of finite discontinuity of the function f(x), then the graph of this function undergoes a jump at $x = a$. The difference R.H.L. – L.H.L. i.e. $f(a^+) - f(a^-)$ is called the jump in the function at $x = a$.
16. The difference between the greatest and least of the three numbers $f(a^+)$, $f(a^-)$, $f(a)$ is the saltus or measure of discontinuity of the function at the point a.
17. A function is said to have an infinite discontinuity at $x = a$ if atleast one of the one-sided limits is infinite.
18. The concept of pole discontinuity is related with infinite limit. For a point $x = a$ to qualify as a pole of a function f, we must have $\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0$.
19. A function is said to have an oscillatory discontinuity at $x = a$ if atleast one of the one-sided limits does not exist because of too much oscillation in the values of the function.
20. If f(x) and g(x) are continuous at $x = a$, then the following functions are also continuous at $x = a$.
    (i) cf(x) is continuous at $x = a$, where c is any constant.
    (ii) $f(x) \pm g(x)$ is continuous at $x = a$.
    (iii) f(x). g(x) is continuous at $x = a$.
    (iv) f(x)/g(x) is continuous at $x = a$, provided $g(a) \neq 0$.
21. If f(x) is continuous at $x = a$ and g(x) is discontinuous at $x = a$, then we have the following results.
    (i) Both the functions $f(x) + g(x)$ and $f(x) - g(x)$ are discontinuous at $x = a$.
    (ii) f(x). g(x) is not necessarily discontinuous at $x = a$.
    (iii) f(x)/g(x) is not necessarily discontinuous at $x = a$.
22. If f(x) and g(x) both are discontinuous at $x = a$, then we have the following results.
    (i) The functions $f(x) + g(x)$ and $f(x) - g(x)$ are not necessarily discontinuous at $x = a$. However, atmost one of $f(x) + g(x)$ or $f(x) - g(x)$ can be continuous at $x = a$. That is, both of them cannot be continuous simultaneously at $x = a$.
    (ii) f(x). g(x) is not necessarily discontinuous at $x = a$.
    (iii) f(x)/g(x) is not necessarily discontinuous at $x = a$.

23. If f(x) is continuous at x = a and g(x) is continuous at x = f(a) then the composite function (gof)(x) is continuous at x = a.

24. Let a function f(x) be continuous at all points in the interval [a, b], and let its range be the interval [A, B] and further a function g(x) is continuous in the interval [A, B], then the composite function (gof)(x) is continuous in the interval [a, b].

25. If the function f is continuous everywhere and the function g is continuous everywhere, then the composition gof is continuous everywhere.

26. All polynomials, trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions are continuous at all points in their domains.

27. If f(x) is continuous, then $|f(x)|$ is also continuous.

28. A root of a continuous function is continuous, wherever it is defined. That is, the composition $h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$ of $f(x) = \sqrt[n]{x}$ and the function g(x) is continuous at a if g is, assuming that $g(a) \geq 0$ if n is even (so that $\sqrt[n]{g(a)}$ is defined).

29. If the function y = f(x) is defined, continuous and strictly monotonic on the interval I, then there exist a single valued inverse function x = g(y) defined, continuous and also strictly monotonic in the range of the function y = f(x).

30. Assume f is integrable on [a, x] for every x in [a, b] and let $A(x) = \int_a^x f(t)\,dt$. Then the integral A is continuous at each point of [a, b]. (At each endpoint we have one-sided continuity)

31. If a function f(x) is continuous at every point of a closed interval [a, b], then the function f(x) is bounded on this interval. Note that the boundedness of a function on the interval [a, b] means that there is a number K > 0 such that $|f(x)| < K$ for all $x \in [a, b]$.

32. If a function is continuous in a closed interval there exists atleast one point at which the function assumes the greatest value and atleast one point at which it assumes the least value on that interval.

33. If f is continuous at c and $f(c) \neq 0$, then there exists an interval $(c - \delta, c + \delta)$ around c such that f(x) has the sign of f(c) for every value of x in this interval.

34. If a function f is continuous on a closed interval [a, b] and the numbers f(a) and f(b) are different from zero and have opposite signs, then there is atleast one point c on the open interval (a, b) such that f(c) = 0.

35. If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is atleast one number c in [a, b] such that f(c) = k.

36. A function continuous on a closed interval [a, b] assumes all the intermediate values lying between its least and greatest values on [a, b].

37. A continuous function whose domain is closed must have a range in a closed interval but it is not necessary that if the domain is open then range is also open (range can be closed).

38. Let f be a continuous strictly increasing function on a closed interval [a, b] and let $\alpha = f(a)$, $\beta = f(b)$. Then (i) the range of f for the closed interval [a, b] is the closed interval $[\alpha, \beta]$, (ii) there exists a function x = g(y), the inverse of f, which is one-valued, strictly increasing and continuous on $[\alpha, \beta]$.

A strictly decreasing function f(x), continuous on [a, b] has an inverse, which is a strictly decreasing continuous function on $[\beta, \alpha]$ where $\alpha = f(a)$, $\beta = f(b)$.

39. Let a strictly increasing function y = f(x) map the closed interval [a, b] onto the closed interval $[\alpha, \beta]$, i.e. $f([a, b]) = [\alpha, \beta]$. Then f is continuous on [a, b].

40. Let f(x) be a function defined and bounded in the interval [a, b], then, if M and m are the bounds of f(x), the number M – m is called the span or oscillation of the function f(x) in the interval.

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. The function

$$f(x) = \frac{1 - \cos x (\cos 2x)^{1/2} (\cos 3x)^{1/3}}{x^2}$$ is not defined at x = 0.

If f(x) is continuous at x = 0 then f(0) equals

(A) 1 (B) 3

(C) 6 (D) –6

2. If $f(x) = \dfrac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$ is continuous at x = 0, then

(A) $f(0) = \dfrac{5}{2}$ (B) $[f(0)] = -2$

(C) $\{f(0)\} = -0.5$ (D) $[f(0)].\{f(0)\} = -1.5$

3. If $f(x) = \begin{cases} |x^2 - 1| - 1 & , x \leq 1 \\ |2x - 3| - |x - 2| & , x > 1 \end{cases}$, then it is continuous for

(A) R (B) R – {0}

(C) R – {1} (D) none of these

4. If f(x) = [x] + $\left[x+\frac{1}{3}\right]+\left[x+\frac{2}{3}\right]$ then number of points of discontinuity of f(x) in [–1, 1] is

(A) 5 (B) 4

(C) 7 (D) none

5. If f(x) = $\begin{cases} \frac{\sin\{x\}}{\{x\}}, & \{x\} \neq 0 \\ K, & \{x\} = 0 \end{cases}$, where [.] denotes fractional part function, then f(x) will be continuous

(A) if K = 0 (B) if K = sin 1

(C) if K = 1 (D) for no value of K

6. If f(x) is a continuous function from R $\to$ R and attains only irrational values, then $\sum_{r=1}^{100} f(r)$ is equal to

(A) $\sum_{r=1}^{99} f(r)$ (B) $\sum_{r=1}^{100} f(2r)$

(C) $\sum_{r=1}^{100} f(2r)+1$ (D) none of these

7. If f(x) = sgn (cos 2x – 2 sin x + 3), then f(x)

(A) is continuous over its domain

(B) has a missing point discontinuity

(C) has isolated point discontinuity

(D) irremovable discontinuity

8. Let f(x) = $(\sin x)^{\frac{1}{\pi-2x}}$, x ≠ π/2 if f(x) is continuous at x = π/2 then f(π/2) is,

(A) e (B) 1

(C) 0 (D) none

9. If f(x) = $\frac{x+1}{x}$, the number of points of discontinuity of the composite function. y = f (f(f(x))) are

(A) 0 (B) 1

(C) 2 (D) 3

10. Let f(x) = $\begin{cases} \frac{a\,|x^2-x-2|}{2+x-x^2}, & x<2 \\ b, & x=2 \\ \frac{x-[x]}{x-2}, & x>2 \end{cases}$

If f(x) is continuous at x = 2 (where [.] denotes greatest integer function) then (a, b) is

(A) (1, 1) (B) (1, 2)

(C) (2, 1) (D) (2, 2)

11. The function f(x) = | 2 sgn 2x | + 2 has

(A) jump discontinuity

(B) removable discontinuity

(C) infinite discontinuity

(D) no discontinuity

12. If g(x) = $\lim_{n\to\infty}\left(\frac{x+x^{2n}\sin x}{1+x^{4n}}\right)$ then

(A) g(x) is continuous at x = 1

(B) g(x) is discontinuous at x = 1

(C) limit does not exist at x = 0

(D) none of these

13. Let f(x) = $\frac{x^3}{4}-a\sin\pi x+3, -4\le x\le 4$. The value of f(x) is $\frac{1999}{199}$ for some x ∈ [–4, 4] This statement is

(A) true (B) false

(C) true only if a ≥ 0 (D) true only if a ∈ [–4, 4]

14. The ordered pair (a, b) such that

$$f(x)=\begin{cases} \frac{be^x-\cos x-x}{x^2}, & x>0 \\ a, & x=0 \\ 2\frac{\left(\tan^{-1}(e^x)-\frac{\pi}{4}\right)}{x}, & x<0 \end{cases}$$

becomes continuous at x = 0 is

(A) (1, 1) (B) $\left(\frac{1}{2},1\right)$

(C) $\left(\frac{1}{2},\frac{1}{2}\right)$ (D) not possible

15. The function f(x) = $\begin{cases} 2x+1, & x\in Q \\ x^2-2x+5, & x\notin Q \end{cases}$ is

(A) continuous no where

(B) continuous at every rational point

(C) continuous at irrational points only

(D) continuous exactly at one point

16. Let [x] denotes the greatest integer less than or equal to x. If f(x) = [x cos x], the f(x) is

(A) continuous at x = 0

(B) continuous in (–1, 0)

(C) discontinuous at x = 1

(D) continuous in (–1, 1)

17. If f(x) = $\frac{\sin^4\left(\frac{1}{x}\right)-\sin^2\left(\frac{1}{x}\right)+1}{\cos^4\left(\frac{1}{x}\right)-\cos^2\left(\frac{1}{x}\right)+1}$ is to be made continuous at x = 0, then f(0) should be equal to

(A) 0 (B) 1

(C) 1/3 (D) 1/2

18. If $f(x) = \lim\limits_{n\to\infty} (1 - \sin^2 x)^n$, $n \in N$, then

(A) $f\left(\frac{\pi}{4}\right)$ is nearly zero

(B) $f\left(\frac{\pi}{4}\right)$ is equal to $\frac{1}{2}$

(C) f(x) is discontinuous at infinite number of points

(D) f(x) is a periodic

19. Let $f(x) = \begin{cases} a\sin^{2n} x & \text{for } x \ge 0 \text{ and } n \to \infty \\ b\cos^{2m} x - 1 & \text{for } x < 0 \text{ and } m \to \infty \end{cases}$ then

(A) $f(0^-) \ne f(0^+)$ (B) $f(0^+) \ne f(0)$

(C) $f(0^-)$ - $f(0)$ (D) f is continuous at $x = 0$

20. f is a continuous function in [a, b]; g is a continuous function in [b, c]

A function h(x) is defined as

$h(x) = f(x)$ for $x \in [a, b)$

$= g(x)$ for $x \in (b, c]$

if f(b) = g(b), then

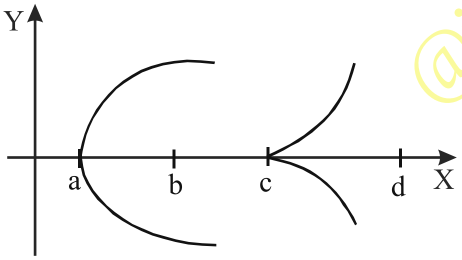
(A) h(x) has a removable discontinuity at x = b.

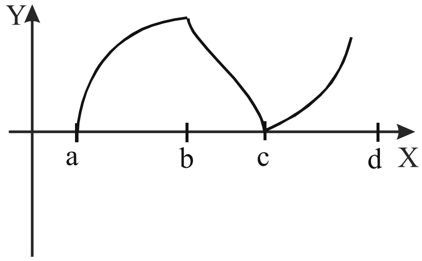
(B) h(x) may or may not be continuous in [a, b]

(C) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$

(D) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

21. If graph of |y| = f(x) and y = |f(x)| is as shown below





Then number of points of discontinuity of f(x) in [a, d] is

(A) 1 (B) 2

(C) 3 (D) 4

22. If $x \in R^+$ and $n \in N$, we can uniquely write $x = mn + r$ where $m \in W$ and $0 \le r < n$. We define x mod n = r. For example 10.3 mod 3 = 1.3. The number of points of discontinuity of the function $f(x) = (x \bmod 2)^2 + (x \bmod 4)$ in the interval $0 < x < 9$ is

(A) 0 (B) 2

(C) 4 (D) none

23. If graph of the function y = f(x) is continuous and passes through point (3, 1) then $\lim\limits_{x\to3} \frac{\ell n(3f(x)-2)}{2(1-f(x))}$ is equal

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$

(C) $-\frac{3}{2}$ (D) $-\frac{1}{2}$

24. Let f(x) = the highest power of $(u^{x^2} + u^2 + 2u + 3)$. Then at $x = \sqrt{2}$, f(x) is

(A) continuous (B) $\lim\limits_{x\to\sqrt{2}} f(x) = -2$

(C) discontinuous (D) none of these

25. Let $f(x) = [\sin x + \cos x]$, $0 < x < 2\pi$, (where [.] denotes the greatest integer function) Then the number of points of discontinuity of f(x) is

(A) 6 (B) 5

(C) 4 (D) 3

26. If α, $\beta (\alpha < \beta)$ are the points of discontinuity of the function f(f(f(x))) where $f(x) = \frac{1}{1-x}$, then the set of values of 'a' for which the points (α, β) and (a, a^2) lie on the same side of the line $x + 2y - 3 = 0$, is

(A) $\left(-\frac{3}{2}, 1\right)$ (B) $\left[-\frac{3}{2}, 1\right]$

(C) $[1, \infty)$ (D) $\left(-\infty, -\frac{3}{2}\right]$

27. Let $f : (0, \infty) \to R$ be a continuous function such $2 \le \frac{f(x)}{x^2} \le 3$ for all x > 0. Then with certainity f(c) = c for atleast one point 'c' belonging to the interval (s)

(A) $\left(\frac{1}{2}, 1\right)$ (B) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(C) $\left(\frac{1}{8}, \frac{1}{4}\right)$ (D) $\left(\frac{1}{4}, 1\right)$

28. If $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi)$, $n \in N$ then

(A) the minimum value of n is 5

(B) the maximum value of n is 6

(C) there are exactly two possible values of n

(D) none of these

29. $f(x) = \lim\limits_{n\to\infty} \sin^{2n}(\pi x) + \left[x + \frac{1}{2}\right]$, where [.] denotes the greatest integer function is

(A) Continuous at x = 1 but discontinous at x = $\frac{3}{2}$

(B) Continuous at x = 1 and x = $\frac{3}{2}$

(C) Discontinuous at x = 1 and x = $\frac{3}{2}$

(D) Discontinuous at x = 1 but continuous at x = $\frac{3}{2}$

30. If f(x) is a continuous function ∀ x ∈ R and the range of f(x) is $(2, \sqrt{26})$ and g(x) = $\left[\frac{f(x)}{a}\right]$ is continuous ∀ x ∈ R (where [.] denotes the greatest integer function), then the least positive integral value of a is

(A) 2 (B) 3
(C) 6 (D) 5

31. Let $f(x) = \begin{cases} \frac{axe^x + b\sin x}{x^3} & , \quad x < 0 \\ c\cos \pi\{x\} & , \quad x \ge 0 \end{cases}$, where {.} represents fractional part function, if f(x) is continuous at x = 0, then the value of c is:

(A) $\frac{4}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{3}$ (D) – 1

32. Let f R → R such that f(x) is continuous and attains only rational values at all real x and f(3) =4. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $\sum_{n=1}^{4} a_r a_{r+1}$ is

(A) f(3). a_1a_5 (B) f(3). a_4a_5
(C) f(3). a_1a_2 (D) f(2). a_1a_3

33. If $f(x) = \begin{cases} 2x - 1 & , \quad -2 \le x < 0 \\ x + 2 & , \quad 0 \le x \le 2 \end{cases}$ and

$g(x) = \begin{cases} [x] & , \quad -4 \le x < -2 \\ x + 2 & , \quad -2 \le x \le 4 \end{cases}$ then

(A) $\lim\limits_{x \to -2^+} f(g(x)) = 2$

(B) f(g(x)) is discontinuous at x = –2
(C) f(g(x)) is not defined at x = 2
(D) none of these

34. The function defined by $f(x) = [x^2 + e^{1/(2-x)}]^{-1}$ when x > 2 and f(x) = k when x = 2 is continuous in interval [2, ∞). Than k is equal to

(A) 0 (B) 1/4
(C) –1/4 (D) None

35. Let $f(x) = \begin{cases} \sin^2 x & , \quad x \text{ rational} \\ -\sin^2 x & , \quad x \text{ irrational} \end{cases}$

Then set of points where f(x) is continuous -

(A) $\left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$
(B) null set
(C) {nπ, n ∈ I}
(D) set of all rational numbers

36. If the graph of the continuous function y = f(x) passes through (a, 0), then

$\lim\limits_{x \to a} \frac{\ln(1 + 6f^2(x) - 3f(x))}{3f(x)}$ is equal to

(A) 1 (B) 0
(C) – 1 (D) None

37. Let f be a continuous function on R such that $f\left(\frac{1}{4^n}\right) = (\sin e^n)\, e^{-n^2} + \frac{n^2}{n^2+1}$. Then f(0) is equal to

(A) 0 (B) 1
(C) 2 (D) none

38. Let $f(x) = [\tan x[\cot x]]$, $x \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right)$ (where [.] represents the greatest integer function) then the number of points, where f(x) is not continuous is

(A) one (B) zero
(C) three (D) infinite

39. If f: R → R and g : R → R such that

$f(x) = \begin{cases} 0 \text{ if } x \text{ rational} \\ 1 \text{ if } x \text{ irrational} \end{cases}$; $g(x) = \begin{cases} 0 \text{ if } x \text{ irrational} \\ 1 \text{ if } x \text{ rational} \end{cases}$ then

(A) f + g is discontinuous
(B) f + g is continuous at rational only
(C) f + g is continuous everywhere
(D) f + g is continuous for irrationals only

40. Let $f(x) = g(x) \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ and x ≠ 0, where g is a continuous function. Then $\lim\limits_{x \to 0} f(x)$ exists if

(A) g(x) is any polynomial
(B) g(x) = x + 4
(C) $g(x) = x^2$
(D) $g(x) = 2 + 3x + 4x^2$

41. The point of discontinuity of the function

$f(x) = \lim\limits_{x \to \infty} \frac{(2\sin x)^{2n}}{3^n - (2\cos x)^{2n}}$ is

(A) nπ ± π/3; n ∈ I (B) nπ ± π/6; n ∈ I
(C) 2nπ∈; n ∈ I (D) none of these

42. The set of all points of discontinuity of the function $f(x) = \left(\dfrac{\tan x \log x}{1-\cos 4x}\right)$ contains

(A) $\left\{\dfrac{n\pi}{2}; n \in I\right\}$ (B) $\left\{\dfrac{n\pi}{2}; n \in Q\right\}$

(C) $(-\infty, 0] \cup \left\{\dfrac{n\pi}{2}; n \in N\right\}$

(D) none of these

43. Let f(x) = sin x if x is rational and f(x) = 1 – 2 cos x if x is irrational then

(A) f(x) is no where continuous

(B) f(x) is continuous at one point only

(C) f(x) is injective

(D) f(x) is continuous at infinite no. of points

44. The function f(x) is defined by

$$f(x) = \begin{cases} \log_{(4x-3)}(x^2-2x+5) & \text{if } \dfrac{3}{4} < x < 1 \,\&\, x > 1 \\ 4 & \text{if } x = 1 \end{cases}$$

(A) is continuous at x = 1

(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists

(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists

(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.

45. The function f defined by

$$f(x) = \lim_{t\to\infty}\left\{\frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}\right\} \text{ is}$$

(A) everywhere continuous

(B) discontinuous at all integer values of x

(C) continuous at x = 0

(D) none of these

46. The function $f : R/\{0\} \to R$ given by

$f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ can be made continuous at x = 0 by defining f(0) as

(A) 2 (B) –1

(C) 0 (D) 1

47. If $f(x) = \begin{cases} \dfrac{1-|x|}{1+x}, & x \neq -1 \\ 1, & x = -1 \end{cases}$, then f([2x]) is where

[ ] represent greatest integer function

(A) continuous at x = –1

(B) continuous at x = 0

(C) discontinuous at x = 1/2

(D) all of these

48. Let $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and g(x) = sin x + cos x, then points of discontinuity of f{g(x)} in (0, 2π) is

(A) $\left\{\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right\}$ (B) $\left\{\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right\}$

(C) $\left\{\dfrac{2\pi}{3}, \dfrac{5\pi}{3}\right\}$ (D) $\left\{\dfrac{5\pi}{4}, \dfrac{7\pi}{3}\right\}$

49. The value of 'a' for which the function $f(x) = \begin{cases} 2x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

$g(x) = \begin{cases} 4\dfrac{\sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$ and $h(x) = \begin{cases} \dfrac{x^2-a^2}{x-a}, & x \ne a \\ 8, & x = a \end{cases}$

are all continuous is

(A) 2 (B) 4

(C) no value of a exists (D) none of these

50. The set of all points of discontinuity of the function $f(x) = \left(\dfrac{\tan x \log x}{1-\cos 4x}\right)$ contains

(A) $\left\{\dfrac{n\pi}{2}, n \in I\right\}$ (B) $\left\{\dfrac{n\pi}{2}, n \in Q\right\}$

(C) $(-\infty, 0] \cup \left\{\dfrac{n\pi}{2}, n \in N\right\}$

(D) none of these

51. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 5x, & \text{if } x \text{ is irrational} \end{cases}$

Then

(A) f (x) is continuous at every rational number.

(B) f (x) is continuous at every irrational number.

(C) f (x) is discontinuous everywhere.

(D) f (x) is continuous only at x = 0.

52. Let $R(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^4 - 4}$

Which of the following statements describes the graph of y = R(x) :

(A) The graph has two vertical asymptotes

(B) The graph has two holes in it

(C) The graph has one hole and one vertical asymptote

(D) The graph has neither holes nor asymptotes

53. Given that $\displaystyle\prod_{n=1}^{n} \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\dfrac{x}{2^n}\right)}$.

Let $f(x) = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{n=1}^{n} \dfrac{1}{2^n} \tan\left(\dfrac{x}{2^n}\right), & x \in (0, \pi) - \left\{\dfrac{\pi}{2}\right\} \\ \dfrac{2}{\pi}, & x = \dfrac{\pi}{2} \end{cases}$

Then which one of the following alternative is true?

(A) f(x) has non-removable discontinuity of finite type at $x = \dfrac{\pi}{2}$.

(B) f(x) has missing point discontinuity at $x = \dfrac{\pi}{2}$.

(C) f(x) is continuous at $x = \dfrac{\pi}{2}$.

(D) f(x) has non-removable discontinuity of infinite type at $x = \dfrac{\pi}{2}$.

54. The value of f(0) such that $f(x) = \dfrac{2 - \sqrt[4]{x^2 + 16}}{\cos 2x - 1}$ is continuous at $x = 0$ is :

(A) 1/64 (B) $-1/64$

(C) 1/32 (D) $-1/32$

55. Consider

$f(x) = \left[\dfrac{2\left(\sin x - \sin^3 x\right) + \left|\sin x - \sin^3 x\right|}{2\left(\sin x - \sin^3 x\right) - \left|\sin x - \sin^3 x\right|}\right]$,

$x \neq \dfrac{\pi}{2}$ for $x \in (0, \pi)$

$f(\pi/2) = 3$ where [.] denotes the greatest integer function then,

(A) f is continuous & differentiable at $x = \pi/2$

(B) f is continuous but not differentiable at $x = \pi/2$

(C) f is neither continuous nor differentiable at $x = \pi/2$

(D) none of these

56. Given $f(x) = b([x]^2 + [x]) + 1$ for $x \geq -1$

$= [\sin(\pi(x+a))]$ for $x < -1$

where [x] denotes the integral part of x, then for what values of a, b the function is continuous at $x = -1$?

(A) $a = 2n + (3/2)$; $b \in R$; $n \in I$

(B) $a = 4n + 2$; $b \in R$; $n \in I$

(C) $a = 4n + (3/2)$; $b \in R^+$; $n \in I$

(D) $a = 4n + 1$; $b \in R^+$; $n \in I$

57. The function $f(x) = \begin{cases} [x] + \sqrt{x - [x]} & \text{for } x \geq 0 \\ \sin x & \text{for } x < 0 \end{cases}$ is

(A) continuous only for all non-negative integers

(B) continuous only for all positive integers

(C) discontinuous only for all negative integers

(D) cont. for all real numbers.

58. The function $f(x) = \dfrac{4 - x^2}{\left|4x - x^3\right|}$ is:

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) none

59. Let $f(x) = \begin{cases} \dfrac{\tan kx}{x} & \text{for } x < 0 \\ 3x + 2k^2 & \text{for } x \geq 0 \end{cases}$. If f (x) is continuous at x = 0 then the number of values of *k* is

(A) more than 2 (B) 1

(C) 2 (D) none

60. If $f(x) = \lim\limits_{n\to\infty} x \tan^{-1}(nx)$; then f(x) is

(A) continuous at $x \in I$

(B) discontinuous at $x \in I$

(C) continuous at all $x \in R$

(D) none of these

61. If $f(x) = 3 + \left(1 + 7^{\frac{1}{1-x}}\right)^{-1}$ then

(A) $\lim\limits_{x\to 1^-} f(x) = 4$

(B) $\lim\limits_{x\to 1^+} f(x) = 3$

(C) $\lim\limits_{x\to 1^+} f(x) = 5$

(D) f has irremovable discontinuity at x = 1

62. Let $f(x) = \begin{cases} 0, & x < 1 \\ 2x - 2, & x \geq 1 \end{cases}$. The number of solutions of the equation $f(f(f(f(x)))) = x$ is

(A) 2 (B) 4

(C) 5 (D) None

63. The function $\dfrac{(3^x - 1)^2}{\sin x \cdot \ell n(1 + x)}$ is not defined for $x = 0$ for the function to be continuous at the point $x = 0$, f(0) must have the value :

(A) e^3 (B) 1

(C) $(ln\,3)^2$ (D) none

64. The value of f(0) so that the function

$f(x) = \dfrac{\sqrt{4 - 2x + x^2} - \sqrt{4 + 2x + x^2}}{\sqrt{2 + x} - \sqrt{2 - x}}$

is continuous at $x = 0$ is:

(A) 2 (B) 1

(C) $-\sqrt{2}$ (D) $-2\sqrt{2}$

65. If $f(x) = \dfrac{\log_{\sin|x|}\cos^3 x}{\log_{\sin|3x|}\cos\frac{x}{2}}$, $|x| < \dfrac{\pi}{3}$, $x \neq 0 = 4$, $x = 0$ then the number of points of discontinuity is

(A) 0 (B) 1

(C) 2 (D) 4

66. Which one of the following functions defined below are discontinuous at the origin?

(A) $f(x) = \begin{cases} x\cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(B) $g(x) = \begin{cases} \dfrac{x^8 + x^4 + 2x^2}{\sin x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(C) $h(x) = \begin{cases} \sin x \,.\, \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(D) $\ell(x) = \begin{cases} \dfrac{x^4 + x^8 + 2x}{\tan x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

67. If $f(x) = \begin{cases} 2x-1 & , \ -2 \le x < 0 \\ x+2 & , \ 0 \le x \le 2 \end{cases}$ and $g(x) = \begin{cases} [x] & , \ -4 \le x < -2 \\ x+2 & , \ -2 \le x \le 4 \end{cases}$ then

(A) $\lim\limits_{x \to -2^+} f\{g(x)\} = 2$

(B) $f(g(x))$ is discontinuous at $x = -2$

(C) $f(g(x))$ is not defined at $x = 2$

(D) None of these

68. The number of points of discontinuity of $f(x) = [4x] + \{3x\}$ in $x \in [0, 5]$ is

(A) 20 points (B) 25 points

(C) 30 points (D) 35 points

69. Let $f(x) = [\tan^2 x][\cot^2 x]$ where [.] denotes greatest integer function then number of points at which function f(x) is discontinuous in $(0, 2\pi)$

(A) 0 (B) 3

(C) 4 (D) 7

70. h(x) is not a constant function and $\lim\limits_{x \to c}[h(x)]$ exists finitely for all values of $c \in R$ (where [.] denotes greatest integer function), then which of the following statement is true ?

(A) $\lim\limits_{x \to c}[h(x)] = [\lim\limits_{x \to c} h(x)], \ \forall \ c \in R$

(B) h(x) can not take any integral value

(C) h(x) can take maximum two integral values

(D) None of these

Multiple Correct Answer Type for JEE Advanced

71. If $f(x) = [x^2] + [x]^2$, then (where [.] denotes greatest integer function

(A) f(x) is discontinuous at $x = \sqrt{2}$

(B) f(x) is continuous at $x = 5^{1/4}$

(C) f(x) is continuous at $x = 3^{1/5}$

(D) f(x) is discontinuous at x = 0

72. Given the function $f(x) = \dfrac{1}{(1-x)}$, the points of discontinuity of the composite function $y = f^{3n}(x)$, where $f^n(x) = fof\ldots of$ (n times) are

(A) 0 (B) 1

(C) 3n (D) 2

73. Let $f(x) = \begin{cases} x\left[\frac{1}{x}\right] + x[x] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ where [x] denotes the greatest integer function, then the correct statements are

(A) Limit exists for x = – 1.

(B) f (x) has a removable discontinuity at x = 1.

(C) f (x) has a non removable discontinuity at x = 2.

(D) f (x) is discontinuous at all positive integers.

74. Let $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$. If f (x) is continuous for all $x \in R$ then

(A) a = 0 (B) b = 0

(C) a = 1 (D) b = 1

75. Which one the following function(s) is/are continuous $\forall \ x \in R$

(A) $\sqrt{2\sin x} + 3$ (B) $\dfrac{e^x + 1}{e^x + 3}$

(C) $\left(\dfrac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ (D) $\sqrt{\text{sgn}\, x + 1}$

76. Which the following equations have roots ?

(A) $\cos x - x + 1 = 0$

(B) $x^5 - 18x + 2 = 0, \ x \in [-1, 1]$

(C) $x2^x = 1, \ x \in (0, 1)$

(D) $x^3 - 3x + 1 = 0, \ x \in [1, 2]$

77. $f(x) = x^4 - 14x^3 + px^2 + qx - 105$
$g(x) = x^4 + ax^3 + bx^2 + cx + 105$
the smallest root of $f(x) = 0$ is α and remaining root are in A.P. If the smallest root is increased by 2 then equation becomes $g(x) = 0$ function $F(x) = \dfrac{f(x)}{g(x)}$ then

(A) Domain of F(x) is $R - \{1, 3, 5, 7\}$ and range is $R - \{1, 2, \frac{3}{2}, \frac{3}{4}\}$
(B) Function F(x) is bijective
(C) Function F(x) has removable discontinuities at $x = 3, 5, 7$
(D) Function F(x) has irremovable discontinuity at $x = 1$

78. Let f(x) and g(x) be defined by $f(x) = [x]$ and $g(x) = \begin{cases} 0 & , \quad x \in I \\ x^2 & , \quad \text{other wise} \end{cases}$, where [.] denotes greatest integer function. Then

(A) gof is continuous for all $x \in R$
(B) $\lim\limits_{x\to 2} . fog(x) = 3$
(C) fog is continuous for all $x \in R$
(D) $\lim\limits_{x\to 5} . fog(x) = 5$

79. If $f(x) = \cos\left[\dfrac{\pi}{x}\right] \cos\left(\dfrac{\pi}{2}(x-1)\right)$ where $[x]$ is the greatest integer function of x, then f(x) is continuous at

(A) $x = 0$ (B) $x = 1$
(C) $x = 2$ (D) none of these

80. Which of the following statement(s) is/are correct?

(A) Let f and g be defined on R and c be any real number. If $\lim\limits_{x\to c} f(x) = b$ and $g(x)$ is continuous at $x = b$ then $\lim\limits_{x\to c} g(f(x)) = g(b)$.
(B) There exist a function $f : [0, 1] \to R$ which is discontinuous at every point in [0, 1] and $|f(x)|$ is continuous at every point in [0, 1].
(C) If f (x) and g (x) are two continuous function defined from $R \to R$ such that $f(r) = g(r)$ for all rational numbers 'r' then $f(x) = g(x)\ \forall\ x \in R$.
(D) If f (a) and f (b) possesses opposite signs then there must exist atleast one solution of the equation $f(x) = 0$ in (a, b) provided f is continuous in [a, b].

81. If
$$f(x) = \begin{cases} \dfrac{\sin ax}{bx} & , \quad x < 0 \\ ax + 1 & , \quad 0 \le x < 1 \\ cx^2 - 2 & , \quad 1 \le x < 2 \\ \dfrac{d(x^2 - 4)}{\sqrt{x}} & , \quad 2 \le x < 4 \\ 12 & , \quad x \ge 4 \end{cases}$$
is continuous $\forall\ x \in R$ then which of the following hold good?

(A) $d = 4c$ (B) $a \ne b$
(C) $a + b + d = -3$ (D) $a + b + c + d = -\dfrac{5}{2}$

82. The function defined as
$$f(x) = \lim_{n\to\infty} \left[\begin{array}{lll} \cos^{2n} x & \text{if} & x < 0 \\ \sqrt[n]{1 + x^n} & \text{if} & 0 \le x \le 1 \\ \dfrac{1}{1 + x^n} & \text{if} & x > 1 \end{array}\right]$$
Which of the following does not hold good?

(A) continuous at $x = 0$ but discontinuous at $x = 1$
(B) continuous at $x = 1$ but discontinuous at $x = 0$
(C) continuous both at $x = 1$ and $x = 0$
(D) discontinuous both at $x = 1$ and $x = 0$

83. Which of the following functions defined below are continuous for every $x \in R$?

(A) $f(x) = \left[\begin{array}{l} \dfrac{\sin x}{x} \text{ if } x < 0 \\ x + 1 \text{ if } x \ge 0 \end{array}\right.$

(B) $g(x) = \left[\begin{array}{ll} x^2 \sin\dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{array}\right.$

(C) $h(x) = \sin |x|$
(D) $k(x) = |1 - x + |x||$

84. Which of the following functions is/are continuous $\forall\ x \in R$?

(A) $f(x) = \cos(x^2 - 2)$

(B) $f(x) = \dfrac{x}{1 - \sin^2 x}$

(C) $f(x) = \begin{cases} \dfrac{\sin^2 x}{x}, & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

(D) $f(x) = \begin{cases} \dfrac{\sin 2x}{\sin 4x}, & \text{if } x \ne 0 \\ 2 & \text{if } x = 0 \end{cases}$

85. Which of the statement(s) is/are incorrect?

(A) If $f + g$ is continuous at $x = a$, then f and g are continuous at $x = a$.
(B) If $\lim\limits_{x\to a} (f\ g)$ exists, then $\lim\limits_{x\to a} f$ and $\lim\limits_{x\to a} g$ both exists.
(C) Discontinuity at $x = a \Rightarrow$ non existence of limit
(D) All functions defined on a closed interval attain a maximum or a minimum value on that interval.

86. Let $f(x) = \begin{bmatrix} |\tan^{-1}(1/x)| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{bmatrix}$

then which of the following do/does not hold good?

(A) f is continuous on $(-\infty, 0) \cup (0, \infty)$.

(B) f has a non removable discontinuity of finite type at x = 0.

(C) f has a non removable discontinuity of oscillatory type at x = 0.

(D) f has a non removable discontinuity of infinite type at x = 0.

87. Let f(x) and g(x) be defined by f(x) = [x] and

$g(x) = \begin{cases} 0 & , \quad x \in I \\ x^2 & , \quad \text{other wise} \end{cases}$, where [.] denotes greatest integer function. Then

(A) gof is continuous for all $x \in R$

(B) $\lim\limits_{x \to 2} . \text{fog}(x) = 3$

(C) fog is continuous for all $x \in R$

(D) $\lim\limits_{x \to 5} . \text{fog}(x) = 5$

88. The function

$f(x) = \max(\{x\}, \{-x\}), x \in (-\infty, \infty)$ is

(A) continuous every where

(B) continuous at x = 0

(C) continuous for all x = n + h where $n \in I$, h = 1/2

(D) continuous every where except all $x \in I$

89. f(x) is continuous at x = 0, then which of the following are always true ?

(A) $\lim\limits_{x \to 0} f(x) = 0$

(B) f (x) is non continuous at x = 1

(C) $g(x) = x^2 f(x)$ is continuous at x = 0

(D) $\lim\limits_{x \to 0^+} (f(x) - f(0)) = 0$

90. The function $f(x) = \begin{cases} \dfrac{e^{[x]} - e^{[x]}}{e^x} & , \quad x < 0 \\ \dfrac{\sin\{x\}}{\{\tan x\}} & , \quad x > 0 \\ 2 & , \quad x = 0 \end{cases}$,

where [.] and { . } represent greatest integer and fractional part functions respectively, is

(A) continuous at x = 0

(B) discontinuous at x = 0

(C) continuous at $x = \dfrac{\pi}{6}$

(D) discontinuous at $x = \dfrac{\pi}{4}$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

91. Let $g(x) = \begin{cases} <1/x>^* & x \neq 0 \\ 0 & x = 0 \end{cases}$, where <r>* is the distance from x to the integer nearest to x then

Assertion (A) : g is discontinuous at x = 0

Reason (R) : Let $x_n = 1/n$ and $x'_n = \dfrac{2}{2n+1}$;

$g(x_n) = <n>^* = 0, \; g(x'_n) = \left\langle \dfrac{2n+1}{2} \right\rangle^*$

$= <n + 1/2>^* = 1/2$

92. Consider the function $f(x) = \begin{bmatrix} x\{x\}+1 & 0 \leq x < 1 \\ 2-\{x\} & 1 \leq x \leq 2 \end{bmatrix}$

where {x} denotes the fractional part function.

Assertion (A) : f(x) is continuous in [0, 2]

Reason (R) : $\lim\limits_{x \to 1} f(x)$ exists

93. **Assertion (A) :** The function $|\ln x|$ and $\ln x$ are both continuous for all x > 0.

Reason (R) : Continuity of $|f(x)| \Rightarrow$ Continuity of f (x).

94. **Assertion (A) :** If f(x) is a continuous function such that f(0) = 1 and $f(x) \neq x$, $\forall x \in R$, then f(f(x)) > x.

Reason (R) : If $f : R \to R$, f(x) is an onto function then f(x) = 0 has atleast one solution.

95. Let $f(x) = \cos\left(x \cos \dfrac{1}{x}\right)$

Assertion (A) : f (x) is discontinuous at x = 0.

Reason (R) : $\lim\limits_{x \to 0^+} f(x)$ does not exist.

96. **Assertion (A) :** Let $f : [0, 1] \to R$ be a continuous function. There cannot exist such a function f which crosses the x axis infinitely often.

Reason (R) : The function $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ crosses the x-axis infinitely often.

97. **Assertion (A) :** Let $f(x) = x^2 + x^4 + x^6 + x^8 + ...$, for all real x such that the sum converges. The number of fixed points of the function is two.

Reason (R) : Since $f(x) = \dfrac{x^2}{1-x^2}$, f(x) = x

$\Rightarrow x^2 + x - 1 = 0$. This gives two values of x.

98. Consider the functions

$f(x) = \text{sgn}(x - 1)$ and $g(x) = \cot^{-1}[x - 1]$

where [.] denotes the greatest integer function.

Assertion (A) : The function F(x) = f(x) . g(x) is discontinuous at x = 1.

Reason (R) : If f(x) is discontinuous at x = a and g(x) is also discontinuous at x = a then the product function f(x). g(x) is discontinuous at x = a.

99. Let $f(x) = \text{sgn}\, x$ and $g(x) = \begin{cases} x-1 & 0 < x \le 2 \\ 1+x^2 & 2 < x \le 4 \end{cases}$, then

Assertion (A) : The function (fog)(x) is continuous at x = 2.

Reason (R) : If (fog) (x) is continuous at x = a then g (x) is continuous at x = a and f (x) is continuous at x = g(a).

100. **Assertion (A) :** f (x) = [1 + cos x] where [x] denotes greatest integer function is discontinuous at x = π

Reason (R) : f (x) = [x] where [x] denotes greatest integer function is discontinuous at all integers.

Comprehension - 1

If $f(x) = \text{maximum}\left(\cos x, \frac{1}{2}, \{\sin x\}\right), 0 \le x \le 2\pi$, where { . } represents the fractional part function, then

101. The number of points where f(x) is equal to 1/2 is

(A) 1 (B) 2
(C) 4 (D) infinite

102. The number of points of discontinuity of f(x) is

(A) 1 (B) 2
(C) 3 (D) 4

103. The number of points where f(x) has non-removable discontinuity is

(A) 0 (B) 1
(C) 2 (D) 3

Comprehension - 2

A strictly monotonic polynomial function f : (0, ∞) → (0, ∞) is such that $f\left(\frac{x^2}{f(x)}\right) \equiv x$.

104. If f(1) = 2 then the value of f(2) is

(A) 8 (B) 4
(C) 16 (D) None

105. The number of fixed points of f (i.e. number of solution of f(x) = x) is

(A) 3 (B) 2
(C) 1 (D) None

106. If $\lim\limits_{x\to 1} \frac{\ln f(x) - \ln f(1)}{\sin^n \pi x}$ exists with nonzero value, then the value of n is

(A) 1 (B) 2
(C) 3 (D) None

Comprehension - 3

Define f : R → R by $f(x) = x^2 + 1$ for $-1 < x \le 1$ and

$f(x+2) = \frac{1}{f(x)}$ for all x ∈ R.

107. The fundamental period of the function f(x) is

(A) 2 (B) 4
(C) 8 (D) None

108. $\lim\limits_{x\to 1^+} f(x)$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{3}{4}$
(C) $\frac{1}{2}$ (D) $\frac{5}{12}$

109. The number of points of discontinuity in the function y = f(x) over the interval [0, 4] is:

(A) 0 (B) 1
(C) 2 (D) 3

Comprehension - 4

Consider $a \in R^+$, $f(x) = |x - a|$

$$g(x) = \begin{cases} f(x)\sec\left(\frac{\pi x}{2a}\right) & \text{for } x < a \\ -\cot\left(\frac{\pi x}{2a}\right)\operatorname{cosec}(f(x)) & \text{for } x > a \end{cases}$$

$$h(x) = \lim_{n\to\infty}\left[2^n \sin\left[\frac{x}{2^n}\right]\right]$$

110. The value of g(a) so that g is continuous at x = a

(A) is $-\frac{2a}{\pi}$ (B) is $\frac{a}{2\pi}$
(C) is 1 (D) can not be determined

111. The number of possible ordered pairs (a, g (a)) is

(A) 0 (B) 1
(C) 2 (D) more than 2

112. If $\lim\limits_{x\to a} (\cos(x-a))^{\frac{6}{(x-a)\sin(2x-2a)}} = e^{-kh([a])}$ then k equals

(where [.] denotes greatest integer function)

(A) 0 (B) 2/3
(C) 3/2 (D) 6

Comprehension - 5

Let the Heaviside step function be defined as $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$

113. The solutions of the equation $x^2 - x\,H(x) - 2 = 0$ are

(A) {–1, 2} (B) $\{-\sqrt{2}, -1, 2\}$
(C) $\{\pm\sqrt{2}, -1, 2\}$ (D) $\{-\sqrt{2}, 2\}$

114. The solution set of the inequality $H(1 - x^2) > |\sin \pi/2\, x|$ is

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(C) $R - \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (D) None of these

MATCH THE COLUMNS FOR JEE ADVANCED

115.

Column - I	**Column - II**
(A) If $f(x)=\begin{cases}\dfrac{1+3\cos x}{x^2}, & x<0\\ b\tan\left(\dfrac{\pi}{[x+3]}\right), & x\ge 0\end{cases}$ is continuous at x = 0, then (where [.] denotes the greatest integer function)	(P) $\lvert a+b\rvert=0$
(B) If $f(x)=\begin{cases}-2\sin x, -\pi\le x\le -\dfrac{\pi}{2}\\ a\sin x+b, -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ \cos x, \dfrac{\pi}{2}\le x\le \pi\end{cases}$ is continuous in [– π, π],then	(Q) $\lvert a-b\rvert=2$
(C) If $f(x)=\begin{cases}\left(\dfrac{3}{2}\right)^{(\cos 3x)/(\cot 2x)}, & 0<x<\dfrac{\pi}{2}\\ b+3, & x=\dfrac{\pi}{2}\\ (1+\lvert\cos x\rvert)^{\left(\frac{a\lvert\tan x\rvert}{b}\right)}, & \dfrac{\pi}{2}<x<\pi\end{cases}$ is continuous at x = $\dfrac{\pi}{2}$, then	(R) $[a-2b]=-2$
(D) If $f(x)=\begin{cases}a+\dfrac{\sin[x]}{x}, & x>0\\ 2, & x=0\\ b+\left[\dfrac{\sin x-x}{x^3}\right], & x<0\end{cases}$	(S) $\lvert a+2b\rvert=4$
(where [.] denotes the greatest integer function), is continuous at x = 0, then b is equal to	(T) $\lvert a-b\rvert=1$

116.

Column – I	**Column – II**
(A) Given $f(u)=\dfrac{1}{u^2+u-2}$, where $u=\dfrac{1}{x-1}$, then f(x) is	(P) continuous at x = 0
(B) If f(x) = sgn x $(1-x^2)$, then f(x) is	(Q) discontinuous at x = 1, 1/2, 2
(C) If $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ exists and f(0) = 0, then f(x) is	(R) discontinuous function
(D) The function $f(x)=\lim\limits_{n\to\infty}\dfrac{nx+\operatorname{sgn} x}{1+n}$ is	(S) discontinuous at x = 2

117.

Column - I	**Column - II**
(A) The number of natural numbers less than the fundamental period of $\sin^2x+\sec^2x-\tan^2x$ is	(P) 1
(B) The number of points of discontinuity of the function $f(x)=[x]+\{2x\}+[3x]$ for $x\in[0,1]$, where [.] and { . } represent greatest integer and fractional part functions is	(Q) 2

(C)	$\left[\lim_{x\to 0}\frac{\sin x(1+\cos x)}{x\cos x}\right]$, where [.] represents greatest integer function, is equal to	(R) 3
(D)	The number of solutions of the equation $\sin^{-1} x - 2\cos^{-1}(1+x) = 0$ is	(S) 4

118.

	Column-I	**Column-II**
(A)	In a ΔABC maximum value of $\cos^2 A + \cos^2 B + \cos^2 C$, is	(P) 3/4
(B)	If a, b are c are positive and $9a + 3b + c = 90$ then the maximum value of $(\log a + \log b + \log c)$ is (base of the logarithm is 10)	(Q) 2
(C)	$\lim_{x\to 0}\frac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3 \cdot \sqrt{x}}$ equals	(R) 3
(D)	If $f(x) = \cos(x\cos\frac{1}{x})$ and $g(x) = \frac{\ln(\sec^2 x)}{x\sin x}$ are both continuous at x = 0 then f (0) + g (0) equals	(S) non existent

119.

	Column I	**Column II**
(A)	If $f(x) = \lim_{n\to\infty}\frac{1}{1+n\sin^2 \pi x}$, then	(P) f(x) is continuous $\forall\, x \in R$
(B)	If $f(x) = \lim_{n\to\infty}\frac{x}{1+(4\sin^2 x)^n}$, then	(Q) f(x) is discontinuous at x = 1
(C)	If $h(x) = \lim_{n\to\infty}\left(\frac{x^{2n-1}+ax^2+bx}{x^{2n}+1}\right)$ is continuous $\forall\, x \in R$, and $f(x) = [\tan(a+b-1)x]$, then	(R) f(x) is discontinuous at $x = 2n\pi + \frac{\pi}{6}$
(D)	If $f(x + y) = f(x).\ f(y), \forall x, y \in R$ and $f(x) = 1 + g(x).G(x)$; $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} G(x)$ is finite real number, then	(S) f(x) is continuous at $x = \frac{5\pi}{6}$

Review Exercises for JEE Advanced

1. Test the continuity of the function

$f(x) = \left|\left(x+\frac{1}{2}\right)[x]\right|$, in the interval $-2 \le x \le 2$. Also draw the graph of y = f(x), where [.] denotes greatest integer function.

2. Discuss the continuity of the function $f(x) = [[x]] - [x-1]$, where [·] denotes the greatest integer function.

3. Let f(x) be defined in the interval [–2, 2] such that

$f(x) = 1,\ -2 \le x \le 0$

$\quad = x - 1,\ 0 < x \le 2$

and $g(x) = f(|x|) + |f(x)|$. Test the continuity of g(x) in [–2, 2].

4. If f(x) be defined as

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1 - x\cos x, & x \le 0 \end{cases}$$

then discuss the continuity y = [f(x)] in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ where [.] denotes G.I.F.

5. Let $f(x) = \frac{x^2}{2} + 1.\ 0 \le x < 1$

$= 2x^2 - 3x + \frac{3}{2},\ 1 \le x \le 2.$

Discuss the continuity of g(x) = f(x) + f(x – 1).

6. Discuss the continuity of the function

$$f(x) = \lim_{n\to\infty} \frac{(1+\cos x)^n + 5\ell nx}{2+(1+\cos x)^n}$$

7. Let $f_n(x) = \cos^n x$ and $g(x) = \lim\limits_{x\to\infty} \sum\limits_{k=0}^{n} f_k\left(\frac{x}{4}\right)$. If g(x) is continuous is (0, c), then find the largest value of c.

8. Examine the function f defined on R by setting

$$f(x) = \frac{e^{1/x}\sin(1/x)}{1+e^{1/x}}, \text{ when } x \neq 0,\ f(0) = 0$$

for points of discontinuity, if any.

9. Examine each of the following functions for points of discontinuity and the nature of discontinuity :

(i) $f(x) = (x - [x])^2$, for all $x \geq 0$

(ii) $f(x) = [x] + (x - [x])^2$, for all $x \geq 0$

10. If the function

$$f(x) = \left[\frac{(x-2)^3}{a}\right]\sin(x-2) + a\cos(x-2) \text{ where } [.]$$

denotes the greater integer function, is continuous in [4, 6], then find the values of a.

11. Find the value of a & b for which

$$f(x) = \begin{cases} (\sin x + \cos x)^{\text{cosec } x} & , \quad -1/2 \leq x < 0 \\ a & , \quad x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{3/x} + be^{3/|x|}} & , \quad 0 < x \leq 1/2 \end{cases}$$

is continuous at x = 0.

12. Prove that the function $f(x) = \begin{cases} 2x, & -1 \leq x \leq 0 \\ \dfrac{2x+1}{2}, & 0 < x \leq 1 \end{cases}$

is discontinuous at x = 0 but still has both maximum and minimum values on [–1, 1].

13. Let $f(x) = \begin{cases} x^n \dfrac{e^{1/x}}{1+e^{1/x}} & , \ x < 0 \\ 0 & , \ x = 0 \\ x^n \sin\dfrac{1}{x} & , \ x > 0 \end{cases}$. Find the smallest

$n \in W$ such that f(x) is continuous.

14. If $f(x) = \begin{cases} x^2 + ax + 1, & x \in Q \\ ax^2 + 2x + b, & x \notin Q \end{cases}$ is continuous at x = 1 & e

then a & b.

15. Let $f(x) = \dfrac{x^2}{e^x}$, $g(x) = \dfrac{2\ln x}{x}$. Prove that there exist a point ‘c’ between 1 and e such that f(c) = g(c).

16. Prove that, if f(x) is continuous on (a, b) and $x_1, x_2, \ldots\ldots\ldots x_n$ are some values of x from this interval, then we can find x = c, c ∈ (a, b) such that $f(c) = (1/n)\,[f(x_1) + f(x_2) + \ldots\ldots\ldots\ldots + f(x_n)]$.

17. Let $f(x) = \begin{cases} |x+1|, & x \leq 0 \\ x, & x > 0 \end{cases}$;

$g(x) = \begin{cases} |x|+1, & x \leq 1 \\ -|x-2|, & x > 1 \end{cases}$.

Discuss the continuity of f + g.

18. Let $f(x) = \begin{cases} 2x-1, & -2 \leq x < 0 \\ x+2, & 0 \leq x \leq 2 \end{cases}$ and

$g(x) = \begin{cases} [x], & -4 \leq x < -2 \\ x+2, & -2 \leq x \leq 4 \end{cases}$

Discuss the continuity of fog(x) over its domain.

19. Discuss the continuity of $f : R^+ \to R$ defined as :

f(x) = x when x is irrational, $f(x) = \left(\dfrac{1+p^2}{1+q^2}\right)^{1/2}$

when x is rational number of the form p/q.

20. Use the Intermediate Value Theorem to show that there is a right circular cylinder of height h and radius less than r whose volume is equal to that of a right circular cone of height h and radius r.

21. Use Intermediate Value Theorem to locate all discontinuties of the function $f(x) = \dfrac{x}{x^3 - 3x + 1}$.

22. Prove that if a and b are positive, then the equation $\dfrac{a}{x-1} + \dfrac{b}{x-3} = 0$ has atleast one solution in the interval (1, 3).

23. Let $f(x) = x(1-x^2)$, x ∈ R and

$g(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Discuss the continuity of (fog) x and (gof) x.

24. Show that any continuous function defined for all real x and satisfying the equation f(x) = f(2x + 1) for all x must be a constant function.

25. Let f(x) be a continuous function in [–1, 1] and satisfies $f(2x^2 - 1) = 2x\, f(x)\ \forall\ x \in [-1, 1]$. Prove that f(x) is identically zero ∀ x ∈ [–1, 1].

26. If $g(x) = \begin{cases} [f(x)], & x \in (0, \pi/2) \cup (\pi/2, \pi) \\ 3, & x = \pi/2 \end{cases}$ and

$$f(x) = \frac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|},\ n \in R$$

where [.] denotes the greatest integer function. Prove that g(x) is continuous at $x = \pi/2$ when $n > 1$.

27. Prove that the function $f(x) = (-1)^{[x^3]}$, where [.] denotes the greatest integer function, is discontinuous for $x = n^{1/3}$, $n \in I$.

28. Let f be a continuous function in [0, 4] and f(0) = f(4). Prove that these exists point $x = c \in [0, 2]$ such that f(c) = f(c + 2).

29. Let f be a function such that $f(xy) = f(x)\,f(y^3)$ for all x and y. If f(x) is continuous at x = 1, show that f(x) is continuous at all $x \neq 0$.

30. If $f(x) = \begin{cases} |x+1|; & x \le 0 \\ x; & x > 0 \end{cases}$ and

$$g(x) = \begin{cases} |x|+1; & x \le 1 \\ -|x-2|; & x > 1 \end{cases}$$

Draw the graph of f + g and discuss its continuity.

31. Let f(x) = x, if x is not the reciprocal of a positive integer; but $f(x) = x^2$ if x = 1/n for some positive integer n. Show that f is continuous at $x_0 = 0$. Is f discontinuous at any point in R?

32. Consider the function $f : R \to R$ defined by

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}$$

in the interval [0, π/2] and explain why the function does not vanish anywhere in this interval, although

f(0) and $f\left(\frac{1}{2}\pi\right)$ differ in sign.

33. A continuous function $f : R \to R$ satisfiesf(x + f(x)) = f(x) $\forall\, x \in [0, 1]$. Prove that f is constant.

34. Let f be a non-zero function whose domain is the set of all real numbers satisfying
$f(x + h) = A\,f(x)\,f(h)$ $(A \neq 0)$. If $f(0) \neq 0$, and f is continuous at x = 0 then show that f is a continuous function.

35. Let f be a continuous function on R. If $f(1/3^n) = (\cos e^n)$

$$3^{-n^2} + \frac{n^2}{n^2+n+1}$$ then find f(0).

Target Exercises for Jee Advanced

1. Let $f(x) = \begin{cases} 1 & , \ |x| \ge 1 \\ \dfrac{1}{n^2} & , \ \dfrac{1}{n} \le |x| < \dfrac{1}{n-1} \\ 0 & , \ x = 0 \end{cases}$, n = 2, 3,..... prove that f(x) is discontinuous at infinitely many points.

2. Let $$f(x) = \begin{cases} \dfrac{(1-x)^{-1/x} - (1+x)^{1/x}}{x} & , \ -1 < x < 0 \\ e, & x = 0 \\ (\log_{\sin x} \sin 2x)^{\ln \sin 2x} & , \ 0 < x < 1 \end{cases}$$

Examine the continuity of f(x) at x = 0.

3. Let $f(x) = x^3 - 9x^2 + 15x + 7$, and

$$g(x) = \begin{cases} (\min f(t) & : \ 0 \le t \le x), \quad 0 \le x \le 6 \\ x - 24 & , \qquad x > 6 \end{cases}$$

Draw the graph of g(x) and discuss the continuity of g(x).

4. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{2(x-x^3)+|x-x^3|}{2(x-x^3)-|x-x^3|} & , \ x \neq 0 \\ \dfrac{1}{3} & , \ x = 0 \\ 3 & , \ x = 1 \end{cases}$$ at x = 0,1.

5. Let f(x) = x when x is rational
= 1–x when x is irrational.
Show that f(x) assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for every value of x except $x = \frac{1}{2}$.

6. Let f be a continuous functions on [–1, 1] such that $(f(x))^2 + x^2 = 1$, for all, $x \in [-1, 1]$. Show that either $f(x) = \sqrt{1-x^2}$ for all x in [–1, 1] or $f(x) = -\sqrt{1-x^2}$ for all x in [–1, 1].

7. Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$

and $y(x) = \lim\limits_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$ (n = 1, 2, 3,.....n) and y(x) at x = 0

8 Show that the function

$f(x) = \frac{1}{2} - x + \frac{1}{2}[2x] - \frac{1}{2}[1 - 2x]$ assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for x = 0, $x = \frac{1}{2}$ and x = 1.

9. Let I be a closed and bounded interval on the line and let f be continuous on I. Suppose that for each $x \in I$, there exists a $y \in I$ such that

$|f(y)| \le \frac{1}{2}|f(x)|$.

Prove the existence of a $t \in I$ such that $f(t) = 0$.

10. Let $f : R \to R$ be a continuous function such that $\lim\limits_{x\to-\infty} f(x) = 0 = \lim\limits_{x\to\infty} f(x)$. Prove that if f is strictly negative somewhere on R then f attains a finite absolute minimum on R, and that if f is strictly positive somewhere on R then f attains a finite absolute maximum on R.

11. Let $f : [0, 1] \to [0, 2]$ be continuous. Show that there exists a point $x \in [0, 1]$ such that $f(x) = 2x$.

12. Let $f : [0, 1] \to R$ be a continuous function such that $f(0) = f(1)$, then prove that there is a solution of the equation $f(x) - f\left(x + \frac{1}{n}\right) = 0$, in $\left[0, \frac{n-1}{n}\right]$ for every natural number n .

13. How many continuous functions are there on R which satisfy $(f(x))^2 = x^2$ for all $x \in R$?

14. Let f be continuous on [a,b], let $f(x) = 0$ for exactly one c in [a, b] and let $f(x) > 0$ for some x such that $c < x \le b$ and let $f(x) < 0$ for some x such that $a \le x < c$. What can we say about f for all x in [a,b].

15. Let f be a function that satisfies the conclusion of the intermediate value theorem on a closed interval I and let f be injective on I then prove that f must be continuous on I.

16. A function f is said to satisfy a Lipschitz condition on a given interval if there is a positive constant M such that $|f(x) - f(y)| < M\,|x - y|$.
for all x and y in the interval (with $x \ne y$). Suppose f satisfied a Lipschitz condition on an interval and let c be a fixed number chosen arbitrarily from the interval. Use limit to prove that f is continuous at c.

17. Find the points of discontinuity of the function $f : (0, \infty) \to R$ where

$$f(x) = \begin{cases} \dfrac{1}{p+q} & \text{if } x \in Q \cap (0,\infty),\ x = \dfrac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \in (0,\infty) - Q \end{cases}$$

18. Let f, g be continuous function from [0, 1] to [0, 1] such that
$f(g(x)) = g(f(x))\ \forall\ x \in [0, 1]$.
Prove that f and g have a common fixed point in [0, 1].

19. Let f be a function such that $|f(u) - f(v)| \le |u - v|$ for all u and v in an interval [a, b].
(i) Prove that f is continuous at each point of [a, b].
(ii) Assume that f is integrable on [a, b] prove that

$$\left|\int_a^b f(x)dx - (b-a)f(a)\right| \le \frac{(b-a)^2}{2}$$

(iii) More generally, prove that for any c in [a, b], we have

$$\left|\int_a^b f(x)dx - (b-a)f(c)\right| \le \frac{(b-a)^2}{2}$$

20. Let f be continuous and strictly monotonic on the positive real axis and let g denote the inverse of f. If $a_1 < a_2 < \ldots < a_n$ are n given positive real numbers, we define their mean value (or average) with respect to a f to be the number M_f defined as follows:

$$M_f = g\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right)$$

Prove that (i) $f(M_f) = \frac{1}{n}\sum\limits_{i=1}^{n} f(a_i)$ (ii) $a_1 < M_f < a_n$

If $h(x) = af(x) + b$, $a \ne 0$, then show that $M_h = M_f$.

21. If f is a function, then by a chord of f we shall mean a line segment whose ends are on the graph of f. Now let f be continuous throughout [0,1] and let $f(0) = f(1) = 0$.

(i) Explain why there is a horizontal chord of f of length $\frac{1}{2}$.

(ii) Explain why there is a horizontal chord of f of length 1/n, where n = 1, 2, 3, 4,....

(iii) Must there exist a horizontal chord of length $\frac{2}{3}$?

(iv) What is the answer to (iii) if it is given that $f(x) \ge 0$ for all x in [0, 1]

22. Classify the points of discontinuity of the function $f(x) = pnx$, where pnx denotes the positive or negative excess of x over the nearest integer; when x exceeds an integer by 1/2 let $pnx = 0$.

23. Discuss the continuity at x = 1 of the functions :

(i) $f(x) = \lim\limits_{n\to\infty} \dfrac{x + x^n \sin x}{1 + x^n}$;

(ii) $\phi(x) = \lim\limits_{n\to\infty} \dfrac{x + x^n}{1 + ax^n}$.

24. If $f(x) = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{[2rx]}{n^2}$, discuss the continuity of f(x) where [.] denotes the greatest integer function.

25. Consider the functions $y(x) = \sin x$ and $z_k(x) = ke^{-x}$. Now define a new function f by $f(k) = \{$smallest positive solution of $y(x) = z_k(x)\}$ Explain why the function f(k) is not continuous on the interval $0 \le k \le 10$.

26. Let f and g be continuous on an interval I, and let $(f(x))^2 (g(x))^2 = 1$ for all $x \in I$. Prove that either $f(x) = 1/g(x)$ for all $x \in I$ or $f(x) = -1/g(x)$ for all $x \in I$.

27. A function f is defined on [0, 1] as follows :

$$f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \ (n \in W),\ f(0) = 0$$

Show that f is discontinuous at the points $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots.$

and examine the nature of discontinuity.

28. $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for > 0

$= \dfrac{\ln(1+x+x^2)+\ln(1-x+x^2)}{\sec x - \cos x}$ for x < 0.

If f is continuous at x = 0, find 'a'.

Now if $g(x) = \ln\left(2 - \frac{x}{a}\right) . \cot(x-a)$ for $x \ne a$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

29. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(i) evaluate h(g(2))

(ii) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

30. Let

$$f(x) = \begin{bmatrix} \dfrac{\left(\dfrac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\cdot \sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)} & \text{for } x \ne 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{bmatrix}$$

where {x} is the fractional part of x.

Consider another function g(x); such that

$g(x) = f(x)$ for $x \ge 0$

$= 2\sqrt{2}\ f(x)$ for $x < 0$

Discuss the continuity of other functions f(x) & g(x) at x = 0.

31. Let f be the function defined on [0,1] by setting

$f(x) = 2rx$, when $\frac{1}{r+1} \le x < \frac{1}{r}$, r = 1, 2, 3, ...

f(0) = 0, f(1) = 1.

Examine for continuity the function f at the points

$1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{r}, \ldots\ldots,$ and 0.

32. Let f be defined on R by setting

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < \frac{1}{2} \\ 0, & \text{if } t = \frac{1}{2} \\ t-1, & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

and f(n+t) = f(t), when n is any integer.

Determine the points of discontinuity of f.

33. Sketch the graph of the function y = f(x), where

$$f(x) = \lim_{n\to\infty} \frac{\ln(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

in the interval $0 \le x \le \frac{1}{2}\pi$ and explain why the function does not vanish anywhere in this interval, although f(0) and $f\left(\frac{1}{2}\pi\right)$ differ in sign.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks:

1. Let $f(x) = \begin{cases} \dfrac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$

If f(x) is continuous for all x, then k =.......

[IIT - 1981]

2. A discontinuous function y = f(x) satisfying $x^2 + y^2 = 4$ is given by f(x) =.......... **[IIT - 1982]**

3. Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function. The domain of f is .... and the points of discontinuity of f in the domain are....... **[IIT - 1996]**

4. Let f(x) be a continuous function defined for $1 \le x \le 3$. If f(x) takes rational values for all x and f(2) = 10, then f(1.5) = ... **[IIT - 1997]**

B. Multiple Choice Questions with ONE correct answer:

5. The function $f(x) = \dfrac{\ln(1+ax) - \ln(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0 so that it is continuous at x = 0, is **[IIT - 1983]**

(A) a – b (B) a + b

(C) ln a – ln b (D) none of these

6. The function $f(x) = [x] \cos\left(\dfrac{2x-1}{2}\right)\pi$, [.] denotes the greatest integer function, is discontinuous at **[IIT - 1995]**

(A) All x (B) All integer points

(C) No x (D) x which is not an integer

7. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at **[IIT - 1999]**

(A) all integers

(B) all integers except 0 and 1

(C) all integers except 0

(D) all integers except 1

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answer :

8. If f (x) = (x – 1)/2, then on the interval [0, π] **[IIT - 1989]**

(A) tan (f(x)) and 1/f(x) are both continuous

(B) tan (f(x)) and 1/f(x) are both discontinuous

(C) tan (f(x)) and $f^{-1}(x)$ are both continuous

(D) tan (f(x)) is continuous but 1/f(x) is not.

9. The following function are continuous on (0, π) **[IIT - 1991]**

(A) tan x

(B) $\displaystyle\int_0^x t \sin\frac{1}{t}\,dt$

(C) $\begin{Bmatrix} 1 \;, & 0 < x \le \dfrac{3\pi}{4} \\ 2\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{Bmatrix}$

(D) $\begin{cases} x\sin x \;, & 0 < x \le \pi/2 \\ \dfrac{\pi}{2}\sin(\pi - x)\;; & \dfrac{\pi}{2} < x < \pi \end{cases}$

10. If $f(x) = \begin{cases} x\sin x \;, & \text{when } 0 < x \le \pi/2 \\ \dfrac{\pi}{2}\sin(\pi + x)\;; & \text{when } \dfrac{\pi}{2} < x < \pi \end{cases}$, then **[IIT - 1991]**

(A) f(x) is discontinuous at $x = \dfrac{\pi}{2}$

(B) f(x) is continuous $x = \dfrac{\pi}{2}$

(C) f(x) is continuous at x = 0

(D) None of these

11. For every integer n, let a_n and b_n be real numbers. Let function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in [2n-1, 2n] \end{cases},$$

for all integers n?

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$

(C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

[IIT - 2012]

12. For every pair of continuous functions f, g:[0, 1] → **R** such that

max {f(x):x ∈ [0, 1]} = max {g(x):x ∈ [0, 1]}, the correct statement(s) is(are):

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0, 1]

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0, 1]

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some c ∈ [0, 1]

(D) $(f(c))^2 = (g(c))^2$ for some c ∈ [0, 1] **[2014]**

13. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function f(x) = x cos(π(x + [x])) is discontinuous?

(A) x = –1 (B) x = 1

(C) x = 0 (D) x = 2 **[2017]**

D. Subjective Problems:

14. Let f(x + y) = f(x) + f(y) for all x and y. If the function f(x) is continuous at x = 0, then show that f(x) is continuous at all x **[IIT - 1981]**

15. Determine the values a, b, c for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & , \quad x < 0 \\ c & , \quad x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & , \quad x > 0 \end{cases}$$

is continuous at x = 0 **[IIT - 1982]**

16. Let $f(x) = \begin{cases} 1+x, & 0 \le x < 2 \\ 3-x, & 2 \le x < 3 \end{cases}$;

Determine the form of g(x) = f[f(x)] and hence find the points of discontinuity of g, If any. **[IIT - 1983]**

17. Let f(x) be a continuous and g(x) be discontinuous function, prove that f(x) + g(x) is discontinuous function. **[IIT - 1987]**

18. Find the value of a and b so that the function **[IIT - 1989]**

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$

19. Let $f(x) = \dfrac{1-\cos 4x}{x^2}$ $\quad x < 0$ **[IIT - 1990]**

$= a \qquad x = 0$

$= \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} \qquad x > 0$

Determine the value of a, if possible, so that the function is continuous at x = 0.

20. Let
$$f(x) = \begin{cases} \{1+|\sin x|\}^{a/|\sin x|} & ; \quad -\dfrac{\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ e^{\tan 2x/\tan 3x} & ; \quad 0 < x < \dfrac{\pi}{6} \end{cases}$$

[IIT - 1994]

Determine a and b such that f(x) is continuous at x = 0.

Previous Year's Questions (JEE Main Papers)

1. If the function

$$f(x) = \begin{cases} \dfrac{\sqrt{2+\cos x} - 1}{(\pi - x)^2}, & x \ne \pi \\ k, & x = \pi \end{cases}$$

is continuous at x = π then k equals

(a) 2 (b) 1/4

(c) 1/2 (d) 0 **[2014, online]**

2. If f(x) is continuous and $f\left(\dfrac{9}{2}\right) = \dfrac{2}{9}$, then

$\lim\limits_{x\to 0} f\left(\dfrac{1-\cos 3x}{x^2}\right)$ is equal to

(a) 9/2 (b) 2/9

(c) 0 (d) 8/9 **[2014, online]**

3. Let k be a non-zero real number. If

$$f(x) = \begin{cases} \dfrac{(e^x - 1)^2}{\sin\left(\dfrac{x}{k}\right)\log\left(1+\dfrac{x}{4}\right)}, & x \ne 0 \\ 12, & x = 0 \end{cases}$$

is a continuous function, then the value of k is

(a) 1 (b) 2

(c) 3 (d) 4 **[2015]**

4. Let a, b ∈ **R** (a ≠ 0). If the function f defined as

$$f(x) = \begin{cases} \dfrac{2x^2}{a}, & 0 \le x < 1 \\ a, & 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^3}, & \sqrt{2} \le x < 8 \end{cases}$$

is continuous in the interval [a, ∞), then an ordered pair (a, b) is

(a) $(-\sqrt{2}, 1-\sqrt{3})$ (b) $(\sqrt{2}, -1+\sqrt{3})$

(c) $(\sqrt{2}, 1-\sqrt{3})$ (d) $(-\sqrt{2}, 1+\sqrt{3})$

[2016, online]

5. If the function f defined as $f(x) = \dfrac{1}{x} - \dfrac{k-1}{e^{2x}-1}, x \ne 0$, is continuous at x = 0, then the ordered pair (k, f(0)) is equal to

(a) (2, 1) (b) (3, 1)

(c) (3, 2) (d) (1/3, 2) **[2018, online]**

6. Let f : **R** → **R** be a function defined as

$$f(x) = \begin{cases} 5 & \text{if } x \le 1 \\ a + bx & \text{if } 1 < x < 3 \\ b + 5x & \text{if } 3 \le x < 5 \\ 30 & \text{if } x \ge 5 \end{cases}$$

then f is

(a) continuous at a = 5 and b = 5

(b) continuous at a = –5 and b = 10

(c) continuous at a = 0 and b = 5

(d) not continuous for any values of a and b **[2019]**

ANSWERS

CONCEPT PROBLEMS—A

2. 1

3. $-\frac{1}{8}$

4. a = 0

5. 6

6. (i) 7/3 (ii) π/4

7. continuous

8. f(0) = e

PRACTICE PROBLEMS—A

9. discontinuous

10. $\frac{5}{6}$

12. 1

13. $a = \frac{\pi}{6}, b = \frac{-\pi}{12}$

14. $a = -\text{In } 3, b = \frac{1}{3}, c = 1$

16. a + b = 0

17. $a = 2/3, b = e^{2/3}$.

18. 1

19. 1

CONCEPT PROBLEMS—B

1. The tangent is continuous everywhere except at $x = \frac{1}{2}\pi + n\pi$, where n is any integer;

the cotangent is continuous everywhere except at x = nπ, where n is any integer.

2. (i) $x = (2n+1)\frac{\pi}{2}, n \in I$ (ii) ±1.

4. $\frac{5}{3}$

5. No

6. $a = (2\cos c - b)/c^2$ if $c \neq 0$; if c = 0 there is no solution unless b = 2, in which case any a will do.

7. (i) a = 16 b = 4

(ii) $a = 2n + (-1)^n \frac{2}{3},\ n \in I, b = \sqrt{3}$

8. (i) $x = \pm\sqrt{n}, n \in N$, (ii) $x = n^2, n \in N$,

(iii) $x = n,\ n \in I$, (iv) $x = n/2,\ n \in I$,

(v) $x \in I$

9. (i) No (ii) No

(iii) $f(x) \to 0$ as $x \to 0$, define f(0) = 0 for continuity at 0

10. True

11. (i) discontinuous at 0, left continuous

(ii) discontinuous at 0, right continuous; discontinuous at 1; left continuous.

12. (i) a + b + 1 = 0 (ii) a = 3/2 + 2n.

13. (i) 0 (ii) nπ/2, n ∈ I.

14. (i) 1 (ii) 0

(iii) yes, define f(0) = 0

15. R

16. (i) Discontinuities at x = 0 and $x = \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \ldots\ldots, \pm\frac{2}{(2n+1)\pi}, \ldots\ldots\ n \in I$.

(ii) Discontinuities at x = –2, –1, 0, 2

(iii) f(x) is discontinuous at $x \in I$.

PRACTICE PROBLEMS—B

17. a = 8

18. $\{1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5\}$

21. $x \in [0, 1) \cup [2, \infty)$

22. 12, 4, 2

23. No

24. all $x \in R$ except $x = n\pi + \frac{\pi}{4}, n \in I$

25. (i) 3 (ii) dne

(iii) no (iv) at x = n + 1/2, n ∈ I.

26. Discontinuous at $x = n\pi \pm \frac{\pi}{3}$, n ∈ I.

27. Discontinuous at all integral values in [–2, 2]

28. Continuous on R.

29. a = 0 , b = 1

30. discontinuous at x = 0.

CONCEPT PROBLEMS—C

1. isolated point removable discontinuity of first kind

2. irremovable finite discontinuity of first kind

4. x = – 2 is a discontinuity of the first kind (the jump being equal to 2)

5. No, it has a discontinuity of second kind.

6. Infinite discontinuity of second kind.

7. No

8. (i) g(x) = x – 4,

(ii) irremovable discontinuity

PRACTICE PROBLEMS—C

9. (i) x = 0 is a point of removable discontinuity,

(ii) x = 0 is a point of discontinuity of the second kind

(iii) $x = k\ (k \in I)$ are points of discontinuity of the first kind,
(iv) at the points $x = 1$ and $x = -1$ the function is continuous and the other points are points of discontinuity of the second kind,
(v) $x = -1$ is a point of discontinuity of the second kind,
(vi) $x = 0$ is a point of discontinuity of the first kind
(vii) $x = 1$ is a point of discontinuity of the first kind
(viii) $x = -1$ and $x = 3$ are points of discontinuity of the second kind,
(ix) $x = 1$ is a point of removable discontinuity,
(x) $x = -1$ is a point of discontinuity of the first kind.

10. (i) missing point discontinuity,
(ii) misolated point discontinuity,
(iii) infinite discontinuity,
(iv) infinite discontinuity,
(v) infinite discontinuity,
(vi) infinite discontinuity,
(vii) irremovable discontinuity

11. No

13. Irremovable discontinuity

14. (i) $x = -2$ is a discontinuity of second kind; $x = 2$ is a removable discontinuity.
(ii) $x = -1$ is a removable discontinuity; $x = 1$ is a discontinuity of first kind.
(iii) $x = 1$ is a discontinuity of first kind.

15. (i) $x = 1$ is a discontinuity of first kind.
(ii) continuous.

CONCEPT PROBLEMS—D

2. (i) Yes, Hint. If the function $\varphi(x) = f(x) + g(x)$ is continuous at the point $x = x_0$, then the function $g(x) = \varphi(x) - f(x)$ is also continuous at this point;
(ii) No. Example: $f(x) = -g(x) = \operatorname{sgn} x$; both function are discontinuous at the point $x = 0$, and their sum is identically equal to zero, and is, hence continuous.

3. (i) No. Example $f(x) = x$ is continuous everywhere, and $g(x) = \sin\dfrac{\pi}{x}$ for $x \neq 0$, $g(0) = 0$ being discontinuous at the point $x = 0$. The product of these function is a function continuous at $x = 0$ since $\lim\limits_{x \to 0} x \sin\dfrac{\pi}{x} = 0$;

(ii) No. Example: $f(x) = -g(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

both functions are discontinuous at the point $x = 0$, their product $f(x)\,g(x) \equiv -1$ being continuous everywhere.

6. No

7. No. Example: $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

8. $f(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$

PRACTICE PROBLEMS—D

11. x^2 if $x \geq 0$; 0 if $x < 0$; h is continuous everywhere.

12. gof(x) is discontinous at $x = -1, 0, 1$

13. 1 if $1 \leq |x| \leq \sqrt{3}$; 0 otherwise. h is continuous everywhere except at $x = \pm 1, \pm\sqrt{3}$.

14. discontinuous at 1 and –1.

15. $\{0, 1\}$

16. $g(x) = \begin{cases} x+2, & 0 \leq x \leq 1 \\ 2-x, & 1 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$ and g(x) is discontinuous at $x = 1$ and $x = 2$

17. continuous everywhere

18. h(x) is discontinuous at $x = 0$

CONCEPT PROBLEMS—E

1. (i) $f(x) = \dfrac{1}{x-1}$; $g(x) = x$
(ii) $f(x) = x$; $g(x) = \dfrac{1}{x^2+1}$
(iii) $f(x) = x^2$; $g(x) = \tan^{-1}x$

5. (i) $(-4, -3), (-2, -1), (-1, 0), (0, 1)$,
(ii) $(-3, -2.5), (-2.5, -2), (0, 0.5), (0.5, 1)$,

PRACTICE PROBLEMS—E

8. (a) yes (b) yes
14. (ii) $f(x) = x^2(\sin^2 x + 2)$

CONCEPT PROBLEMS—F

2. (i) $c = 3$ (ii) $c = 2$
(iii) $c = 3$

5. 1.53 **10.** Yes

14. f must be a constant function.

PRACTICE PROBLEMS—F

24. $(-4, -3), (0, 1), (4, 5)$ **26.** Yes
30. 0.6 approx. **34.** False
37. $\sqrt{5}$
38. (i) $[1, 2]$ (ii) $[0, 1]$
(iii) $\left[0, \dfrac{\pi}{2}\right]$ (iv) $\left[-\dfrac{\pi}{4}, 0\right]$

41. $f(x) = 0$

OBJECTIVE EXERCISES

1. B
2. D
3. D
4. C
5. D
6. B
7. C
8. B
9. D
10. A
11. B
12. B
13. A
14. A
15. D
16. B
17. B
18. C
19. A
20. C
21. A
22. C
23. C
24. A
25. B
26. A
27. D
28. C
29. A
30. B
31. B
32. A
33. B
34. B
35. C
36. C
37. B
38. C
39. C
40. C
41. B
42. C
43. D
44. D
45. B
46. D
47. D
48. B
49. B
50. C
51. D
52. A
53. C
54. A
55. A
56. A
57. D
58. C
59. C
60. C
61. D
62. A
63. C
64. C
65. C
66. D
67. A
68. C
69. C
70. D
71. ABCD
72. AB
73. ABCD
74. AD
75. BC
76. ABCD
77. ABCD
78. AB
79. BC
80. ABCD
81. ACD
82. ABC
83. ABCD
84. AC
85. ABCD
86. BCD
87. AB
88. CD
89. CD
90. BCD
91. A
92. D
93. C
94. B
95. C
96. D
97. C
98. C
99. C
100. D
101. D
102. B
103. B
104. B
105. D
106. A
107. B
108. C
109. C
110. C
111. B
112. C
113. D
114. A
115. (A)–(R); (B)–(PQ); (C)–(QS); (D)–(T)
116. (A)–(QRS); (B)–(R); (C)–(P); (D)–(P)
117. (A)–(R); (B)–(S); (C)–(Q); (D)–(P)
118. (A)–(S); (B)–(R); (C)–(P); (D)–(Q)
119. (A)–(Q); (B)–(R); (C)–(PS); (D)–(PS)

REVIEW EXERCISES for JEE ADVANCED

1. discontinuous at $x = -1, 0, 1, 2$.
2. continuous in R
3. discontinuous at $x = 0$.
4. discontinuous at $x = 0, \pi$.
5. g is continuous on (1, 2].
6. discontinuous at all positive odd multiples of $\pi/2$
7. 4π
8. discontinuity of the second kind at $x = 0$.
9. (i) discontinuity of the first kind from left at $x = 1, 2, 3, ...$;
(ii) continuous for all $x \geq 0$.
10. $a > 64$
11. $a = e, b = \frac{1}{e} - e$
13. $n = 1$
14. $a = 1 - \frac{1}{e}, b = 0$
17. Discontinuous at 0, 1
18. continuous in domain
19. Discontinuous at all positive rational number except 1, continuous otherwise.
21. discontinuties lie one in each interval : $(-2, -1), (0, 1), (1, 2)$
23. (fog) x is continuous everywhere and (gof) x is discontinuous at $x = 0, \pm 1$.
30. discontinuous at $x = 0$.
35. 1

TARGET EXERCISES for JEE ADVANCED

2. discontinuous
3. continuous everywhere.
4. discontinuous at $x = 0, 1$.
7. $y_n(x)$ is continuous at $x = 0$ for all n and $y(x)$ is discontinuous at $x = 0$
8. $f(x) = 0$ when $x = 0$,
$= \frac{1}{2} - x$ when $0 < x < \frac{1}{2}$,
$= \frac{1}{2}$ when $x = \frac{1}{2}$,
$= \frac{3}{2} - x$ when $\frac{1}{2} < x < 1$,
$= 1$ when $x = 1$,
13. 4

14. $f(x) > 0$ for $c < x \le b$, and $f(x) < 0$ for $a \le x < c$.

17. f is continuous at every irrational in $(0, \infty)$ and discontinuous at every rational in $(0, \infty)$

21. (iii) No (iv) yes

22. finite discontinuity at $x = n + 1/2$, $n \in I$.

23. both are discontinuous; $\phi(x)$ is continuous if $a = 1$.

24. continuous everywhere.

27. Jump discontinuity.

29. (i) $4 - 3\sqrt{2} + a$, (ii) $a = 3$

30. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ f is discontinuous at $x = 0$; $g(0^+) = g(0^-) = g(0) = \pi/2$ $\Rightarrow$ g is cont. at $x = 0$.

31. discontinuity of the first kind at $1/r$, $r = 2, 3, \ldots$; discontinuity of the first kind from left at $x = 1$; discontinuity of the first kind from right at $x = 0$.

32. $n + 1/2$, $n \in I$.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $k = 7$

2. $f(x) = \sqrt{4 - x^2}$, $-2 \le x \le 0 = -\sqrt{4 - x^2}$, $0 \le x \le 2$

3. $(-\infty, -1) \cup [0, \infty)$, $I - \{0\}$ where I is the set of integer except $n = -1$

4. 10

5. B

6. C

7. D

8. CD

9. BCD

10. A

11. B, D

12. A, B

13. A, B

14. $a = -\frac{3}{2}$, $b \in R^+$, $c = \frac{1}{2}$

16. $$g(x) = \begin{cases} 2 + x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 4 - x. & 2x < x \le 3 \end{cases}$$

discontinuity at $x = 1, 2$

17. $a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$

18. $a = 8$

19. $a = \frac{2}{3}$, $b = e^{2/3}$

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. B

2. B

3. D

4. C

5. B

6. D

HINTS & SOLUTIONS

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. Correction $\rightarrow f(x)=\dfrac{1-\cos x(\cos 2x)^{1/2}\cos(3x)^{1/3}}{x^2}$

Sol using L-Hospital's Rule

$$\lim_{x\to 0} f(x)=\frac{\lim\limits_{x\to 0}(\sin x)(\cos 2x)^{1/2}(\cos 3x)^{1/3}+\cos x(\cos 3x)^{1/3}\dfrac{\sin 2x}{\sqrt{\cos 2x}}+\cos x(\cos 2x)^{1/3}\dfrac{\sin 3x}{(\cos 3x)^{2/3}}}{2}$$

$$=\lim_{x\to 0}\left[\frac{1}{2}+1+\frac{3}{2}\right]=3$$

Hence, option B is correct.

2. $\lim\limits_{x\to 0}\dfrac{x-e^x+\cos 2x}{x^2}$

$$\lim_{x\to 0}\frac{1-e^x-2\sin 2x}{2x}=\frac{-1}{2}-2=-\frac{5}{2}$$

Hence, D is correct.

3. $f(x)=\begin{cases}\left|x^2-1\right|-1, & x\le 1\\ |2x-3|-|x-2|, & x>1\end{cases}$

$$=\begin{cases}x^2-2 & x\le -1\\ -x^2 & -1<x\le 1\\ 1-x & 1<x\le 3/2\\ 3x-5 & 3/2<x<2\\ x-1 & x\ge 2\end{cases}$$

Hence, C is correct.

4. $f(x)=[x]+\left[x+\dfrac{1}{3}\right]+\left[x+\dfrac{2}{3}\right]$

In $[-1,1]$, $f(x)$ is discontinuous at 7 points i.e.,

$0,\ \pm\dfrac{1}{3},\ \pm\dfrac{2}{3},\ \pm 1$

Hence, C is correct.

5. $f(x)=\begin{cases}\dfrac{\sin\{x\}}{\{x\}}, & \{x\}\ne 0\\ k & \{x\}=0\end{cases}$

$f(x)$ can never be continuous for any value of K.

Hence, D is correct.

6. If $f(x)$ take only irrational values, then, $f(x)$ is a constant function.

$$\Rightarrow \sum_{r=1}^{100} f(r)=\sum_{r=1}^{100} f(2r)$$

Hence, B is correct.

7. $f(x)=\operatorname{sgn}(\cos 2x-2\sin x+3)$

$$=\operatorname{sgn}\left(4-2\sin^2 x-2\sin x\right)$$

$$=\operatorname{sgn}\left(-2\left(\sin^2 x+\sin x-2\right)\right)$$

$$=\operatorname{sgn}\left(-2\left(\left(\sin x+\frac{1}{2}\right)^2-\frac{9}{4}\right)\right)$$

For $x\ne(4n+1)\dfrac{\pi}{2}$,

$f(x)=1$

For $x=(4n+1)\dfrac{\pi}{2}$

$f(x)=0$

Hence, option C is correct.

8. $f(x)=(\sin x)^{\frac{1}{\pi-2x}},\qquad x\ne\dfrac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{\pi-2x}}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\pi-2x}}$$

$$=1$$

Hence, B is correct.

9. $f(x)=\dfrac{x+1}{x},\qquad x\ne 0$

$$f\circ f(x)=\frac{\dfrac{x+1}{x}+1}{\dfrac{x+1}{x}}=\frac{2x+1}{x+1},\quad x\ne -1$$

$$f\circ f\circ f(x)=\frac{\dfrac{2x+1}{x+1}+1}{\dfrac{2x+1}{x+1}}$$

$$=\frac{3x+2}{2x+1},\qquad x\ne\frac{-1}{2}$$

So, there are three points of discontinuity

Hence, D is correct.

10. $\lim\limits_{x\to 2^+}\dfrac{x-[x]}{x-2}=\lim\limits_{x\to 2^+}\dfrac{\{x\}}{\{x\}}=1$

So, b = 1

$$\lim_{x\to 2^-}\frac{a\left|x^2-x-2\right|}{2+x-x^2}=\lim_{x\to 2^-}\frac{a\left(2+x-x^2\right)}{2+x-x^2}$$

= a

So, a = 1

Hence, A is correct.

11. $f(x)=|2\,\text{sgn}\,2x|+2.$

$f(0)=2$

$f(0^+)=4$ $\quad f(0^-)=4$

Hence, B is correct.

12. $$\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}\lim_{n\to\infty}\frac{x+x^{2n}\sin x}{1+x^{4n}}$$

$$=\lim_{x\to 1^+}\lim_{n\to\infty}\frac{x^{-4n+1}+x^{-2n}\sin x}{x^{-4n}+1}$$

$= 0$

$$\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-}\lim_{n\to\infty}\frac{x+x^{2n}\sin x}{1+x^{4n}}=1$$

Hence, B is correct.

13. $f(x)=\dfrac{x^3}{4}-a\sin\pi x+3,\quad f(4)=19,\quad f(-4)=-13$

As $f(x)$ is continuous & $-13<\dfrac{1999}{199}<19.$

Using IVT, f(x) is $\dfrac{1999}{199}$ for some $x\in[-4,4]$

Hence, C is correct.

14. $$\lim_{x\to 0^+}\frac{be^x-\cos x-x}{x^2}$$

For limit to exist, b = 1

$$\lim_{x\to 0^+}\frac{e^x-\cos x-x}{x^2}=1=a$$

$$\lim_{x\to 0^-}\frac{2\left(\tan^{+}e^x-\frac{\pi}{4}\right)}{x}=1$$

Hence, A is correct.

15. For f(x) to be continuous,

$2x+1=x^2-2x+5$

$\Rightarrow x^2-4x+4=0$

$\Rightarrow (x-2)^2=0$

$\Rightarrow x=2.$

Hence, f(x) is continuous at only x = 2.

Hence, D is correct.

16. $f(x)=[x\cos x]$

$\lim_{x\to 0^+} f(x)=0,\quad \lim_{x\to 0^-} f(x)=-1$

$\lim_{x\to 1^+} f(x)=0,\quad \lim_{x\to 1^-} f(x)=0,\quad f(1)=0$

Hence, B is correct.

17. $$\lim_{x\to 0}\frac{\sin^4\left(\frac{1}{x}\right)-\sin^2\left(\frac{1}{x}\right)+1}{\cos^4\left(\frac{1}{x}\right)-\cos^2\left(\frac{1}{x}\right)+1}$$

$$\lim_{x\to 0}\frac{-\sin^2\left(\frac{1}{x}\right)\cos^2\left(\frac{1}{x}\right)+1}{-\cos^2\left(\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+1}=1$$

Hence, B is correct.

18. $f(x)=\lim_{n\to\infty}\left(1-\sin^2 x\right)^n=\lim_{n\to\infty}\cos^{2n}x$

$f\left(\dfrac{\pi}{4}\right)=0$

& f(x) discontinuous is at x = nπ.

Hence, A or C is correct.

19. $f(0^-)=\lim_{x\to 0^-} b\cos^{2m}x-1=-1$

$f(0^+)=\lim_{x\to 0^+} a\sin^{2n}x=0$

Hence, A is correct.

20. As f(b) = g(b) & both are Continuous function, then

$h(b^-)=g(b^+)$ & $h(b^+)=f(b^-)$

Hence, C is correct.

21. $f(x)$ is discontinous at x = b.

Hence, A is correct.

22. $f(x)=(x\bmod 2)^2+(x\bmod 4)$

Points of discontinuity are x = 2, 4, 6, 8

Hence, C is correct.

23. $$\lim_{x\to 3}\frac{\ln(3f(x)-2)}{2(1-f(x))}=\lim_{x\to 3}\frac{\ln(1+3(f(x)-1))}{\frac{2}{3}3(1-f(x)}$$

$$=\frac{-3}{2}$$

Hence, C is correct.

24. $f(x) =$ highest power of $(u^{x^2} + u^2 + 2u + 3)$

$$\lim_{x \to \sqrt{2}^+} f(x) = 2 \qquad f(\sqrt{2}) = 2$$

$$\lim_{x \to \sqrt{2}^-} f(x) = 2$$

Hence, A is correct.

25. $f(x) = [\sin x + \cos x]$

$$= \left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right]$$

So, f(x) is discontinuous at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

Hence, B is correct.

26. $f(x) = \dfrac{1}{1-x},\ x \neq 1$

$$f \circ f(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{-x},\ x \neq 0$$

$$f \circ f \circ f(x) = \frac{1}{1 + \frac{1-x}{x}} = x$$

So, $\alpha = 0,\quad \beta = 1$

$a + 2a^2 - 3 < 0$

$\Rightarrow 2a^2 + a - 3 < 0$

$\Rightarrow 2a^2 + 3a - 2a - 3 < 0$

$\Rightarrow (2a+3)(a-1) < 0$

$a \in \left(-\frac{3}{2}, 1\right)$

Hence, A is correct.

27. $2 \le \dfrac{f(x)}{x^2} \le 3$

$2x^2 \le f(x) \le 3x^2$

If $x = \frac{1}{2}$

$$\to \frac{1}{2} \le f\left(\frac{1}{2}\right) \le \frac{3}{4}$$

If $x = 1$

$$\to 2 \le f(1) \le 3$$

If $x = \frac{1}{3}$

$$\to \frac{2}{9} \le f\left(\frac{1}{3}\right) \le \frac{1}{3}$$

If $x = \frac{1}{4}$

$$\to \frac{1}{8} \le f\left(\frac{1}{4}\right) \le \frac{3}{16}$$

If $x = \frac{1}{8}$

$$\to \frac{1}{32} \le f\left(\frac{1}{8}\right) \le \frac{3}{64}$$

Hence, D is correct.

28. $f(x) = \operatorname{sgn}\left(\sin^2 x - \sin x - 1\right)$

$$= \operatorname{sgn}\left(\left(\sin x - \frac{1}{2}\right)^2 - \frac{5}{4}\right)$$

For f(x) to be discontinuous,

$$\Rightarrow \left(\sin x - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\sin x - \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$

$$\sin x = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \sin x = \frac{1 - \sqrt{5}}{2}$$

Hence, C is correct.

29. $f(x) = \lim\limits_{n \to \infty} \sin^{2n}(\pi x) + \left[x + \frac{1}{2}\right]$

$f(1) = 1$

$$\lim_{x \to 1^+} f(x) = 1, \quad \lim_{x \to 1^-} f(x) = 1$$

$$f\left(\frac{3}{2}\right) = 3$$

$$\lim_{x \to \frac{3}{2}^+} f(x) = 2, \quad \lim_{x \to \frac{3}{2}^-} f(x) = 1$$

Hence, A is correct.

30. For g(x) to be continuous, then
a > range of f(x)

$\Rightarrow\ a > \sqrt{26}$

least positive integral value of *a* = 3
Hence, B is correct.

31. As f(x) is continuous at *x* = 0
f(0) = c

$$\text{LHL at } x = 0 = \lim_{x \to 0} \frac{axe^x + b\sin x}{x^3}$$

Applying L-Hospital Rule,

$$\lim_{x\to 0}\frac{ae^x+axe^x+b\cos x}{3x^2}.$$

For limit to exist, a + b = 0

Applying L-Hospital again, $\lim\limits_{x\to 0}\dfrac{2ae^x+axe^x-b\sin x}{6x}$

32. If f(x) attains only rational values for all real x, then f(x) =4.

a_1, a_2, a_3, a_4, a_5 are in H.P.

Then, $a_r\cdot a_{r+1}=\dfrac{1}{a+(r-1)d}\cdot\dfrac{1}{(a+rd)}$

$$=\frac{1}{d}\left[\frac{1}{a+(r-1)d}-\frac{1}{a+rd}\right]$$

$$\sum_{r=1}^{4}a_ra_{r+1}=\frac{1}{d}\left[\left(\frac{1}{a}-\frac{1}{a+d}\right)+\left(\frac{1}{a+d}-\frac{1}{a+2d}\right)\right.$$

$$\left.+\cdots\left(\frac{1}{a+3d}-\frac{1}{a+4d}\right)\right]$$

$$=\frac{1}{d}\left[\frac{1}{a}-\frac{1}{a+4d}\right]$$

$$=\frac{4}{a\cdot a+4d}=4a_1a_5$$

Hence, A is correct.

33. $f(x)=\begin{cases}2x-1, & -2\le x<0\\ x+2, & 0\le x\le 2\end{cases}$

$g(x)=\begin{cases}[x], & -4\le x<-2\\ x+2, & -2\le x\le 4.\end{cases}$

$\lim\limits_{x\to -2^+} f(g(x))=2$

$\lim\limits_{x\to -2^-} f(g(x)=$ Not defined

Hence, A & B is correct.

34. $f(x)=\begin{cases}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}, & x>2\\ k, & x=2.\end{cases}$

$$\lim_{x\to 2^+}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}=\frac{1}{4}$$

Hence B is correct.

35. $f(x)=\begin{cases}\sin^2 x, & x \text{ is rational}\\ -\sin^2 x, & x \text{ is irrational}\end{cases}$

If f(x) is continuous, Then,

$\sin^2 x=-\sin^2 x$

$\Rightarrow 2\sin^2 x=0$

$\Rightarrow x=n\pi$

Hence, C is correct.

36. $\lim\limits_{x\to a}\dfrac{\ln\left(1+6f^2(x)-3f(x)\right)}{3f(x)}$

$$\lim_{x\to 0}\frac{\ln\left(1+6f^2(x)-3f(x)\right)}{\left(6f^2(x)-3f(x)\right)}\times\frac{(6f^2(x)-3f(x))}{3f(x)}$$

$=-1$

Hence, C is correct.

37. $f\left(\dfrac{1}{4^n}\right)=\left(\sin e^n\right)e^{-n^2}+\dfrac{n^2}{n^2+1}$

$$f(0)=\lim_{n\to\infty}\left(\left(\sin e^n\right)e^{-n^2}+\frac{n^2}{n^2+1}\right)$$

$=1$

Hence, B is correct.

38. $f(x)=[\tan x[\cot x]],\ x\in\left[\dfrac{\pi}{12},\dfrac{\pi}{2}\right)$

As $\cot\dfrac{\pi}{12}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\approx 3.7$

$\cot\dfrac{\pi}{2}=0$

$[\cot x]=0,1,2,3$

Hence, f(x) is discontinuous whenever

$[\cot x]=1,2,3$

Hence, C is correct.

39. $f(x)+g(x)=\{1,\ \forall x\in R$

Hence, C is correct.

40. $\lim\limits_{x\to 0} g(x)\cdot\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} g(x)$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} g(x)$

Hence, C is correct.

41. $f(x)=\lim\limits_{n\to\infty}\dfrac{(2\sin x)^{2n}}{3^n-(2\cos x)^{2n}}$

$$\Rightarrow \lim_{n\to\infty} \frac{(2\sin x)^{2n}}{1-\left(\frac{2}{\sqrt{3}}\cos x\right)^{2n}}$$

f(x) is discontinuous whenever

$$2\sin x = \pm 1 \,\&\, \frac{2}{\sqrt{3}}\cos x = \pm 1$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{6}$$

Hence, B is correct.

42. $f(x) = \frac{\tan x \log x}{1-\cos 4x}$

Firstly, $x \ne (-\infty, 0]$

& $1-\cos 4x \ne 0$

$\cos 4x \ne 1$

$\Rightarrow 4x \ne 2n\pi$

$x \ne \frac{n\pi}{2}$

Hence, C is correct.

43. $f(x) = \begin{cases} \sin x, & x \in \text{Rational} \\ 1-2\cos x, & x \in \text{Irrational} \end{cases}$

For continuity,

$\sin x = 1-2\cos x$

$\sin x + 2\cos x = 1$

$\frac{1}{\sqrt{5}}\sin x + \frac{2}{\sqrt{5}}\cos x = \frac{1}{\sqrt{5}}$

$\sin(x+\alpha) = \frac{1}{\sqrt{5}}$, where $\cos\alpha = \frac{1}{\sqrt{5}}$

There are infinite values of ‘x’ which satisfies the eqn.

Hence, D is correct.

44. $f(x) = \begin{cases} \log_{(4x-3)}\left(x^2-2x+5\right), & \frac{3}{4} < x < 1 \,\&\, x > 1 \\ 4, & x = 1 \end{cases}$

$$\lim_{x\to 1^+} \frac{\log\left(x^2-2x+5\right)}{\log(4x-3)}$$

$$\lim_{h\to 0} \frac{\log\left((1+h)^2-2(1+h)+5\right)}{\log(1+4h)} \times \frac{4h}{4h}$$

= Does not exist

Similarly, $\lim_{x\to 1^-} f(x) =$ Does not exist

Hence, D is correct.

45. $f(x) = \lim_{t\to\infty} \frac{\left((1+\sin \pi x)^t - 1\right)}{\left((1+\sin \pi x)^t + 1\right)}$

Case-I If *x* is integer

LHL of f(x) = $\lim_{t\to\infty} f(x) \to 1$

RHL of f(x) = $\lim_{t\to\infty} f(x) \to -1$

Hence, B is correct.

46. $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$

$$f(0) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{e^{2x}-1}\right) = \lim_{x\to 0} \frac{e^{2x}-1-2x}{x\left(e^{2x}-1\right)}$$

$$= \lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2} \times \frac{x}{e^{2x}-1}$$

=1

Hence, D is correct.

47. $f(x) = \begin{cases} \frac{1-|x|}{1+x}, & x \ne -1 \\ 1, & x = -1 \end{cases}$

$$f([2x]) = \begin{cases} \frac{1-|[2x]|}{1+[2x]}, & [2x] \ne -1 \\ 1 & [2x] = -1 \end{cases}$$

$$= \begin{cases} \frac{1-|[2x]|}{1+[2x]} & x \in R - \left[\frac{-1}{2}, 0\right) \\ 1 & -\frac{1}{2} \le x < 0 \end{cases}$$

At x = –1

$\text{LHL} = \frac{-2}{-2} = 1, \quad \text{RHL} = 1, \quad f(-1) = 1$

At $x = 0$

$\text{LHL} = \text{RHL} = f(0) = 1$

At $x = \frac{1}{2}$

$\text{LHL} = 1, \quad \text{RHL} = 0, f\left(\frac{1}{2}\right) = 0$

Hence, D is correct.

48. $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0, \\ 1, & x > 0 \end{cases} \quad g(x) = \sin x + \cos x$

$$f\circ g(x)=\begin{cases}-1, & \sin x+\cos x<0\\ 0, & \sin x+\cos x=0\\ 1 & \sin x+\cos x>0\end{cases}$$

$$=\begin{cases}-1 & \sin\left(\frac{\pi}{4}+x\right)<0\\ 0 & \sin\left(\frac{\pi}{4}+x\right)=0\\ 1 & \sin\left(\frac{\pi}{4}+x\right)>0\end{cases}$$

Hence, B is correct.

49. For continuity of $f(x)\rightarrow$

$4a=16$

$a=4$

For $g(x)\Rightarrow 4=a$

For $h(x)\Rightarrow a=4$

Hence, B is correct.

50. $f(x)=\dfrac{\tan x\log x}{1-\cos 4x}$

$=\dfrac{\tan x\log x}{2\sin^2 2x}$

As $\log x$ is not defined for $(-\infty,0]$ & $\tan x$ is not defined

for $(2n+1)\dfrac{\pi}{2}$

& $\dfrac{1}{\sin 2x}$ is not defined for $n\dfrac{\pi}{2}$

Hence, C is correct.

51. $f(x)$ is continuous only if $0=5x$

$\Rightarrow x=0$

Hence, D is correct.

52. $R(x)=\dfrac{x^3-2x^2-9x+18}{x^4-4}$

For vertical asymptote, $\lim\limits_{x\to a}R(x)\rightarrow\pm\infty$

$$\lim_{x\to a}R(x)=\lim_{x\to a}\frac{(x-2)\left(x^2-9\right)}{\left(x^2-2\right)\left(x^2+2\right)}$$

So, at $x=\pm\sqrt{2}$, there are two vertical asymptotes

Hence, A is correct.

53. $\displaystyle\prod_{n=1}^{n}\cos\left(\frac{x}{2^n}\right)=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$

Taking log & differentiate w.rt. x.

$$\sum_{n=1}^{n}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\frac{\cos\left(x/2^n\right)}{\sin\left(\frac{x}{2^n}\right)}-\cot x.$$

now, $\displaystyle\lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{x}-\cot x.$

$$\lim_{x\to\frac{\pi}{2}}\lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{2}{\pi}$$

Hence, C is correct.

54. $\displaystyle\lim_{x\to 0}\frac{2-\sqrt[4]{x^2+16}}{\cos 2x-1}$

$$\lim_{x\to 0}\frac{4-\sqrt[2]{x^2+16}}{\left(2+\sqrt[4]{x^2+16}\right)(\cos 2x-1)}$$

$$\lim_{x\to 0}\frac{-x^2}{\left(2+\sqrt[4]{x^2+16}\right)\left(4+\sqrt[2]{x^2+16}\right)(\cos 2x-1)}$$

$=\dfrac{1}{64}$

Hence, A is correct.

55. $\displaystyle\lim_{x\to\frac{\pi^+}{2}}\left[\frac{2\left(\sin x-\sin^3 x\right)+\sin x-\sin^3 x}{2\left(\sin x-\sin^3 x\right)-\sin x+\sin^3 x}\right]$

$$\lim_{n\to\frac{\pi^+}{2}}\left[\frac{3\left(\sin x-\sin^3 x\right)}{\left(\sin x-\sin^3 x\right)}\right]=3$$

$$\lim_{x\to\frac{\pi^+}{2}}\left[\frac{2\left(\sin x-\sin^3 x\right)+\sin x-\sin^3 x}{2\left(\sin x-\sin^3 x\right)-\sin x+\sin^3 x}\right]$$

$=3$

So, $f(x)$ is continuous at $x=\dfrac{\pi}{2}$

LHD = RHD = 0 (By first principle)

Hence, A is correct.

56. $\displaystyle\lim_{x\to-1^+}b\left([x]^2+[x]\right)+1$

$\Rightarrow 1\quad\Rightarrow\quad b\in R$

$$\lim_{x\to -1^-}[\sin\pi(x+a)]=1$$

$$\Rightarrow \quad \pi(a-1)=(4n+1)\frac{\pi}{2}$$

$$\Rightarrow a-1=2n+\frac{1}{2}$$

$$a = 2n+\frac{3}{2}$$

Hence, A is correct.

57. $f(x)=\begin{cases}\dfrac{[x]+\sqrt{x-[x]}}{\sin x}, & x\ge 0\\ \sin x, & x<0\end{cases}$

If x is +ve integer = k

$f(x)=k$

$\lim\limits_{x\to k^+} f(x)=k, \quad \lim\limits_{x\to k} f(x)=(k-1)+1=k$

If x is –ve integer = K

f(x) = sin k

LHL = RHL=sin K

If x= 0

f(0) = 0 = RHL, LHL = sin 0 = 0

If x is any + real no, = K

$f(x)=[k]+\sqrt{\{k\}}$

$\text{RHL}=[k]+\sqrt{\{k\}}, \quad \text{LHL}=[K]+\sqrt{\{k\}}$

If x is any – Real no = K

f(x) = sin K = RHL = LHL

Hence, D is correct.

58.

$$f(x)=\frac{4-x^2}{|4x-x^3|}$$

$$=\begin{cases}\dfrac{-(4-x^2)}{(4x-x^3)}, & x>2\\ \dfrac{4-x^2}{4x-x^3}, & 0<x<2\\ \dfrac{4-x^2}{4x-x^3}, & -2<x<0\\ \dfrac{-(4-x^2)}{4x-x^3}, & x<-2\end{cases}$$

$$=\begin{cases}-\dfrac{1}{x}, & x>2 \text{ or } -2<x<0\\ \dfrac{1}{x}, & 0<x<2<x<-2\end{cases}$$

So, f(x) is discontinuous at x = 0, +2, –2.

Hence, C is correct.

59. $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{\tan kx}{x}=k$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\left(3+2k^2\right)=2k^2$

For continuity, $k = 2k^2$

$\Rightarrow K = 0 \text{ or } \dfrac{1}{2}$

Hence, C is correct.

60. $f(x)=\lim\limits_{x\to\infty} x\tan^{-1} nx$

$$\begin{cases}\dfrac{\pi}{2}x, & x>0\\ 0, & x=0\\ \dfrac{-\pi}{2}x, & x<0\end{cases}$$

Hence, C is correct.

61. $f(x)=3+\left(1+7^{\frac{1}{1-x}}\right)^{-1}$

$f(1)=$ Not defined

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(3+\left(1+7^{\frac{1}{1-x}}\right)^{-1}\right)=4$$

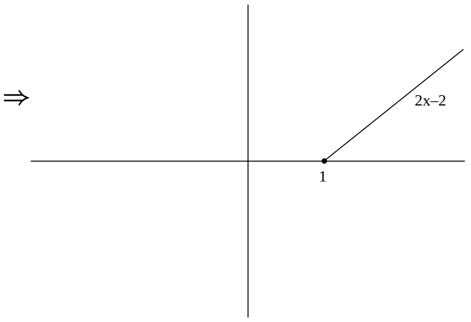
$$\lim_{x\to 1^-} f(x)=3+\lim_{x\to 1^-}\left(1+7^{\frac{1}{1-x}}\right)^{-1}$$

$=3$

Hence, option D is correct.

62. fofofof(x) = x

Y = f(x) ⇒



Now, from the graph, roots of fofofof(x) = x is only possible at x = 0 & 2.

Hence, A is correct.

63. $\lim\limits_{x\to 0}\dfrac{\left(3^x-1\right)^2}{(\sin x)\ln(1+x)}$

$$=\lim_{x\to 0}\frac{\left(3^x-1\right)^2}{x^2}\times\frac{x}{\sin x}\times\frac{x}{\ln(1+x)}$$

$$=(\ln 3)^2$$

Hence, C is correct.

64. $$\lim_{x\to0}\frac{\sqrt{4-2x+x^2}-\sqrt{4+2x+x^2}}{\sqrt{2+x}-\sqrt{2-x}}$$

Rationalize,

$$\lim_{x\to0}\frac{(-4x)\left(\sqrt{2+x}+\sqrt{2-x}\right)}{(2x)\left(\sqrt{4-2x+x^2}+\sqrt{4+2x+x^2}\right)}$$

$\Rightarrow -\sqrt{2}$

Hence, C is correct

65. $$f(x)=\frac{\log_{\sin x|}\cos^3 x}{\log_{\sin|3x|}\cos\frac{x}{2}},\quad x\in\left(-\frac{\pi}{3},\frac{\pi}{3}\right),\quad x\ne0$$

$$=\frac{3\log(1+\cos x-1)}{\log\left(1+\cos\frac{x}{2}-1\right)}\times\frac{\log(1+\sin|3x|-1)}{\log(1+\sin|x|-1)}$$

f(x) is discontinuous at $x=\pm\frac{\pi}{6}$

Hence, C is correct.

66. $$l(x)=\frac{x^4+x^8+2x}{\tan x}$$

$\lim_{x\to0} l(x)=2\ne l(0)$

Hence, D is correct.

67. $\lim_{x\to-2^+} f(g(x))=\lim_{x\to-2^+} f\left(0^+\right)=2$

Hence, A is correct.

68. f(x) = [4x] + {3x} of (4x)

pt of discontinuous are $\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2,\frac{9}{4}$

$\frac{5}{2},\frac{11}{4},3,\frac{13}{4},\frac{7}{2},\frac{15}{4},4,\frac{17}{4},\frac{9}{2}$

Pd of discontinuous for {3x} are

$\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\frac{5}{3},2,\frac{7}{3},\frac{8}{3},3,\frac{10}{3},\frac{11}{3},4,\frac{13}{3},\frac{14}{3}$

There are 30pts in total.

Hence, C is correct.

69. $f(x)=\left[\tan^2 x\right]\left[\cot^2 x\right]$

$f(x)$ is discontinous at $\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4}$

Hence, D is correct.

70. $\lim_{x\to c}[h(x)]\ne\left[\lim_{x\to c}h(x)\right]$

& h(x) can take any number of integral values.

Hence, D is correct.

MULTI CORRECT ANSWER TYPE FOR JEE ADVANCED

71. $f(x)=[x^2]+[x]^2$

As [x] is discontinuous whenever x is an integer.

Hence, f(x) will be discontinuous at $x=\sqrt{2}$ & 0.

Hence, A, B, C, D are correct.

72. $$f(x)=\frac{1}{1-x}$$

$$f^2(x)=\frac{1}{1-\frac{1}{1-x}}=\frac{1-x}{-x}$$

$$f^3(x)=\frac{1}{1+\frac{1-x}{x}}=x$$

So, pattern will repeat after an interval of three

$f^{3n}(x)=x.\qquad \forall x$ except $\{0,1\}$

Hence, A & B are correct.

73. $$f(x)=\begin{cases}x\left[\frac{1}{x}\right]+x[x], & x\ne0\\ 0, & x=0\end{cases}$$

$\lim_{x\to-1^+} f(x)=3,\quad \lim_{x\to-1^-} f(x)=3$

$\lim_{x\to1^+} f(x)=1,\quad \lim_{x\to1^-} f(x)=1,\quad f(1)=2$

$\lim_{x\to2^+} f(x)=4,\qquad \lim_{x\to2^-} f(x)=2$

Hence, A, B, C, D are correct.

74. $$f(x)=\lim_{n\to\infty}\frac{x^{2n-1}+ax^2+bx}{x^{2n}+1}$$

$\lim_{x\to1^+} f(x)=1$

$\lim_{x\to1^-} f(x)=a+b$

$$f(1)=\frac{a+b+1}{2}$$

$\Rightarrow a+b=1.$

75. $f(x)=\sqrt{2\sin x}+3,\ \frac{e^x+1}{e^x+3},\ \left(\frac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ are continuous for $x\in R$.

and

$f(x)=\sqrt{\text{sgn}(x)+1}$ is not continuous at x = 0.

Hence, A, B, C are correct.

76. D) $y = x^3 - 3x + 1 \quad \forall\ x \in [1, 2]$

$$\frac{dy}{dx} = 3x^2 - 3 = 3\left(x^2 - 1\right) > 0$$

$f(1) = -1,\ f(-1) = 3$

$\Rightarrow$ it will have a root.

Hence, D is correct.

77. $f(x) = x^4 - 14x^3 + px^2 + qx - 105$

Sum of roots are 14 & product is -105.

Hence, roots are $-1, 3, 5, 7$.

$g(x) = (x-1)(x-3)(x-5)(x-7)$

$$F(x) = \frac{(x+1)(x-3)(x-5)(x-7)}{(x-1)(x-3)(x-5)(x-7)}$$

$$= \frac{x+1}{x-1} \quad \forall x \neq 1, 3, 5, 7$$

Hence, ABCD are correct.

78. $f(x) = [x] \quad \& \quad g(x) = \begin{cases} 0, & x \in I \\ x^2, & \text{otherwise} \end{cases}$

$g \circ f(x) = 0, \qquad x \in R$

$\lim\limits_{x\to 2^+} f \circ g(x) = 4, \quad \lim\limits_{x\to 2^-} f \circ g(x) = 3 \quad f \circ (g(2) = 0$

$\lim\limits_{x\to 5^+} f \circ g(x) = 25, \quad \lim\limits_{x\to 5^-} f \circ g(x) = 24$

Hence, A is correct.

79. $f(x) = \cos\left[\frac{\pi}{x}\right]\cos\left(\frac{\pi}{2}(x-1)\right)$

$\underset{x\to 0^-}{\text{LHL}} = \underset{x\to 0^+}{\text{RHL}} = f(0) = 0$

$\lim\limits_{x\to 1^+} f(x) = \cos 3, \quad \lim\limits_{x\to 1^-} f(x) = \cos 3, \quad f(1) = \cos 3$

$\lim\limits_{x\to 2^+} f(x) = 0, \quad \lim\limits_{x\to 2^-} f(x) = 0, \quad f(2) = 0$

Hence, A, B, C are correct.

80. Conceptual question

Hence, A, B, C, D are correct.

81. $\lim\limits_{x\to 0^-} \frac{\sin ax}{bx} = \frac{a}{b}$

$\lim\limits_{x\to 0^+} (ax+1) = 1$

$\lim\limits_{x\to 1^-} (ax+1) = a+1 \quad \lim\limits_{x\to 2^-} (cx^2 - 2) = 4c - 2$

$\lim\limits_{x\to 1^+} (cx^2 - 2) = c - 2 \quad \lim\limits_{x\to 2^+} \frac{d(x^2-4)}{\sqrt{x}} = 0$

$\lim\limits_{x\to 4^-} \frac{d\left(x^2-4\right)}{\sqrt{x}} = {}^{-}6d, \qquad \lim\limits_{x\to 4^+} 12 = 12$

$a = \frac{-3}{2},\ b = \frac{-3}{2},\ c = \frac{1}{2},\ d = -2.$

Hence, A, C, D are correct.

82. $\lim\limits_{n\to\infty} \lim\limits_{x\to 0^-} \cos^{2n} x = 0$

$\lim\limits_{n\to\infty} \lim\limits_{x\to 0^+} \sqrt[n]{1+x^n} = 1$

$\lim\limits_{n\to\infty} \lim\limits_{x\to 1^-} \sqrt[n]{1+x^n} = 1$

$\lim\limits_{n\to\infty} \lim\limits_{x\to 1^+} \frac{1}{1+x^n} = 0$

Hence, A, B, C are correct.

83. All are continuous $\forall x \in R$

Hence, A, B, C, D are correct.

84. B & D are discontinuous function

Hence, A & C are correct.

85. Conceptual question

Hence, A, B, C, D are correct.

86. $f(x) = \begin{cases} \left|\tan^{-1}\frac{1}{x}\right|, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = \frac{\pi}{2}, \quad \lim\limits_{x\to 0^-} f(x) = \frac{\pi}{2}, \quad f(0) = 1$

Hence, B, C, D are correct.

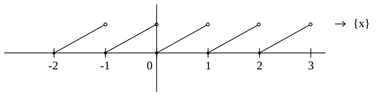
87. $f(x) = [x]\ \&\ g(x) = \begin{cases} 0, & x \in I \\ x^2, & \text{otherwise} \end{cases}$

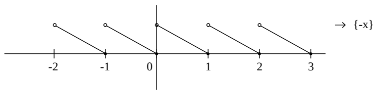
$g \circ f(x) = 0, \qquad x \in R$

$f \circ g(x) = \begin{cases} 0, & x \in I \\ [x^2] & \text{otherwise} \end{cases}$

Hence C, D are correct.

88. $f(x) = \max(\{x\}, -\{-x\})$





Hence, C, D are correct.

89. Conceptual problem.

Hence, C, D are correct.

90. $\lim\limits_{x\to 0^-} \frac{e^x - e^{[x]}}{e^x} = 1 - e$.

$$\lim_{x\to 0^+}\frac{\sin\{x\}}{\{\tan x\}}=1$$

$$\lim_{x\to \frac{\pi}{6}^+}\frac{\sin\{x\}}{\{\tan x\}}=\frac{\sin\pi/6}{\tan\pi/6}=\frac{\sqrt{3}}{2}$$

$$\lim_{x\to \frac{\pi}{6}^-}\frac{\sin\{x\}}{\{\tan x\}}=\frac{\sqrt{3}}{2},\ f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$\lim_{x\to \frac{\pi}{4}^+}\frac{\sin\{x\}}{\{\tan x\}}=\frac{\sin\pi/4}{\tan\frac{\pi}{4}-1}=\infty$$

Hence, B, C, D are correct.

91. Both A & R are correct statements.

Hence, A is correct.

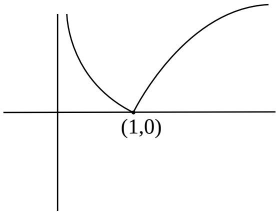
92. $\lim\limits_{x\to 0^+} f(x)=1,\quad \lim\limits_{x\to 1^-} f(x)=2$

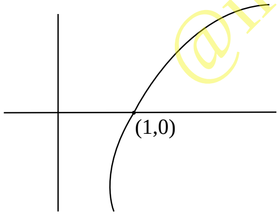
$\lim\limits_{x\to 1^+} 2-\{x\}=2,\quad \lim\limits_{x\to 2^-} f(x)=1,\quad f(2)=2.$

Hence, D is correct.

93. $y=|\ln x|\qquad y=\ln x$

Hence, C is correct.





94. Conceptual

Hence, B is correct.

95. $\lim\limits_{x\to 0^+}\cos\left(x\cos\frac{1}{x}\right)=1\qquad$ *f*(0) is not defined.

$$\lim_{x\to 0^-}\cos\left(x\cos\frac{1}{x}\right)=1$$

Hence, C is correct.

96. Conceptual.

Hence, D is correct.

97. $f(x)=\dfrac{x^2\left(x^{2n}-1\right)}{x^2-1}$

98. $\lim\limits_{x\to 1^+}F(x)=\lim\limits_{x\to 1^+}\operatorname{sgn}(x-1)\cot^{-1}[x-1]$

$$=\frac{\pi}{2}$$

$$\lim_{x\to 1^-}F(x)=\lim_{x\to 1^-}\operatorname{sgn}(x-1)\cot^{-1}[x-1]$$

$$=-\frac{3\pi}{4}$$

Hence, C is correct.

99. $\lim\limits_{x\to 2^+} f\circ g(x)=1,\ \lim\limits_{x\to 2^-} f\circ g(x)=1,\ f\circ g(2)=1$

A is true statement, but R is not correct.

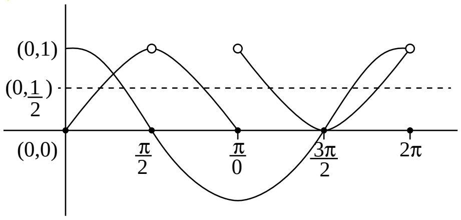
Hence, C is correct.

100. $\lim\limits_{x\to \pi^+}[1+\cos x]=0,\ \lim\limits_{x\to \pi^-}[1+\cos x]=0,\ f(\pi)=0$

Hence, D is correct.

Comprehension 1

$$f(x)=\max\left(\cos x,\frac{1}{2},\{\sin x\}\right),\ 0\le x\le 2\pi$$



$$f(x)=\begin{cases}\cos x, & 0\le x<\frac{\pi}{4},\ \frac{11\pi}{6}\le x\le 2\pi\\ \sin x, & \frac{\pi}{4}\le x<\frac{\pi}{2},\ \frac{\pi}{2}<x\le\frac{5\pi}{6}\\ \frac{1}{2}, & x=\frac{\pi}{2},\ \pi,\ \frac{7\pi}{6}<x<\frac{11\pi}{6}\\ 1+\sin x, & \frac{5\pi}{6}<x\le\frac{7\pi}{6}\end{cases}$$

101. Hence, D is correct.

102. Hence, B is correct.

103. Hence, B is correct.

Comprehension 2

104. Put x = 2

$$f\left(\frac{4}{f(2)}\right)=2$$

$$\Rightarrow\frac{4}{f(2)}=1\Rightarrow f(2)=4$$

Hence, B are correct.

105. $f(x) = x \,\&\, f\left(\frac{x^2}{f(x)}\right) = x$

$\Rightarrow x = \frac{x^2}{f(x)} \Rightarrow x(f(x) - x) = 0$

$\Rightarrow x = 0 \text{ or } f(x) = x$

Hence, D is correct.

106. $\lim_{x\to 1} \frac{\ln(fx) - \ln f(1)}{\sin^n zx}$

Comprehension 3

$f(x) = x^2 + 1, -1 < x \le 1 \,\&\, f(x+2) = \frac{1}{f(x)}$

107. As f(x) is symmetric about origin in [-1,1]. Now, value of f(x) in (1,2) is reciprocal of values of f(x) is (0,1). Values of f(x) in (2,4) is reciprocal of values of f(x) is (0, 2) Hence, fundamental period is 4.

Hence, B is correct.

108. $\lim_{x\to 1^+} f(x) = \lim_{x\to -1} \frac{1}{x^2+1} = \frac{1}{2}$

Hence, C is correct.

109. No of discontinuity are x = 1 & 3.

Hence, C is correct.

Comprehension 4

As $g(x) = \begin{cases} (a-x)\sec\frac{\pi x}{2a}, & x < a \\ -\cot\frac{\pi x}{2a}\operatorname{cosec}(x-a), & x > a \end{cases}$

110. $\lim_{x\to a^-} g(x) = \lim_{x\to a^-} \frac{(a-x)}{\cos\left(\frac{\pi x}{2a}\right)}$

$$= \lim_{h\to 0} \frac{h}{\sin\left(\frac{\pi}{2}\frac{h}{a}\right)} \Rightarrow \frac{2a}{\pi}$$

$$\lim_{x\to a^+} g(x) = \lim_{h\to 0}\left(-\cot\left(\frac{\pi}{2} + \frac{\pi}{2a}h\right)\operatorname{cosec}(h)\right)$$

$$= \lim_{h\to 0} + \frac{\tan\left(\frac{\pi}{2a}h\right)}{\sin h} \times \frac{h}{\frac{\pi}{2a}h} \times \frac{\pi}{2a} = \frac{\pi}{2a}$$

Hence, D is correct.

111. For g(x) to be continuous, $\frac{2a}{\pi} = \frac{\pi}{2a} \Rightarrow 2a = \pi$

Hence, B is correct.

112. $\lim_{x\to a} (\cos(x-a))^{\frac{6}{(x-a)2\sin(x-a)\cos(x-a)}}$

$$\Rightarrow e^{-\lim_{x\to a} 2\sin^2\frac{(x-a)}{2} \times \frac{6}{(x-a)\sin(x-a)\cos(x-a)}}$$

$$\Rightarrow e^{-3/2}$$

Now, $h(x) = \lim_{n\to\infty} \frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}} \times x = x$

$h(a) = a$

$\Rightarrow k = \frac{3}{2}$

Hence, C is correct.

Comprehension 5

$H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$

113. If $x \ge 0$

$x^2 - x + 1 - 2 = 0$

$x^2 - x - 1 = 0$

$x = 1 \pm \sqrt{5} \Rightarrow x = \frac{1+\sqrt{5}}{2}$

If $x < 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \text{ or } -1$

so, $x = -1$

No options matches.

114. $H(1-x^2) = \begin{cases} 1, & x \in [-1,1] \\ 0, & x \ne [-1,1] \end{cases}$

Now, $H(1-x^2) > \left|\sin\frac{\pi}{2}x\right|$

MATCH THE COLUMNS FOR JEE ADVANCED

115.

A) $f(x)=\begin{cases}\dfrac{1+a\cos x}{x^2}, & x<0\\ b\tan\left(\dfrac{\pi}{[x+3]}\right), & x\ge 0\end{cases}$

$$\lim_{x\to 0^-}\frac{1+a\cos x}{x^2}$$

Exists if $a=-1$. Then limit is $\frac{1}{2}$.

$$\lim_{x\to 0^+} b+a\frac{\pi}{3}=\frac{1}{2}\Rightarrow b=\frac{1}{2\sqrt{3}}\Rightarrow[a-2b]=-2$$

Hence, R is correct.

B) $\lim\limits_{x\to -\frac{\pi^-}{2}}(-2\sin x)=2$

$$\lim_{x\to -\frac{\pi^+}{2}}(a\sin x+b)=+b-a$$

$$\lim_{x\to \frac{\pi^-}{2}}(a\sin x+b)=a+b$$

$$\lim_{x\to \frac{\pi^+}{2}}\cos x=0$$

Hence P, Q is correct.

C) $\lim\limits_{x\to\frac{\pi^-}{2}}\left(\dfrac{3}{2}\right)^{\frac{\cos 3x}{\cot 2x}}=1$

$$\lim_{x\to\frac{\pi^+}{2}}(1+|\cos x|)^{\frac{a|\tan x|}{b}}=e^{\lim\limits_{x\to\frac{\pi^+}{2}}\frac{a|\sin x|}{b}}=e^{\frac{a}{b}}$$

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+3$$

$\Rightarrow\quad a=0$

D) $\lim\limits_{x\to 0^+} a+\dfrac{\sin[x]}{x}=a.$

$$\lim_{x\to 0^-} b+\left[\frac{\sin x-x}{x^3}\right]=b-1$$

$f(0)=2$

$\Rightarrow a=2,\quad b=3$

So, T is correct.

116.

A) $f(x)=\dfrac{1}{\dfrac{1}{(x-1)^2}+\dfrac{1}{x-1}-2}$

$$=\frac{(x-1)^2}{1+x-1-2(x-1)^2}=\frac{-(x-1)^2}{2x^2-5x+2}$$

$$=\frac{-(x-1)^2}{(x-2)(2x-1)}$$

So, Q, R, S are correct.

B) $f(x)=\operatorname{sgn} x(1-x^2)$

So, R is correct as it is discontinuous at $x=0,\pm 1$

So, R is correct.

C) $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ exist & $f(0)=0$

$\Rightarrow f(x)$ is continuous at $x=0$

So, P is correct.

D) $f(x)=\lim\limits_{n\to\infty} x+\dfrac{\frac{\operatorname{sgn} x}{n}}{1+\frac{1}{n}}=x$

So, P is correct.

117.

A) Fundamental period $=\pi=3.14$

Hence, R is correct.

B) $f(x)=[x]+\{2x\}+[3x]$

f(x) is discontinuous at $x=0,\ \frac{1}{2},\ \frac{1}{3},\ 1$

So, S is correct.

C) $\left[\lim\limits_{x\to 0}\dfrac{\sin x(1+\cos x)}{x\cos x}\right]=\left[\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)\left(1+\dfrac{1}{\cos x}\right)\right]=2$

So, Q is correct.

D) $\sin^{-1}x=+2\cos^{-1}(1+x)$

Domain $-1\le x\le 0$

LHS ≤ 0, RHS ≥ 0

It holds only if both
sides are simultaneously
zero, which happens
for $x=0$.

So, P is correct.

118. Using AM – GM inequality

A) $\text{Cos}^2 A + \text{Cos}^2 B + \text{Cos}^2 C \le 1 + 1 + 1$

In a triangle this is positive if $A \to 0$, $B \to 0$ and $C \to \pi$.

Hence the maximum value of 3 is not attained.

Hence $\text{Cos}^2 A + \text{Cos}^2 B + \text{Cos}^2 C < 3$

So, P is correct.

B) $\frac{9a + 3b + c}{3} \ge \sqrt[3]{27abc}$

$abc \le (10)^3$

Now, $\Rightarrow \log a + \log b + \log c = \log abc$

$\le \log(10)^3 = 3$

So, R is correct.

C) $\lim\limits_{x \to 0} \frac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3\sqrt{x}}$

$\lim\limits_{x \to 0} \frac{\sin x\sqrt{\sin x}\,(1 - (\cos x)^{3/2})}{(\cos x)^{3/2} \cdot x^{3/2} \times x^2}$

$= \frac{3}{4}$

So, P is correct.

D) $\lim\limits_{x \to 0} \cos\left(x \cos\frac{1}{x}\right) = 1$

$\lim\limits_{x \to 0} \frac{\ln\left(\sec^2 x\right)}{x \sin x} = \lim\limits_{x \to 0} - \frac{\ln\left(1 - \sin^2 x\right)}{x^2 \frac{\sin x}{x}} = 1$

So, $f(0) + g(0) = 2$

So, Q is correct.

119.

A) $f(x) = \lim\limits_{n \to \infty} \frac{1}{1 + n\sin^2 \pi x}$

It is discontinuous at x =1

So, Q is correct.

B) $f(x) = \lim\limits_{n \to \infty} \frac{x}{1 + \left(4\sin^2 x\right)^n}$

It is discontinuous when $\sin x = \pm\frac{1}{2}$

So, R is correct.

C) $h(x) = \lim\limits_{n \to \infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$

$\lim\limits_{x \to 1^+} \lim\limits_{n \to \infty} \frac{1 + a/x^{2n-3} + b/x^{2n-2}}{x + \frac{1}{x^{2n-1}}} = 1.$

$\lim\limits_{x \to 1^-} \lim\limits_{n \to \infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1} = a + b$

$\Rightarrow a + b = 1.$

$f(x) = [\tan(a + b - 1)x] = 0$

So, P, S is correct.

D) $f(x + y) = f(x) \cdot f(y)$

$\Rightarrow \lim\limits_{x \to 0} f(x) = 1 + \lim\limits_{x \to 0} g(x) \cdot \lim\limits_{x \to 0} a(x)$

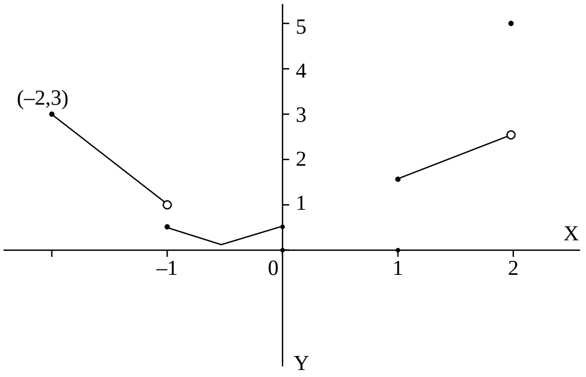
$\Rightarrow f(0)\lim\limits_{h \to 0} f(h) = 1 \;\&\; f(0) = 1$

$\lim\limits_{h \to 0} f(h) = 0$

So, P, S are correct.

Review Exercises Solution

1. $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right| \quad \forall x \in [-2, 2]$



$\lim\limits_{x \to -2^+} f(x) = 3, \lim\limits_{x \to -1^-} f(x) = 1, \lim\limits_{x \to 1^+} f(x) = \frac{1}{2}, f(-1) = \frac{1}{2}$

$\lim\limits_{x \to 0^-} f(x) = \frac{1}{2}, \lim\limits_{x \to 0^+} 0, f(0) = 0, \lim\limits_{x \to 1^-} f(x) = 0$

$\lim\limits_{x \to 1^+} f(x) = \frac{3}{2}, f(1) = \frac{3}{2}, \quad \lim\limits_{x \to 2^-} f(x) = \frac{5}{2}, \lim\limits_{x \to 2^+} f(x) = 5$

$f(2) = 5$

$f(x)$ is discontinous at $x = -1, 0, 1, 2$

2. $f(x) = [[x]] - [x] + 1$

$f(x) = 1$

So, $f(x)$ is continous $\forall x \in R$

3. $f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$

$f(|x|) = \begin{cases} 1, & -2 \le |x| \le 0 \\ |x|-1, & 0 < |x| \le 2 \end{cases}$

$= \begin{cases} 1, & x = 0 \\ |x|-1; & x \in [-2,2] \sim \{0\} \end{cases}$

$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ 1-x, & 0 < x \le 1 \\ x-1, & 1 < x \le 2 \end{cases}$

$g(x) = f(|x|) + |f(x)|$

$= \begin{cases} |x|, & -2 \le x < 0 \\ 2, & x = 0 \\ |x|-x, & 0 < x \le 1 \\ x+|x|-2, & 1 < x \le 2 \end{cases}$

So, g(x) is discontinuous at x = 0.

4. $y = \begin{cases} \left[\dfrac{\sin x}{x}\right], & x > 0 \\ 1-[x\cos x], & x \le 0 \end{cases}$

$= \begin{cases} 0, & 0 < x \le \pi \\ -1, & \pi < x \le 3\pi/2 \\ 1, & x = 0 \\ 2 & -\dfrac{\pi}{2} \le x < 0 \end{cases}$

So, discontinuous at $x = 0$ & π

5. $f(x) = \begin{cases} \dfrac{x^2}{2}+1, & 0 \le x < 1 \\ 2x^2-3x+\dfrac{3}{2}, & 1 \le x \le 2 \end{cases}$

$f(x-1) = \begin{cases} \dfrac{(x-1)^2}{2}+1, & 1 \le x < 2 \\ 2(x-1)^2-3(x-1)+\dfrac{3}{2}, & 2 \le x \le 3 \end{cases}$

$g(x) = \begin{cases} 2x^2-3x+\dfrac{3}{2}+\dfrac{(x-1)^2}{2}+1, & 1 \le x < 2 \\ 2x^2-3x+\dfrac{3}{2}+2(x-1)^2-3x+\dfrac{9}{2}, & x = 2 \end{cases}$

g(x) is discontinuous at x = 2 only

6. $f(x) = \lim\limits_{n\to\infty} \dfrac{(1+\cos x)^n + 5\ln x}{2+(1+\cos x)^n}$

$f(x) = \begin{cases} \dfrac{5\ln x}{2}, & x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right) \\ \dfrac{5\ln x+1}{3}, & x = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \\ 1, & x \in [0, \pi/2) \\ & \text{or } x \in \left(\dfrac{3\pi}{2}, 2\pi\right) \end{cases}$

Hence, f(x) is discontinuous at all positive odd Multiples of $\dfrac{\pi}{2}$

7. $g(x) = 1+\cos\left(\dfrac{x}{4}\right)+\cos^2\left(\dfrac{x}{4}\right)+\ldots\ldots\infty$

$= \dfrac{1}{1-\cos x/4}$

If g(x) is continuous from (0, c)

$\Rightarrow \dfrac{x}{4} = \pi$ ($\therefore$ common ratio of Infinite GP becomes -1)

$\Rightarrow x = 4\pi$ Hence $c = 4\pi$.

8. $\lim\limits_{x\to 0^-} \dfrac{e^{1/x}\sin(1/x)}{1+e^{1/x}} = 0$

$\lim\limits_{x\to 0^+} \dfrac{\sin(1/x)}{1+e^{-1/x}} =$ Not defined

Hence, f(x) is discontinuous at $x = 0$.

9.

i) $f(x) = (x-[x])^2 = \{x\}^2$

Discontinuity of first kind for all $x \in I$

ii) $f(x) = [x]+(x-[x])^2$

$= [x]+\{x\}^2$

For $x = k$, $k \in I$

$\lim\limits_{x\to k+h} f(x) = k$, $\lim\limits_{x\to k^-} f(x) = k-1+1 = k, f(x) = k$

So, f(x) is continuous for all $x \ge 0$.

10. $$f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$$

Max value of $(x-2)^3 \ \forall\ x\in[4,6]$ is 64

Hence, $a>64$ for f(x) to be continuous.

11. $$\lim_{x\to 0^-}(\sin x+\cos x)^{\operatorname{cosec} x}$$

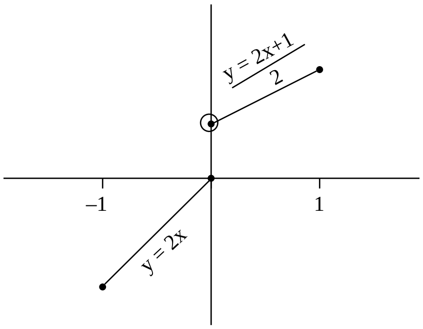
$$e^{\lim\limits_{x\to 0^-}\frac{(\sin x+\cos x-1)}{\sin x}}=e=a$$

$$\lim_{x\to 0^+}\frac{e^{1/x}+e^{2/x}+e^{3/x}}{e^{1+3/x}+be^{3/x}}=$$

$$\lim_{x\to 0^+}\frac{e^{-2/x}+e^{-1/x}+1}{e+b}=\frac{1}{b+e}$$

$$b=\frac{1}{e}-e$$

12.



$f(x)$ is discontinuous at $x=0$ but has maximum value of $\frac{3}{2}$ at $x=1$ & minimum value as

-2 at $x=-1$.

13. $f(x)$ is continuous for all $n\in N$

Smallest value of 'n' is 1.

14. If $f(x)$ is continuous at $x=1$

$$a+2=a+b+2$$
$$b=0$$

If $f(x)$ is continuous at $x=e$

$$e^2+ae+1=ae^2+2e+b$$

$$\Rightarrow a=\frac{e^2-2e+1}{e^2-e}=\frac{e-1}{e}$$

15. Assume a $h(x)=f(x)-g(x)$

$$=\frac{x^2}{e^x}-\frac{2\ln x}{x}$$

$$h(1)=\frac{1}{e}>0$$

$$h(e)=e^{2-e}-\frac{2}{e}=e^{2-e}-2e^{-1}<0$$

as h(x) is continuous between (1, e) & it has opposite signs, Then, there will be atleast one 'c' where $c\in(1,e)$ where h(c) = 0

$\Rightarrow f(c)=g(c)$

16. Use intermediate value Theorem.

17. $$f(x)=\begin{cases}-(x+1), & x\le -1\\ x+1, & -1<x\le 0\\ x, & x>0\end{cases}$$

$$g(x)=\begin{cases}1-x & x\le 0\\ x+1 & 0<x\le 1\\ x-2 & 1<x<2\\ 2-x & x\ge 2\end{cases}$$

$$f+g=\begin{cases}-2x, & x\le -1\\ 2 & -1<x\le 0\\ 2x+1 & 0<x\le 1\\ 2x-2 & 1<x<2\\ 2 & x\ge 2\end{cases}$$

So, $f+g$ is discontinuous at $x=0,1$

18. $$f\circ g(x)=\begin{cases}2g(x)-1, & -2\le g(x)<0\\ g(x)+2, & 0\le g(x)\le 2\end{cases}$$

$$=x+4 \quad -2\le x\le 0$$

$f\circ g(x)$ is continuous over its domain.

19. for $f(x)$ to be continuous.

$$\frac{p}{q}=\left(\frac{1+p^2}{1+q^2}\right)^{1/2}$$

$$\Rightarrow p^2\left(1+q^2\right)=q^2\left(1+p^2\right)$$

$$\Rightarrow \quad p^2=q^2$$

$$\Rightarrow p=q,\ x>0$$

$$\Rightarrow x=1$$

Hence, $f(x)$ is continuous only at $x=1$.

20. Volume of cone $= \frac{1}{3}\pi r^2 h$.

Volume of right cylinder $= \pi r^2 h = f(r)$

$f(0) = 0$

$f(r) = \pi r^2 h$

Using Intermediate value theorem, there exist $c \in (0, r)$ where $f(c) = \frac{1}{3}\pi r^2 h$.

21. $f(x) = \frac{x}{x^3 - 3x + 1}$ is discontinuous wherever

$x^3 - 3x + 1 = 0$.

Now, $g(x) = x^3 - 3x + 1$

$g'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$g(-2) = -1, \; g(-1) = 3, \; g(0) = 1, \; g(1) = -1, \; g(z) = 3$

Using IVT, $f(x)$ is discontinuous in $(-2, -1)$, (0, 1) & $(k, 2)$.

22. $f(x) = \frac{a}{x-1} + \frac{b}{x-3}$

$f(1^+) = \infty, \quad f(3^-) = -\infty$

& f(x) is continuous in (1, 3).

Using IVT, $f(x)$ will be zero for atleast one value of $x \in (1, 3)$.

23. $f \circ g(x) = g(x)(1 - g^2(x))$

$$= \begin{cases} 0 & x > 0 \\ 0 & x < 0 \\ 0 & x = 0 \end{cases}$$

$f \circ g(x)$ is always continuous.

$$g \circ f(x) = \begin{cases} -1 & x > 1 \\ 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \\ 1 & x < -1 \\ 0 & x = 0, \pm 1 \end{cases}$$

$g \circ f(x)$ is discontinuous at $x = 0, \pm 1$.

24. As $f(x) = f(2x+1)$

$\Rightarrow f(x) = f\left(\frac{x}{2} - 1\right) = f\left(\frac{x}{2^2} - \frac{1}{2} - 1\right)$

$= f\left(\frac{x}{23} - \frac{1}{2^2} - \frac{1}{2} - 1\right)$

As $n \to \infty$

$f(x) = f\left(\frac{x}{2^n} - \frac{1}{2^{n-1}} - \frac{1}{2^{n-2}} \cdots 1\right)$

$f(x) = f(-2)$

Hence, f(x) is a constant function.

25. As $f(2x^2 - 1) = 2x f(x)$

Put $x = \cos\theta$

$f(\cos 2\theta) = 2\cos\theta \, f(\cos\theta)$

$= 2^2 \cos\theta \cos\frac{\theta}{2} f\left(\cos\frac{\theta}{2}\right)$

$= 2^3 \cos\theta \cos\frac{\theta}{2} \cos\frac{\theta}{4} f\left(\cos\frac{\theta}{4}\right)$

$= 2^{n+1} \cos\theta \cos\frac{\theta}{2} \cos\frac{\theta}{4} .... \cos\left(\frac{\theta}{2^n}\right) f\left(\cos\frac{\theta}{2^n}\right)$

As $n \to \infty, f(\cos 2\theta) = \frac{\sin 2\theta}{\left(\sin\frac{\theta}{2\pi}\right)} f\left(\cos\frac{\theta}{2^n}\right)$

$f(\cos\theta) = 0$ For all values of θ

26. $\lim_{x \to \frac{\pi^-}{2}} \left[\frac{2(\sin x - \sin^n x) + \sin x - \sin^n x}{2(\sin x - \sin^n x) - \sin x + \sin^n x}\right]$

$\lim_{x \to \frac{\pi^-}{2}} \left[\frac{3(\sin x - \sin^n x)}{(\sin x - \sin^n x)}\right] = 3$

$\lim_{x \to \frac{\pi^+}{2}} [f(x)] = 3$ & $g\left(\frac{\pi}{2}\right) = 3$.

Hence, $g(x)$ is continuous at $x = \frac{\pi}{2}$ for $n > 1$

27. $f(x) = (-1)^{[x^3]}$

$\lim_{x \to n^{1/3}+} f(x) = (-1)^n, \; \lim_{x \to n^{1/3}-} f(x) = (-1)^{n-1}$

Now if $n \in I$, LHL $\neq$ RHL.

So, $f(x)$ is discontinuous for $x = n^{1/3}$.

28. $g(x) = f(x) - f(x+2)$.

$g(0) = f(0) - f(2)$

$g(2) = f(2) - f(4) = -\left(f(0) - f(2)\right)$

$g(0) = -g(2)$

$\Rightarrow$ Using IVT, there exist point $x = c \in (0, 2)$

Where $f(c) = f(c+2)$

29. $f(xy) = f(x) f\left(y^3\right)$

Put $x = 1, y = 1 \Rightarrow f(1) = 1$

As $f(x)$ is continous at $x = 1$

$\lim\limits_{x\to 1} f(1) = f(1) = 1$

Now, for any $x = x_0$

$\lim\limits_{x_0\to 1} f(yx_0) = f(x) f\left(y^3\right)$

$\Rightarrow \lim\limits_{x\to x_0} f(x) = f\left(x_0\right)$

So, it is continous for all $x \neq 0$.

30. Similar to Ques 17.

31. $f(x) = \begin{cases} x, & x \neq \dfrac{1}{n} \\ x^2, & x = \dfrac{1}{n} \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = 0$

$\lim\limits_{x\to 0^-} f(x) = 0 \qquad f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$

At integral points $\Rightarrow x = x^2$

$x(x-1) = 0$

$x = 0$ or 1

Hence, $f(x)$ is discontinuous at all positive integer except 1.

32. $f(x) = \begin{cases} \log(2+x), & x < 1 \\ \dfrac{\log 3 - \sin 1}{2} & x = 1 \\ -\sin x & x > 1 \end{cases}$

$f(x)$ is not continuous in $\left(0, \dfrac{\pi}{2}\right)$.

33. $f(x + f(x)) = f(x)$

Let $f(x) = y$

$f(x+y) = y$

As $y \to 0$

$\lim\limits_{y\to 0} f(x+y) = 0$

As $f(x)$ is a continous for all x

$\Rightarrow f(x) = \lim\limits_{y\to 0} f(x+y) = 0$

Hence, $f(x)$ is a constant function.

34. $f(x+h) = Af(x)f(h)$

Put $x = h = 0$

$\Rightarrow f(0) = \dfrac{1}{A}$ as $f(0) \neq 0$

As $f(x)$ is continous at $x = 0$

$\lim\limits_{x\to 0} f(x+h) = f(h)$

$\Rightarrow f(x)$ is continous for all x

35. As $f\left(\dfrac{1}{3^n}\right) = \dfrac{\left(\cos e^n\right)}{3^{n^2}} + \dfrac{n^2}{n^2+n+1}$

As $n \to \infty$

$\lim\limits_{n\to\infty} f\left(\dfrac{1}{3^n}\right) = \lim\limits_{n\to\infty}\left(\dfrac{\cos e^n}{3^{n^2}} + \dfrac{n^2}{n^2+n+1}\right)$

$\Rightarrow f(0) = 1$. As $f(x)$ is continous on $x \in R$

Target Exercises Solution

1. $$f(x)=\begin{cases}1, & |x|\ge 1\\ \dfrac{1}{n^2}, & \dfrac{1}{n}\le |x|<\dfrac{1}{n-1},\quad n=2,3,\ldots\\ 0, & x=0\end{cases}$$

$$\lim_{x\to\frac{1}{n}} f(x)=\frac{1}{(n)^2}$$

$$\lim_{x\to\frac{1}{n^-}} f(x)=\frac{1}{(n-1)^2}$$

So, $f(x)$ is not continous at $x=\dfrac{1}{n}$.

2. $$f(x)=\begin{cases}\dfrac{(1-x)^{-1/x}-(1+x)^{1/x}}{x}, & -1<x<0\\ e\quad , & x=0\\ \left(\log_{\sin x}\sin 2x\right)^{\ln\sin 2x}, & 0<x<1\end{cases}$$

$$\lim_{x\to 0^+}\left(\log_{\sin x}\sin 2x\right)^{\ln\sin 2x}=2$$

$f(0)=e$

Hence, f(x) is discontinuous at x = 0.

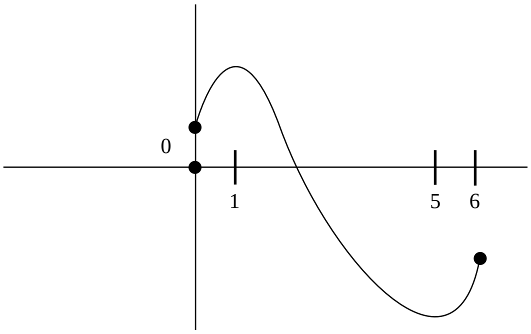
3. $f(x)=x^3-9x^2+15x+7$

$f'(x)=3x^2-18x+15$

$=3\left(x^2-6x+5\right)$

$=3(x-5)(x-1)$

Graph of f(x)



As $f(0)=7$

$f(1)=14$

$f(5)=-18$

$f(6)=-11$

$$g(x)=\begin{cases}7, & 0\le x<\alpha\\ x^3-9x^2+15x+7, & \alpha\le x<5\\ -18, & 5\le x<6\end{cases}$$

Where $\alpha^3-9\alpha^2+15\alpha+7=0\quad\Rightarrow\alpha=\dfrac{9-\sqrt{21}}{2}$

Hence, g(x) is always continuous.

4. $$\lim_{x\to 0^+}\frac{2\left(x-x^3\right)+\left|x-x^3\right|}{2\left(x-x^3\right)-\left|x-x^3\right|}$$

$$\lim_{x\to 0^+}\frac{3\left(x-x^3\right)}{x-x^3}=3$$

$$\lim_{x\to 0^-}\frac{2\left(x-x^3\right)+\left|x-x^3\right|}{2\left(x-x^3\right)-\left|x-x^3\right|}=\frac{1}{3}$$

So, discontinuous at x = 0

$$\lim_{x\to 1^+}\frac{2\left(x-x^3\right)+\left|x-x^3\right|}{2\left(x-x^3\right)-\left|x-x^3\right|}=\frac{1}{3}$$

$$\lim_{x\to 1^-}\frac{2\left(x-x^3\right)+\left|x-x^3\right|}{2\left(x-x^3\right)-\left|x-x^3\right|}=3$$

So, discontinuous at x = 1.

5. As $f(x)=\begin{cases}x, & \text{when } x \text{ is rational}\\ 1-x, & \text{when } x \text{ is irrational}\end{cases}$

For limit to exist, $x=1-x\Rightarrow x=\dfrac{1}{2}$

So, f(x) is continuous at $x=\dfrac{1}{2}$ only.

6. $(f(x))^2+x^2=1\quad\forall x\in[-1,1]$

$\Rightarrow (f(x))^2=1-x^2$

$\Rightarrow f(x)=\pm\sqrt{1-x^2}$

As $f(x)$ is continous function in $[-1,1]$

$\Rightarrow f(x)$ is either $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

7. $$y_n(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\cdots$$

As $n\to\infty$

$y_n(x)$ is infinite G.P with $|r|<1$

$$\Rightarrow \lim_{n\to\infty} y_n(x)=\frac{x^2}{1-\dfrac{1}{1+x^2}}=1+x^2$$

$$\Rightarrow y(x)=1+x^2$$

So, y(x) is continuous everywhere except x = 0.

$$y_n(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\cdots$$

$y_n(0)=0$ LHL = RHL = 0

So, $y_n(x)$ is continuous at x = 0.

8. $f(x)=\frac{1}{2}-x+\frac{1}{2}[2x]-\frac{1}{2}-\frac{1}{2}[-2x]$

$=-x+\frac{1}{2}([2x]-[-2x])$

For x = 0, f(0) = 0

RHL $=\frac{1}{2}$, Hence, discontinuous at x = 0.

For $x=\frac{1}{2}$, $f\left(\frac{1}{2}\right)=\frac{1}{2}$

RK = 1. Hence, discontinuous at $x=\frac{1}{2}$.

At x = 1, f(1) = 1

RHL $=\frac{3}{2}$. So, discontinuous at x = 1.

Now as f(0) = 0 & f(1) = 1,
Using IVT, f(x) will assume all values between (0,1).

9. As for each $x\in I, |f(y)|\le\frac{1}{2}|f(x)|$

$\Rightarrow |f(y_1)|\le\frac{1}{2}|f(y)|\le\frac{1}{4}|f(x)|$

Repeating this process n times,

$\left|f(y_n)\right|\le\frac{1}{2^n}\left|f(x)\right|$

As $n\to\infty,\left|f(y_n)\right|\le 0$

$\Rightarrow f(y_n)=0$

⇒ there is some values of x for which f(x) = 0.

10. If f(x) is strictly negative somewhere on R & f(x) is continuous on R → R, then, f'(x) = 0 at some point and it will be a finite absolute point of minima. Similarly, if f(x) is strictly positive somewhere on R. Then, it will have finite absolute point of Maxima.

11. As $0\le f(0)\le 2$ & $0\le f(1)\le 2$.

As x ∈ [0,1], y = f(x) will cut y = 2x atleast at one point since f(x) is continuous function

⇒ f(x) = 2x atleast at one point in [0,1]

12. Let $g(x)=f\left(x+\frac{1}{n}\right)-f(x)$

Assume g(x) is always +ve,

$\Rightarrow f(0)<f\left(\frac{1}{n}\right)<f\left(\frac{2}{n}\right)<f\left(\frac{3}{n}\right)=\cdots<f(1)$

But f(0) = f (1)

Hence, it is a contradiction.

Assume g(x) is always -ve.

$\Rightarrow f(0)>f\left(\frac{1}{n}\right)>f\left(\frac{2}{n}\right)\cdots>f(1)$

Which is also a contradiction

⇒ g(x) should be equal to zero for $x\in\left[0,1-\frac{1}{n}\right]$

$\Rightarrow f(x)=f\left(x+\frac{1}{n}\right)$ for some value of $x\in\left[0,1+\frac{1}{n}\right]$

Hence proved.

13. If f(x) is continuous, then
f(x) = ±x or ±|x|.
So, there are 4 continuous function possible.

14. If f(x) = 0 exist for only one C ∈ [a,b] & f(x) > 0 ∀ some x ∈ (c,b]
⇒ f(x) is always +ve for x ∈ (c,b]
& If f(x) < 0 ∀ some x ∈ [a,c)
⇒ f(x) is always -ve for x ∈ [a,c)

15. For an injective function, following IVT Theorem,
⇒ f(x) is a monotonic function ∀ x ∈ I
⇒ f(x) needs to be a continuous function in x ∈ I

16. As $|f(x)-f(y)|<M|x-y|$

Put $x=c+h, y=c, h\to 0$

$\lim\limits_{h\to 0}\left|f(c+h)-f(c)\right|\le 0$

$\Rightarrow \lim\limits_{h\to 0}f(c+h)=f(c)$ I

Put x = c − h, y = c, h → 0

$\lim\limits_{h\to 0}\left|f(c-h)-f(c)\right|\le 0$

$\Rightarrow \lim\limits_{h\to 0}\left|f(c-h)-f(c)\right|$ II

From 1 & II $\Rightarrow \lim\limits_{h\to 0}\left|f(c+h)=\lim\limits_{h\to 0}f(c-h)=f(c)\right.$.

17. If x is rational $=\frac{p}{q}$

For $x+h=\frac{p}{q}+\frac{1}{n}$ where h → 0, ⇒ n → ∞

$$\text{RHL}=\lim_{n\to\infty}f\left(x+\frac{1}{n}\right)=f\left(\frac{np+q}{nq}\right)=\frac{1}{np+q+nq}\to 0$$

$$\text{LHL}=\lim_{n\to\infty}f\left(x-\frac{1}{n}\right)=f\left(\frac{np-q}{nq}\right)=\frac{1}{np-q+nq}\to 0$$

Hence, f(x) is discontinuous for rational numbers.
If x is irrational, then if x ± h = rational ⇒ q ⇒ ∞

$$\Rightarrow \frac{1}{p+q} = 0$$

Hence, f(x) is continuous for irrational numbers.

18. As f(g(x)) = gf(x) ⇒ For a point (a,b) lying on f(x), there is a point (b,a) lying on g(x).
Now Let h(x) = f(x) – g(x)
h(a) = f(a) – g(a) = b – g(a) = b – b = 0
Hence, f(x) & g(x) have a common point.

19. (i)

$$\text{As } |f(u)-f(v)| \leq |u-v|$$

$$\text{Put } u = x+h, v = x, \quad h \to 0$$

$$\Rightarrow \lim_{h\to 0} |f(x+h)-f(x)| \leq 0$$

$$\Rightarrow \lim_{h\to 0} f(x+h) = f(x)$$

Similarly,

$$u = x-h, v = x, h \to 0$$

$$\lim_{h\to 0} \left|f(x-h)-f(x)\right| \leq 0$$

$$\Rightarrow \lim_{h\to 0} f(x-h) = f(x)$$

Hence, f(x) is continuous in x ∈ [a,b].

20. (i) g(x) is inverse of f(x) ⇒ f(o) g(x) = x or g(o) f(x) = x.

$$\text{Now, As } M_f = g\left(\frac{1}{n}\sum f(a_i)\right)$$

$$\Rightarrow f(M_f) = f\left(g\left(\frac{1}{n}\sum f(a_i)\right)\right)$$

$$= \frac{1}{n}\sum f(a_i)$$

Hence proved.

(ii) As $f(M_f) = \dfrac{f(a_1)+\cdots+f(a_n)}{n}$

Using Averages, $a_i < M_f < a_n$.
Hence proved.

(iii) If $h(x) = af(x)+b$

$$M_f = g\left(\frac{1}{n}\sum f(a_i)\right)$$

$$M_h = g\left(\frac{1}{n}\sum_{n}(h(i)) = g\left(\frac{1}{n}\sum^{a} b^{(a_i)} + b\right)\right)$$

$$= g\left(\frac{1}{n}\left(\varepsilon_b(a_i)+b_n\right)\right)$$

$$= g\left(a\sum \frac{f(a_i+b)}{n}\right)$$

$$= g\left(\frac{1}{n}\sum f/a_i\right)$$

$$= M_f$$

Hence proved.

21. Let us assume $g(x) = f\left(x+\frac{1}{k}\right) - f(x)$ in $\left[0, 1-\frac{1}{k}\right]$

For a horizontal chord of length $\frac{1}{k}$, g(x) should be zero in $\left[0, 1-\frac{1}{k}\right]$.

Now Let us assume g(x) is always +ve,

$$\Rightarrow \quad f(0) < b\left(\frac{1}{k}\right) < b\left(\frac{2}{k}\right) < \cdots < \left(\frac{k}{k}\right) = f(1)$$

Which is a contradiction

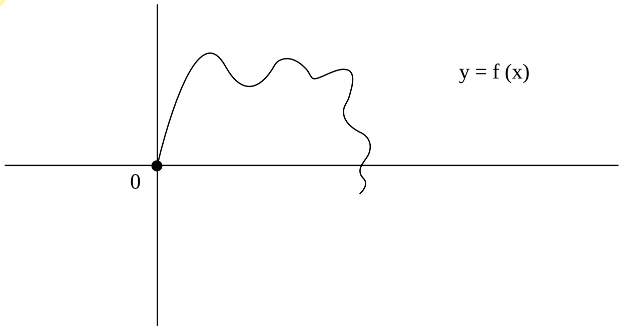
& Similarly, if g(x) is always -ve,

$$\Rightarrow \quad f(0) < b\left(\frac{1}{k}\right) < b\left(\frac{2}{k}\right) < \cdots > \left(\frac{k}{k}\right) = f(1)$$

Which is a contradiction. Hence, g(x) should be zero somewhere.

⇒ there is a horizontal chord of length '$\frac{1}{k}$'

(iii) No



(iv) If $f(x) \geq 0$
Then, g(x) will have horizontal length from 0 to 1.

22. As f(x) = pnx, excess of x over nearest integer

At $x = x + \frac{1}{2}$, n being integer

$$f(x) = \frac{1}{2}$$

$$\lim_{x\to n+\frac{1}{2}^{+}} f(x) = 0$$

$$\lim_{n\to n+\frac{1}{2}^{-}} f(x) = \frac{1}{2}$$

Hence, f(x) is discontinuous at $x = n + \frac{1}{2}$.

23.

(i) $$f(x)=\lim_{h\to\infty}\frac{x+x^n\sin x}{1+x^n}$$

$$f(1)=\frac{1+\sin 1}{2}$$

$$\text{RHL}=\lim_{n\to\infty}\frac{x^n\left(\dfrac{x}{x^n}+\sin x\right)}{x^n\left(1+\dfrac{1}{x^n}\right)}=\sin 1$$

Hence, f(x) is discontinuous at x = 1.

(ii)

$$\phi(x)=\lim_{h\to\infty}\frac{x+x^n}{1+ax^n}$$

$$\phi(1)=\frac{2}{1+a}$$

$$\text{RHL}=\lim_{n\to\infty}\frac{x^n\left(\dfrac{x}{x^n}+1\right)}{x^n\left(\dfrac{1}{x^n}+a\right)}=\frac{1}{a},\ \text{LHL}=\lim_{n\to\infty}\frac{x+x^n}{1+ax^n}=1$$

Hence, $\phi(x)$ is discontinuous at x = 1, but $\phi(x)$ can be continuous if a = 1.

24. $$f(x)=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{[2rx]}{n^2}$$

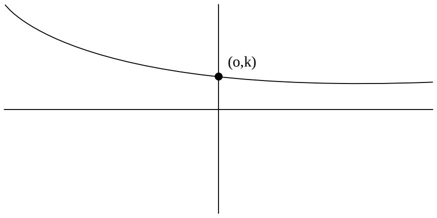
$$=\lim_{n\to\infty}\sum\frac{2rx}{n^2}-\sum\frac{\{2rx\}}{n^2}$$

$$=\lim_{n\to\infty}\frac{2x(n)(n+1)}{2n^2}-\sum\frac{\{2rx\}}{n^2}$$

$$=x-0=x$$

Hence, f(x) is continuous everywhere.

25. Graph of $zx(x)=ke^{-x}$



If k = 0, $Z_K(x)=0$

f(0) = z

$f(0^+)=0^+$

So, discontinuous at x = 0

Hence, f(x) is discontinuous is $0\le x\le 10$.

26. As $$f^2(x)=\frac{1}{g^2(x)}$$

Taking square root,

$$f(x)=\frac{1}{g(x)}\ \text{or}\ f(x)=\frac{-1}{g(x)}$$

27. $$f(x)=\frac{1}{2^n},\ \frac{1}{2^{n+1}}<x\le\frac{1}{2^n}$$

$$\text{As } f\left(\frac{1}{2^n}\right)=\frac{1}{2^n}$$

$$\text{RHL}=f\left(\frac{1}{2^n}+n\right)=\frac{1}{2^{n-1}}$$

$$\text{LHL}=f\left(\frac{1}{2^n}-n\right)=\frac{1}{2^{n-1}}$$

Hence, f(x) is discontinuous at $x=\dfrac{1}{2^n}$ & it is a jump discontinuity.

28. $$\text{Now, }\lim_{x\to 0^-}\frac{\left(\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)\right)\cos x}{\left(1-\cos^2 x\right)}$$

$$=\lim_{x\to 0^-}\frac{\ln\left(1+x^2+x^4\right)}{\left(1-\cos^2 x\right)}\cos x$$

$$\text{Now, }\lim_{x\to 0^+}\frac{a^{\sin x}-a^{\tan x}}{\tan x-\sin x}$$

$$=\lim_{x\to 0^+}-a^{\sin x}\frac{\left(a^{\tan x-\sin x}-1\right)}{(\tan x-\sin x)}$$

$$=-\ln a=1$$

$$\Rightarrow a=e^{-1}$$

$$\text{Now, } g(x)=\ln\left(2-\frac{x}{a}\right)\cot(x-a),\ x\ne a.$$

As g(x) is continous at x = a

$$\Rightarrow\lim_{x\to a}\frac{\ln\left(2-\dfrac{x}{a}\right)}{\tan(x-a)}$$

$$\Rightarrow\lim_{x\to a}\ln\frac{\left(1+\left(1-\dfrac{x}{a}\right)\right)}{\left(1-\dfrac{x}{a}\right)}\times\frac{(a-x)}{a(\tan(x-a)}$$

$$\Rightarrow-\frac{1}{a}=-e$$

$$\Rightarrow g\left(e^{-1}\right)=-e$$

Hence proved.

29.

(i) $h(g(2)) = h(\sqrt{2}) = 4-3\sqrt{2}+a$

(ii) $\lim\limits_{x\to1^-} f(x) = 2$

$\lim\limits_{x\to1^+} f(x) = a-1$

For f(x) to be continuous

$\Rightarrow a-1=2$

$\Rightarrow a=3$

30. $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{\cos^{-1}\left(1-x^2\right)\sin^{-1}(1-x)}{\sqrt{2}x\left(1-x^2\right)}$

$$= \lim_{x\to0^+} \frac{\sin^{-1}(1-x)}{\sqrt{2}x\left(1-x^2\right)}\sin^{-1}\left(\sqrt{2x^2-x^4}\right)$$

$$= \frac{\pi}{2}$$

Hence, f(x) is discontinuous at x = 0.

As $g(x) = \begin{cases} f(x), & x \geqslant 0 \\ 2\sqrt{2}f(x), & x<0 \end{cases}$

$g(0) = \dfrac{\pi}{2}$

$\lim\limits_{x\to0^+} g(x) = \dfrac{\pi}{2}$, $\quad \lim\limits_{x\to0^-} g(x) = \dfrac{\pi}{2}$

Hence, g(x) is continuous at x = 0.

31. $f\left(\dfrac{1}{r}\right) = 2(r-1)x$

$\lim\limits_{x\to\frac{1}{r}^{+}} f(x) = 2(r-1)x$

$\lim\limits_{x\to\frac{1}{r}^{-}} f(x) = 2rx$

Hence, f(x) is discontinuous at $x = \dfrac{1}{r}$.

32. $f(t) = \begin{cases} t, & 0 \le t < \dfrac{1}{2} \\ 0, & t = \dfrac{1}{2} \\ t-1, & \dfrac{1}{2} < t \le 1 \end{cases}$

& f(n + t) = f(t), where n is integer.

$f\left(\dfrac{1}{2}\right) = 0, \qquad f\left(\dfrac{1^-}{2}\right) = \dfrac{1}{2}, \quad f\left(\dfrac{1^+}{2}\right) = \dfrac{-1}{2}$

f(t) is discontinuous at $t = \dfrac{1}{2}$

As f(x+t) = f(t) ⇒ f(x) is discontinuous for all $x = n+\dfrac{1}{2}$, where n is an integer.

33.

$f(x) = \lim\limits_{n\to\infty} \dfrac{\ln(2+x) - x^{2n}\sin x}{1+x^{2n}}$

$f(1) = \dfrac{\ln(2+x)-\sin x}{2}$

$\lim\limits_{x\to1^+} f(x) = -\sin x$

$\lim\limits_{x\to1^-} f(x) = \ln(2+x)$

For all x ∈ [0,1), $\qquad$ f(x) = ln(2 + x)

For all $x \in \left(1, \dfrac{\pi}{2}\right]$, f(x) = -sin x.

$$\Rightarrow f(x) = \begin{cases} \ln(2+x), & 0 \le x < 1 \\ \dfrac{\ln(2+x)-\sin x}{2}, & x=1 \\ -\sin x, & 1 < x \le \dfrac{\pi}{2} \end{cases}$$

As f(x) is discontinuous at x = 1 ⇒ IVT is not applicable for f(x) where $x \in \left[0, \dfrac{\pi}{2}\right]$ ⇒ f(x) doesn't vanishes in $x \in \left[0, \dfrac{\pi}{2}\right]$.

Previous Year's Questions (JEE Advanced)

1. $\lim\limits_{x\to2} \dfrac{(x^3+x^2-16x+20)}{(x-2)^2}$

Apply L-hospital rule

$\lim\limits_{x\to2} \dfrac{3x^2+2x-16}{2(x-2)} = \lim\limits_{x\to2} \dfrac{6x+2}{2} = 7$

$\Rightarrow k = 7$

2. $y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

$$\Rightarrow f(x) = \begin{cases} \sqrt{4-x^2}, & -2 \le x \le 0 \\ -\sqrt{4-x^2}, & 0 \le x \le 2 \end{cases}$$

3. $f(x)=[x]\sin\left(\frac{\pi}{[x]+1}\right)$

Domain $\Rightarrow [x]\neq -1$

$x\neq[-1,0)$

$\Rightarrow \quad x\in(-\infty,-1)\cup[0,\infty)$

Discontinuity → at all integers in domain except 0

4. If f(2) = 10 & f(x) is continuous & take all rational values for all x.
This is true only if f(x) = constant function
f(1.5) = 10

5. $\lim\limits_{x\to 0}\frac{\ln(1+ax)-\ln(1-bx)}{x}$

$=\lim\limits_{x\to 0}\frac{a\ln(1+ax)}{ax}+\frac{b\ln(1-bx)}{-bx}$

$\Rightarrow$ a + b Hence, B is correct.

6. $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$

At all integral points, $\cos\frac{(2x-1)\pi}{2}$ is always zero.

Hence, f(x) will always be continuous.

Hence, C is correct.

7. $f(x)=[x]^2-\left[x^2\right]$

$\lim\limits_{x\to 0^+}f(x)=0,\quad \lim\limits_{x\to 0^-}f(x)=1,\quad f(0)=0$

$\lim\limits_{x\to 1^+}f(x)=0,\quad \lim\limits_{x\to 1^-}f(x)=0,\quad f(1)=0$

Hence, D is correct.

8. $\tan f(x)=\tan\left(\frac{x-1}{2}\right)$

As tan x is discontinuous at odd multiple of $\frac{\pi}{2}$

in $[0,\pi]$, $\frac{x-1}{2}$ will vary from $\left[\frac{-1}{2},\frac{\pi-1}{2}\right]$

So, tan f(x) is continuous.

$f^{-1}(x)=2x+1$ is also continuous.

$\frac{1}{f(x)}=\frac{2}{x-1}$ is discontinuous at x=1

Hence, C, D are correct.

9. A) tan x is discontinuous at $\frac{\pi}{2}$

B) $\int\limits_0^x t\sin\left(\frac{1}{t}\right)dt=\frac{t^2}{2}\sin\frac{1}{t}+\int\frac{1}{2}\cos\frac{1}{t}dt$

Put $t=\frac{1}{y}$ and use IBP

$\int\limits_0^x t\sin\left(\frac{1}{t}\right)dt=\frac{t^2}{2}\sin\left(\frac{1}{t}\right)+\frac{t}{2}\cos\left(\frac{1}{t}\right)$

It is continuous for $x\in(0,\pi)$

C) $\lim\limits_{x\to\frac{3\pi}{4}}2\sin\frac{2x}{9}=1$

Hence, it is continuous for $x\in(0,\pi)$

D) $\lim\limits_{x\to\frac{\pi^-}{2}}x\sin x=\frac{\pi}{2}$

$\lim\limits_{x\to\frac{\pi^+}{2}}\frac{\pi}{2}\sin(\pi-x)=\frac{\pi}{2}$

It is continuous for $x\in(0,\pi)$

Hence, B C D are correct.

10. $\lim\limits_{x\to\frac{\pi-}{2}}x\sin x=\frac{\pi}{2}$

$\lim\limits_{x\to\frac{\pi+}{2}}\frac{\pi}{2}\sin(\pi+x)=-\frac{\pi}{2}$

Hence, A is correct.

11. $\lim\limits_{x\to 2n^+}\left(a_n+\sin\pi x\right)=a_n$

$\lim\limits_{x\to 2n^-}\left(b_n+\cos\pi x\right)=b_n+1$

$a_n=1+b_n\Rightarrow a_n-b_n=1$

Similarly, $a_{n-1}+1=+b_n$

$\Rightarrow a_{n-1}-b_n=-1$

Hence, B & D are correct.

12. We take two cases.

Case 1: Let f and g attain their common maximum value at p

$\Rightarrow f(P)=g(P)$

where $p\in[0,1]$

Case 2: Let f and g attain their common maximum value at different point.

$\Rightarrow f(a)=M$ and $g(b)=M$

$f(a)-g(a)>0$ and $f(b)-g(b)<0$

$\Rightarrow f(c)-g(c)=0$ for some $c\in[0,1]$ as f and g are continous functions.

$\Rightarrow f(c)-g(c)=0$ for some $c\in[0,1]$ for all cases.

Option A $\Rightarrow f^2(c)-g^2(c)+3[f(c)-g(c)]=0$

Which is true for Eq I

Option D $\Rightarrow\ f^2(c)-g^2(c)=0$ which is true

Now, if we take $f(x)=1$ and $g(x)=1 \quad \forall\ x \in [0,1]$

options b an c does not hold, Hence.

A and D are correct.

13. $f(x)=x\cos(\pi(x+[x]))$

At x = 0

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x\cos(\pi(x+\sqrt{x}))=0$

and $f(x)=0$

$\therefore$ Its continous at x = 0 & clearly discontinous at other integers

Hence ans is $\to$ A, B, D

14. At x = t

$\lim\limits_{x\to t^+} f(t)+f(h)=f(t)+f\left(0^+\right)$

$\lim\limits_{x\to t^-} f(t)+f(-h)=f(t)+f\left(0^-\right)$

$f(t)=f(t)+f(0)$

If f(x) is continuous at x=0 =>f(x) is continuous for all x.

15. $\lim\limits_{x\to 0^-}\dfrac{\sin(a+1)x+\sin x}{x}=(a+1)+1 \quad = a+2$

$$\lim_{x\to 0^+}\frac{\left(x+bx^2\right)^{1/2}-x^{1/2}}{bx^{3/2}}=\lim_{x\to 0}\frac{(1+bx)^{1/2}-1^{1/2}}{bx}$$

$$=\frac{1}{2}$$

$\Rightarrow c=\dfrac{1}{2}, a=-\dfrac{3}{2}, b=R^+$

16. $$f(x)=\begin{cases}1+x & ,0\le x<2\\ 3-x & ,2\le x<3\end{cases}$$

$$[f(x)]=\begin{cases}1 & ,0\le x<1\\ 2 & ,1\le x<2\\ 1 & ,x=2\\ 0 & ,2<x<3\end{cases}$$

$$g(x)=f[f(x)]=\begin{cases}2 & ,0\le x<1\\ 1 & ,1\le x<2\\ 2 & ,x=2\\ 1 & ,2<x<3\end{cases}$$

Hence, g(x) is discontinuous at x = 1, 2

17. Let $h(x)=f(x)+g(x)$

$\lim\limits_{x\to t^+} h(x)=f(t+h)+g(t+h)=LHL$

$\lim\limits_{x\to t^-} h(x)=f(t-h)+g(t-h)=RHL$

$h(t)=f(t)+g(t)$

As, f(x) is continuous $LHL \ne RHL$ for h(x)

Hence, h(x) is a discontinuous function.

18. $\lim\limits_{x\to\frac{\pi^-}{4}} f(x)=a+\dfrac{\pi}{4}$

$\lim\limits_{x\to\frac{\pi^+}{4}} f(x)=\dfrac{\pi}{2}+b$

$\lim\limits_{x\to\frac{\pi^-}{2}} f(x)=b$

$\lim\limits_{x\to\frac{\pi^+}{2}} f(x)=-a-b$

Now,

$a=\dfrac{\pi}{4}+b \quad \& \quad a+2b=0$

$b=-\dfrac{\pi}{12}, \quad a=+\dfrac{\pi}{6}$

19. $\lim\limits_{x\to 0^-} f(x)=8$

$\Rightarrow a=8$

20. $\lim\limits_{x\to 0^-} f(x)=e^{\lim\limits_{x\to 0^-}|\sin x|\times\frac{a}{|\sin a|}}=e^a$

$\lim\limits_{x\to 0^+} f(x)=e^{2/3}$

$\Rightarrow e^a=b=e^{2/3}$

$\Rightarrow a=\dfrac{2}{3}$ & $b=e^{2/3}$

Previous Year's Questions (AIEEE/JEE MAINS)

1. $$f(x)=\begin{cases}\dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k, & x=\pi\end{cases}$$

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}\times\frac{(\sqrt{2+\cos x}+1)}{(\sqrt{2+\cos x}+1)}$$

$$\lim_{x\to\pi}\frac{1+\cos x}{(\pi-x)^2}\times\frac{1}{(\sqrt{2+\cos x}+1)}$$

$$\lim_{h\to0}\left(\frac{1-\cos h}{h^2}\right)\times\frac{1}{(\sqrt{2-\cos h}+1)}=\frac{1}{4}$$

For $f(x)$ to be continous, $k=\frac{1}{4}$

Hence, B is correct.

2. $$\lim_{x\to0}\frac{f(1-\cos 3x)}{x^2}$$

As $$\lim_{x\to0}\frac{1-\cos 3x}{x^2}=\lim_{x\to0}\frac{2\sin^2\frac{3x}{2}}{\frac{9}{4}x^2}\times\frac{9}{4}=\frac{9}{2}$$

As $f(x)$ is continuous, $f\left(\frac{9}{2}\right)=\lim_{x\to0}f\left(\frac{1-\cos 3x}{x^2}\right)=\frac{2}{9}$

Hence, B is correct

3. $$f(x)=\begin{cases}\dfrac{\left(e^x-1\right)^2}{\sin\left(\frac{x}{k}\right)\log\left(1+\frac{x}{4}\right)}, & x\neq0\\ 12, & x=0\end{cases}$$

$$\lim_{x\to0}\frac{\left(\frac{e^x-1}{x}\right)^2}{\frac{\sin\left(\frac{x}{k}\right)}{x}\log\frac{\left(1+\frac{x}{4}\right)}{x}}=4k$$

As $f(x)$ is continous, $4k=12$

$\Rightarrow k=3$

Hence, c is correct

4. $$f(x)=\begin{cases}\dfrac{2x^2}{a}, & 0<x<1\\ a, & 1\le x<\sqrt{2}\\ \dfrac{2b^2-4b}{x^3}, & \sqrt{2}\le x<8\end{cases}$$

$$\lim_{x\to1^-}f(x)=\frac{2}{a},\ \lim_{x\to1^+}f(x)=a\Rightarrow a^2=2\Rightarrow a=\pm\sqrt{2}$$

$$\lim_{x\to\sqrt{2}^-}f(x)=a$$

$$\lim_{x\to\sqrt{2}^+}\frac{2b^2-4b}{x^3}=\frac{2b^2-4b}{2\sqrt{2}}=\sqrt{2}$$

$\Rightarrow b^2-2b=2\quad\Rightarrow b^2-2b-2=0$

$$\Rightarrow b=\frac{2\pm\sqrt{12}}{2}=1\pm\sqrt{3}$$

Hence, c is correct

5. $$f(x)=\frac{1}{x}-\frac{k-1}{e^{2x}-1},\ x\neq0$$

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{e^{2x}-1-(k-1)x}{x\left(e^{2x}-1\right)}$$

Apply L-Hospital Rule,

$$\lim_{x\to0}\frac{2e^{2x}-(k-1)}{\left(e^{2x}-1\right)+2xe^{2x}}$$

For limit to exist, $k=3$

$$\lim_{x\to0}\frac{2\left(e^{2x}-1\right)}{\left(e^{2x}-1\right)+2xe^{2x}}=1=f(0)$$

Hence, B is correct

6. $$f(x)=\begin{cases}5 & \text{if}\quad x\le1\\ a+bx, & 1<x<3\\ b+5x, & 3\le x<5\\ 30, & x\ge5\end{cases}$$

$$\lim_{x\to1^+}f(x)=a+b,\ \lim_{x\to3^-}f(x)=a+3b$$

$$\lim_{x\to3^+}f(x)=b+15,\ \lim_{x\to5^-}f(x)=b+25,\ \lim_{x\to5^+}f(x)=30$$

As $f(x)$ is continous, $b+25=30$

$\Rightarrow b=5$

Now, $b+15=a+3b$

$\Rightarrow a=5$

but it will be discontinous at $x=1$

Hence, D is correct.

3

DIFFERENTIABILITY

3.1 Introduction

Our first application of the concept of limit will be to the problem of determining the instantaneous rate of change of a function. Geometrically, this problem is equivalent to that of finding a tangent line to the graph of the function. Both of these problems are solved by finding the derivative of the given function, as will be described in this chapter.

The Definition of Derivative

Let there be a function y = f(x) defined in a certain interval. The function y = f(x) has a definite value for each value of the argument x in this interval. Let the argument x receive a certain increment Δx (it is immaterial whether it is positive or negative). Then the function y will receive a certain increment Δy.

Thus, for the value of the argument x we will have

$$y = f(x),$$

for the value of the argument $x + \Delta x$ we will have

$$y + \Delta y = f(x + \Delta x).$$

Let us find the increment of the function Δy

$$\Delta y = f(x + \Delta x) - f(x) \qquad \text{...(1)}$$

Forming the ratio of the increment of the function to the increment of the argument, we get the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{...(2)}$$

We then find the limit of this ratio as $\Delta x \to 0$. If this limit exists, it is called the derivative or differential coefficient of the given function f(x) and is denoted f ′(x). Thus, by definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{...(3)}$$

Note: Δx is a single symbol and does not mean delta times x. Do not forget that as $\Delta x \to 0$, Δx is getting close to 0, but is not equal to 0.

Consequently, the derivative of a given function y = f(x) with respect to the argument x is the limit of the ratio of the increment in the function Δy to the increment in the argument Δx, when the latter approaches zero, provided the limit exists.

Symbolically, the above equation is written as

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x)$$

where Δx represents finite change and dx represents infinitesimally small change in x.

It is convenient to write Δx as h while calculating the derivative. With such notation the derivative of the function f with respect to the variable x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{...(4)}$$

provided the limit exists.

Usually, when we say that a function f has a derivative f ′(x) at a point x, it is implied that this derivative is finite, i.e. the above limit is finite. But it may occur that there exists an infinite limit equal to ∞ or $-\infty$. In such cases it is useful to say that the function f has at point x an infinite derivative (equal to ∞ or $-\infty$).

It should be emphasized that although the notation $\frac{df}{dx}$ suggest a ratio, the derivative as we have defined it, is not a ratio – even though it is the limit of one. $\frac{df}{dx}$ is simply an abbreviation of f ′.

It will be noted that in the general case, the derivative f ′(x) has a definite value for each value of x. If we associate with each value of x the value of the derivative of f(x), if it exists, we obtain a new function of x, called simply the derivative of f(x) with respect to x. This is denoted by $f'(x)$.

The domain of f', the set of points in the domain of f for which the limit (3) exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f has a derivative (or f is differentiable) at x. $f'(x)$ often has the same domain of definition as f(x), but not always.

The operation of finding the derivative of a function f(x) is called differentiation of the function. If we can find a mathematical expression for $f'(x)$, we can then find the value of the derivative for a particular value of x by mere substitution.

Example 1: Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Solution: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)\,(2+x)-(1-x)\,(2+x+h)}{h(2+x+h)\,(2+x)}$$

$$= \lim_{h\to 0} \frac{(2-x-2h-x^2-xh)-(2-x+h-x^2-xh)}{h(2+x+h)\,(2+x)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2+x+h)\,(2+x)}$$

$$= \lim_{h\to 0} \frac{-3}{(2+x+h)\,(2+x)} = -\frac{3}{(2+x)^2}\,.$$

Example 2: If $f(x) = x^3 - 1$, plot the graph of the derived function f'.

Solution: For any real number x,

$$f'(x) = \lim_{t\to 0} \frac{f(x+h)-f(x)}{h}\,.$$

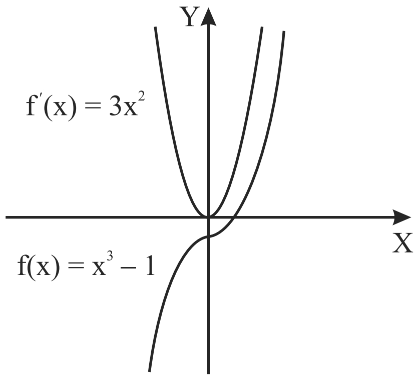
We have $f(x + h) - f(x) = ((x + h)^3 - 1) - (x^3 - 1)$

$$= 3x^2h + 3xh^2 + h^3,$$

and so $\dfrac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2$

Consequently, $f'(x) = \lim\limits_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$.

The graph of the function $f'(x) = 3x^2$ is the parabola shown in figure, on which the graph of the original function $f(x) = x^3 - 1$ has also been drawn.



Here, we considered the derivative of a function f at a general point x. If we change our point of view and let the number x be a fixed number a then we replace x in Equation (4) by a. We obtain

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad ...(5)$$

Note that we can also define

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$ provided the limit exists.

Thus, the derivative of a function f is the function f' defined by

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\,.$$

Instantaneous Rate of Change – The Physical Meaning of Derivative

Suppose y is a function of x, say y = f(x). Corresponding to a change from x to x + Δx, the variable y changes from f(x) to f(x + Δx).

The change in y is $\Delta y = f(x + \Delta x) - f(x)$, and the average rate of change of y with respect to x over the interval from x to x + Δx is

Average rate of change

$$= \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

For example, let $y = f(x) = x^2 + 2x$. Starting at $x_0 = 1$, change x to 1.5. Then Δx = 0.5. The corresponding change in y is $\Delta y = f(1.5) - f(1) = 5.25 - 3 = 2.25$. Hence the average rate of change of y on the interval between

x = 1 and x = 1.5 is $\dfrac{\Delta y}{\Delta x} = \dfrac{2.25}{0.5} = 4.5$.

As the interval over which we are averaging becomes shorter (that is, as $\Delta x \to 0$), the average rate of change approaches what we would intuitively call the instantaneous rate of change of y with respect to x, and the difference quotient approaches the derivative $\dfrac{dy}{dx}$.

Thus, the instantaneous rate of change at x

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f\,'(x)$$

provided that the limit exists.

It is conventional to use the word instantaneous even when x does not represent time. The word is, however, frequently omitted. When we say rate of change, we mean instantaneous rate of change.

If the dependence upon time t of the distance s of a moving point is expressed by the formula

$$s = f(t)$$

The velocity v at time t is expressed by the formula

$$v = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$

Hence $v = s_t{}' = f'(t)$

or, the velocity is equal to the derivative of the distance with respect to the time.

In general, suppose an object moves along a straight line according to an equation of motion s = f(t), where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function of the object. In the time interval from t = a to t = a + h the change in position is f(a + h) – f(a). The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Now suppose we compute the average velocities over shorter and shorter time intervals [a, a + h]. In other words, we let h approach 0. We define the velocity (or instantaneous velocity) v(a) at time t = a to be the limit of these average velocities :

$$v(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Example 3: The position of a particle is given by the equation of motion s = f(t) = 1/(1 + t), where t is measured in seconds and s in metres. Find the velocity and the speed after 2 seconds.

Solution: The derivative of f when t = 2 is

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\frac{1}{1+(2+h)}-\frac{1}{1+2}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{\frac{3-(3+h)}{3\,(3+h)}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{3(3+h)h} = \lim_{h\to 0}\frac{-1}{3(3+h)} = -\frac{1}{9}.$$

Thus, the velocity after 2 second is $f'(2) = -\frac{1}{9}$ m/s, and the speed is $|f'(2)| = \left|-\frac{1}{9}\right| = \frac{1}{9}$ m/s.

Example 4: Suppose that a ball is dropped from the top of a tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball travelling when it hits the ground?

Solution: We first use the equation of motion

$$s = f(t) = \frac{1}{2}(9.8)\,t^2 = 4.9t^2$$

to find the velocity v(a) after a seconds :

$$v(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{4.9(a+h)^2-4.9a^2}{h}$$

$$= \lim_{h\to 0}\frac{4.9(a^2+2ah+h^2-a^2)}{h} = \lim_{h\to 0}\frac{4.9(2ah+h^2)}{h}$$

$$= \lim_{h\to 0}4.9(2a+h) = 9.8a$$

(a) The velocity after 5s is v(5) = (9.8)(5) = 49 m/s.

(b) Since the height of tower is 450 m above the ground, the ball will hit the ground at the time a when $s(t_1) = 450$, that is, $4.9a^2 = 450$

This gives $a^2 = \frac{450}{4.9}$ and $a = \sqrt{\frac{450}{4.9}} \approx 9.6$ s.

The velocity of the ball as it hits the ground is therefore

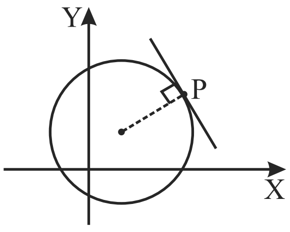
$$v(a) = 9.8a = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}.$$

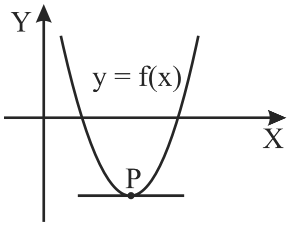
Slope of Tangent – The Geometrical Meaning of Derivative

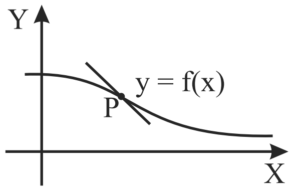
The concept of the line tangent to a curve at a point is an important one in geometry. We shall now consider the problem of defining the tangent line to the graph of a function f at a point. We will soon find that the concept of tangent can be expressed in purely analytic terms involving the function f. In fact, the problem leads directly to the definition of the derivative of a function, the central idea in differential calculus.

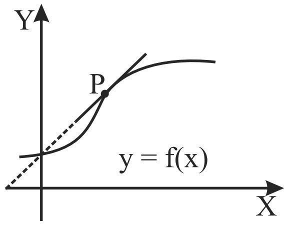
What does it mean to say that a line is tangent to a curve at a point ? School geometry defines the tangent to a circle as a straight line lying in the plane containing the circle and having a single point in common with it. In other words, the tangent line to a circle at point P is the line that is perpendicular to the radial line at point P, as shown in the first figure below. However, it is not so simple an idea as it may first appear.

For example, how would one define the tangent lines shown in the other figures? We might say that a line is tangent to a curve at point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in the figure, but not for the second. Or we might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point – this definition would work for a circle but not for more general curves, as the third curve in the figure shows. Here, the tangent "cuts" the curve at the point of tangency instead of "touching" it. In the fourth curve, the tangent "touches" the curve at P, but also "cuts" it again on the y-axis.







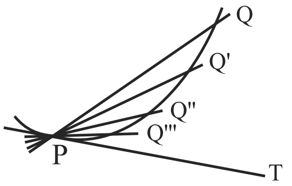


As another example, both x and y-axis have one common point each with the parabola $y = x^2$, but the x-axis is tangent to the parabola, while the y-axis is not. Thus, we see that for more complicated curves, more careful definitions must be made.

If we have a simple continuous curve, our intuition and experience tell us that a point which moves along this curve constantly changes its direction of motion, but that at each point P of the curve there is a definite straight line which gives the direction of motion and closely approximates the curve near P. The line is called the tangent at P and may be approximated in drawing by turning a ruler about P until it appears to have the proper position.

In the figure below, Q, Q′, Q″, Q‴ are several successive positions of the point Q as it approaches P. Evidently the secant PQ rotates about P and approaches a limiting position PT, which is the tangent.

Note: A secant line is a line that intersects a curve in two or more points.



Example 5: Find the slope of the secant line joining the points (1, – 2) and (1.2, – 0.56) of the parabola

$y = x^2 + 5x - 8$.

Solution: Given $y = f(x) = x^2 + 5x - 8$, we find Δy and $\Delta y/\Delta x$ as x changes from $x_0 = 1$ to

$x_1 = x_0 + \Delta x = 1.2$

$\Delta x = x_1 - x_0 = 1.2 - 1 = 0.2$ and

$\Delta y = f(x_0 + \Delta x) - f(x_0) = f(1.2) - f(1)$

$= -0.56 - (-2) = 1.44.$

So $\dfrac{\Delta y}{\Delta x} = \dfrac{1.44}{0.2} = 7.2$

The slope of the secant line joining the points (1, – 2) and (1.2, – 0.56) is 7.2.

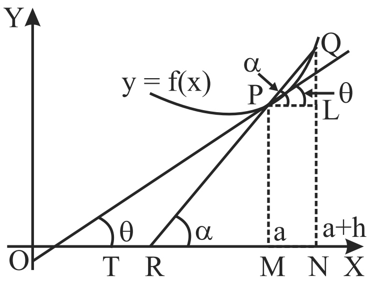
Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the slope of the tangent line at P. We can approximate this slope using a secant line through the point of tangency and a second point on the curve. This method of approximation leads to the following definition.

Definition The tangent to the curve at the point P is the limit of a secant line through P and another point Q on the curve as Q approaches P.

Let P(a, f(a)) and Q (a + h, f(a + h)) be two points very close to each other on the curve y = f(a). Draw PM and QN perpendiculars from P and Q on x-axis, and draw PL as perpendicular from P on QN. Let the chord PQ produced meet the x-axis at R and ∠QPL = ∠QRN = α. Now in right angled triangle QLP

$$\tan\alpha = \frac{QL}{PL} = \frac{NQ - NL}{MN} = \frac{NQ - MP}{ON - OM}$$

$$= \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad \ldots(1)$$



When h → 0, the point Q moving along the curve tends to P, i.e., Q → P. The chord PQ approaches the tangent line PT at the point P and then α → θ. Now applying

$$\lim_{h\to 0}\tan\alpha = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

$$\tan\theta = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = m = f'(a)$$

If the limit exists, then the function f(x) is said to have a finite derivative at the point x = a and the line passing through (a, f(a)) with slope m is the tangent line to the graph of f at the point (a, f(a)).

The slope of the tangent line to the graph of f at the point (a, f(a)) is also called the slope of the graph of f at x = a.

If we use the point slope form of the equation of a line, we can write an equation of the tangent line to the curve y = f(x) at the point (a, f(a)) as :

$y - f(a) = f'(a)(x - a)$

Further, if $\lim\limits_{x\to a} f'(x)$ does not exist in the ordinary sense, but $\lim\limits_{x\to a} |f'(x)| = \infty$ then the secant line gets steeper and steeper as x approaches a. In this case the line tangent to the graph of the function $y = f(x)$ at the point $(a, f(a))$ is perpendicular to the x-axis and we call such a line as a **vertical tangent**. The equation of the vertical tangent is $x = a$. Here, the function $f(x)$ is said to have an infinite derivative at the point $x = a$.

Note that not every curve has a tangent at any of its points because the existence of tangent depends on the existence of the above limit.

Definition The tangent line to the graph of f at the point $P(a,f(a))$ is

(i) the line $y - f(a) = f'(a)(x - a)$ through P with slope $f'(a)$, if $f'(a)$ exists;

(ii) the line $x = a$ if $\lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}\right| = \infty$.

If neither (i) nor (ii) holds, then the graph of f does not have a tangent line at the point $P(a, f(a))$.

Example 6: Find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Solution: We have $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0}\frac{[(a+h)^2-8(a+h)+9]-[a^2-8a+9]}{h}$$

$$= \lim_{h\to 0}\frac{a^2+2ah+h^2-8a-8h+9-a^2+8a-9}{h}$$

$$= \lim_{h\to 0}\frac{2ah+h^2-8h}{h} = \lim_{h\to 0}(2a+h-8) = 2a-8.$$

Since the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$, the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$.

Thus, the equation of the tangent line is

$$y - (-6) = (-2)(x - 3) \quad \text{or} \quad y = -2x.$$

Example 7: Let $f(x) = x^2 - 4x + 7$.

(i) Find the average rate of change of f with respect to x between $x = 3$ and 5.

(ii) Find the instantaneous rate of change of f at $x = 3$. Interpret the result geometrically.

Solution:

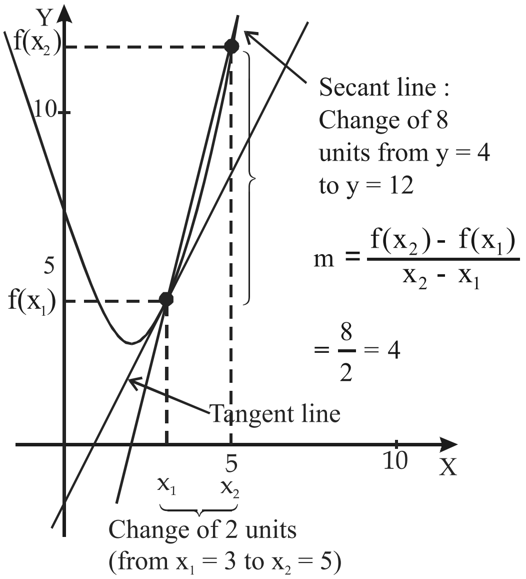
(i) The (average) rate of change from $x = 3$ to $x = 5$ is found by dividing the change in f by the change in x. The change in f from $x = 3$ to $x = 5$ is

$f(5) - f(3) = [5^2 - 4(5) + 7] - [3^2 - 4(3) + 7] = 8.$

Thus, the average rate of change is

$$\frac{f(5)-f(3)}{5-3} = \frac{8}{2} = 4.$$

The slope of the secant line is 4, as shown in the figure.



(ii) $f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$$= \lim_{h\to 0}\frac{[(3+h)^2-4(3+h)+7]-[3^2-8.3+7]}{h}$$

$$= \lim_{h\to 0}\frac{2h+h^2}{h} = 2.$$

The derivative of the function at $x = 3$ is 2.

Thus, the instantaneous rate of change of f at $x = 3$ is $f'(3) = 2$.

The tangent line at $x = 3$ has slope 2, as shown in the figure.

Example 8: Find equation of the tangent line to the graph of the function $f(x) = \sqrt[3]{x}$ at the origin.

Solution: We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt[3]{x}}{x} = \lim_{x\to 0}\frac{1}{x^{2/3}} = \infty\,.$$

Therefore, the tangent is the vertical line $x = 0$.

Example 9: Show that the following functions do not have finite derivative at the indicated points :

(a) $y = \sqrt[5]{x^2}$ at $x = 0$ (b) $y = \sqrt[5]{x-1}$ at $x = 1$

Solution: (a) $f'(0) = \lim_{\Delta x\to 0}\frac{\sqrt[5]{(\Delta x)^2}}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\sqrt[5]{(\Delta x)^3}}$.

Now the left hand limit is $-\infty$ and right hand limit is ∞. Hence, the limit does not exists and there is no derivative at $x = 0$.

(b) $f'(1) = \lim_{\Delta x\to 0}\frac{\sqrt[5]{1+\Delta x-1}}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\sqrt[5]{(\Delta x)^4}} = \infty.$

Essentially, both the functions donot have a finite derivative at the given points.

Meaning of Sign of Derivative

Remember that a positive derivative signals an increasing quantity and that a negative derivative signals a decreasing quantity.

This is obvious also from the geometrical interpretation of $\frac{dy}{dx}$.

For, if x and y are increasing together $\frac{dy}{dx}$ is the tangent of an acute angle and therefore positive, while if, as x increases y decreases, $\frac{dy}{dx}$ represents the tangent of an obtuse angle and is negative.

This aspect will be studied in detail in the chapter of monotonicity.

3.2 Differentiability

The derivative of the function f with respect to the variable x at x = a is given by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

If the derivative of f(x) exists for x = a, we say that f(x) is differentiable (derivable) for x = a, otherwise, it is non-differentiable.

If a function f(x) is differentiable at x = a, the graph of f(x) will be such that there is a tangent with finite slope to the graph at the corresponding point. But if f(x) is non-differentiable at x = a, there will be either vertical tangent or no tangent with finite slope at the corresponding point of the graph.

If f(x) is differentiable for every value of x in an interval I, we say that f(x) is differentiable in the interval I.

If f is differentiable at every point in its domain, it is simply called a differentiable function.

If f is differentiable on (– ∞, ∞) we will say that f is differentiable everywhere.

Example 1: If g(x) = 1/x, find the derivative g′(x). Also find the domain of g′(x).

Solution: $g'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$$= \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)} \qquad = \lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Evidently, the domain of g ′(x) is the set of all non-zero numbers, just as is the domain of g(x). That is, g is differentiable for all x except x = 0.

One-Sided Derivatives

For the derivative of f at a point x to exist, it is necessary that the function f be defined in a certain neighbourhood of the point x including the point x itself. Then the expression $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ is defined for all sufficiently small Δx different from zero.

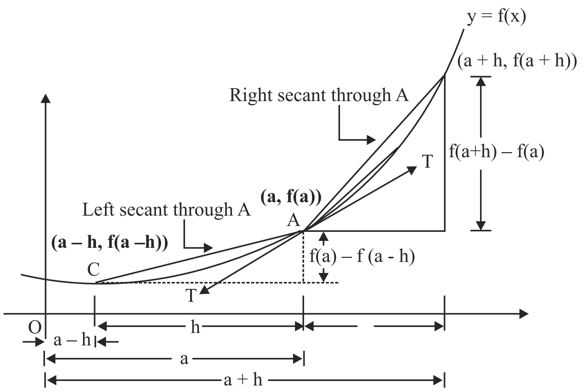
Now, the limit $f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ does not always exist for any function f defined in a neighbourhood of the point x.

The one-sided limit form of the derivative is useful in investigating the existence of derivative.

If in the above limit it is assumed that Δx tends to zero attaining only positive values (Δx > 0), then the corresponding limit (wherever it exists) is called the right hand derivative of the function f at a point x. We shall denote it as $f'(x^+)$.

Analogously, the limit, when Δx tends to zero running through negative values (Δx < 0), is termed as the left hand derivative of f at x (denoted as $f'(x^-)$.

Of course, to compute $f'(x^+)$ (or $f'(x^-)$) it is only necessary that the function f be defined at the point x and on the right of it in a certain neighbourhood (or at x and on the left of x). Consider the existence of derivative of y = f(x) at the point A(a, f(a)).



Let us choose a point B to the right of A. If the slope of tangent at A is evaluated by allowing B to approach A along the curve, then we have

$$\text{Slope of right tangent} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a^+)$$

which is called the **Right Hand Derivative (R.H.D.)** of f at x, provided the limit exists..

Let us choose a point C to the left of A. If the slope of tangent at A is evaluated by allowing C to approach P along the curve, then we have

$$\text{Slope of left tangent} = \lim_{h\to 0}\frac{f(a)-f(a-h)}{h} = f'(a^-)$$

which is called the **Left Hand Derivative (L.H.D.)** of f at x, provided the limit exists..

Sometimes, we prefer to write the left hand derivative as

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}.$$

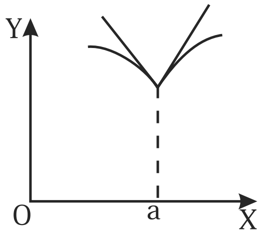
Differentiability at an Interior Point

A function f(x) is said to be differentiable at some interior point x = a in its domain if both L.H.D. and R.H.D. exist at that point, and are equal, that is, if $f'(a^+) = f'(a^-)$ = a finite quantity.

This geometrically means that a unique tangent with finite slope can be drawn at x=a as shown in the figure.

There are cases of failure of existence of $f'(a)$ and even of $f'(a^-)$ and $f'(a^+)$ at the point x = a, i.e. when the graph of the function has neither a right, nor a left tangent at the given point.

It may happen that f has at x = a, left and right derivatives different from each other : i.e. $f'(a^-) \neq f'(a^+)$.



Then such a point is called a **sharp corner**. In this case the tangent line does not exist at the point, but we say that there exist right and left tangents with different slopes at that point.

Differentiability at Endpoints

At the left endpoint of domain, the existence of R.H.D. is a sufficient condition for differentiability and at the right endpoint of domain, the existence of L.H.D. is a sufficient condition for differentiability.

If x = a is the left endpoint of the domain of the function f, then f is differentiable at x = a if the right hand derivative

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.}$$

If x = b is the right endpoint of the domain of the function f, then f is differentiable at x = b if the left hand derivative

$$\lim_{h\to 0}\frac{f(b-h)-f(b)}{-h} \text{ exists.}$$

Example 2: Examine the differentiability of the function f(x) = *ln*²x at x = 1.

Solution: $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{\ln^2(1-h)-0}{-h} = 0.$$

$$\text{R.H.D.} = f(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\ln^2(1+h)-0}{h} = 0.$$

Since L.H.D.= R.H.D. f(x) is differentiable at x = 1 and $f'(1) = 0$.

Example 3: Examine the differentiability of the function f(x) = | *ln* x | at x = 1.

Solution: $f(x) = |\ln x| = \begin{cases}\ln x & \text{if } \ln x \geq 0\\ -\ln x & \text{if } \ln x < 0\end{cases}$

$$= \begin{cases}\ln x & \text{if } x \geq 1\\ -\ln x & \text{if } 0 < x < 1\end{cases}$$

$$\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{-\ln(1-h)-0}{-h} = -1.$$

$$\text{R.H.D.} = f(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\ln(1+h)-0}{h} = 1.$$

Since L.H.D. ≠ R.H.D., f(x) is not differentiable at x = 1.

Example 4: Let $f(x) = \begin{cases}\dfrac{\sin x^2}{x} & \text{if } x \neq 0\\ 0 & \text{if } x = 0\end{cases}$

Find the slope of tangent at x = 0, if it exists.

Solution: $\text{L.H.D.} = f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{\sin h^2}{+h^2} - 0 = 1.$$

$$\text{R.H.D.} = f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\sin h^2}{h^2} - 0 = 1.$$

Since L.H.D.= R.H.D., f(x) is differentiable at x = 0 and the slope of tangent at x = 0 is 1.

Example 5: Comment on the differentiability of

$$f(x) = \begin{cases} x\ , & x < 1\\ x^2\ , & x \geq 1\end{cases} \text{ at } x = 1.$$

Solution: $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{1-h-1}{-h} = 1.$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^2-1}{h} = \lim_{h\to 0}\frac{1+h^2+2h-1}{h}$$

$$= \lim_{h\to 0}(h+2) \qquad = 2.$$

Since L.H.D. ≠ R.H.D., $f(x)$ is not differentiable at $x = 1$.

Example 6: If $f(x) \;= A + Bx^2,\; x < 1$
$\qquad\qquad\;\;= 3Ax - B + 2,\; x \geq 1$

then find A and B so that $f(x)$ becomes differentiable at $x = 1$.

Solution: $\text{L.H.D.} = f'(1^-) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{A+B(1-h)^2-3A+B-2}{-h}$$

$$= \lim_{h\to 0}\frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$$

Hence, for this limit to exist $-2A + 2B - 2 = 0$

i.e., $B = A + 1$. Then,

$$f'(1^-) = \lim_{h\to 0}(Bh - 2B) = 2B$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{3A(1+h)-B+2-3A+B-2}{h}$$

$$= \lim_{h\to 0}\frac{3Ah}{h} = 3A.$$

Since f is differentiable at $x = 1$, we have $f'(1^-) = f(1^-)$

$\Rightarrow \quad 3A = 2B \quad \Rightarrow \quad 3A = 2(A + 1)$

$\Rightarrow \quad A = 2, B = 3.$

Example 7: Let $f(x) = \begin{cases}[\cos \pi x] & x \leq 1\\ 2\{x\}-1 & x > 1\end{cases}$. Comment on the derivability at $x = 1$.

Solution: $f(1^-) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{\cos(\pi-\pi h)+1}{-h} = \lim_{h\to 0}\frac{1-\cos \pi h}{-h}$$

$$= \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{-h} = 0.$$

$$f(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2\{1+h\}+1-1}{h} = \lim_{h\to 0} 2h = 2.$$

Since L.H.D. ≠ R.H.D., $f(x)$ is not derivable at $x = 1$.

Example 8: A function $f(x)$ is such that

$$f\left(x+\frac{\pi}{2}\right) = \frac{\pi}{2} - |x| \;\forall\; x. \text{ Find } f'\left(\frac{\pi}{2}\right), \text{ if its exists.}$$

Solution: Given that $f\left(x+\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} - |x|$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h} = -1.$$

$$f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h} = 1.$$

$\Rightarrow \quad f'\left(\dfrac{\pi}{2}\right)$ does not exist.

Example 9: Show that the function $f(x) = |\cos x|$ does not have derivative at $x = \dfrac{2k+1}{2}\pi,\; k \in I.$

Solution: $f'\left(\dfrac{2k+1}{2}\pi^-\right) = \lim\limits_{\Delta x\to 0^-}\dfrac{\left|\cos\left(\dfrac{2k+1}{2}\pi+\Delta x\right)\right|}{\Delta x}$

$$= \lim_{\Delta x\to 0^-}\frac{|\sin \Delta x|}{\Delta x} = -1$$

$$f'\left(\frac{2k+1}{2}\pi^+\right) = \lim_{\Delta x\to 0^+}\frac{|\sin \Delta x|}{\Delta x} = 1.$$

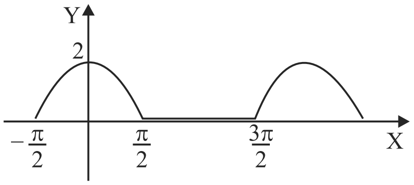
Since L.H.D. ≠ R.H.D., $f(x)$ is not derivable at

$$x = \frac{2k+1}{2}\pi,\; k \in I.$$

Example 10: Examine the differentiability of the function $f(x) = \cos x + |\cos x|$ at $x = \dfrac{\pi}{2}$.

Solution:

$$f(x) = \cos x + |\cos x| = \begin{cases} 2\cos x & \text{if } 0 \leq x < \dfrac{\pi}{2}\\[2mm] 0 & \text{if } \dfrac{\pi}{2} \leq x \leq \pi\end{cases}$$



$$f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0}\frac{2\sin h-0}{h} = -2$$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

Since the left and right derivatives are different from each other, the graph of the function has a corner point at $x = \frac{\pi}{2}$ and f(x) is not differentiable here.

Example 11: Given $f(x) = [x]\tan(\pi x)$ where [.] denotes greatest integer function, find the LHD and RHD at x = k, where $k \in I$.

Solution: $f(x) = [x]\tan(\pi x)$

$$f'(k^+) = \lim_{h\to 0}\frac{f(k+h)-f(k)}{h}$$

$$= \lim_{h\to 0}\frac{[k+h]\cdot\tan(k\pi+\pi h)-0}{h}$$

$$= \lim_{h\to 0}\frac{\pi k\tan(\pi h)}{\pi h} = k\pi$$

$$f'(k^-) = \lim_{h\to 0}\frac{f(k-h)-f(k)}{-h}$$

$$= \lim_{h\to 0}\frac{[k-h]\cdot\tan(k\pi-\pi h)-0}{-h}$$

$$= \lim_{h\to 0}\frac{-(k-1)\tan(\pi h)}{-h}$$

$= \pi(k-1)$.

Hence R.H.D. $= f(k^+) = k\pi$ and L.H.D. $= f'(k^-) = (k-1)\pi$.

Example 12:

Let $g(x) = \begin{cases} \dfrac{x^2 + x\tan x - x\tan 2x}{ax+\tan x-\tan 3x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

If g′ (0) exists and is equal to a non-zero value b, then find the value of $\frac{b}{a}$.

Solution: $g'(0) = b = \lim_{x\to 0}\dfrac{x^2+x\tan x - x\tan 2x}{x(ax+\tan x-\tan 3x)}$

$$= \lim_{x\to 0}\frac{x+\tan x-\tan 2x}{ax+\tan x-\tan 3x}$$

$$= \lim_{x\to 0}\frac{x+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots..\right) - \left(2x+\frac{8x^3}{3}+\frac{2}{15}\cdot 32x^5+\ldots\ldots.\right)}{ax+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots..\right) - \left(3x+\frac{27x^3}{3}+\frac{2}{15}243x^5+\ldots..\right)}$$

$$= \lim_{x\to 0}\frac{x^3\left(-\frac{7}{3}+\frac{-62}{15}x^2+\ldots\right)}{(a+1-3)x+\left(\frac{1}{3}-9\right)x^3+\frac{2}{15}(-242)x^5+\ldots}$$

limit b can be non - zero if a + 1 – 3 = 0

$$\therefore\ a = 2 \text{ and hence } b = \frac{-\frac{7}{3}}{\frac{1}{3}-9} = \left(\frac{-7}{3}\right)\left(\frac{3}{-26}\right)$$

$$= \frac{7}{26} \Rightarrow \frac{b}{a} = \frac{7}{52}.$$

Example 13: Let $f(x) = \begin{cases} ax^2 - bx + 2 & \text{if} \quad x < 3 \\ bx^2 - 3 & \text{if} \quad x \ge 3 \end{cases}$.

Find the values of a and b so that f(x) is differentiable everywhere.

Solution: $f'(3^-)$ and $f'(3^+)$ exist finitely.

$$\Rightarrow f'(3^-) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$$

$$= \lim_{h\to 0}\frac{a(3-h)^2-b(3-h)+2-(9b-3)}{-h}$$

$$= \lim_{h\to 0}\frac{ah^2+h(b-6a)+9a-12b+5}{(-h)} \qquad \ldots(1)$$

$\because$ $f'(3^-)$ exists, the numerator must tend to zero as $h \to 0$

$\therefore$ 9a – 12b + 5 = 0 ...(2)

Now from (1) and (2)

$$f'(3^-) = \lim_{h\to 0}\frac{ah^2+h(b-6a)}{-h} = 6a-b,$$

and $f'(3^+) = \lim_{h\to 0}\dfrac{f(3-h)-f(3)}{h}$

$$= \lim_{h\to 0}\frac{b(3+h)^2-3-(9b-3)}{h}$$

$$= b\lim_{x\to 0}\frac{(3+h)^2-9}{h} = 6b. \qquad \ldots(3)$$

Given $f'(3^-) = f'(3^+)$

$\Rightarrow$ $6a - b = 6b \Rightarrow b = \frac{6a}{7}$. ...(4)

From (2) and (4),

$$9a - \frac{72a}{7} + 5 = 0 \qquad \Rightarrow -\frac{9a}{7} + 5 = 0$$

$$\therefore\ a = \frac{35}{9},\ b = \frac{10}{3}.$$

Example 14: A function f is defined by $f(x^2) = x^3$ for all x > 0. Show that f is differentiable at 4.

Solution:

$$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{f\left((\sqrt{4+h})^2\right)-f\left(2^2\right)}{h}$$

$$=\lim_{h\to 0}\frac{(4+h)^{3/2}-8}{h}=\lim_{h\to 0}\frac{8[(1+h/4)^{3/2}-1]}{h}$$

$$=\lim_{h\to 0}\frac{8\left[1+\frac{3}{2}\frac{h}{4}+....-1\right]}{h}=\lim_{h\to 0}\frac{8\left[\frac{3}{8}h+\frac{3}{8}\left(\frac{h^2}{16}\right)+...\right]}{h}$$

$$= 3.$$

Hence f is differentiable at 4.

Example 15: Let f be differentiable at x = a and let f(a) ≠ 0. Evaluate $\lim\limits_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$.

Solution: $l = \lim\limits_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$ (1^∞ form)

$$l = (\exp.)\left(\lim_{n\to\infty} n\left\{\frac{f(a+1/n)-f(a)}{f(a)}\right\}\right)$$

put $n = \frac{1}{h}$

$$= (\exp.)\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{f(a)}\right)$$

$$= (\exp.)\left(\frac{f'(a)}{f(a)}\right) = e^{\frac{f'(a)}{f(a)}}.$$

Example 16: Let f : R → R satisfying $|f(x)| \le x^2$, ∀ x ∈ R then show f(x) is differentiable at x = 0.

Solution: Since, $|f(x)| \le x^2$, ∀ x ∈ R

at x = 0, $|f(0)| \le 0 \Rightarrow f(0) = 0$ ...(1)

$$\therefore \quad f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} \qquad ....(2)$$

{f(0) = 0 from (1)}

Now, $\left|\frac{f(h)}{h}\right| \le |h| \quad \Rightarrow \ -|h| \le \frac{f(h)}{h} \le |h|$

$$\Rightarrow \quad \lim_{h\to 0}\frac{f(h)}{h} = 0 \qquad ...(3)$$

{using Sandwich theorem}

∴ from (2) and (3) , we get f ′(0) = 0.

i.e. f(x) is differentiable at x = 0.

Concept Problems A

1. Examine the differentiability of the function $f(x) = e^{-|x|}$ at x = 0.
2. Let $f(x) = \begin{cases} \sin x & \text{if } x < \pi \\ mx+n & \text{if } x \ge \pi \end{cases}$ where m and n are constants. Determine m and n such that f is derivable at x = π.
3. Discuss the differentiability of f at x = 1
   $f(x) = 3^x, \quad -1 \le x \le 1$
   $= 4 - x, \quad 1 < x < 4$
4. (a) If $g(x) = x^{2/3}$, show that g ′(0) does not exist.
   (b) If a ≠ 0, find g ′(a).
   (c) Show that $y=x^{2/3}$ has a vertical tangent line at(0, 0).
   (d) Illustrate part (c) by graphing $y = x^{2/3}$.
5. Compute the difference quotient for the function defined by $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$
   Do you think f(x) is differentiable at x = 0? If so, what is the equation of the tangent line at x = 0 ?
6. Let $f(x) = \begin{cases} x\tan^{-1}1/x, & x \ne 0 \\ 0 \quad , & x = 0 \end{cases}$.
   Comment on the derivative of f(x) at x = 0.

Practice Problems A

7. Find the slope of the secant to the parabola $y = 2x - x^2$, if the abscissas of the points of intersection are equal to: $x_1 = 1$, $x_2 = 1 + h$. To what limit does the slope of the secant tend if h → 0 ?
8. Find $f'(0^+)$ and $f'(0^-)$ if
   (a) $f(x) = \sqrt{\sin(x^2)}$
   (b) $f(x) = \sin^{-1}\frac{a^2-x^2}{a^2+x^2}$
   (c) $f(x) = \frac{x}{1+e^{\frac{1}{x}}}$, $x \ne 0$; f(0) = 0
   (d) $f(x) = x^2 \sin\frac{1}{x}$, $x \ne 0$; f(0) = 0.

9. Discuss the differentiability of the function

$$f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x \neq 0 \\ = 0, & x = 0 \end{cases}$$

at x = 0.

10. Prove that the function

$$f(x) = \begin{cases} |x|^{3/2} \sin\dfrac{1}{x} & \text{if} \quad x \neq 0, x \in R, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is differentiable at the point x = 0, and $f'(0) = 0$.

11. It is known that $f(0) = 0$ and there exists a limit of the expression $\dfrac{f(x)}{x}$ as $x \to 0$. Prove that the limit is equal to $f'(0)$.

12. Prove that if f(x) and φ(x) are equal to zero for x = 0 and have derivatives at x = 0, φ ′(0) being not equal to zero, then $\lim\limits_{x\to 0} \dfrac{f(x)}{\varphi(x)} = \dfrac{f'(0)}{\varphi'(0)}$.

13. Prove that if f(x) has a derivative at x = a, then

$$\lim_{x\to a} \frac{xf(a) - af(x)}{x-a} = f(a) - af'(a).$$

14. Let f(x) be an even function defined for $x \in R$. If $f'(0)$ exists, find its value.

15. Let $f(x) = \dfrac{x\,g(x)}{|x|}$, $g(0) = g'(0) = 0$ and f(x) be continuous at x = 0. Find f'(0), if it exists.

3.3 Reasons of Non-Differentiability

We now find the reasons because of which a function does not have a derivative at a point.

A function has a derivative at a point x = a if the slopes of the secant lines through P(a, f(a)) and a nearby point Q on the graph approach a limit, as Q approaches P. However, a function will fail to have a derivative at a point where the graph has

(i) a corner, where the one-sided derivatives differ.

(ii) an oscillation point, where the one-sided derivative(s) does (do) not exist.

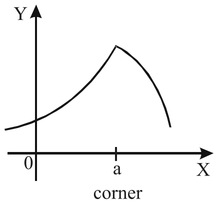
(iii) a vertical tangent, where the absolute value of slope of PQ approaches ∞.

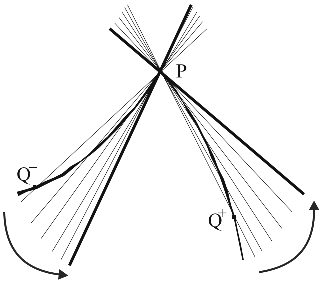
(iv) a discontinuity.

Such slopeless points may be roughly classified as "sharp points", "oscillation points", "steep points", and "points of discontinuity".

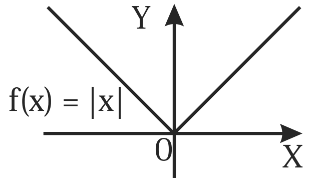
CASE I: Sharp Corner or Kink

If it happens that in a continuous function f at x = a, the left hand and right hand derivatives are different from each other i.e. $f'(a^-) \neq f'(a^+)$, then such a point is called a sharp corner or kink.





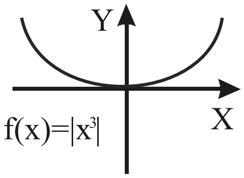
For example, we see that the function y = |x| as shown in the figure has a graph which changes its direction abruptly at x = 0. The point x = 0 is called a corner and the function y = |x| is not differentiable at 0.



In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. This can be verified by computing the left and right hand derivatives at the point.

Caution

But one should be careful about predicting a corner at a point. For instance, one may feel that the function $f(x) = |x^3|$ has a corner at x = 0 and hence non-differentiable there. This is found to be wrong.



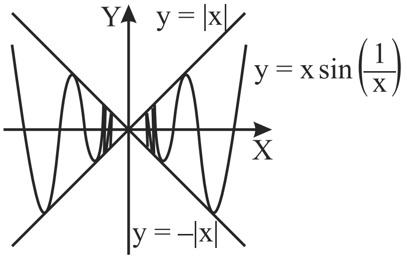
On computing, we find that $f'(a^-) = 0 = f'(a^+)$.
So, f is differentiable at x = 0.

CASE II : Oscillation Point

If in a continuous function f, either the left hand derivative or right hand derivative does not exist at x = a, because of too much oscillation of the graph in the neighbourhood of the point, then such a point is called an oscillation point.

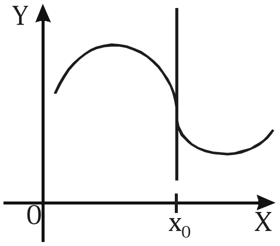
For example, the function $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

has an oscillation point at x = 0.
This is illustrated in the following figure.



CASE III : Vertical Tangent

A continuous function may also fail to be differentiable at $x = x_0$ if the absolute value of its difference quotient approaches infinity. In this case, the function is said to have a vertical tangent at $x = x_0$.



Vertical Tangent

Definition The curve y = f(x) has a vertical tangent line at the point $(x_0, f(x_0))$ provided that f is continuous at a and

$$|f'(x)| \to \infty \quad \text{as} \quad x \to x_0 \qquad ...(1)$$

Note that the requirement that f be continuous at $x = x_0$ implies that $f(x_0)$ must be defined. Thus, it would be pointless to ask about a line (vertical or not) tangent to the curve y = 1/x where x = 0.

If f is defined on only one side of $x = x_0$, we mean by (1) that $|f'(x)| \to \infty$ as x approaches x_0 from that side.

For example, the function $f(x) = x^{1/3}$ is continuous at x = 0, and

$$\text{L.H.D.} = \lim_{h\to 0} \frac{(0-h)^{1/3} - 0}{-h} = \infty$$

$$\text{R.H.D} = \lim_{h\to 0} \frac{(0+h)^{1/3} - 0}{h} = \infty$$

Since both L.H.D. and R.H.D are infinite, the function has a vertical tangent at x = 0.

∴ The y-axis i.e. x = 0 is the vertical tangent to $f(x) = x^{1/3}$ at origin. Note that f is not differentiable at x = 0.

The graph of a continuous function f having a vertical tangent at $(x_0, f(x_0))$ can have

(a) $f'(x)$ approaching either ∞ or –∞ as $x \to x_0^-$ and as $x \to x_0^+$, i.e. where the slope of the secant line approaches ∞ from both sides or approaches – ∞ from both sides, or,

(b) $f'(x)$ approaching ∞ from one side and –∞ from the other side. In such a case, the function f is said to have a **cusp** at x_0.

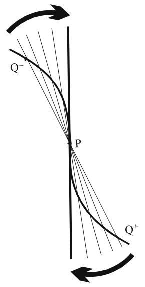


Figure (a)

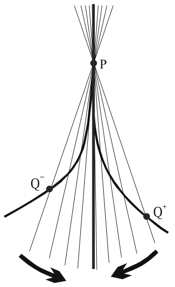


Figure (b)

In figure (a), the slope of PQ approaches –∞ from both sides. In figure (b), the slope of PQ approaches ∞ from left side and –∞ from right side.

Thus, the graph of the continuous function $f(x) = x^{2/3}$ has a vertical tangent line at the origin, even though f is not differentiable at x = 0, since, the function $f(x) = x^{2/3}$ is continuous at x = 0, and has infinite derivative at x = 0:

$$\text{L.H.D.} = \lim_{h\to 0} \frac{(0-h)^{2/3} - 0}{-h} = -\infty$$

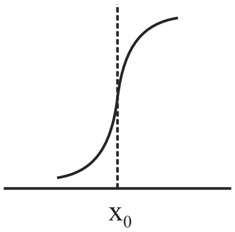
$$\text{R.H.D} = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0}{h} = \infty.$$

Further, since the one-sided derivatives are infinity opposite signs, we say that its graph has a cusp at x = 0.

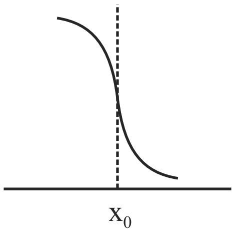
The graph of the function $f(x) = 1 - \sqrt[5]{x^2}$, has a cusp"(rather than a corner) at the point (0,1) because the derivative $f'(x) = -\dfrac{2}{5}x^{-3/5}$ approaches ∞ as $x \to 0^-$ and approaches –∞ as $x \to 0^+$. The curve also has a vertical tangent at the point.

Note: A function cannot have a vertical tangent at a point of discontinuity. For example, y = sgn (x) does not have a vertical tangent at x = 0 even if both the one- sided derivatives at x = 0 are infinite.

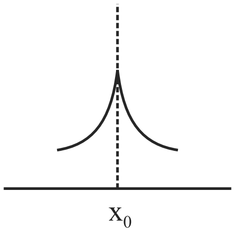
The figures give below show four curve elements that are commonly found in graphs of functions that involve radicals or fractional exponents. In all four cases, the function is not differentiable at x_0 because the secant line through $(x_0, f(x_0))$ and (x, f(x)) approaches a vertical position as x approaches x_0 from either side. Thus, in each case, the curve has vertical tangent line at $(x_0, f(x_0))$.



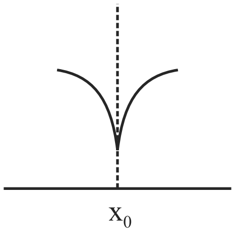
$\lim\limits_{x\to x_0^-} f'(x) = \infty$, $\lim\limits_{x\to x_0^+} f'(x) = \infty$



$\lim\limits_{x\to x_0^-} f'(x) = -\infty$, $\lim\limits_{x\to x_0^+} f'(x) = -\infty$



$\lim\limits_{x\to x_0^-} f'(x) = \infty$, $\lim\limits_{x\to x_0^+} f'(x) = -\infty$



$\lim\limits_{x\to x_0^-} f'(x) = -\infty$, $\lim\limits_{x\to x_0^+} f'(x) = \infty$

Note that the third and fourth figures represent a cusp.

Consider $f(x) = (x-4)^{2/3}$.

$$f'(x) = \frac{2}{3}(x-4)^{1/3} = \frac{2}{3(x-4)^{1/3}}$$

There is a vertical tangent and cusp at x = 4 since $f(x) = (x-4)^{2/3}$ is continuous at x = 4 and

$$\lim_{x\to 4^+} f'(x) = \lim_{x\to 4^+} \frac{2}{3(x-4)^{1/3}} = \infty$$

$$\lim_{x\to 4^-} f'(x) = \lim_{x\to 4^-} \frac{2}{3(x-4)^{1/3}} = -\infty.$$

Now consider $f(x) = 6x^{1/3} + 3x^{4/3}$.

$$f'(x) = 2x^{-2/3} + 4x^{1/3} = 2x^{-2/3}(1+2x) = \frac{2(2x+1)}{x^{2/3}}.$$

There is point of vertical tangency at x = 0, since

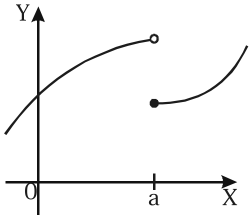
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2(2x+1)}{x^{2/3}} = \infty$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \frac{2(2x+1)}{x^{2/3}} = \infty.$$

It does not have a cusp at x = 0.

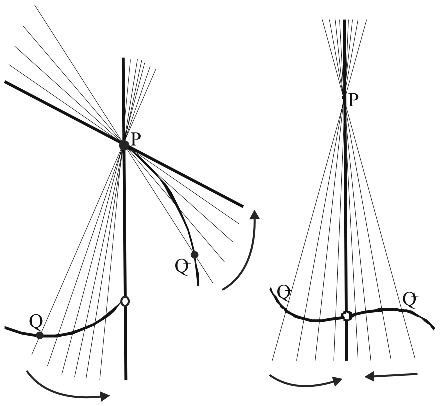
CASE IV: Discontinuity

There is another way for a function not to have a derivative. We shall prove in the next section that if f is discontinuous at x = a, then f is not differentiable at x = a. So, at any discontinuity, f fails to be differentiable.



Discontinuity

The following figures of discontinuous functions suggest that the secant lines donot approach a finite slope from atleast one side.



3.4 Relation between Continuity and Differentiability

Theorem If a function f is differentiable at x = a then it must be continuous at x = a.

Proof Since f is differentiable at x = a,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.} \quad \text{...(1)}$$

To prove that f is continuous at x = a, we must prove that $\lim\limits_{h\to 0}\left(f(a+h)-f(a)\right)=0$.

$$\text{Now } \lim_{h\to 0}\left(f(a+h)-f(a)\right)=\lim_{h\to 0}\frac{\left(f(a+h)-f(a)\right)}{h}.h$$

$$=\lim_{h\to 0}\frac{\left(f(a+h)-f(a)\right)}{h}.\lim_{h\to 0}h \;=\; f'(a).0 \;=\; 0$$

{using (1)}

$$\Rightarrow \lim_{h\to 0}f(a+h) \;=\; f(a) \Rightarrow f \text{ is continuous at } x = a.$$

Further, if f is differentiable at every point of its domain, then it is continuous in that domain.

Example 1: Examine the differentiability of the function $f(x)=\begin{cases}\dfrac{\cos x-1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ at x = 0 and comment on continuity at that point.

Solution: We have

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\frac{\cos x-1}{x}-0}{x-0}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{x^2}=\lim_{x\to 0}\frac{-\sin^2 x}{x^2(\cos x+1)}$$

$$=-\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\right]\left[\lim_{x\to 0}\frac{1}{\cos x+1}\right]=-\frac{1}{2}.$$

Since, f is differentiable at x = 0, and using the above theorem, it is also continuous at x = 0.

Example 2: Given $f(x) = x^2 \,.\, \text{sgn}(x)$, examine the continuity and derivability at x = 0.

Solution: $f(x)=\begin{cases}x^2 & \text{if} \quad x>0\\ 0 & \text{if} \quad x=0\\ -x^2 & \text{if} \quad x<0\end{cases}$

$$f'(0^+)=\lim_{h\to 0}\frac{h^2-0}{h}=0$$

$$f'(0^-)=\lim_{h\to 0}-\frac{h^2-0}{h}=0$$

⇒ f is derivable at x = 0.

⇒ f is continuous at x = 0.

The Converse of the Above Theorem Is Not Necessarily True

If f is continuous at x = a, then f may or may not be differentiable at x = a.

For example, the functions f(x) = $|x|$ and

$g(x) = x\sin\dfrac{1}{x}$ if $x\neq 0$; $g(0)=0$ are

continuous at x = 0 but not differentiable at x = 0.

Since f is continuous at x = a, we have

$$\lim_{h\to 0}\left(f(a+h)-f(a)\right)=0. \qquad \text{...(1)}$$

We need to check whether

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.} \qquad \text{...(2)}$$

Now (1) suggests that the limit (2) is in $\left(\dfrac{0}{0}\right)$ form, but we know that only some of these limits exist and hence we are not certain about the existence of the derivative.

In fact, continuity of a function at a point is a necessary condition for its differentiability at this point. However, it should be noted that the continuity of a function at a point is not a sufficient condition for the derivative of this function to exist at that point, i.e. the continuity of a function at a point does not necessarily mean its differentiability at that point.

Thus, differentiability of a function is a stronger condition than continuity alone.

Example 3: Let $f(x)=\begin{cases}x^p\sin\dfrac{1}{x} & , \quad x\neq 0\\ 0 & , \quad x=0\end{cases}$.

Find the values of p for which (i) f is continuous but not differentiable at x = 0 (ii) f is continuous and differentiable at x = 0.

Solution: f(0) = 0

For continuity, $\lim\limits_{x\to 0} f(x) = 0$

$$\therefore \quad \lim_{x\to 0} x^p\sin\frac{1}{x}=0.$$

This is possible only when p > 0. ...(1)

For differentiability,

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^p\sin\frac{1}{h}-0}{h} \text{ should}$$

exist i.e. $\lim\limits_{h\to 0}h^{p-1}\sin\dfrac{1}{h}$ should exist.

Thus, f′(0) will exist only when p > 1

∴ f(x) will not be differentiable if p ≤ 1 ...(2)

From (1) and (2),

(i) f is continuous but not differentiable at x = 0 if 0 < p ≤ 1.

(ii) f is continuous and differentiable at x = 0 if p > 1.

Example 4: If $f(x)=\begin{cases}ax+b \text{ for} & \text{for } x\le -1\\ ax^3+x+2b & x>-1\end{cases}$

is differentiable for at x = – 1, find a and b.

Solution: For differentiability, continuity is a necessary condition. Hence L.H.L. = R.H.L. at x = – 1.

$\Rightarrow -a+1=-a-1+2b$

$\Rightarrow -a-1+b+a=0 \qquad \Rightarrow b=1.$

$$\text{Now, } f'(-1^-)=\lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$=\lim_{h\to 0}\frac{a(+1+h)-b+(b-a)}{+h}$$

$$=\lim_{h\to 0}\frac{a+ah-a}{h}=a$$

$$\text{and, } f'(-1^+)=\lim_{h\to 0}\frac{[a(-1+h)^3+(-1+h)+2b]-[b-a]}{h}$$

$$=\lim_{h\to 0}\frac{ah(h^2-3h+3)+h+b-1}{h} \quad \{\text{using } b=1\}$$

$= 3a + 1.$

Hence $f'(-1^+)=f'(-1^-) \Rightarrow 3a+1=a \Rightarrow a=-\frac{1}{2}.$

Thus, $a=-\frac{1}{2}$, $b = 1$.

Example 5: A function f is defined as,

$$f(x)=\begin{cases}\dfrac{1}{|x|} & \text{if } |x|\geq\dfrac{1}{2}\\ a+bx^2 & \text{if } |x|<\dfrac{1}{2}\end{cases}.$$

If f (x) is derivable at x = 1/2 find the values of a and b.

Solution:
$$f(x)=\begin{cases}\dfrac{1}{|x|}, & x\leq-\dfrac{1}{2},\ x\geq\dfrac{1}{2}\\ a+bx^2, & -\dfrac{1}{2}<x<\dfrac{1}{2}\end{cases}$$

$$=\begin{cases}\dfrac{1}{-x}, & x\leq-\dfrac{1}{2}\\ \dfrac{1}{x}, & x\geq\dfrac{1}{2}\\ a+bx^2, & -\dfrac{1}{2}<x<\dfrac{1}{2}\end{cases}$$

$$f'\left(\frac{1^+}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{\left(\frac{1}{2}+h\right)}-2}{h}$$

$$=\lim_{h\to 0}\frac{1-2\left(\frac{1}{2}+h\right)}{h\left(\frac{1}{2}+h\right)}=\lim_{h\to 0}\frac{-2h}{h\left(\frac{1}{2}+h\right)}=-4.$$

$$f'\left(\frac{1^-}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$$

$$=\lim_{h\to 0}\frac{a+b\left(\frac{1}{2}-h\right)^2-2}{-h}=\lim_{h\to 0}\frac{\left(a+\frac{b}{4}-2\right)-bh+h^2}{-h}$$

For existence of this limit $a+\frac{b}{4}=2$ ....(1)

$\Rightarrow$ f must be continuous at x = ± 1/2.

Then, $f'\left(\frac{1^-}{2}\right)=b.$

Now $f'\left(\frac{1^-}{2}\right)=f'\left(\frac{1^+}{2}\right)$

$\Rightarrow \quad b=-4.$

Using (1), $a-1=2 \Rightarrow a=3.$

Hence, a = 3 and b = – 4.

Theorem If a function f is discontinuous at x = a then it is non-differentiable at x = a.

Proof If a function f is discontinuous at x = a then the equation $\lim_{h\to 0}\left(f(a+h)-f(a)\right)=0$ does not hold true.

For f to differentiable at x = a,

$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ must exist.

We know that this limit can only exist in $\left(\frac{0}{0}\right)$ form, which is not possible because the numerator does not approach to 0 because the condition

$$\lim_{h\to 0}\left(f(a+h)-f(a)\right)=0 \text{ does not hold true.}$$

Hence, f is not differentiable at x = a.

Example 6: Comment on the differentiability of

$$f(x)=\begin{cases}x, & x<1\\ x+2, & x\geq 1\end{cases} \text{ at } x=1.$$

Solution: L.H.D. $=f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$=\lim_{h\to 0}\frac{1-h-3}{-h}=\infty.$$

$$\text{R.H.D.}=f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h+2)-3}{h}=\lim_{h\to 0}\frac{h}{h}=1.$$

Since L.H.D. ≠ R.H.D. f(x) is not differentiable at x = 1.

If we use the previous theorem, we see that

L.H.L. = $f(1^-)$ = 1 and R.H.L. = $f(1^+)$ = 3

f(1) = 3. Thus, the function is discontinuous and hence non-differentiable. We need not find L.H.D. and R.H.D. here to find differentiability.

Theorem If a function f is non-differentiable at $x = a$ but both the one-sided derivatives exist (though being unequal), then f is continuous at $x = a$.

Proof To prove that f is continuous at $x = a$, we must prove that $\lim\limits_{h\to 0}\left(f(a-h)-f(a)\right)=0$ ...(1)

and $\lim\limits_{h\to 0}\left(f(a+h)-f(a)\right)=0$ ...(2)

Since L.H.D. exists at $x = a$,

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \text{ exists.} \quad ...(3)$$

$$\text{Now } \lim_{h\to 0}\left(f(a-h)-f(a)\right)=\lim_{h\to 0}\frac{\left(f(a-h)-f(a)\right)}{-h}.(-h)$$

$$=\lim_{h\to 0}\frac{\left(f(a-h)-f(a)\right)}{-h}.\lim_{h\to 0}(-h) = f'(a^-).0 = 0.$$

{using (3)}

Since R.H.D. exists at $x = a$,

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.} \quad ...(4)$$

$$\text{Now } \lim_{h\to 0}\left(f(a+h)-f(a)\right)=\lim_{h\to 0}\frac{\left(f(a+h)-f(a)\right)}{h}.h$$

$$=\lim_{h\to 0}\frac{\left(f(a+h)-f(a)\right)}{h}.\lim_{h\to 0} h = f'(a).0 = 0$$

{using (4)}

Thus, both the conditions (1) and (2) are established. Hence, the function is continuous at $x = a$.

Note that if any of the one-sided derivatives at $x = a$ does not exist then, f may or maynot be continuous at $x = a$.

Study Tip

Differentiable $\Rightarrow$ Continuous

Continuous $\not\Rightarrow$ Differentiable

Discontinuous $\Rightarrow$ Non-differentiable

Both one-sided derivatives exist $\Rightarrow$ Continuous Non-differentiable $\not\Rightarrow$ Discontinuous

Example 7: If $f(x) = \dfrac{xe^{1/x}}{1+e^{1/x}}$, $x \neq 0$, $f(0) = 0$, examine the continuity and differentiability at $x = 0$.

Solution: We check the differentiability first.

$$f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-0}\frac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x}$$

$$= \lim_{x\to 0-0}\frac{e^{1/x}}{1+e^{1/x}} = 0.$$

$$f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x}$$

$$= \lim_{x\to 0^+}\frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+}\frac{1}{e^{-1/x}+1} = 1.$$

Since $f'(0^-) \neq f'(0^+)$, $f(x)$ is non-differentiable at $x = 0$.

To check continuity, we use the previous theorem.

Since both the one-sided derivatives exist (though they are unequal), f is continuous at $x = 0$.

We verify this result below :

$$\text{We have } f(0^-) = \lim_{x\to 0^-}\frac{xe^{1/x}}{1+e^{1/x}} = 0$$

$$f'(0^+) = \lim_{x\to 0^+}\frac{xe^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+}\frac{x}{1+e^{-1/x}} = 0$$

Also $f(0) = 0$

$\therefore$ $f(0^-) = f(0^+) = f(0) \Rightarrow$ f is continuous at $x = 0$.

Example 8: Show that the function $f : R \to R$ defined by $f(x) = x\left\{1+\frac{1}{3}\sin(\ln x^2)\right\}$, $x \neq 0$ and $f(0) = 0$ is everywhere continuous, but has no differential coefficient at the origin.

Solution: Let us first check the differentiability

$$f'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{(0-h)\left\{1+\frac{1}{3}\sin(\ln h^2)\right\}-0}{(-h)}$$

$$= \lim_{h\to 0}\left\{1+\frac{1}{3}\sin(\ln h^2)\right\}$$

= does not exist.

Hence f(x) is not differentiable at $x = 0$. We cannot say anything about continuity immediately.

Now we check the continuity at $x = 0$:

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0}\left\{1+\frac{1}{3}\sin(\ln h^2)\right\} = 0$$

Finally f(x) is continuous at $x = 0$ but not differentiable at $x = 0$.

Example 9: If $f(x) = \begin{cases} x^2\,\text{sgn}[x]+\{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$, comment on the continuity and differentiability of f(x) at $x = 1, 2$.

Solution: Looking at the kind of functions involved we check the continuity first.

Continuity at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 \operatorname{sgn}[x] + \{x\})$$

$$= 1 \operatorname{sgn}(0) + 1 \quad = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 \operatorname{sgn}[x] + \{x\})$$

$$= 1 + 0 = 1$$

f(1) = 1.

Since, L.H.L = R.H.L = f(1), f(x) is continuous at x = 1.

Now, differentiability at x = 1,

$$f'(1^-) = \lim_{h\to 0^-} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0^+} \frac{(1-h)^2 \operatorname{sgn}[1-h] + 1 - h - 1}{-h} = 1$$

$$f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 \operatorname{sgn}[1+h] + \{1+h\} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 + h - 1}{h} = \lim_{h\to 0^+} \frac{h^2+3h}{h} = 3.$$

Here, $f'(1^+) \neq f'(1^-)$.

Hence f(x) is non-differentiable at x = 1.

Now at x = 2,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 \operatorname{sgn}[x] + \{x\} = 4 \,.\, 0 + 1 = 1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (\sin x + |x-3|) = 1 + \sin 2$$

Since, L.H.L ≠ R.H.L, f(x) is discontinuous at x = 2 and hence it is non-differentiable at x = 2.

Note: It can be proved that a function f is continuous from the left at those points where it is differentiable from the left, and f is continuous from the right at those points where it is differentiable from the right.

Now, consider the function,

$$f(x) = 2x\sqrt{x^3-1} + 5\sqrt{x}\sqrt{1-x^4} + 7x^2\sqrt{x-1} + 3x$$

Here the domain of the function is x ∈ {1}

Since, it is a point function, there is no question of continuity and hence differentiability at x = 1.

∴ f(x) is neither continuous nor differentiable at x = 1.

Suppose that the function f is continuous everywhere. At how many points do you suspect that f can fail to be differentiable ? One might think that if a function is continuous on an interval, then it might fail to be derivable at finitely many points at the most. This, however, is far from the truth.

What 's the worst such function you can think of ? Can a continuous function fail to have a derivative at every point? The answer, surprisingly enough, is yes.

There exist functions which are continuous on R but which are not derivable at any point whatsoever. It did give a jolt to mathematicians when, Weierstrass gave an example of such a function.

Consider one of his functions

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, the formula produces a graph that is too bumpy in the limit to have a tangent anywhere.

Another example of such a function defined on R is

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x), \text{ for all } x \in R,$$

It can be shown that f is continuous everywhere but is not derivable anywhere. The proof of these assertions is, however, beyond the scope of the present book.

Remark In the definition of derivative we are regarding the derivative $f'(x_0)$ as a real number associated with the difference quotient $(f(x) - f(x_0))/(x - x_0)$ considered as a function which has values for each x in the domain of f with the exception of x_0. In fact, the definition merely equates the ideas of derivability of f at x_0 with the notion of continuous extension of the difference quotient to x_0.

For if we write $Q(x) = (f(x) - f(x_0))/(x - x_0)$, the domain of Q(x) is $D_f - \{x_0\}$ so that x_0 is a removable discontinuity for Q(x), and we have the continuous extension Q*(x) defined to be Q(x) for $x \neq x_0$ with $Q^*(x_0) = f'(x_0)$. In symbols,

$$Q^*(x) = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & , \quad x \in D_f - \{x_0\} \\ f'(x_0) & , \quad x = x_0 \end{cases}$$

Thus, for instance, if f is continuous on its domain, then Q(x) is continuous on the same domain minus x_0, but the continuous extension Q*(x) would be continuous on the whole domain of f. The latter idea is expressed geometrically by rotating secant line through a given point on a curve continuously to a tangent line as a limiting line.

Theorem Suppose f is differentiable at x = c. Then $f(x) = f(c) + f'(c)(x - c) + e(x)(x - c)$,

where $e(x) \to 0$ as $x \to c$ and e(c) = 0.

Conversely, let f be defined on an interval D and let c ∈ D. Suppose there is a constant K such that $f(x) = f(c) + K(x - c) + e(x)(x - c)$, where $e(x) \to 0$ as $x \to c$. Then f is differentiable at c and f′(c) = K.

Proof If $\lim_{x\to c} \dfrac{f(x)-f(x)}{x-c} = f'(c)$ exists, then

$$e(x) = \frac{f(x)-f(c)}{x-c} - f'(c) \to 0 \quad \text{as } x \to c$$

Conversely, given f(x) = f(c) + K(x – c) + e(x)(x – c), where e(x) → 0, then

$$\frac{f(x)-f(c)}{x-c} = K + e(x) \to K$$

Hence the derivative exists and equals K.

Example 10: Check the continuity and differentiability of f(x) at x = 1, where f(x) = |x – 1| ([x] – {x}).

Solution: $f'(1^+) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{|h|([1+h]-\{1+h\})-0}{h} = \lim_{h\to 0} \frac{h(1-h)}{h} = 1.$$

$$f'(1^-) = \lim_{h\to 0} \frac{f(1)-f(1-h)}{h}$$

$$= \lim_{h\to 0} \frac{0-|-h|([1-h]-\{1-h\})}{h}$$

$$= \lim_{h\to 0} \frac{-h[0-(1-h)]}{h} = -1$$

Since $f'(1^+)$ and $f'(1^-)$ are not equal, hence f is not differentiable at x = 1. Since, the one-sided derivatives exist, f is continuous at x = 1.

Example 11: Let
$$f(x) = \begin{cases} e^{\frac{x^2+1}{x}}, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0 \end{cases}$$
Discuss continuity and differentiability of f (x) at x = 0.

Solution: At x = 0

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{e^{-\left(h+\frac{1}{h}\right)}-0}{-h} = \lim_{h\to 0} e^{-h}\cdot \frac{e^{-\frac{1}{h}}}{-h}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h}}}{-h} \text{ put } t = -\frac{1}{h} \text{ we get}$$

$$= \lim_{t\to -\infty} t\,e^{t} = \lim_{t\to -\infty} \frac{t}{e^{-t}} = \lim_{t\to -\infty} \frac{t}{-e^{-t}}$$

(applying L' Hospital rule)

= 0.

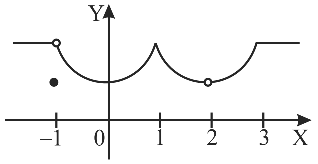
$$f'(0^+) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2-0}{h}$$

$$= \lim_{h\to 0} h = 0.$$

Hence, function is continuous and differentiable at x = 0.

Concept Problems B

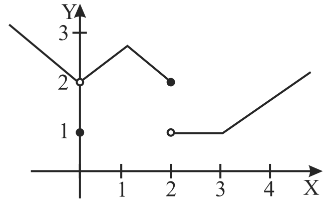
1. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x\to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

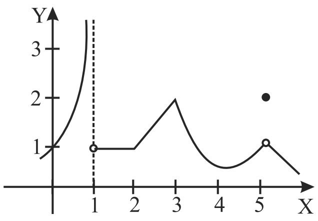
2. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x\to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

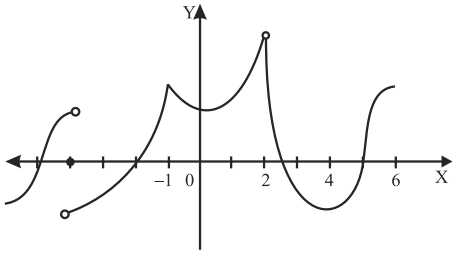
3. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x\to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

4. The graph of f is shown. State, with reasons, the number at which f is not differentiable.



5. Let $f(x) = \begin{cases} 2-x & \text{if } x \le 1 \\ x^2-2x+2 & \text{if } x > 1 \end{cases}$ Is f differentiable at 1? Sketch the graphs of f and f′.

6. Let $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ \sqrt{x}-3 & \text{if } x \ge 1 \end{cases}$. Show that f is continuous but not differentiable at x = 1.

7. If $f(x) = \begin{cases} ax^2+1, & x \le 1 \\ x^2+ax+b, & x > 1 \end{cases}$ is differentiable at x = 1, then find a and b.

8. If $f(x) = \begin{cases} \frac{1}{x}\sin x^2, & x \ne 0 \\ 0, & x = 0 \end{cases}$, discuss the continuity and differentiability of f(x) at x = 0.

9. A function f is defined as follows :

$$f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+\sin x & \text{for } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}.$$

Discuss the continuity and differentiability at x = 0 and x = π/2 .

10. If $f(x) = \begin{cases} e^{x^2+x}, & x > 0 \\ ax+b, & x \le 0 \end{cases}$ is differentiable at x = 0, then find a and b.

11. If $f(x) = \begin{cases} [2x]+x, & x < 1 \\ \{x\}+1, & x \ge 1 \end{cases}$ comment on the continuity and differentiable at x = 1.

Practice Problems B

12. Check the continuity and differentiability of

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x^3-x+1, & x > 1 \end{cases} \quad \text{at } x = 0 \text{ and } x = 1.$$

13. Test the continuity and differentiability of the function

$$f(x) = \begin{cases} \dfrac{x(2^{1/x}-2^{-1/x})}{(2^{1/x}+2^{-1/x})}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at x = 0.

14. Find the value of 'a' for which

$f(x) = \begin{cases} \operatorname{sgn}(\cos x), & x < \pi/2 \\ a-\sin x, & x \ge \pi/2 \end{cases}$ is differentiable at x = π/2.

15. Check the differentiability of the function

$$f(x) = \sin^{-1}\left(\frac{2\cos x}{1+\cos^2 x}\right) \text{ at } x = 0, \pi.$$

16. Examine for continuity and differentiability at the points x = 1 and x = 2, the function f defined by

$$f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)[x], & 2 \le x \le 3 \end{cases}$$

where [x] = greatest integer less than or equal to x.

17. Show that the function

$$f(x) = \begin{cases} \sin x.\cos(1/x) & \text{at } x \ne 0, \\ 0 & \text{at } x = 0. \end{cases}$$

is continuous at the point x = 0, but does not have even one sided derivatives.

18. If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [2x-3]x, & x \ge 1 \end{cases}$, where [.] denotes the greatest integer function, then find whether it is differentiable at x = 1.

3.5 Derivability at End Points

Till now, the derivative $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ was calculated when the function f was defined in a neighbourhood of the point x, where Δx tends to zero attaining both positive and negative values.

Suppose that a function is defined only in the left neighbourhood of x, then the above limit can be calculated when Δx tends to zero running through negative values (Δx < 0) only. This is termed as the left hand derivative of f at x (denoted as $f'(x^-)$). If this exists, then it is considered as the derivative of the function at x.

Similarly, if a function is defined only in the right neighbourhood of x, then the above limit can be calculated when Δx tends to zero running through positive values (Δx > 0) only. This is termed as the right hand derivative of f at x (denoted as $f'(x^+)$). If this exists, then it is considered as the derivative of the function at x.

Definition If a function f is defined only in the left neighbourhood of a point x = a (for $x \le a$), it is said to be differentiable at a if

L.H.D. = $f'(a^-) = \lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h}$ exists.

In such a case $f'(a) = f'(a^-)$.

Similarly, if a function f is defined only in the right neighbourhood of a point x = a (for $x \ge a$), it is said to be differentiable at a if

R.H.D. = $f'(a^+) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists.

In such a case $f'(a) = f'(a^+)$.

For example, $f(x) = x^{3/2}$ is differentiable at x = 0 but $g(x) = \sqrt{x}$ is not differentiable at x = 0.

Since, f and g are defined only in the right neighbourhood of a point x = 0, we calculate only R.H.D.

$$f'(0^+) = \lim_{h\to 0}\frac{h^{3/2}-0}{h} = \lim_{h\to 0} h^{1/2} = 0 \text{ exists.}$$

Hence $f'(0) = 0$.

$$g'(0^+) = \lim_{h\to 0}\frac{h^{1/2}-0}{h} = \lim_{h\to 0} h^{-1/2} = \infty$$

does not exist.

Hence, $\sqrt{x}$ is not differentiable at x = 0.

Note: The four inverse trigonometric functions $\sin^{-1}x$, $\cos^{-1}x$, $\text{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points $x = \pm 1$.

Consider $f(x) = \sin^{-1}x$.

At x = 1, f is defined only in the left neighbourhood.

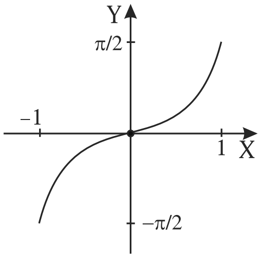
Hence, $f'(1^-) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-h)-\sin^{-1}1}{-h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-h)-\pi/2}{-h} = \lim_{h\to 0}\frac{-\cos^{-1}(1-h)}{-h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h-h^2}}{h} = \infty$$

Thus, $f(x) = \sin^{-1}x$ is not differentiable at x = 1.



There is a vertical tangent to the graph of f at x = 1.

Example 1: Discuss the differentiability of the function $f(x) = x\left(\sqrt{x}-\sqrt{(x+1)}\right)$ at x = 0.

Solution: Since domain of f(x) is $[0, \infty)$, we find the R.H.D.

$$\therefore \quad f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h(\sqrt{h}-\sqrt{h+1})-0}{h}$$

$$= \lim_{h\to 0}(\sqrt{h}-\sqrt{h+1}) = -1$$

Hence, f is differentiable at x = 0.

3.6 Differentiability over an Interval

f (x) is said to be differentiable in an interval if it is differentiable at each and every point of the interval.

Differentiability Over an Open Interval

A function f(x) is said to be differentiable in an open interval (a, b) if it is differentiable at all interior points in (a, b).

For example, $f(x) = |x-1|$ is differentiable in (0, 1).

$y = \sin^{-1}x$ is differentiable in (–1, 1).

Similarly, a function f(x) is said to be differentiable in the open interval (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$ if it is differentiable at all interior points in the interval.

For example, $f(x) = \sqrt{x}$ is differentiable in $(0, \infty)$.

Differentiability Over a Closed Interval

A function f(x) is said to be differentiable in a closed interval [a, b] if

(i) it is differentiable at all interior points in (a, b).

(ii) R.H.D. = $f'(a^+)$ exists at the left endpoint a.

(iii) L.H.D. = $f'(a^-)$ exists at the right endpoint b.

If a function is defined in a closed interval then, we by derivatives at the end points of the interval, we always mean the one-sided derivatives : the right hand derivative at the left end point and the left hand derivative at the right end point. This interpretation of differentiability at the end points of the interval of definition is analogous to that of continuity.

For example, $f(x) = x^{3/2}$ is differentiable in [0, 1].

$f(x) = \sin x$ is differentiable in $[0, 2\pi]$.

It should be noted that not all the elementary functions are differentiable everywhere in their domain.

Note:

1. All polynomial, exponential, logarithmic and trigonometric functions (inverse trigonometric not included) are differentiable at each point in their domain.

2. Modulus function and signum function are non differentiable at x = 0. Hence, y = | f(x) | and y = sgn(f(x)) should be checked at points where f(x) = 0.
3. The inverse trigonometric functions y = $\sin^{-1}x$, $\cos^{-1}x$, $\text{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points x = ± 1. Hence, y = $\sin^{-1}(f(x))$, $\cos^{-1}(f(x))$, $\text{cosec}^{-1}(f(x))$, and $\sec^{-1}(f(x))$ should be checked at points where f(x) = ± 1.
4. Greatest integer function and fractional part functions are non differentiable at all integral x. Hence, y = [f(x)] and y = {f(x)} should be checked at points where f(x) = n, n ∈ I.
5. Further, a function should be checked at all those points where discontinuity may arise.

Example 1: Show that $f(x) = \dfrac{x}{1+|x|}$ is differentiable for all x ∈ R.

Solution: Since $f(x) = \begin{cases} \dfrac{x}{1+x} & , \quad x > 0 \\ 0 & , \quad x = 0 \\ \dfrac{x}{1-x} & , \quad x < 0 \end{cases}$,

clearly f is differentiable for all $x \in R - \{0\}$. We need to check at x = 0.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{h}{1+h}-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+h} = 1$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}$$

$$= \lim_{h\to 0}\frac{1}{1+h} = 1$$

⇒ $f'(0^+) = f'(0^-)$. Hence f is differentiable at x = 0.

Finally, f is differentiable for all x ∈ R.

Example 2:

$$\text{Let } f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & 0 \le x < 2 \\ [2x]\,\text{sgn}\left(x-\frac{4}{3}\right) & 1 \le x \le 2 \end{cases}$$

where [.] and {.} denote the greatest integer function and fractional part function respectively. Find the points at which continuity and differentiability should be checked. Also check the continuity and differentiability of f(x) at x = 1.

Solution: $f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & 0 \le x < 1 \\ [2x]\,\text{sgn}\left(x-\frac{4}{3}\right) & 1 \le x \le 2 \end{cases}$

The suspicious points for continuity and differentiability are obtained when:

(i) $x + \frac{1}{3}$ becomes an integer for $0 \le x < 1$

$\Rightarrow \quad x = \frac{2}{3}$.

(ii) sin πx becomes an integer for $0 \le x < 1$

$\Rightarrow \quad x = 0, \frac{1}{2}$

(iii) The point of change in definition is x = 1.

(iv) 2x becomes an integer for $1 \le x \le 2$

$\Rightarrow \quad x = 1, \frac{3}{2}, 2$.

(v) $x - \frac{4}{3}$ becomes zero for $1 \le x \le 2 \Rightarrow x = \frac{4}{3}$.

(vi) The end points of closed domain [0, 2] ⇒ x = 0, 2.

Hence, the points where we should check the continuity and differentiability are

$x = 0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2$.

We check continuity at x = 1.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left\{x+\frac{1}{3}\right\}[\sin \pi x] = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}[2x]\,\text{sgn}\left\{x+\frac{1}{3}\right\} = 2\,\{-1\} = -2$$

Since, L.H.L ≠ R.H.L. f(x) is discontinuous at x = 1 and hence it is non-differentiable at x = 1.

Example 3: Check differentiability of f(x) in [–2, 2], if

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(|x| - \{x\}) & x < 1 \\ \sqrt{4x^2 - 12x + 9\{x\}} & x \ge 1 \end{cases}$$

where {.} denotes the fractional part function.

Solution: We have

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(|x| - \{x\}) & x < 1 \\ \sqrt{4x^2 - 12x + 9\{x\}} & x \ge 1 \end{cases}$$

$$\text{i.e.} \quad f(x) = \begin{cases} \cos\frac{\pi}{2}(|x| - \{x\}) & x < 1 \\ |2x - 3|\{x\} & x \ge 1 \end{cases}$$

Now, we have

$$\begin{aligned} |x| - \{x\} &= |x| - x + [x] = -2x - 2, && -2 \le x < -1 \\ &= -2x - 1, && -1 \le x < 0 \\ &= 0, && 0 \le x < 1 \end{aligned}$$

and $|2x - 3|\{x\} = |2x-3|(x - [x])$

$$\begin{aligned} &= (3-2x)(x-1), && 1 \le x < 3/2 \\ &= (2x-3)(x-1), && 3/2 \le x < 2 \\ &= 0, && x = 2 \end{aligned}$$

Thus, we have

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(2x+2), & -2 \le x < -1 \\ \cos\frac{\pi}{2}(2x+1), & -1 \le x < 0 \\ \cos 0, & 0 \le x < 1 \\ (3-2x)(x-1), & 1 \le x < 3/2 \\ (2x-3)(x-1), & 3/2 \le x < 2 \\ 0, & x = 2 \end{cases}$$

We need to check differentiability of f(x) at $x = -2, -1, 0, 1, 3/2, 2$. We first check continuity.

At $x = 2$, we have $f(2) = 0$ and $f(2^-) = (2 \,.\, 2 - 3)(2 - 1) = 1$.

$\Rightarrow$ discontinuity at $x = 2$.

At $x = 3/2$, we have

$$f\left(\frac{3^+}{2}\right) = f\left(\frac{3}{2}\right) = 0$$

and $$f\left(\frac{3^-}{2}\right) = \left(3 - 2.\frac{3}{2}\right)\left(\frac{3}{2} - 1\right) = 0$$

$\Rightarrow$ continuity at $x = 3/2$.

At $x = 1$, we have $f(1^+) = f(1) = 0$ and $f(1^-) = \cos 0 = 1$

$\Rightarrow$ discontinuity at $x = 1$.

At $x = 0$, we have

$$f(0^+) = f(0) = \cos 0 = 1 \text{ and } f(0^-) = \cos\frac{\pi}{2} = 0$$

$\Rightarrow$ discontinuity at $x = 0$.

At $x = -1$, we have

$$f(-1^+) = f(-1) = \cos\frac{\pi}{2} = 0 \text{ and } f(-1^-) = \cos 0 = 1$$

$\Rightarrow$ discontinuity at $x = -1$.

At $x = -2$, we have $f(-2^+) = f(-2)$

$\Rightarrow$ continuity at $x = -2$.

Hence, we need to check differentiability only at $x = -2, 3/2$.

At $x = -2$ which is the left endpoint, we have

$$f'(-2^+) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h\to 0} \frac{\cos\frac{\pi}{2}(-4+2h+2) - \cos\frac{\pi}{2}(-4+2)}{h}$$

$$= \lim_{h\to 0} \frac{\cos\pi(1-h)+1}{h} = \lim_{h\to 0} \frac{\pi\sin(\pi - \pi h)}{1} = 0$$

$\Rightarrow$ f is differentiable at $x = -2$.

At $x = 3/2$, we have

$$f'\left(\frac{3^+}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{3}{2}+h\right) - f\left(\frac{3}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2h\left(\frac{3}{2}+h-1\right)-1}{h} = 1$$

$$f'\left(\frac{3^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{3}{2}\right) - f\left(\frac{3}{2}-h\right)}{h}$$

$$= \lim_{h\to 0} \frac{0 - 2h\left(\frac{3}{2}-h-1\right)}{h} = -1$$

$\Rightarrow$ non-differentiable at $x = 3/2$

Hence, f is differentiable on $[-2,2]$ except at $x = -1, 0, 1, 3/2, 2$.

Example 4: If $x + 4|y| = 6y$, then find whether y as a function of x is continuous and derivable.

Solution: We have $x + 4|y| = 6y$

$$\Rightarrow \quad y = f(x) = \begin{cases} \frac{1}{2}x, & x \ge 0 \\ \frac{1}{10}x, & x < 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{1}{2}h = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1}{10}(0-h) = 0$$

$\Rightarrow$ f is continuous at $x = 0$ and hence for all $x \in R$.

$$f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h}{2} - 0}{h} = \frac{1}{2}$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-\frac{h}{10} - 0}{-h} = \frac{1}{10}$$

$\therefore$ f is not derivable at $x = 0$.

$\Rightarrow$ f is derivable at all x except $x = 0$.

Example 5:

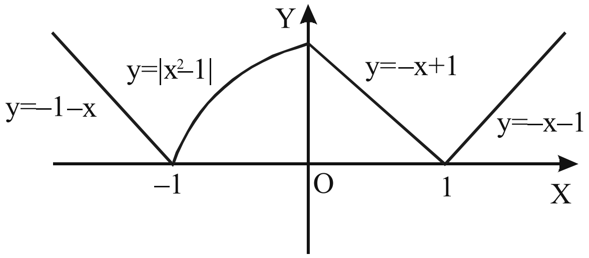
$$\text{If } f(x) = \begin{cases} -1-x, & x \le -1 \\ |x^2 - 1|, & -1 < x \le 0 \\ k(-x+1), & 0 < x \le 1 \\ |x-1|, & x > 1 \end{cases},$$

then find the value of k so that f(x) becomes continuous at x = 0. Hence, find all the points where the functions is non-differentiable.

Solution: The only possible value of k is 1 which is obtained by equating the one-sided limits at x = 0.

Now, we draw the graph of y = f(x).

From the graph, it is clear that the function is non-differentiable at x = 0, ±1 because at each of these points the graph is having a corner.

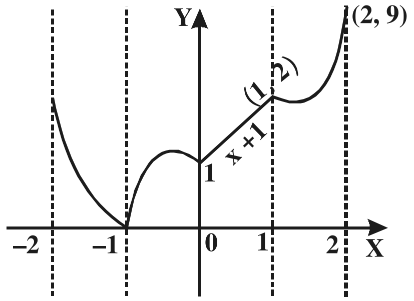


Example 6: If f(x) = |x + 1| (|x| + |x – 1|), then draw the graph of f(x) in the interval [–2, 2] and discuss the continuity and differentiability in [–2, 2]

Solution: Here, f(x) = |x + 1| (|x| + |x – 1|)

$$= \begin{cases} (x+1)(2x-1), & -2 \le x < -1 \\ -(x+1)(2x-1), & -1 \le x < 0 \\ (x+1), & 0 \le x < 1 \\ (x+1)(2x-1), & 1 \le x \le 2 \end{cases}$$

The graph of f(x) is



which is clearly, continuous for x ∈ R.

The graph has corners at x = –1, 0, 1.

Hence, f(x) is differentiable for x ∈ R – {–1, 0, 1}.

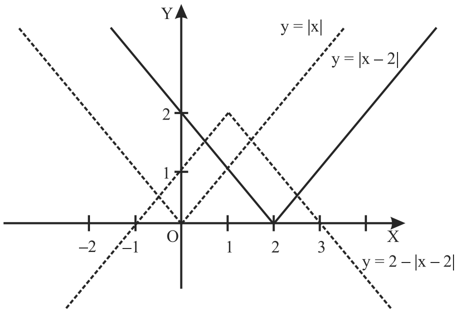
Example 7: If f(x) = Min {|x|, |x – 2|, 2 – |x – 1|}, then draw the graph of f(x) and also discuss its continuity and differentiability.

Solution: f(x) = Min {|x|, |x – 2|, 2 – |x – 1|}

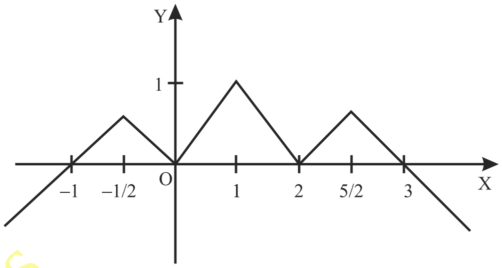
We compare the graphs of

y = |x|, y = |x – 2| and y = 2 – |x – 1| and

choose the least value at each x.



Graph of y = Min {|x|, |x – 2|, 2 – |x – 1| :

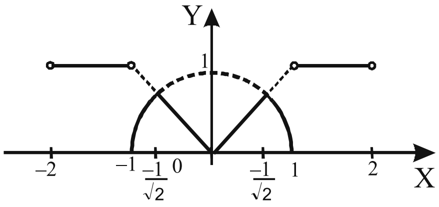


It is clear from the figure that f(x) is continuous ∀ x ∈ R and non-differentiable at x = $-\frac{1}{2}$, 0, 1, 2, $\frac{5}{2}$.

Example 8: Consider the function

$$f(x) = \min(|x|, \sqrt{1-x^2}), \qquad -1 \le x \le 1$$
$$= [|x|], \qquad 1 < |x| < 2$$

Plot the curve and discuss its continuity and differentiability.



Solution: y = [|x|], x ∈ (–2, –1) ∪ (1, 2)

⇒ y = 1.

The curve shown by thick lines represents the curve. From the graph, we can see that f(x) is

continuous on x ∈ $(-2, 2) - \{\pm 1\}$ and

differentiable on x ∈ $(-2, 2) - \left\{\pm 1, \frac{\pm 1}{\sqrt{2}}, 0\right\}$.

Example 9: Let f (x) = x³ – x² + x + 1 and

$$g(x) = \begin{cases} \max\{f(t);\ 0 \le t \le x\}, & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}.$$

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

Solution: $f(t) = t^3 - t^2 + t + 1$

$$\therefore\ f'(t) = 3t^2 - 2t + 1$$

Its discriminant $= (-2)^2 - 4.3.1 = -8 < 0$
and coefficient of $t^2 = 3 > 0$

Hence $f'(t) > 0$ for all real t.
⇒ f(t) is strictly increasing
Thus f(t) is maximum when t is maximum and $t_{max} = x$
∴ max f(t) = f(x)

$$\Rightarrow\quad g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

Now it can be easily seen that f(x) is continuous in (0, 2) and differentiable in (0, 2) except at x = 1 because at x = 1. LHD > 0 while RHD = – 1 < 0.

Example 10: Check the differentiability of the function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}$.

Solution: $\sin^{-1}|\sin x|$ is periodic with period π

$$\Rightarrow\quad \sin^{-1}|\sin x| = \begin{cases} x & , n\pi \le x \le n\pi + \frac{\pi}{2} \\ \pi - x, & n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$$

Also $\cos^{-1}|\sin x| = \frac{\pi}{2} - \sin^{-1}|\sin x|$

$$\Rightarrow\quad f(x) = \max\begin{cases} \left\{x, \frac{\pi}{2} - x\right\} & , \quad n\pi \le x \le n\pi + \frac{\pi}{2} \\ \left\{\pi - x, x - \frac{\pi}{2}\right\} & , \quad n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$$

$$\Rightarrow\quad f(x) = \begin{cases} \frac{\pi}{2} - x, & n\pi \le x \le n\pi + \frac{\pi}{4} \\ x, & n\pi + \frac{\pi}{4} < n\pi + \frac{\pi}{2} \\ \pi - x, & n\pi + \frac{\pi}{2} < x \le n\pi + \frac{3\pi}{4} \\ x - \frac{\pi}{2}, & n\pi + \frac{3\pi}{4} < x \le n\pi + \pi \end{cases}$$

⇒ f (x) is not differentiable at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\ldots$

⇒ f (x) is not differentiable at $x = \frac{n\pi}{4}$.

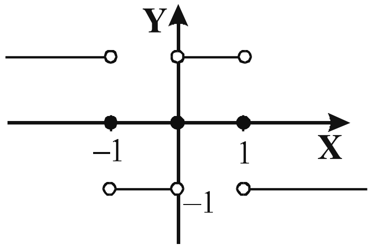
Example 11: Let f (x) = sgn x and g (x) = x (1 – x^2). Investigate the composite functions f(g(x)) and g(f(x)) for continuity and differentiability.

Solution: Since $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$f(g(x)) = \begin{cases} 1 & \text{if } g(x) > 0 \\ 0 & \text{if } g(x) = 0 \\ -1 & \text{if } g(x) < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in (-\infty, -1) \cup (0, 1) \\ 0 & \text{if } x = 0, 1, -1 \\ -1 & \text{if } x \in (-1, 0) \cup (1, \infty) \end{cases}$$

The graph of y = f(g(x)) is shown below



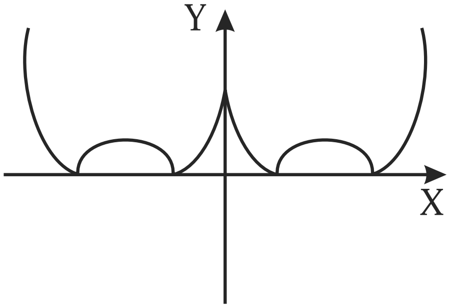
From the graph we can easily conclude that f(g(x)) is discontinuous and non derivable at – 1, 0, 1.
Now, $g(f(x)) = f(x)\,(1 - f^2(x))$
$= \text{sgn}\,x\,(1 - \text{sgn}^2 x) = 0$ for all x,
Hence g (f(x)) is continuous and derivable for all x.

Example 12: Find the interval of values of k for which the function $f(x) = |x^2 + (k - 1)|x| - k|$ is non differentiable at five points.

Solution:

$$f(x) = |x^2 + (k - 1)|x| - k| = |(|x| - 1)(|x| + k)|$$

Also f(x) is an even function and f(x) is not differentiable at five points. So |(x – 1) (x + k)| is non differentiable for two positive values of x.



⇒ Both the roots of (x – 1) (x + k) = 0 are positive.
⇒ k < 0 ⇒ k ∈ (–∞, 0).

Concept Problems C

1. Indicate the points at which the following functions have no finite derivative :

(a) $y = \cos^{-1}\dfrac{x+1}{2}$ (b) $y = \sin^{-1}\dfrac{1}{x}$

2. Show that for any positive integer n, the root function f(x) = $x^{1/n}$ is not derivable at the origin. Is it continuous at the origin ?

3. (a) Find $f'(0^-)$ and $f'(0^+)$ given $f(x) = |x|$.

(b) The function $f(x) = |3x - 10|$ is differentiable except at a single point. What is this point, and what are the values of its left hand and right hand derivatives of f there ?

4. Sketch the graph of the function $f(x) = x + |x|$. Then investigate its differentiability. Find the derivative $f'(x)$ where it exists. Also find the one-sided derivatives at the points where $f'(x)$ does not exist.

5. Investigate the differentiability of the functions:

(i) $f(x) = |x^3|$

(ii) $f(x) = \begin{cases} \dfrac{12}{(5-x)^2} & \text{if } x < 3, \\ x^2 - 3x + 3 & \text{if } x \ge 3 \end{cases}$

6. Discuss the continuity and differentiability of the function $f(x) = \sin x + \sin|x|$, $x \in R$. Draw a rough sketch of the graph of f(x) .

7. (a) Sketch the graph of the function $f(x) = x|x|$.

(b) For what value of x is f differentiable?

(c) Find a formula for $f'(x)$.

8. If $f(x) = [x] + [1 - x] = -1 \le x \le 3$, Draw its graph and comment on the continuity and differentiability of f(x).

9. Show that the Dirichlet function is nowhere derivable.

10. Examine the differentiability of

$$f(x) = \begin{cases} xe^{|x|} & , \ x < 1 \\ ex^2 & , \ x \ge 1 \end{cases}.$$

Practice Problems C

11. Let $f(x) = \dfrac{\sqrt{x+1}-1}{\sqrt{x}}$ for $x \ne 0$, $f(0) = 0$. Is the function f(x) continuous and differentiable at x = 0?

12. Investigate the differentiability of the function

$$f(x) = \begin{cases} 11 + 6x - x^2 & \text{if } x < 3 \\ x^2 - 6x + 29 & \text{if } x \ge 3 \end{cases}.$$

13. If $f(x) = \begin{cases} \min\{x, x^2\}, x \ge 0 \\ \max\{2x, x^2 - 3\}, \ x < 0 \end{cases}$ then find the points of non differentiability of f(x).

14. Find a and b, given that the derivative of f(x) exists everywhere, when

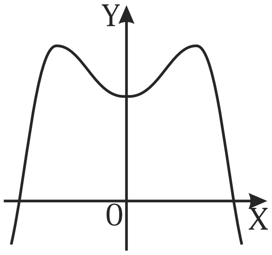
$$f(x) = \begin{cases} ax^2 - bx + 2, \ x < 3 \\ bx^2 - 3, x \ge 3 \end{cases}.$$

15. Check differentiability of f(x) at x = 0,

$$\text{where } f(x) = x\left[1 + \frac{1}{3}\sin(\ln x^2)\right], \quad x \ne 0$$
$$= 0, \quad x = 0$$

where [.] represents the greatest integer.

16. Copy the graph of the function. Then sketch a graph of its derivative directly beneath.



17. $f(x) = \begin{cases} 1-x & , \ (0 \le x \le 1) \\ x+2 & , \ (1 < x < 2) \\ 4-x & , \ (2 \le x \le 4) \end{cases}$.

Discuss the continuity and differentiability of $y = f(f(x))$ for $0 \le x \le 4$.

18. Draw the graph of $f(x) = \max.\{\sin x, \cos x, 1 - \cos x\}$ and find the number of points belonging to $(0, 2\pi)$ where f(x) is non differentiable.

19. Let $f(x) = [x]$ and $g(x) = \begin{cases} 0 \text{ if } x \text{ is an integer} \\ x^2 \text{ otherwise} \end{cases}$ then define g o f and f o g and examine their differentiability.

20. Let f(x) = cos x and

$$g(x) = \begin{cases} \text{minimum } \{f(t)\,;\, 0 \le t \le x\}, \ x \in [0, \pi] \\ \sin x - 1, \quad x > \pi \end{cases}.$$

Draw the graph of g(x) and comment on differentiability.

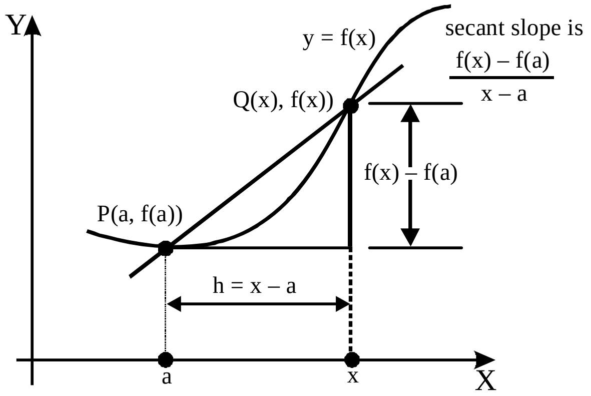
3.7 Alternative Limit Form of the Derivative

The following is an alternative limit form of the derivative. The derivative of f at x = a is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to c}\frac{f(x)-f(a)}{x-a}$$

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

The use of this formula simplifies some derivative calculations.



The one-sided derivatives of f at a are given by

$$f'(a^+) = \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} \text{ and } f'(a^-) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}.$$

f ′(a) exists if they are equal.

Example 1: If f is a differentiable function with $f'(2)=3$ then find the value of $\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}$.

Solution: $\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}\cdot\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

$$= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\left(\sqrt{x}+\sqrt{2}\right)$$

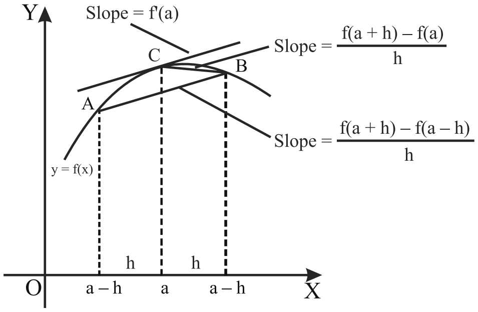
$$= f'(2)\cdot 2\sqrt{2} = 6\sqrt{2}.$$

Centred Difference Quotient

The centred difference quotient

$$\frac{f(a+h)-f(a-h)}{2h}$$

is used to approximate f ′(a) in numerical work because (i) its limit as h → 0 equals f′(a) when f′(a) exists, and (ii) it usually gives a better approximation of f′(a) for a given value of h than Fermat's difference quotient $\frac{f(a+h)-f(a)}{h}$


Y
Slope = f'(a)
C
B
A
Slope = $\frac{f(a + h) - f(a)}{h}$
Slope = $\frac{f(a + h) - f(a - h)}{h}$
y = f(x)
h
h
O
a – h
a
a – h
X

Caution

The quotient $\frac{f(a+h)-f(a-h)}{2h}$ may have a limit as h → 0 when f has no derivative at x = a.

For instance, if f(x) = |x| then

$$\lim_{h\to 0}\frac{|0+h|-(0-h|}{2h} = 0.$$

As we can see, the limit exists even though f(x) = |x| has no derivative at x = 0.

In fact, $\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}+\frac{f(a-h)-f(a)}{-h}\right]$$

$$= \frac{1}{2}(f'(a^+)+f'(a^-)).$$

We notice that it is equal to f′(a), if f is differentiable at x = a. Hence, if f ′(a) exists, then

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

Example 2: If f ′(5) = 7 then find the value of

$$\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}.$$

Solution: $\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}$

$$= 7\,.\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}$$

$$= \lim_{t\to 0}\left[\frac{f(5+t)-f(5)}{2t}+\frac{f(5-t)-f(5)}{-2t}\right]$$

$$= \frac{f'(5^+)}{2}+\frac{f'(5^-)}{2} = f'(5) = 7.$$

Note:

(i) If f ′(a) exists and ψ(h) → 0 as h → 0, then

$$\lim_{h\to 0}\frac{f(a+\psi(h))-f(a)}{\psi(h)} = f'(a).$$

(ii) If f′(a) exists, and ψ(h) → 0 and ϕ(h) → 0 as h → 0,

then $$\lim_{h\to0}\frac{f(a+\psi(h))-f(a+\phi(h))}{\psi(h)-\phi(h)} = f'(a).$$

Example 3: If f′(a) = 3, evaluate the limit

$$\lim_{h\to0}\frac{f(a+2h)-f(a-3h)}{h}.$$

Solution: $\lim_{h\to0}\frac{f(a+2h)-f(a-3h)}{h}$

$$= 5\,.\lim_{h\to0}\frac{f(a+2h)-f(a-3h)}{5h}$$

$$= f'(a)\times5 = 3\times5 = 15.$$

Example 4: If f is a differentiable function and

$$\lim_{h\to0}\frac{f\left((\pi+h)^3\right)-f(\pi^3)}{h} = \pi,$$

then find the value of $f'(\pi^3)$.

Solution: $$\pi = \lim_{h\to0}\frac{f\left((\pi+h)^3\right)-f(\pi^3)}{h} \quad ...(1)$$

$$= \lim_{h\to0}\frac{f\left(\pi^3+(3h^2\pi+3\pi^2h+h^3)\right)-f(\pi^3)}{h}$$

where $(3h^2\pi+3\pi^2h+h^3)\to0$ as $h\to0$.

$$= \lim_{h\to0}\frac{f((\pi+h)^3)-f(\pi^3)}{(3h^2\pi+3\pi^2h+h^3)}\times\lim_{h\to0}\frac{(3h^2\pi+3\pi^2h+h^3)}{h}$$

$$= f'(\pi^3)\cdot3\pi^2.$$

$$\Rightarrow \pi = f'(\pi^3)\cdot3\pi^2 \Rightarrow f'(\pi^3) = \frac{1}{3\pi}.$$

Alternative:

Using L′Hospital′s rule in (1), we get

$$\lim_{h\to0} f'\left((\pi+h)^3\right)\cdot3(\pi+h)^2 = \pi$$

$$\Rightarrow\ f'(\pi^3)\cdot3\pi^2 = \pi$$

$$f'(\pi^3) = \frac{\pi}{3\pi^2} = \frac{1}{3\pi}.$$

Example 5: Given f′(2) = 6 and f′(1) = 4, evaluate

$$\lim_{h\to0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}.$$

Solution: $\lim_{h\to0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$$= \lim_{h\to0}\frac{f(2+2h+h^2)-f(2)}{2h+h^2}\times\frac{h(2+h)}{h(1-h)}\times\frac{(1+h-h^2)-1}{f(1+h-h^2)-f(1)}$$

$$= f'(2)\times\lim_{h\to0}\frac{2+h}{1-h}\times\frac{1}{f'(1)} = 6\times2\times\frac{1}{4} = 3.$$

Example 6: If f′(2) = 4 then find the value of

$$\lim_{h\to0}\frac{f(2+h^2)-f(1+\cos h)}{h^2}.$$

Solution: $\lim_{h\to0}\frac{f(2+h^2)-f(1+\cos h)}{h^2}$

$$= \lim_{h\to0}\frac{f(2+h^2)-f(2+\cos h-1)}{h^2-(\cos h-1)}\cdot\frac{h^2-(\cos h-1)}{h^2}$$

now as h → 0, $h^2\to0$ and $(\cos h-1)\to0$

$$= f'(2)\times\lim_{h\to0}\frac{h^2-(\cos h-1)}{h^2}$$

$$= 4\times\frac{3}{2} = 6.$$

Differentiability of Parametric Functions

Let the function y = f(x) be defined by
x= x(t) and y= y(t), where t is the parameter.

Suppose we are interested in the derivative of y = f(x) and the possibility of its existence.

For parametric functions, the derivative is defined as

$$f'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The issue of differentiability is investigated using the existence of the limit :

$$\frac{dy}{dx} = \lim_{\delta\to0}\frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}} = \lim_{\delta\to0}\frac{y(t+\delta)-y(t)}{x(t+\delta)-x(t)}$$

The one-sided derivatives of f at x = x(t) are given by

$$\text{L.H.D.} = f'(x^-) = \lim_{\delta\to0}\frac{\frac{y(t-\delta)-y(t)}{-\delta}}{\frac{x(t-\delta)-x(t)}{-\delta}}$$

$$= \lim_{\delta\to0}\frac{y(t-\delta)-y(t)}{x(t-\delta)-x(t)} \quad ...(1)$$

$$\text{R.H.D.} = f'(x^+) = \lim_{\delta\to0}\frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}}$$

$$= \lim_{\delta\to0}\frac{y(t+\delta)-y(t)}{x(t+\delta)-x(t)} \quad ...(2)$$

provided x = x(t) increases with increase in t.

If $x = x(t)$ decreases with increase in t, then (1) evaluates the R.H.D. and (2) evaluates the L.H.D.

As before, $f'(x)$ exists when both (1) and (2) exist and are equal.

Another way is to eliminate t to find the function y in terms of x and then proceed as usual.

Example 7: The function f is defined by $y = f(x)$ where $x = 2t - |t-1|$, $y = 2t^2 - t|t|$, $t \in R$. Discuss the differentiability of $y = f(x)$ at $x = 2$.

Solution: We solve the equation $x = 2$ to get the corresponding value of t.

$$2t - |t-1| = 2 \quad \Rightarrow \quad t = 1.$$

We notice that at $t = 1$, $x = x(t)$ increases with increase in t. Hence,

$$\text{L.H.D.} = f'(2^-) = \lim_{\delta\to0}\frac{y(1-\delta)-y(1)}{x(1-\delta)-x(1)}$$

$$= \lim_{\delta\to0}\frac{2(1-\delta)^2-(1-\delta)|1-\delta|-1}{2(1-\delta)-|1-\delta-1|-2}$$

$$= \lim_{\delta\to0}\frac{-2\delta+\delta^2}{-3\delta} = \frac{2}{3}.$$

$$\text{R.H.D.} = f'(2^+) = \lim_{\delta\to0}\frac{y(1+\delta)-y(1)}{x(1+\delta)-x(1)}$$

$$= \lim_{\delta\to0}\frac{2(1+\delta)^2-(1+\delta)|1+\delta|-1}{2(1+\delta)-|1+\delta-1|-2}$$

$$= \lim_{\delta\to0}\frac{2\delta+\delta^2}{\delta} = 2.$$

$\therefore$ $f'(2^-) \neq f'(2^+)$ and hence $f'(2)$ does not exist.

Example 8: The function f is defined by $y = f(x)$ where $x = 1 - t^3$, $y = [t]t^3$, $t \in R$. Discuss the differentiability of $y = f(x)$ at $x = 1$.

Solution: The corresponding value of t is,

$$1 - t^3 = 1 \quad \Rightarrow \quad t = 0$$

We notice that at $t = 0$, $x = 1 - t^3$ decreases with increase in t. Hence,

$$\text{R.H.D.} = f'(1^+) = \lim_{\delta\to0}\frac{y(0-\delta)-y(0)}{x(0-\delta)-x(0)}$$

$$= \lim_{\delta\to0}\frac{[-\delta](-\delta)^3-0}{1-(-\delta)^3-1} = 1$$

$$\text{L.H.D.} = f'(1^-) = \lim_{\delta\to0}\frac{y(0+\delta)-y(0)}{x(0+\delta)-x(0)}$$

$$= \lim_{\delta\to0}\frac{[\delta]\delta^3-0}{1-\delta^3-1} = 0.$$

Hence, f is non-differentiable at $x = 1$.

Example 9: The function f is defined by $y = f(x)$ where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$. Discuss the continuity and differentiability at $x = 0$.

Solution: Here, we eliminate t and get the function $y = f(x)$ first.

If $t \geq 0$ we have $x = t$, $y = 2t^2$,

hence $y = 2x^2$, $x \geq 0$

and for $t < 0$, $x = 3t$, $y = 0$, $x < 0$

$\therefore$ The function is defined as

$$f(x) = \begin{cases} 2x^2 & , \text{ if } 0 \leq x \leq 1 \\ 0 & , \text{ if } -1 \leq x < 0 \end{cases}$$

$$f'(0^-) = 0$$

$$f'(0^+) = \lim_{h\to0}\frac{2h^2-0}{h} = 0.$$

It is clear that f(x) is differentiable and hence continuous at $x = 0$.

Differentiability Using Derivative of Functions

Consider the function $f(x) = x^{4/3}e^x$. Let us find $f'(0)$ from the formula of its derivative. We differentiate f(x) using product rule (to be discussed later).

$$f'(x) = x^{4/3}e^x + \frac{4}{3}x^{1/3}e^x \qquad ...(1)$$

To find $f'(0)$, even when we donot know whether it exists, we put $x = 0$ into the formula (1). We get $f'(0) = 0$. This provides the result in a simple way

Now, if $f(x) = x^{1/3}\sin x$,

$$f'(x) = x^{1/3}\cos x + \frac{1}{3}\sin x \cdot x^{-2/3}$$

$$= x^{1/3}\cos x + \frac{\sin x}{3x^{2/3}} \qquad ...(2)$$

Here, $f'(x)$ is not defined at $x = 0$. Do we conclude that f(x) is not differentiable at $x = 0$? The answer is no.

In such cases, we should try to find the derivative using its basic definition (i.e. first principles). The derivative may exist even when the formula $f'(x)$ is undefined.

$$\text{Now, } f'(0) = \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to0}\frac{h^{1/3}\sin h - 0}{h} = 0.$$

$\therefore$ $f'(0) = 0$. The function f is differentiable at $x = 0$.

There is another point to be noted. We find the limit of $f'(x)$ given by (2) as $x \to 0$,

$$\lim_{x\to0} f'(x) = \lim_{x\to0}\left(x^{1/3}\cos x + \frac{\sin x}{3x^{2/3}}\right) = 0.$$

We are now tempted to take the limit of the derivative ($\lim\limits_{x\to0} f'(x)$) as the value of the derivative ($f'(0)$). Both are equal to 0 in this case.

Example 10: Suppose that the real valued function f is continuous on R and differentiable on R–{0}, and that

$$\lim_{x\to0} f'(x) = L.$$

Prove that f is differentiable at 0 and that $f'(0) = L$.

Solution: We are required to prove that

$$\lim_{x\to0}\frac{f(x)-f(0)}{x-0} \text{ exists and equals L.}$$

Let $F(x) = f(x) - f(0)$ and $G(x) = x$.

Then, since f is continuous, $\lim\limits_{x\to0} F(x) = 0$, and of course

$$\lim_{x\to0} G(x) = 0 \text{ too.}$$

However, $F'(x) = f'(x)$ and $G'(x) = 1$,

$$\text{so} \quad \lim_{x\to0}\frac{F'(x)}{G'(x)} \text{ exists and equals L.}$$

By L′ Hospital′s rule, $\lim\limits_{x\to0}\dfrac{F'(x)}{G'(x)} = L$ too,

$$\text{i.e.} \quad \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = L, \text{ as required.}$$

In general, the limit of the derivative may not exist, but when it exists then it is equal to the value of the derivative.

In such a case, we say that the derived function $f'(x)$ is continuous, or the function f(x) is **continuously differentiable**.

In most of the differentiable functions used in routine, the limit of the derivative exists and this gives us an opportunity to find the value of the derivative using the limit of the derivative.

But, it must be checked before differentiating the function that it is continuous.

Example 11: If $f(x)=x|x|$, then find its derivative.

Solution: $f(x) = x\,|x|$

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \ge 0 \end{cases} \qquad \text{...(1)}$$

We see that f is continuous.

$$f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x > 0 \end{cases} \qquad \text{...(2)}$$

Note that the equality sign at x = 0 has been removed because the existence of $f'(0)$ is in doubt.

We can see that $\lim\limits_{x\to0^-} f'(x) = \lim\limits_{x\to0^-}(-2x) = 0$

and $\lim\limits_{x\to0^+} f'(x) = \lim\limits_{x\to0^+} 2x = 0$.

Thus, at x = 0, L.H.D. = R.H.D. = 0,
so we have $f'(0) = 0$.
Now, we extend the definition of $f'(x)$ at x = 0.

$$\text{Hence, } f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x \ge 0 \end{cases}$$

$$\Rightarrow \quad f'(x) = 2\,|x|.$$

Note:

If we consider the function

$$g(x) = \begin{cases} -x^2 & x < 0 \\ x^2+1 & x \ge 0 \end{cases}$$

and proceed as above, we get

$$g'(x) = \begin{cases} -2x & x < 0 \\ 2x & x > 0 \end{cases}$$

Here, also we get $\lim\limits_{x\to0^-} g'(x) = 0$ and $\lim\limits_{x\to0^+} g'(x) = 0$.

Since the function is discontinuous at x = 0, there is no question of differentiability. However, if the above limits are considered as the L.H.D. and R.H.D. then it would be misleading.

In fact when we calculate L.H.D. and R.H.D. using first principles, we get L.H.D. = ∞ and R.H.D. = 0.

Hence, we should check the continuity of the function in advance.

Using the following example, we now show that the derivative of a continuous function is not always a continuous function.

Example 12: If $f(x) = x^2 \sin(1/x)$ when $x \ne 0$ and $f(0) = 0$, show that f is differentiable for every value of x but the derivative is not continuous for x = 0.

Solution: For $x \ne 0$,

$$f'(x) = 2x\sin\frac{1}{x} + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

At x = 0, we have

$$f'(0) = \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{x^2\sin\frac{1}{x}}{x}$$

$$= \lim_{x\to0} x\sin\frac{1}{x} = 0.$$

Thus the function possesses a derivative for every value of x given by

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \text{ when } x \ne 0,\ f'(0) = 0.$$

We now show that f' is not continuous for x = 0.

Here $\lim\limits_{x\to0} f'(x)$, i.e., $\lim\limits_{x\to0}\left(2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}\right)$ does not exist since $\lim\limits_{x\to 0}\left(2x\sin\frac{1}{x}\right)=0$ exists but $\lim\limits_{x\to 0}\left(\cos\frac{1}{x}\right)$ does not exist.

Since, $\lim\limits_{x\to 0} f'(x)$ does not exist, f′ is discontinuous at x = 0.

In other words, f(x) is differentiable but not continuously differentiable.

If the limit of the derivative does not exist, does it mean that the function f(x) is not differentiable ? The answer is no. This is proved by the above example.

So, the question arises that in what circumstances can we use limit of the derivative in deciding the differentiability of the function ?

Assuming that the function f(x) is continuous at x = a, two cases arise:

(i) If $\lim\limits_{x\to a^-} f'(x)$ and $\lim\limits_{x\to a^+} f'(x)$ both exist or are infinite then

$\lim\limits_{x\to a^-} f'(x) = f'(a^-)$

i.e L.H.L. of f ′(x) = L.H.D. of f(x) at x = a

and $\lim\limits_{x\to a^+} f'(x) = f'(a^+)$

i.e R.H.L. of f ′(x) = R.H.D. of f(x) at x = a.

(a) If $\lim\limits_{x\to a^-} f'(x) = \lim\limits_{x\to a^+} f'(x)$, then f(x) is differentiable and its derivative f ′(x) is continuous at x = a.

Example 13:

If $f(x) = \begin{cases} x\tan^{-1}x + \sec^{-1}(1/x), & x\in(-1,1)-\{0\} \\ \pi/2 \quad \text{if} \quad x=0 \end{cases}$, then find f ′(0), if it exists.

Solution:

$$f'(x) = \frac{x}{1+x^2} + \tan^{-1}x + \frac{1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{x}{1+x^2} + \tan^{-1}x + \frac{|x|^2}{\sqrt{1-x^2}}\left(-\frac{1}{x^2}\right)$$

$\lim\limits_{x\to 0^-} f'(x) = -1$ and $\lim\limits_{x\to 0^+} f'(x) = -1$

Hence f ′(0) = – 1

Alternative:
Using first principles:

$$f'(0^+) = \lim_{h\to 0}\frac{h\tan^{-1}(h)+\sec^{-1}(1/h)-\pi/2}{h}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(h)-\pi/2}{h} = \lim_{h\to 0}\frac{-\sin^{-1}(h)}{h} = -1$$

Similarly, f ′(0⁺) = – 1
Hence, f ′(0) = – 1.

(b) If $\lim\limits_{x\to a^-} f'(x) \neq \lim\limits_{x\to a^+} f'(x)$, then f(x) is non-differentiable and its derivative f ′(x) is discontinuous at x = a.

Example 14: Let $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$. Define f ′(x) stating clearly the points where f(x) is not differentiable.

Solution: $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $x \in R$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$f'(x) = -\frac{2(1-x^2)}{|1-x^2|(1+x^2)}$$

$\lim\limits_{x\to 1^-} f'(x) = -1$ and $\lim\limits_{x\to 1^+} f'(x) = 1$

Since $\lim\limits_{x\to 1^-} f'(x)$ and $\lim\limits_{x\to 1^+} f'(x)$ are unequal, f′(1) does not exist.

Similarly f′(–1) does not exist.

$$\text{Hence, } f'(x) = \begin{cases} -\dfrac{2}{1+x^2} & \text{if } -1<x<1 \\ \text{non existent} & \text{if } x=\pm 1 \\ \dfrac{2}{1+x^2} & \text{if } x>1 \text{ or } x<-1 \end{cases}$$

(c) If $\lim\limits_{x\to a^-} f'(x)$ and $\lim\limits_{x\to a^-} f'(x)$ are infinite, then f(x) has an infinite derivative at x = a and hence it is non-differentiable there and f ′(x) is discontinuous at x = a.

Example 15: Suppose $f(x) = \begin{cases} |x|^x, & x\neq 0 \\ 1, & x=0 \end{cases}$, then find whether f(x) is differentiable at x = 0.

Solution:
For x > 0, f (x) = x^x
and f ′(x) = $x^x(1 + ln\ x)$ ...(1)
For x < 0, f (x) = $(-x)^x = e^{x\ ln(-x)}$
and f ′(x) = $(-x)^x[1 + ln(-x)]$ ...(2)
It can be shown using L'Hospital's rule that

$\lim\limits_{x\to 0} x^x = 1$ and $\lim\limits_{x\to 0}(-x)^x = 1$

Now, $\lim\limits_{x\to 0^-} f'(x) = -\infty$ and $\lim\limits_{x\to 0^+} f'(x) = -\infty$

$\lim\limits_{x\to 0} f'(x) = -\infty$

Hence f is not differentiable at x = 0.

(ii) If any of the limits $\lim\limits_{x\to a^-} f'(x)$ or $\lim\limits_{x\to a^+} f'(x)$ does not exist, then we cannot conclude anything about the differentiability of the function. In such a case, we should try to find the derivative using its basic definition (i.e. first principles).

In the previous example, we have seen this situation :

$f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$,

For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

Here $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist.

However, using first principles, we have

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

Hence, the function f is differentiable at x = 0, even if the limit of its derivative does not exist.

Example 16: Calculate the derivative of the function

$f(x) = \sqrt{x^2 - 2x + 1}$ on the interval [0, 2].

Solution: Since the expression under the radical sign is a perfect square, we can, according to the definition of the modulus, represent the given function in the following form :

$$f(x) = |x-1| = \begin{cases} 1-x, & 0 \le x < 1, \\ x-1, & 1 \le x \le 2 \end{cases}$$

Differentiating f(x) separately on the intervals [0, 1) and (1, 2], we obtain

$$f'(x) = \begin{cases} -1, & 0 \le x < 1, \\ 1, & 1 < x \le 2 \end{cases}$$

$$\lim_{x\to 1^-} f'(x) = -1 \text{ and } \lim_{x\to 1^+} f'(x) = 1$$

Since the left and right derivatives of f (x) at the point x = 1 do not coincide, the derivative does not exist at x = 1. We take the values of the left hand derivative of function at the point 2 and of its right hand derivative at the point 0 as the values of f ′(x) at the endpoints of the interval [0, 2].

$$\text{Thus } f'(x) = \begin{cases} -1, & x \in [0,1), \\ 1, & x \in (1,2] \end{cases}$$

Example 17: Check the differentiability of the function $f(x) = \sqrt{1-\sqrt{1-x^2}}$.

Solution: The function $f(x) = \sqrt{1-\sqrt{1-x^2}}$ is defined for $x \in [-1, 1]$. It is continuous in its domain.

Differentiating w.r.t. x, we have

$$f'(x) = \frac{1}{2\sqrt{1-\sqrt{1-x^2}}} \cdot \frac{-1}{2\sqrt{1-x^2}} \cdot -2x$$

$$= \frac{x}{2\sqrt{1-x^2}\sqrt{1-\sqrt{1-x^2}}}$$

The formula of f ′(x) is undefined at x = – 1, 0, 1.

The differentiability of the function should be checked at x = – 1, 0, 1.

We can see that

$$\lim_{x\to -1^+} f'(x) = -\infty \text{ and } \lim_{x\to -1^-} f'(x) = \infty .$$

Hence, f has an infinite derivative at x = – 1 and 1.

To check the existence of f ′(x) at x = 0, we find whether $\lim\limits_{x\to 0} f'(x)$ exists or not.

$$\text{We have } \lim_{x\to 0}\frac{x}{\sqrt{1-x^2}\sqrt{1-\sqrt{1-x^2}}} = \lim_{x\to 0}\frac{x}{\sqrt{1-\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0} x\sqrt{\frac{1+\sqrt{1-x^2}}{x^2}}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}x}{|x|} = \begin{cases} -\sqrt{2} & \text{as } x\to 0^- \\ \sqrt{2} & \text{as } x \to 0^+ \end{cases}$$

= does not exist.

Hence, f is non-differentiable at x = 0.

Thus, the function f is differentiable in its domain except at x = –1, 0, 1.

Alternative approach at x = 0 using first principles.

$$f'(0^-) = \lim_{h\to 0}\left\{\frac{f(0-h)-f(0)}{-h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sqrt{1-\sqrt{(1-h^2)}}-0}{-h}\right\}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{(1-h^2)}}}{-h} \times \frac{\sqrt{1+(1-h^2)}}{1+\sqrt{1-h^2}}$$

$$= \lim_{h\to 0}\frac{h}{(-h)\sqrt{1+\sqrt{(1-h^2)}}} = -\frac{1}{\sqrt{2}}$$

$$\text{and } f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{(1-h^2)}}-0}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-\sqrt{(1-h^2)}}}{h} \times \frac{\sqrt{1+(1-h^2)}}{\sqrt{1+\sqrt{1-h^2}}}$$

$$= \lim_{h\to 0} \frac{h}{h\sqrt{1+\sqrt{(1-h^2)}}} = \frac{1}{\sqrt{2}}$$

∵ $f'(0^-) \neq f'(0^+)$, f(x) is not differentiable at x = 0.

Example 18: If $f(x) = \begin{cases} A + Bx^2 & , \quad x < 1 \\ 3Ax - B + 2, & \quad x \geq 1 \end{cases}$ find the value of A and B so that f(x) is differentiable at x = 1.

Solution: Since f(x) will be differentiable at x = 1, it must be continuous,

∴ $3A - B + 2 = A + B$

∴ $A - B + 1 = 0$ ...(1)

Now, $f'(x) = \begin{cases} 2Bx, & x < 1 \\ 3A \quad , & x > 1 \end{cases}$

$$\lim_{x\to 1^-} f'(x) = 2B \text{ and } \lim_{x\to 1^+} f'(x) = 3A$$

∴ $3A = 2B$ ...(2)

From (1) and (2), $\frac{2B}{3} - B + 1 = 0$

$\Rightarrow \quad -\frac{B}{3} = -1 \quad \Rightarrow \quad B = 3$

$\Rightarrow \quad A = 2.$

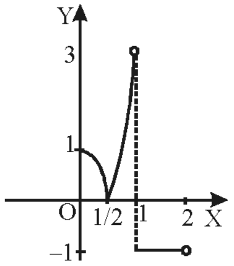
Example 19: If $f(x) = \begin{cases} |1-4x^2| & , \quad 0 \leq x < 1 \\ [x^2 - 2x] & , \quad 1 \leq x < 2 \end{cases}$

where [.] denotes the greatest integer function.
Discuss the continuity and differentiability of f(x) in [0, 2).

Solution: Since $1 \leq x < 2$ we have$0 \leq x - 1 < 1$,
and $[x^2 - 2x] = [(x - 1)^2 - 1] = [(x - 1)^2] - 1$
$= 0 - 1 = -1$

$$\therefore \quad f(x) = \begin{cases} 1-4x^2 & , \quad 0 \leq x < \frac{1}{2} \\ 4x^2 - 1 & , \quad \frac{1}{2} \leq x < 1 \\ -1 & , \quad 1 \leq x < 2 \end{cases}$$

Hence, the graph of f(x) is:



It is clear from the graph that f(x) is discontinuous at x = 1 and not differentiable at $x = \frac{1}{2}$ and x = 1. To verify these details we calculate the derivative of f:

$$f'(x) = \begin{cases} -8x, & 0 \leq x < 1/2 \\ 8x, & 1/2 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

This shows that f(x) is not differentiable

at x = 1/2 (as RHD = 4 and LHD = –4)
and x = 1 (as RHD = 0 and LHD = 8)

Therefore, f(x) is differentiable ∀ $x \in [0, 2) - \{1/2, 1\}$.

Concept Problems D

1. Evaluate $\lim_{x\to 0} \frac{\sin(3+x)^2 - \sin 9}{x}$.
2. Find a function f and a number a such that $\lim_{h\to 0} \frac{(2+h)^6 - 64}{h} = f'(a)$.
3. Each of the following limit represents the derivative of some function f at some number a. State such an f and a in each case.

   (i) $\lim_{h\to 0} \frac{2^x - 32}{x-5}$ (ii) $\lim_{h\to 0} \frac{\cos(\pi + h) + 1}{h}$

4. Let $f(x) = \begin{cases} x^2 & x \leq 0 \\ -x^2 & x > 0 \end{cases}$.

   Prove that f is differentiable. Is f′ is continuous?

5. If $f'(a) = \frac{1}{4}$, find $\lim_{h\to 0} \frac{f(a+2h^2) - f(a-2h^2)}{h^2}$.
6. If $f'(2) = 4$ then find the value of

   $$\lim_{h\to 0} \frac{f(2+h) - f(2+\sin h)}{h.\sin h.\tan h}.$$

7. Discuss the differentiability of the function y = f(x) defined as $x = 2t + |t|$ and $y = [t]t^2$ at x = 0.
8. Let $f(x) = \begin{cases} 3x^2, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$. Find the values of a and b so that f will be differentiable at x = 1.
9. Let $f(x) = \begin{cases} x^2, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases}$. Show that

   $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ but that f′(0) does not exist.

3.8 Derivatives of Higher order

Because the derivative of a function is a function, differentiation can be applied over and over, as long as the derivative itself is a differentiable function.

Let a function y = f(x) be defined on an open interval (a, b). Its derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] and is called the first derivative of y w. r. t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w. r. t. x and is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Once we have found the derivative f′ of any function f, we can go on and find the derivative of f′.

$$f''(a) = \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h}.$$

Similarly, the third order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y'''.

The third derivative**,** written f''', is the derivative of the second derivative, and, in principle, we can go on forever and form derivatives of higher order. We adopt the alternative notation $f^{(n)}$ for the nth derivative of f. Notice also that for derivatives higher than the third, the parentheses distinguish a derivative from a power. For example, $f^4 \neq f^{(4)}$.

Twice Differentiability

Suppose that a function y = f(x) defined in an interval has the derivative $f'(x)$ with respect to the independent variable x. If the function $f'(x)$ is further differentiable, its derivative is called the second derivative (or the derivative of the second order) of the original function f(x) and is denoted $f''(x)$.

A function f(x) is twice differentiable at x = a if its derivative $f'(x)$ is differentiable at x = a i.e.

the limit $f''(a) = \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h}$ exists.

Alternatively, $f''(a) = \lim_{x\to a}\frac{f'(x)-f'(a)}{x-a}$.

The value $f''(x)$ of the second derivative at a point x characterizes the rate of change of $f'(x)$ at that point, that is the rate of change of the rate of change of f(x). By analogy with mechanics, we can say that $f''(x)$ is the acceleration of the change of the function f(x) at the given point x.

The geometrical meaning of the second derivative will be discussed later.

Further, the nth derivative of f(x) at x = a exists if

$$f^{(n)}(a) = [f^{(n-1)}]'(a) = \lim_{x\to a}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a} \text{ exists.}$$

Example 1: Find whether the function $f(x) = x^{4/3}\sin x$ is twice differentiable at x = 0.

Solution: Let us find $f'(x)$ first. We differentiate f(x) using product rule.

$$f'(x) = x^{4/3}\cos x + \frac{4}{3}x^{1/3}\sin x \quad ...(1)$$

Now, $f''(0) = \lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}$

$$= \lim_{h\to 0}\frac{h^{4/3}\cos h+\frac{4}{3}h^{1/3}\sin h-0}{h}$$

$$= \lim_{h\to 0}\left(h^{1/3}\cos h+\frac{4h^{1/3}\sin h}{3h}\right) = 0.$$

$\therefore$ $f''(0) = 0$.

Hence, the function f is twice differentiable at x = 0.

Example 2: Find whether

$$f(x) = \begin{cases} -x^2+x & x<0 \\ \sin x+x^2 & x\ge 0\end{cases}$$

is twice differentiable and find its second derivative.

Solution: $f(x) = \begin{cases} x^2+x & x<0 \\ \sin x+x^2 & x\ge 0\end{cases}$ ..(1)

We see that f is continuous.

$$f'(x) = \begin{cases} 2x+1 & x<0 \\ \cos x+2x & x>0\end{cases} \quad ...(2)$$

Note that the equality sign at x = 0 has been removed because the existence of $f'(0)$ is in doubt.

We can see that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 1$.

So we have $f'(0) = 1$.

Now, we extend the definition of $f'(x)$ at x = 0.

Hence, $f'(x) = \begin{cases} 2x+1 & x<0 \\ \cos x+2x & x\ge 0\end{cases}$

Now $f''(x) = \begin{cases} 2 & x<0 \\ -\sin x+2 & x>0\end{cases}$

We find that $\lim_{x\to 0^-} f''(x) = \lim_{x\to 0^+} f''(x) = 2$.

$\therefore$ $f''(0) = 2$ and the function f is twice differentiable at x = 0. We can easily see that f is twice differentiable at other values of x.

Example 3: Let f be defined in a neighbourhood of x and $f''(x)$ exists.

Show that $\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$

Show by an example that the limit may exist ever if $f''(x)$ does not exist.

Solution: $f''(x) = \lim\limits_{h\to 0} \dfrac{f'(x+h) - f'(x)}{h}$

$$= \lim_{h\to 0} \frac{\lim\limits_{k\to 0}\left[\dfrac{f(x+h+k)-f(x+h)}{k} - \dfrac{f(x+k)-f(x)}{k}\right]}{h}$$

Let $k = -h$

$$\Rightarrow \quad f''(x) = -\lim_{h\to 0} \frac{f(x)-f(x+h)-2(x-h)+f(x)}{h^2}$$

$$= \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

Note: The above limit cannot be used as the test for twice differentiability of f. One can verify this by using $f(x) = x|x|$.
Let us calculate the limit first:

$$\lim_{h\to 0} \frac{f(0+h)+f(0-h)-2f(0)}{h^2}$$

$$= \lim_{h\to 0} \frac{h|h|+(-h)|-h|-0}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0.$$

In the previous section we had calculated

$$f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x \ge 0 \end{cases}$$

from where $f''(0^-) = -2$ and $f''(0^+) = 2$.
Hence $f''(0)$ does not exist. However, the given limit exists.

Example 4: If $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then examine whether $f'(x)$ is continuous and differentiable at $x = 0$.

Solution: $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$

f(x) is continuous at x = 0.

$$f'(x) = \begin{cases} xe^x + e^x, & x < 0 \\ 1 + 2x - 3x^2, & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f'(x) = 1 = \lim_{x\to 0^+} f'(x)$$

Hence f(x) is differentiable at x = 0 and thus

$$f'(x) = \begin{cases} xe^x + e^x, & x \le 0 \\ 1 + 2x - 3x^2, & x > 0 \end{cases} \text{ is continuous.}$$

$$\text{Now, } f''(x) = \begin{cases} e^x + (x+1)e^x, & x < 0 \\ 2 - 6x, & x > 0 \end{cases}$$

$$\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^-} f''(x) = 2$$

$\Rightarrow \quad f'(x)$ is also differentiable at x = 0.

Example 5: Let $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \ge 1 \end{cases}.$

If $f''(1)$ exist then find the value of $a^2 + b^2 + c^2$.

Solution: For continuity at x = 1 we must have

$f(1^+) = f(1^-)$

$\Rightarrow \quad a + b + c = 1$ ...(1)

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 2ax + b & \text{if } x > 1 \end{cases}$$

For continuity of $f'(x)$ at x = 1

$f'(1^+) = f'(1^-)\ 2a + b$

$\Rightarrow \quad 2a + b = 3$ ...(2)

$$f''(x) = \begin{cases} 6x & \text{if } x < 1 \\ 2a & \text{if } x > 1 \end{cases}$$

$f''(1^-) = 6, \; f''(1^+) = 2a$

Hence, $2a = 6 \quad \Rightarrow a = 3$
From (1) and (2), $b = -3$, $c = 1$
Hence, the value of $a^2 + b^2 + c^2 = 19$.

Example 6: Let $f(x) = \begin{cases} 2x^2 \sin \pi x & x \le 1 \\ x^3 + ax^2 + b & x > 1 \end{cases}$ be a differentiable function. Examine whether it is twice differentiable in R.

Solution: For continuity at x = 1 we must have

$a + b + 1 = 0$

$$f'(x) = \begin{cases} 4x \sin \pi x + 2\pi x^2 \cos \pi x & x < 1 \\ 3x^2 + 2ax & x > 1 \end{cases}$$

For differentiability of f :

$f'(1^+) = 3 + 2a$, $f'(1^-) = -2\pi$ should be equal.

$$\Rightarrow \quad a = -\frac{2\pi + 3}{2} \quad \text{and} \quad b = \frac{2\pi + 3}{2}$$

$$\text{Thus } f'(x) = \begin{cases} 4x \sin \pi x + 2\pi x^2 \cos \pi x & , x \le 1 \\ 3x^2 - (3 + 2\pi)x & , x > 1 \end{cases} \text{ and}$$

$$f''(x) = \begin{cases} \left(4 - 2\pi^2 x^2\right) \sin \pi x + 8\pi x \cos \pi x & , \; x < 1 \\ 6x - 3 - 2\pi & , \; x > 1 \end{cases}$$

$f''(1^-) = -8\pi$ and $f''(1^+) = 3 - 2\pi$

$\Rightarrow \quad f''(x)$ does not exist at x = 1.
However, f(x) is twice differentiable for x other than x = 1

Example 7: Find

$$f''(x) \text{ if } f(x) = \begin{cases} x^3 \sin(1/x) & \text{for } x \ne 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and find whether $f''(x)$ is continuous at the point x = 0.

Solution: For $x \neq 0$ the derivative $f'(x)$ can be found by differentiating the function $x^3 \sin(1/x)$ according to the rule of differentiation of a product. This yields

$$f'(x) = 3x^2 \sin(1/x) - x \cos(1/x) \quad (x \neq 0).$$

$$\lim_{x\to 0^-} f'(x) = 0 = \lim_{x\to 0^+} f'(x)$$

This means that $f'(0)$ exists

Thus $f'(x)$ exists at all points:

$$f'(x) = \begin{cases} 3x^2 \sin(1/x) - x\cos(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Now, $f''(x) = 6x \sin(1/x) - 4\cos(1/x) - (1/x)\sin(1/x) \quad (x \neq 0).$

$\lim_{x\to 0} f''(x)$ does not exist.

To test whether $f''(x)$ exists at $x = 0$, we apply first principles on $f'(x)$

$$f''(x) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h}$$

$$= \lim_{h\to 0} \frac{3h^2 \sin\frac{1}{h} - h\cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0}\left(3h\sin\frac{1}{h} - \cos\frac{1}{h}\right)$$

which does not exist.

Thus f is twice differentiable for all x except $x = 0$.
Further, $f''(x)$ is discontinuous at the point $x = 0$.

Example 8: Let f, g be twice differentiable functions from R to R, and suppose that $f(x) = g(x)/x$ for all $x \neq 0$. Given that $g(0) = g'(0) = 0$ and that $g''(0) = 6$, determine $f(0)$ and $f'(0)$.

Solution: By the definition of $g'(0)$, we have

$$0 = g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{g(x)}{x} \quad (\text{since } g(0) = g'(0) = 0)$$

$$= \lim_{x\to 0} f(x) \quad \left(\text{since } f(x) = \frac{g(x)}{x} \text{ for } x \neq 0\right),$$

$$= f(0) \qquad (\text{since f is continuous})$$

So $f(0) = 0$. To find $f'(0)$, note that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{g(x)}{x^2}$$

$$\left(\text{since } f(0) = 0 \text{ and } f(x) = \frac{g(x)}{x} \text{ for } x \neq 0\right).$$

To determine this latter limit, we can use L′Hospital ′s rule. Certainly $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} x^2 = 0$.

So we look at $\frac{g'(x)}{2x}$ (differentiating top and bottom).

But $g'(0) = 0$, so $\lim_{x\to 0} \frac{g'(x)}{x} = g''(0) = 6$, and so $\lim_{x\to 0} \frac{g'(x)}{2x} = 3$.

Thus by L ′Hospital ′s rule,

$$\lim_{x\to 0} \frac{g(x)}{x^2} = 3 \text{ too, and this gives } f'(0) = 3.$$

Concept Problems

E

1. Let $f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$ Show that $f'(0)$ exists but $f''(0)$ does not.

2. Let f be the function defined by

$$f(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } x \ge 0, \\ -\dfrac{x^2}{2} & \text{if } x \le 0. \end{cases}$$

   (a) Compute f'.
   (b) Is f a differentiable function?
   (c) Show that $f''(0)$ does not exist, and compute $f''(x)$ for $x \neq 0$.

3. Let $f(x) = x^{4/3}$
   (a) Compute f'.
   (b) Is f a differentiable function?
   (c) Show that $f''(0)$ does not exist, and compute $f''(x)$ for $x \neq 0$.

4. Let the function g be defined by

$$g(x) = \begin{cases} x^2 & x \le 0, \\ 2x - 1 & x > 1. \end{cases}$$

   (a) Compute g' and g''.
   (b) Are g and g' differentiable functions ?

5. Evaluate $\lim_{x\to 0} \dfrac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

Practice Problems D

6. Find y" for the following functions:

(a) $y = |x^3|$

(b) $y = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$

Also find y" (0) if it exists.

7. A function f is defined as

$$f(x) = \begin{cases} \frac{1}{2}(b^2 - a^2) & \text{for } 0 \le x \le a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^2}{3} & \text{for } a < x \le b \\ \frac{1}{2}\left(\frac{b^3 - a^3}{x}\right) & \text{for } x > b \end{cases}$$

Prove that f and f′ are continuous but f″ is discontinuous.

8. Evaluate $\lim_{x \to 0} \frac{\tan(a + 2x) - 2\tan(a + x) + \tan a}{x^2}$.

9. (a) Show that $f(x) = x^{7/3}$ is twice differentiable at 0, but not three times differentiable at 0.

(b) Find an exponent k such that $f(x) = x^k$ is (n – 1) times differentiable at 0, but not n times differentiable at 0.

3.9 Algebra of Differentiable Functions

We can deduce from the theorems on limits that the sum, product, difference or quotient of two functions which are differentiable at a certain point are themselves differentiable at that point (except that, in the case of the quotient, the denominator must not vanish at the point in question). Further it is true that composition of a differentiable function with a differentiable function is a differentiable function.

1. If f(x) and g(x) are differentiable at x = a, then the following functions are also differentiable at x = a.
   (i) cf(x) is differentiable at x = a, where c is any constant.
   (ii) f(x) ± g(x) is differentiable at x = a.
   (iii) f(x). g(x) is differentiable at x = a.
   (iv) f(x)/g(x) is differentiable at x = a, provided g(a) ≠ 0.

Here, we prove some of these results:

Theorem If f and g are differentiable at x = a, then so is their product f + g.

Proof We apply the definition of derivative to F(x) = f(x) + g(x) as follows:

$$F'(a) = \lim_{h \to 0} \frac{(f(a+h) + g(a+h)) - (f(a) + g(a))}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}\right]$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= f'(a) + g'(a).$$

Therefore, f(x) + g(x) is differentiable at x = a.

Note: The sum of a finite number of functions differentiable at a point is a differentiable function at the point.

Theorem If f and g are differentiable at x = a, then so is their product f. g.

Proof Let F(x) = f(x). g(x)

$$F'(a) = \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}$$

$$= \lim_{h \to 0} \left[f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h}\right]$$

$$= \lim_{h \to 0} f(a+h) \cdot \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$+ g(a) \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

As h approaches zero, f(a+h) approaches f(a) because f, being differentiable at a, is continuous at a.

$= f(a)\, g'(a) + f'(a)\, g(a).$

Therefore, f(x). g(x) is differentiable at x = a.

Note: The product of a finite number of functions differentiable at a point is a differentiable function at that point.

By the above theorems we recognize that the functions

$y = x^2 + 3x + 2$

$y = \sin^3 x - x \sin x - (x^4 - 1) \cos x$, and

$y = (\sin x - 2x)/(x^2 + 1)$

are differentiable at every point in a domain common to all functions involved.

2. If f(x) is differentiable at x = a and g(x) is non-differentiable at x = a, then we have the following results:

(i) Both the functions f(x) + g(x) and f(x) – g(x) are non-differentiable at x = a.

For example, consider, f(x) = x and g(x) = $|x|$.

Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0. Both the sum function $x + |x|$ and the difference function $x - |x|$ are non-differentiable at x = 0.

(ii) f(x). g(x) is not necessarily non-differentiable at x = a.

We need to find the result by first principles.

For example, consider, f(x) = x^3 and g(x) = sgn(x).

Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0. But the product function

$$f(x)g(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$$

is differentiable at x = 0.

As another example, the product of the functions

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at x = 0 even when f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0.

However, the product of the functions f(x) = x and g(x) = [x] is non-differentiable at x = 1.

(iii) f(x)/g(x) is not necessarily non-differentiable at x = a.

Here also we need to work on the function f(x)/g(x) to get the result.

Let f(x) = $x^2(x^2 - 1)$ and $g(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$.

Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0.

$$f(x)/g(x) = \begin{cases} x^2(x-1), & x \geq 0 \\ x^2(x+1), & x < 0 \end{cases}$$

We can check that f(x)/g(x) is differentiable at x = 0.

Example 1: Discuss the differentiability of f(x) = [x] + | 2x – 1 |.

Solution: Let us study the functions y = [x] and y = | 2x – 1 |. It is a clear that f(x) = [x] is non-differentiable at all integral points and g(x) = |2x – 1| is differentiable for all x ∈ R – {1/2}. The sum of a non-differentiable and a differentiable function is non-differentiable. Hence f(x) + g(x) is non-differentiable at all integral points and x = 1/2.

3. If f(x) and g(x) both are non-differentiable at x = a, then we have the following results.

(i) The functions f(x) + g(x) and f(x) – g(x) are not necessarily non-differentiable at x = a. However, atmost one of f(x) + g(x) or f(x) – g(x) can be differentiable at x = a. That is, both of them cannot be differentiable simultaneously at x = a.

Let us assume that both f(x) + g(x) and f(x) – g(x) are differentiable. Then the sum of functions (f(x) + g(x)) + (f(x) – g(x)) = 2f(x) must be differentiable at x = a, which is wrong as it is given that f(x) is non-differentiable at x = a. Hence our assumption is wrong. So, both the functions cannot be differentiable simultaneously at x = a.

For example, consider, f(x) = [x] and g(x) = {x}.

Here both f(x) and g(x) are non-differentiable at x = 0. The sum function [x] + {x} being equal to x is differentiable at x = 0. The difference function [x] – {x} however is non-differentiable at x = 0.

But this does not mean that one of the functions f(x) + g(x) or f(x) – g(x) must be differentiable. We can have both the functions non-differentiable. For example, if f(x) = 2[x] and g(x) = {x}, then both the functions f(x) + g(x) or f(x) – g(x) are non-differentiable simultaneously at x = 0 .

(ii) f(x). g(x) is not necessarily non-differentiable at x = a.

We need to find the result by applying first principles on the product f(x). g(x).

For example, consider, f(x) = [x] and g(x) = [–x].

Here both f(x) and g(x) are non-differentiable at x = 0 but, the product function [x].[–x] is differentiable at x = 0.

Further, f(x) = [x] and g(x) = {x} are both non-differentiable at x = 0 and, the product function [x].{x} is non-differentiable at x = 0.

Hence, we cannot comment in advance when both the functions are non-differentiable at x = a.

(iii) f(x)/g(x) is not necessarily non-differentiable at x = a.

Here also we need to work on the function f(x)/g(x) to get the result.

Let $f(x) = \begin{cases} x^2 - 1, & x \geq 0 \\ x+1, & x < 0 \end{cases}$ and

$$g(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}.$$

Here both f(x) and g(x) are non-differentiable at x = 0. But we find that f(x)/g(x) is differentiable at x = 0.

Differentiability of Composite Functions

Theorem If f(x) is differentiable at x = a and g(x) is differentiable at x = f(a) then the composite function (gof)(x) is differentiable at x = a.

Note: A function of a function composed of a finite number of differentiable functions is a differentiable function.

For example, f(x) = sin x is differentiable at $x = \frac{\pi}{2}$ and $g(x) = \begin{cases} x^2 - 1, & x \le 1 \\ 2x - 2, & x > 1 \end{cases}$ is differentiable at $x = f(\frac{\pi}{2}) = 1$. Hence the composite function (gof)(x) is differentiable at $x = \frac{\pi}{2}$.

Note:

1. Let a function f(x) be differentiable at all points in the interval [a, b], and let its range be the interval [A, B] and further a function g(x) is differentiable in the interval [A, B], then the composite function (gof)(x) is differentiable in the interval [a, b].
2. If the function f is differentiable everywhere and the function g is differentiable everywhere, then the composition gof is differentiable everywhere.
3. If f(x) is differentiable, then $|(f(x))^p|$, p > 1 is also differentiable.

   For example, $f(x) = \frac{\sin x}{x^2 + 1}$ and $g(x) = x|x|$ are differentiable for all x. Hence, the composite function $(gof)(x) = \frac{\sin x}{x^2+1}\left|\frac{\sin x}{x^2+1}\right|$ is also differentiable for all x.

Example 2: Check the differentiability of

$$f(x) = \frac{x^3}{\sin(\cos x)}$$

Solution: The numerator is differentiable for all x. As far as the denominator is concerned, according to the theorem on differentiability of a composite function, it is differentiable at points where the function u = cos x is differentiable, since the function sin u is differentiable everywhere.

We must exclude the points at which sin(cos x) = 0. i.e. the points at which cos x = kπ (k ∈ I), or cos x = 0. Thus, the function f(x) is differentiable everywhere except at the points x = (2n + 1)π/2 (n ∈ I).

Example 3: What can you say about the differentiability of the function $f(x) = \sqrt{9 - x^2}$?

Solution: Because the natural domain of this function is the closed interval [–3, 3], we will need to investigate the differentiability of f on the open interval (–3, 3) and at the two endpoints. If c is any number in the interval (–3, 3), then f is differentiable at c. The function f is non-differentiable at the endpoints since

$$f'(3^-) = \lim_{h\to 0}\frac{\sqrt{9-(3-h)^2} - 0}{-h}$$

$$= \lim_{h\to 0}\frac{\sqrt{6h - h^2}}{-h} = -\infty$$

$$f'(-3^+) = \lim_{h\to 0}\frac{\sqrt{9-(-3+h)^2} - 0}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{6h - h^2}}{h} = \infty$$

Thus, f is differentiable on the open interval (–3, 3).

Note: The nth-root function $f(x) = \sqrt[n]{x}$ is non-differentiable at x = 0.

We may combine this result with the previous theorem. Then we see that a root of a differentiable function is differentiable except possibly at the points where the given function is zero. That is, the composition

$$h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$$

of $f(x) = \sqrt[n]{x}$ and the differentiable function g(x) is differentiable at x = a if g(a) ≠ 0.

Example 4: Show that the function

$$f(x) = \left(\frac{x+1}{x^2 + 2x + 2}\right)^{2/3}$$

is differentiable everywhere except at one point.

Solution: Note first that the denominator

$x^2 + 2x + 2 = (x + 1)^2 + 1$ is never zero.

Hence the rational function

$$r(x) = \frac{x+1}{x^2 + 2x + 2}$$

is defined and differentiable everywhere. It then follows from the theorem and the differentiability of the cube root function that $f(x) = [r(x)]^{2/3} = \sqrt[3]{[r(x)]^2}$ is differentiable everywhere except when r(x) = 0.

Here r(x) = 0 at x = – 1.

$$f'(-1^-) = \lim_{h\to 0}\frac{\left(\frac{-h}{h^2+1}\right)^{2/3} - 0}{-h} = -\infty$$

$$f'(-1^+) = \lim_{h\to 0}\frac{\left(\frac{h}{h^2+1}\right)^{2/3} - 0}{h} = \infty$$

Hence, f is differentiable everywhere except at x = – 1.

Example 5: Let f(x) = [sin x] + [cos x], x ∈ [0, 2π], where [.] denotes the greatest integer function.
Find the number of points where f(x) is non-differentiable.

Solution: [sin x] is non-differentiable at $x = \frac{\pi}{2}, \pi, 2\pi$ and [cos x] is non-differentiable at $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$.

Thus, f(x) is definitely non-differentiable at

$$x = 0, \pi, \frac{3\pi}{2}.$$

We need to check at $x = \frac{\pi}{2}$ and 2π, since both the functions are non-differentiable at these points.

$$f\left(\frac{\pi}{2}\right) = 1,\ f\left(\frac{\pi}{2} - 0\right) = 0$$

and, $f(2\pi) = 1$, $f(2\pi - 0) = -1$.

Thus, f(x) is discontinuous at $x = \frac{\pi}{2}$ and 2π and hence non-differentiable at these points.

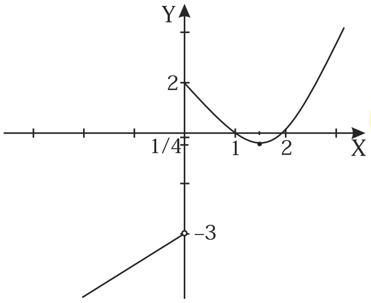
Finally, f is non-differentiable at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

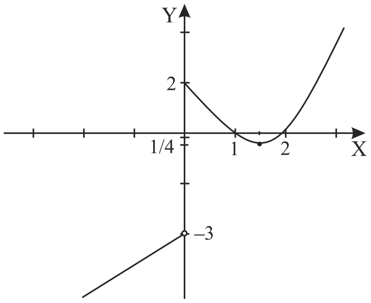
i.e. at 5 points.

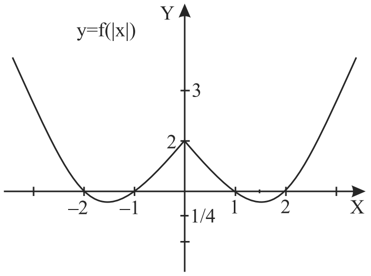
Example 6: If $f(x) = \begin{cases} x-3 & x<0 \\ x^2-3x+2 & x \ge 0 \end{cases}$

and $g(x) = f(|x|) + |f(x)|$, then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)|.

Solution: Graph of y = f(x)







Continuity of g(x):

g(x) is continuous at all those points where both f(|x|) and |f(x)| are continuous.

At x = 0, f(|x|) is continuous but |f(x)| is discontinuous.

$\therefore$ g(x) is discontinuous at x = 0 and hence non-differentiable.

At x = 1, f(|x|) is differentiable but |f(x)| is non-differentiable because of corner. Hence, g(x) is non-differentiable at x = 1.

Similarly g(x) is non differentiable at x = 2.

Finally, g(x) is discontinuous at x = 0 and it is non differentiable at x = 0, 1, 2.

Theorem Let f(x) and g(x) be defined on an open interval containing the point x = a where f is differentiable at a, f(a) = 0 and g is continuous at a, then the product f.g is differentiable at a.

$$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - 0.g(a)}{h} \quad \text{[since f(a) = 0]}$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.g(a+h)$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.\lim_{h\to 0} g(a+h) = f'(a).\ g(a)$$

As h approaches zero, g(a+h) approaches g(a) because g is continuous at a.

Therefore, f(x). g(x) is differentiable at x = a.

For example, f(x) = sin x.(|x – π| + 1) is differentiable at x = π, since sin x is differentiable at x = π, sin π = 0 and (|x – π| + 1) is continuous at x = π.

Corollary. If f(x) is differentiable at x = a, then the product f(x).|f(x)| is also differentiable at x = a.

For example, f(x) = sin x|sin x| is differentiable at x = 0, since sin x is differentiable at x = 0.

Example 7: Let $f(x) = e^{(x-1)} - ax^2 + b$ and

$$g(x) = \begin{cases} e^{x-1}, & x \le 1 \\ x^2, & x > 1 \end{cases} \text{ with } f'(1) = 2. \text{ Find the values}$$

of a and b so that the function $h(x) = f(x) \cdot g(x)$, is differentiable at x = 1.

Solution: Note that g is continuous at x = 1, but g′(1) does not exist.

Also $f'(x) = e^{x-1} - 2ax$

$$f'(1) = 1 - 2a = 2 \Rightarrow a = -\frac{1}{2}$$

In such a situation, for $h(x) = f(x) \cdot g(x)$ to be differentiable at x = 1, f(1) should be 0, using the theorem given above.

$$f(1) = 0 \Rightarrow 1 - a + b = 0$$

$$\Rightarrow \quad b = -\frac{3}{2}.$$

Concept Problems F

1. If it reasonable to assert that the sum $F(x) = f(x) + g(x)$ has no derivative at the point $x = x_0$ if :
   (a) The function $f(x)$ has a derivative at the point x_0, and the function $g(x)$ has no derivative at this point?
   (b) Neither function has a derivative at the point x_0?

2. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composite functions $(f \circ g)(x) = |x|^2 = x^2$ and $(g \circ f)(x) = |x^2| = x^2$ are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the chain rule? Explain.

3. Consider the functions:
   (i) $f(x) = x$, $g(x) = |x|$;
   (ii) $f(x) = |x|$, $g(x) = |x|$. Is it reasonable to assert that the product $F(x) = f(x)\,g(x)$ has no derivative at the point $x = x_0$ if :
   (a) The function $f(x)$ has a derivative at the point x_0, and the function $g(x)$ has no derivative at this point ?
   (b) Neither function has a derivative at the point x_0?

4. Find the one-sided derivatives of the function $f(x) = |x - x_0|\, g(x)$ at the point x_0, where $g(x)$ is a function continuous at the point x_0. Does the function $f(x)$ possess a derivative at the point x_0?

5. If $f(x)$ is differentiable at $x = a$, then prove that $F(x) = (f(x) - f(a)).|x - a|$ is also differentiable at $x = a$.

Practice Problems E

6. If $f(x) = \tan \pi x$ and $g(x) = \left|x - 1\right|$. Is the product function $F(x) = f(x) . g(x)$ differentiable at $x = 1$?

7. If $f(x) = = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and $g(x) = \frac{1}{x}$. Is the product function $F(x) = f(x) . g(x)$ differentiable for all real x ?

8. Let $g(x)$ be derivable on R and $g(x_0) \neq 0$.
   (a) Show that if $f(x) = (x - x_0)^2\, g(x)$, then
   $f(x_0) = f'(x_0) = 0$ and $f''(x_0) \neq 0$.
   (b What can you say about $f(x) = (x - x_0)^3\, g(x)$?
   (c) Generalize to $f(x) = (x - x_0)^n\, g(x)$.

9. Suppose $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

10. Let $f(x) = \begin{cases} 1 - x^2 & , \; x < 0 \\ 2x + 1 & , \; 0 \le x < 1 \\ x^2 + 2 & , \; x \ge 1 \end{cases}$ without finding (fof) (x) explicitly, find whether fof is differentiable at $x = 0$.

11. Let $f(x) = \begin{cases} 1 - x & 0 \le x \le 1 \\ x + 2 & 1 < x < 2 \\ 4 - x & 2 \le x \le 4 \end{cases}$, Discuss the continuity and differentiability of fof

12. Let $f(x) = \begin{cases} x - 1, & -1 \le x < 0 \\ x^2 \;, & 0 \le x \le 1 \end{cases}$, Discuss the continuity and differentiability of $h(x) = f(|\sin x|) + |f(\sin x)|$ in $[0, 2\pi]$.

13. Let $f(x) = x^p \dfrac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}}$ when $x \neq 0$ and $f(0) = 0$, with p, $a > 0$. Find all possible values of p so that $f(x)$ is differentiable at $x = 0$.

14. Discuss the differentiability of the function$f(x) = (x^2 - a)\,|x^2 - 5x + 6| + (\sin x)\,|\sin x|$ for all $a \in R$.

3.10 Functional Equations

We follow the following steps to determine the functions which are differentiable(or which can be proved to be differentiable) and satisfying a given functional rule :

(i) First we write down the expression for $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(ii) We manipulate $f(x+h) - f(x)$ in such a way that the given functional rule is applicable. Now we apply the functional rule and simplify the R.H.S. to get $f'(x)$ as a function of x.

(iii) Then we integrate $f'(x)$ to get $f(x)$ as a function of x and a constant of integration. In some cases a differential equation in formed which can be solved to get $f(x)$.

(iv) Finally we apply the boundary conditions to determine the value of the constant of integration.

Example 1: If $f(x + y) = f(x) \cdot f(y)$, $\forall\; x, y \in R$ and $f(x)$ is a differentiable function, then find $f(x)$.

Solution: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x)\frac{[f(h)-1]}{h} \qquad ...(1)$$

Put $x = 0, y = 0;\ f(0) = f^2(0)$

$\Rightarrow\quad f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$ then put $y = 0$ in the rule

$f(x) = f(x)\cdot f(0) = 0$

$\Rightarrow\quad f(x) = 0.\ f(x)$

$\Rightarrow\quad f(x) = 0.$

If $f(0) = 1$, then we proceed from (1) as follows :

$$f'(x) = f(x)\frac{[f(h)-f(0)]}{h} = f(x)\cdot f'(0)$$

[Let $k = f'(0)$]

$$\Rightarrow\quad \frac{f'(x)}{f(x)} = k$$

$\Rightarrow\quad \ln|f(x)| = kx + c$

$|f(x)| = e^{kx+c} = e^{kx}\cdot e^{c}$

$f(x) = \pm\, e^{kx}\cdot e^{c}$

$f(x) = a.e^{kx}$ where $a = \pm\, e^{c}$

If $x = 0,\ f(0) = 1 \Rightarrow a = 1$

$\Rightarrow f(x) = e^{kx}$ where $k = f'(0)$.

Hence, the functions are $f(x) = 0$ and $f(x) = e^{kx}$.

Example 2: If $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$ and $f(x)$ is a differentiable function, then prove that

$f(kx) = k\, f(x)$ for $\forall\ k, x \in R$.

Solution: Given $f(x + y) = f(x) + f(y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0). \qquad [\text{as } f(0) = 0]$$

$\Rightarrow\quad f'(x) = f'(0)$

$\therefore\quad f(x) = f'(0)x + c$

If $\quad x = 0,\ f(0) = 0 \Rightarrow c = 0$

$\therefore\quad f(x) = f'(0)x$

$f(x) = ax,$ where $a = f'(0)$

$f(kx) = a\,kx = k\,ax = kf(x)$

Hence, $f(kx) = kf(x)$.

Example 3: A function $f : (0, \infty) \to R$ satisfies the equation $f(x/y) = f(x) - f(y)$. If $f(x)$ is differentiable on $(0, \infty)$ and $\lim\limits_{x\to 0}\dfrac{f(1+x)}{x} = 3$, then determine $f(x)$.

Solution: We have $f(x/y) = f(x) - f(y)$ ...(1)

Putting $x = 1$ and $y = 1$ in equation (1), we have

$f(1) = f(1) - f(1) \Rightarrow f(1) = 0.$

We have $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} \qquad [\text{from } (1)]$$

$$= \lim_{h\to 0}\frac{1}{x}.\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x}.\lim_{\delta\to 0}\frac{f(1+\delta)}{\delta}$$

$$\left[\because \frac{h}{x}\to 0 \text{ as } x\to 0\ \forall\ x\in(0,\infty)\right]$$

$$= \frac{3}{x} \qquad \left[\text{from } \lim_{x\to 0}\frac{f(1+x)}{x} = 3\right]$$

To find $f(x)$, write the above equation as

$$\frac{df}{dx} = \frac{3}{x}$$

On integrating both sides w.r.t. x we get

$f(x) = 3\ln x + c$, where c is a constant.

Now, using the condition $f(1) = 0$, we have

$f(1) = c$ which gives $c = 0$

Hence, we have $f(x) = 3\ln x$.

Example 4: A differentiable function satisfies the relation $f(x + y) = f(x) + f(y) + 2xy - 1\ \ \forall\ x, y \in R$.

If $f'(0) = \sqrt{3+a-a^2}$, find $f(x)$ and prove that $f(x) > 0\ \forall\ x \in R$.

Solution: $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+2xh-1-f(x)}{h}$$

$$= 2x + \lim_{h\to 0}\frac{f(h)-1}{h} \quad [\text{Put } x = 0, y = 0 \text{ to get } f(0) = 1]$$

$$= 2x + \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$\Rightarrow\quad f'(x) = 2x + f'(0)$

Integrating, $f(x) = x^2 + f'(0)x + c$

If $x = 0$; $f(0) = 1 \Rightarrow c = 1$

$\therefore\quad f(x) = x^2 + \left(\sqrt{3+a-a^2}\right)x + 1$

Now $D = 3 + a - a^2 - 4 = -(a^2 - a + 1) < 0$

$\Rightarrow\quad f(x) > 0\ \ \forall\ x \in R.$

Example 5: Given a function g which has derivative $g'(x)$ for all x satisfying $g'(0) = 2$ and $g(x + y) = e^{y}\, g(x) + e^{x}\, g(y)$ for all $x, y \in R$, $g(5) = 32$. Find the value of $g'(5) - 2e^{5}$.

Solution: Putting $x = y = 0$ in

$g(x + y) = e^{y}\, g(x) + e^{x}\, g(y)$ we get $g(0) = 2g(0)$

$\Rightarrow\quad g(0) = 0$

So $2 = g'(0) = \lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0}\frac{g(h)}{h}$

Also $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$$= \lim_{h\to 0}\frac{e^x g(h)+e^h g(x)-g(x)}{h}$$

$$= \lim_{h\to 0}\left(e^x\frac{g(h)}{h}+\frac{e^h-1}{h}g(x)\right)$$

$$= e^x\lim_{h\to 0}\frac{g(h)}{h}+1.\,g(x)$$

$= g(x) + 2e^x$.

Thus, $g'(5) - 2e^5 = g(5) = 32$.

Example 6: A differentiable function f(x) satisfies the condition $f(x+y) = f(x) + f(y) + xy$ for all $x, y \in R$ and $\lim_{h\to 0}\frac{1}{h}f(h) = 3$ then find the least value of f(x).

Solution: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+hx-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h} + x = 3 + x$$

Integrating $f(x) = 3x + \frac{x^2}{2} + k$ ...(1)

Putting x = 0, y = 0 in the given relation

$f(0) = f(0) + f(0) + 0 \Rightarrow f(0) = 0$

Now from (1) we have $f(0) = 0 + k \Rightarrow k = 0$

$\therefore \quad f(x) = 3x + \frac{x^2}{2}$.

It is a quadratic function whose least value occurs at $x = -3$. The least value is $-\frac{9}{2}$.

Example 7: Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0) = -1$ and $f(0) = 1$, then find f(2).

Solution: Given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Replacing x by 2x and y by 0, then $f(x) = \frac{f(2x)+f(0)}{2}$

$\Rightarrow \quad f(2x) + f(0) = 2f(x) \Rightarrow f(2x) - 2f(x) = -f(0)$ ...(1)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\left\{\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{f(2x)+f(2h)-2f(x)}{2h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{f(2h)-f(0)}{2h}\right\} \qquad \text{(from (1))}$$

$= f'(0)$

$= -1 \ \forall\ x \in R$ (given)

Integrating, we get $f(x) = -x + c$

Putting x = 0, then $f(0) = 0 + c = 1$ (given)

$\therefore \quad c = 1$ then $f(x) = 1 - x$

$\therefore \quad f(2) = 1 - 2 = -1$.

Alternative 1:

$\because \quad f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Differentiating both sides w.r.t x treating y as constant.

$\therefore \quad f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2} = \frac{f'(x)+0}{2} \Rightarrow f'\left(\frac{x+y}{2}\right) = f'(x)$.

Replacing x by 0 and y by 2x,

then $f'(x) = f'(0) = -1$ (given)

Integrating, we have $f(x) = -x + c$.

Putting x = 0, $f(x) = 0 + c = 1$ (given)

$\therefore \quad c = 1$

Hence, $f(x) = -x + 1$ then $f(2) = -2 + 1 = -1$.

Alternative 2:

Suppose A(x, f(x)) and B(y, f(y)) be any two points on the curve y = f(x).

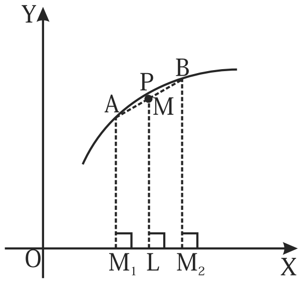
If M is the mid point of AB, then coordinates of M are

$$\left(\frac{x+y}{2}\,.\,\frac{f(x)+f(y)}{2}\right)$$

According to the graph, coordinates of P are

$$\left(\frac{x+y}{2}\,.\,f\left(\frac{x+y}{2}\right)\right)$$

and PL > ML $\Rightarrow \left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$



But given $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ which is possible when P → M, i.e., P lies on AB. Hence, y = f(x) must be a linear function.

Let $f(x) = ax + b \Rightarrow f(0) = 0 + b = 1$ (given)
and $f'(x) = a \Rightarrow f'(0) = a = -1$ (given)
$\therefore\ f(x) = -x + 1$
$\therefore\ f(2) = -2 + 1 = -1.$

Example 8: If $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$ then determine y = f(x).

Solution: $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ ...(1)

Replacing x by 3x and y by 0 then

$$f(x) = \frac{2+f(3x)+f(0)}{3}$$

$\Rightarrow\ f(3x) - 3f(x) + 2 = -f(0)$ ...(2)
Putting x = 0 and y = 0 in (1),
we get f(0) = 2 ...(3)

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3}-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(3x)-3f(x)+f(3h)+2}{3h}$$

$$= \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} \quad \{\text{from } 2)\}$$

$= f'(0) = c$ (say)
$\therefore\ f'(x) = c$
At $x = 2, f'(2) = c = 2$ (given)
$\therefore\ f'(x) = 2$

Integrating both sides, we get f(x) = 2x + a
Putting x = 0 then
$f(0) = 0 + a = 2$ {from (2)}
$\therefore\ a = 2$
then f(x) = 2x + 2.

Alternative:

We have $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$

Differentiating both sides w.r.t. x treating y as constant,

we get $f'\left(\frac{x+y}{3}\right)\left(\frac{1}{3}\right)=\frac{2+f'(x)+0}{3}$

Now replacing x by 0 and y by 3x, then

$f'(x) = f'(0) = c$ (say)
At $x = 2,$
$f'(2) = c = 2$ (given)
$\therefore\ f'(x) = 2$
On integrating we get
$f(x) = 2x + a$
Putting x = 0, then f(0) = 0 + a = 2 {from (1)}
$\therefore\ f(x) = 2x + 2.$

Example 9: Let f(xy) = xf(y) + yf(x) for all $x, y \in R^+$ and f(x) be differentiable in $(0, \infty)$ then determine f(x).

Solution: Given f(xy)= xf(y) + yf(x)
Replacing x by 1 and y by x then we get x f(1) = 0
$\therefore\ f(1) = 0, x \neq 0$ $(\because\ x, y, \in R^+)$

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)}+\lim_{h\to 0}\frac{f(x)}{x}$$

$$= f'(1) + \frac{f(x)}{x}$$

$$\Rightarrow\ f'(x) - \frac{f(x)}{x} = f'(1)$$

$$\Rightarrow\ \frac{xf'(x)-f(x)}{x^2} = \frac{f'(1)}{x}$$

$$\Rightarrow\ \frac{f(x)}{x}\left\{\frac{f(x)}{x}\right\} = \frac{f'(1)}{x}$$

On integrating w.r.t.x and taking limit 1 to x,

we have $\frac{f(x)}{x}-\frac{f(1)}{1} = f'(1)(\ln x - \ln 1)$

$\Rightarrow\ \frac{f(x)}{x} - 0 = f'(1)\ln x$ $(\because\ f(1) = 0)$

$\therefore\ f(x) = f'(1)(x \ln x).$

Alternative:

Given f(xy) = xf(y) + yf(x)
Differentiating both sides w.r.t. x treating y as constant,
$f'(xy)\,.\,y = f(y) + y f'(x)$

Putting y = x and x = 1, then

$f'(xy).\ x = f(x) + xf'(x)$

$$\Rightarrow \quad \frac{xf'(x) - f(x)}{x^2} = \frac{f'(1)}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(1)}{x}$$

Integrating both sides w.r.t. x taking limit 1 to x,

$$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\}$$

$$\Rightarrow \quad \frac{f(x)}{x} - 0 = f'(1)\ln x \qquad (\because\ f(1) = 0)$$

Hence, $f(x) = f'(1)(x \ln x)$.

Example 10: If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ $(xy < 1)$ and $\lim\limits_{x\to 0}\frac{f(x)}{x} = 2$. Find $f\left(\frac{1}{\sqrt{3}}\right)$ and $f'(1)$.

Solution: $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ ...(1)

Putting x = y = 0, we get f(0) = 0.

Putting y = – x, we get f(x) + f(–x) = f(0)

$\Rightarrow \quad f(-x) = -f(x)$ ...(2)

Also, $\lim\limits_{x\to 0}\frac{f(x)}{x} = 2$

Now, $(x) = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$ ...(3)

$= \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$ (using (2))

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h} \qquad \text{(using (1))}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h}\right)}{h}\right]$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \left(\frac{1}{1+xh+x^2}\right)$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \lim_{h\to 0}\frac{1}{1+xh+x^2}$$

$$\left(\text{using } \lim_{x\to 0}\frac{f(x)}{x} = 2\right)$$

$$\Rightarrow \quad f'(x) = 2 \times \frac{1}{1+x^2} \qquad \Rightarrow f'(x) = \frac{2}{1+x^2}$$

Integrating both sides, we get

$f(x) = 2\tan^{-1}(x) + ck$, where $f(0) = 0 \Rightarrow c = 0$

Thus, $f(x) = 2\tan^{-1} x$.

Hence, $f\left(\frac{1}{\sqrt{3}}\right) = 2\ \frac{\pi}{6} = \frac{\pi}{3}$, and

$$f'(1) = \frac{2}{1+1^2} = \frac{2}{2} = 1.$$

Example 11: If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)$ ∀ x, $y \in R^+$, and $f'(1) = e$, determine f(x).

Solution: Given $e^{-xy} f(xy) = e^{-x}f(x)\ e^{-y}f(y)$ ...(1)

Putting x = y = 1 in (1), we get, f(1) = 0 ...(2)

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x.1)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h}\left\{e^{-x}f(x) + e^{-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\} - 2^{x}(e^{-x}f(x) + e^{-1}f(1))}{h}$$

$$= \lim_{h\to 0}\frac{e^{h}f(x) + e^{x+h-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right) - f(x) - e^{x-1}f(1)}{h}$$

$$= f(x)\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right) + e^{(x-1)}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}}$$

$(\because\ f(1) = 0)$

$$= f(x)\ .\ 1 + e^{x-1}.\ \frac{f'(1)}{x}$$

$$= f(x) + \frac{e^{x-1}.e}{x} \qquad (\because\ f'(1) = e)$$

$$f'(x) = f(x) + \frac{e^x}{x} \quad \Rightarrow \quad e^{-x}f'(x) - e^{-x}f(x) = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}(e^{-x}f(x)) = \frac{1}{x}$$

On integrating we have

$e^{-x}f(x) = \ln x + c$ at x = 1, c = 0

∴ $f(x) = e^x \ln x$.

Example 12: A function $f : (-1, 1) \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation

$$f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2}).$$

(i) Show that f(x) is odd.

(ii) If f(x) is differentiable on (–1, 1) and $f'(0) = 1$, then show that $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

(iii) Hence, determine f(x).

Solution:

(i) $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$. ...(1)

Putting y = –x in equation (1), we have

$$f(x) + f(-x) = f(x\sqrt{1-x^2} - x\sqrt{1-x^2})$$

$\Rightarrow$ $f(x) + f(-x) = f(0)$

Putting x = 0 and y = 0 in equation (1), we have

$f(0) + f(0) = f(0) \Rightarrow f(0) = 0.$

Hence, $f(x) + f(-x) = 0$ ...(2)

Thus f(x) is odd.

(ii) Now, $f'(0) = 1$

$$\Rightarrow \lim_{\delta\to 0}\frac{f(\delta)-f(0)}{\delta} = 1 \qquad [\text{using } f(0) = 0]$$

$$\Rightarrow \lim_{\delta\to 0}\frac{f(\delta)}{\delta} = 1$$

$$\text{Now } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)+f(-x)}{h} \qquad [\text{using}(2)]$$

$$= \lim_{h\to 0}\frac{f\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\}}{h}$$

[using (1)]

$$= \lim_{\delta\to 0}\frac{f(\delta)}{\delta}.\lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h}$$

assuming $\delta = (x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}$

$$= \lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h}$$

$$= \lim_{h\to 0}\frac{x(\sqrt{1-x^2} - x\sqrt{1-(x+h)^2})}{h} + \frac{h\sqrt{1-x^2}}{h}$$

$$= \sqrt{1-x^2} + \lim_{h\to 0}\frac{x(1-x^2-1+(x+h)^2)}{h(\sqrt{1-x^2} + \sqrt{1-(x+h)^2})}$$

$$= \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{Hence, } f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad ...(3)$$

(iii) Integrating both sides of (3) gives $f(x) = \sin^{-1}x + c$ (c is a constant).

Using the condition f(0) = 0 gives c = 0, and hence we have $f(x) = \sin^{-1}x$.

Example 13: A differentiable function f satisfies the relation $f(x + y) - 2f(x - y) + f(x) - 2f(y) = y - 2$ $\forall$ x , y $\in$ R. Find f (x).

Solution: $f(x + y) - 2f(x - y) + f(x) - 2f(y) = y - 2$ ...(1)

Put x = y = 0 in (1)

$\Rightarrow$ f(0) = 1

Put y = h in (1)

$f(x + h) - 2f(x - h) + f(x) - 2f(x) = h - 2$

$\Rightarrow f(x + h) - f(x) - 2[(f(x - h) - f(x)] - 2(f(h) - 1) = h$

Dividing by h on both sides and applying limit h$\to$ 0

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \frac{2(f(x-h)-f(x))}{-h} - \left(\frac{2f(h)-1}{h}\right) = 1$$

$\Rightarrow$ $f'(x) + 2f'(x) - 2f'(0) = 1$ [$\because$ f is differentiable]

$\Rightarrow$ $3f'(x) = 1 + 2f'(0)$ ...(2)

Putting x = 0 in (2) we get $f'(0) = 1$.

$\Rightarrow$ $f'(x) = 1$

$\therefore$ f(x) = x + c

Since f(0) = 1, c = 1, we have f(x) = x + 1.

Example 14: A differentiable function f satisfies $f(x + y) + f(x - y) - (y + 2) f(x) + y(x^2 - 2y) = 0$ $\forall$ x , y $\in$ R. Find f (x).

Solution: Put y = h

$f(x + h) - f(x) + f(x - h) - f(x) - hf(x) + h(x^2 - 2h) = 0$

Dividing by h on both sides

$$\frac{f(x+h)-f(x)}{h} + \frac{f(x-h)-f(x)}{h} - f(x) + x^2 - 2h = 0 \text{ and}$$

applying limit h$\to$ 0 on both sides, we get

$f'(x) - f'(x) - f(x) + x^2 = 0$

$\Rightarrow$ $f(x) = x^2$.

Example 15: A twice differentiable function f satisfies the relation $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$ $\forall$ x, y $\in$ R. If f(0) = 0 and $f''(0) = 2$, find f(x).

Solution: $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$

Put y = h

$f(x^2 + h^2) = f(x^2 - h^2) + f(2xh)$

Dividing by h^2 on both sides

$$\frac{f(x^2+h^2)-f(x^2)}{h^2} = \frac{f(x^2-h^2)-f(x^2)}{h^2} + \frac{f(2xh)}{h^2} \qquad ...(1)$$

Now $\lim\limits_{h\to 0}\dfrac{f(2xh)}{h^2} = \lim\limits_{h\to 0}\dfrac{f'(2xh).2x}{2h}$
(using L′Hospital 's rule)

$= \lim\limits_{h\to 0}\dfrac{xf''(2xh).2x}{1}$

$= 2x^2f''(0).$

Applying limit $h \to 0$ on both sides of (1), we get

$f'(x^2) = -f'(x^2) + 2x^2f''(0)$

$2f'(x^2) = 2x^2f''(0)$

Replacing x^2 by t

$2f'(t) = 2t\,f''(0) \quad \Rightarrow \quad f'(t) = 2t$

Integrating both sides w.r.t. t

$\int f'(t)\,dt = \int 2t\ dt$

$f(t) = t^2 + c$

Since $f(0) = 0$, $c = 0$.

$\therefore \quad f(t) = t^2$ or, $f(x) = x^2$.

Example 16: f(x) is a differentiable function satisfy the relationship $f^2(x) + f^2(y) + 2(xy - 1) = f^2(x + y)$ $\forall$ x, $y \in R$. Also $f(x) > 0$ $\forall$ $x \in R$, and $f(\sqrt{2}) = 2$. Determine f (x).

Solution: Put $x = 0$ and $y = 0 \Rightarrow f^2(0) = 2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{[f(x+h)+f(x)].h}$$

$$= \lim_{h\to 0}\frac{f^2(h)+2(xh-1)}{2f(x)h}$$

$$= \frac{1}{2f(x)}\lim_{h\to 0}\left[\frac{2xh}{h}+\frac{f^2(h)-2}{h}\right]$$

$$= \frac{1}{2f(x)}\left[2x+\lim_{h\to 0}\frac{f^2(h)-f^2(0)}{h}\right]$$

$$= \frac{1}{2f(x)}\left[2x+\lim_{h\to 0}\frac{f(h)-f(0)}{h}.(f(h)+f(0))\right]$$

$$f'(x) = \frac{1}{\cancel{2}f(x)}[\cancel{2}x+\cancel{2}f(0).f'(0)]$$

$\therefore \quad f(x).f'(x) = x + f(0).f'(0)$

$\Rightarrow \quad f(x).f'(x) = x + \lambda$, where $\lambda = f(0)\cdot f'(0)$

Integrating both sides,

$$\frac{f^2(x)}{2} = \frac{x^2}{2} + \lambda x + c$$

$f^2(x) = x^2 + 2\lambda x + c$

at $x = 0$, $f^2(0) = 2 \quad \Rightarrow \quad c = 2$

at $x = \sqrt{2}$, $f^2(\sqrt{2}) = 4 \quad \Rightarrow \lambda = 0$

$\therefore \quad f^2(x) = x^2 + 2$

$\Rightarrow \quad f(x) = \sqrt{x^2+2} \qquad (\text{as } f(x) > 0)$

Example 17: function $f: (0, \infty) \to R$ satisfies the equation $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$. If f is differentiable on R^+ and $f(1) = 0$, $f'(1) = 1$, then show that

(i) $f(y) = -f\left(\frac{1}{y}\right)$ (ii) $f(x) + f(y) = f\left(\frac{x}{y}\right)$

and hence determine f(x).

Solution: We have $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$ ...(1)

(i) Putting x = 1 in (1), we have

$2f(1) = f(y) - \left(\frac{1}{y}\right)$

$\Rightarrow \quad f(y) = -f\left(\frac{1}{y}\right)$ [since f(1) = 0] ...(2)

(ii) In (1) we exchange x and y to get

$f(xy) = 2f(y) - f\left(\frac{y}{x}\right)$

$\Rightarrow \quad f(xy) = 2f(y) + f\left(\frac{x}{y}\right)$ [using (2)] ...(3)

Now we subtract (1) from (3)

$0 = 2f(y) - 2f(x) + 2f\left(\frac{x}{y}\right)$

$\Rightarrow \quad f(x) - f(y) = f\left(\frac{x}{y}\right)$ ...(4)

We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} \qquad \text{[using (4)]}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}} = \frac{f'(1)}{x}$$

i.e. $f'(x) = \dfrac{1}{x}$

This gives $f(x) = \ln x + c$

Now, using the condition $f(1) = 0$ gives $c = 0$.

Hence, we have $f(x) = \ln x$.

Example 18: A differentiable function f satisfies the relation $f(x + y) + f(xy - 1) = f(x) + f(y) + f(xy)$ $\forall$ x, $y \in R$. If $f(1) = 2$, $f'(0) = 1$ and $f'(-1) = -1$, find f(x).

Solution: Putting x = 0, y = 0 in the given rule,

$f(0) + f(-1) = 3f(0)$

$\Rightarrow \quad f(-1) = 2f(0)$ ...(1)

Now, putting y = h

$f(x + h) + f(xh - 1) = f(x) + f(h) + f(xh)$

$\Rightarrow\quad f(x+h) - f(x) + f(-1+xh) - f(-1)$
$\quad = f(h) + f(xh) - f(-1)$

$\Rightarrow\quad f(x+h) - f(x) + f(-1+xh) - f(-1)$
$\quad = f(h) - f(0) + f(xh) - f(0)$ [using(1)]

Dividing both sides by h,

$$\frac{f(x+h)-f(x)}{h} + \frac{f(-1+xh)-f(-1)}{xh}.x$$

$$= \frac{f(h)-f(0)}{h} + \frac{f(xh)-f(0)}{xh}.x$$

Now, applying limit $h \to 0$ on both sides , we get

$f'(x) + xf'(-1) = f'(0) + x f'(0)$

Given that $f'(0) = 1$, $f'(-1) = -1$, we have

$f'(x) = 2x + 1$

$f(x) = x^2 + x + c$

$f(1) = 2 \Rightarrow c = 0$

$\therefore\ f(x) = x^2 + x.$

Example 19: Let $f : R \to R$ be such that for all x and y in R, $|f(x) - f(y)| \le |x - y|^3$. Prove that f(x) is a constant function.

Solution: We are given that

$|f(x) - f(y)| \le |x - y|^3$ ...(1)

Let x be any real number and let y be chosen arbitrarily close to x but not equal to x. Then writing (1) as

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$$

and letting $y \to x$, we get

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2 \qquad ...(2)$$

Since $\lim\limits_{y\to x}\dfrac{f(x)-f(y)}{x-y} = f'(x)$, we see from (2) that

$|f'(x)| \le 0 \quad \Rightarrow \quad f'(x) = 0$

$\Rightarrow\ f(x) = c.$

Hence f(x) is a constant function.

Example 20: Suppose

$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n.$

If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that

$|a_1 + 2a_2 + ... + na_n| \le 1.$

Solution: Given $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

$\therefore\quad p'(x) = 0 + a_1 + 2a_2x + ... + na_nx^{n-1}$

$\Rightarrow\quad p'(1) = a_1 + 2a_2 + ... + na_n$

Now, $|p(1)| \le |e^{1-1} - 1| = |e^0 - 1| = 0$

$\Rightarrow\quad |p(1)| \le 0 \Rightarrow p(1) = 0$

As $\quad |p(x)| \le |e^{x-1} - 1|$

we get $|p(1+h)| \le |e^h - 1|\ \forall\ h > -1, h \ne 0$

$\Rightarrow\quad |p(1+h) - p(1)| \le |e^h - 1| \qquad (\because p(1) = 0)$

$$\Rightarrow\quad \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{p^h-1}{h}\right|$$

Taking limit as $h \to 0$ on both sides, then

$$\Rightarrow\quad \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h-1}{h}\right|$$

$$\Rightarrow\quad \left|\lim_{h\to 0}\frac{p(1+h)-p(1)}{h}\right| \le \left|\lim_{h\to 0}\frac{e^h-1}{h}\right|$$

$\Rightarrow\quad |p'(1)| \le 1$

$\Rightarrow\quad |a_1 + 2a_2 + ... + na_n| \le 1 \qquad$ {from (1)}

Practice Problems F

1. A function $f : R \to R+$ satisfies $f(x + y) = f(x)$. $f(y)\ \forall\ x \in R$. If $f'(0) = 2$, then show that $f'(x) = 2f(x)$.
2. Let $f(xy) = f(x) f(y)\ \forall\ x, y \in R$ and $f(1) \ne 0$, $f'(1) = 1$, prove that f is differentiable for all $x \ne 0$. Hence determine f(x).
3. Let $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x)+f(y)}{2}$ for real x and y. If $f'(0)$ exists and equals -1 and $f(0) = 1$ then find the value of f(2)
4. If f(x) satisfies $f(1 - x) = f(x)\ \forall\ x \in R$ and $f'(1) = 0$ then find $f'(0)$ if it exists.
5. Let $f\left(\dfrac{x+y}{n}\right) = \dfrac{f(x)+f(y)}{n}\ \forall\ x, y \in R$, $n > 2$ and $f'(0) = 2$ then find f(x).
6. Let $f\left(\dfrac{xy}{2}\right) = \dfrac{f(x)f(y)}{6}$ for all real x and y. If $f(1) = f'(1) = 3$, then prove that one of the functions f satisfies $f(x) + f(1 - x) = 3$ for all non-zero real x.
7. If $2f(x) = f(xy) + f\left(\dfrac{x}{y}\right)\ \forall\ x, y \in R^+$, $f(1) = 0$ and $f'(1) = 1$ then find $f'(2)$ and f(2).
8. Let $f : R \to R$ such that $f(x + y) = f(x) + f(y) + e^{x+y}(x + y) - xe^x - ye^y + 2xy\ \forall\ x, y \in R$ then find f(x) given that $f'(0) = 1$.
9. If $f\left(\dfrac{2x+3y}{5}\right) = \dfrac{2f(x)+3f(y)}{5}\ \forall\ x, y \in R$, $f(0) = 1$, $f'(0) = -1$, prove that

$$\sum_{r=2}^{n+1}(f(r))^3 = -\frac{n^2(n+1)^2}{4}.$$

10. A differentiable function f satisfies the relation f(x + y) [f(x) – f(y)] = f(x –y)[f(x) + f(y)], where f(1) = 2. Find f(x).

Target Problems for JEE Advanced

Problem 1: Check the differentiability of f(x) at x = 0, where

$$f(x) = x\left(\frac{2-e^{|x|+\{x\}}}{|x|+\{x\}}\right), \qquad x \neq 0$$

$$= \frac{1}{2}, \quad x = 0$$

where { } represents the fractional part function.

Solution:

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\left(\frac{2-e^{|h|+\{h\}}}{|h|+\{h\}}\right)-\frac{1}{2}}{h}$$

$$= \lim_{h\to 0}\frac{h\left(\frac{2-e^{2h}}{2h}\right)-\frac{1}{2}}{h}$$

[∵ |h| + {h} = h + h = 2h for small h > 0]

$$= \lim_{h\to 0}\frac{1-e^{2h}}{2h} = -\ln e = -1.$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0) = f(0-h)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2}-(-h)\left(\frac{2-e^{|-h|+\{-h\}}}{|-h|+\{-h\}}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2}+h(2-e)}{h}$$

[∵ |–h| + {–h} = h + 1– h = 1 for small h > 0]

$$= \lim_{h\to 0}\frac{1}{2h} = \infty$$

Hence, f is non-differentiable at x = 0.

Problem 2: Find f ′(0) if f(x) = h(x)cos(1/x), x ≠ 0, f(0) = 0, where h(x) is an even function differentiable at x = 0 and h(0) = 0.

Solution: The existence of h′(0) implies that

$$\lim_{\delta\to 0}\frac{h(\delta)-h(0)}{\delta} = \lim_{\delta\to 0}\frac{h(0)-h(-\delta)}{\delta}$$

$$\Rightarrow \lim_{\delta\to 0}\frac{h(\delta)}{\delta} = \lim_{\delta\to 0}\frac{-h(\delta)}{\delta}$$

$$\Rightarrow 2\lim_{\delta\to 0}\frac{h(\delta)}{\delta} = 0 \qquad \text{...(1)}$$

Now consider f ′(x) at x = 0.

$$\text{R.H.D.} = \lim_{\delta\to 0}\frac{f(\delta)-f(0)}{\delta} = \lim_{\delta\to 0}\frac{h(\delta)\cos(1/\delta)}{\delta}$$

$$= \lim_{\delta\to 0}\frac{h(\delta)}{\delta}.\lim_{\delta\to 0}\cos(1/\delta)$$

$$= 0 \qquad [\text{ using (1)}] \qquad \text{...(2)}$$

$$\text{L.H.D.} = \lim_{\delta\to 0}\frac{f(0)-f(-\delta)}{\delta} = \lim_{\delta\to 0}\frac{-h(-\delta)\cos(-1/\delta)}{\delta}$$

$$= \lim_{\delta\to 0}\frac{-h(\delta)}{\delta}.\lim_{\delta\to 0}\cos(1/\delta)$$

$$= 0 \qquad [\text{ using (1)}] \qquad \text{...(3)}$$

Since R.H.D. = L.H.D. = 0, we have f ′(0) = 0.

Problem 3: If $f(x) = \begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{|x|}\right)}, & x \neq 0, \\ a, & x = 0 \end{cases}$

find the value of ‘a’ such that f(x) is differentiable at x= 0.

Solution:

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h\,e^{-\left(\frac{1}{h}+\frac{1}{|h|}\right)}-a}{h}$$

$$f'(0^+) = \lim_{h\to 0}\frac{h\,e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}-a}{h}$$

$$= \lim_{h\to 0}\frac{h\,.\,e^{-2/h}-a}{h} = 0-\lim_{h\to 0}\frac{a}{h} = 0, \text{ provided } a = 0.$$

$$\text{Also } f'(0^-) = \lim_{h\to 0}\frac{-h\,.\,e^{-\left(-\frac{1}{h}+\frac{1}{h}\right)}-a}{-h}$$

$$= 1 + \lim_{h\to 0}\frac{a}{h} = 1, \text{ provided } a = 0.$$

Hence, a = 0.

Problem 4: Let $f(x) = \begin{cases} |x-3|\,[x], & x \geq 1 \\ \sin\left(\frac{\pi x}{2}\right), & x < 1 \end{cases}$

where [.] denotes the greatest integer function.

Find whether f(x) is

(i) is continuous at x = 0,

(ii) is differentiable at x = 0,

(iii) is continuous but not differentiable at $x = 1$,
(iv) is continuous but not differentiable at $x = 3/2$.

Solution:

(i) $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \sin\left(\frac{-\pi h}{2}\right) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \sin\frac{\pi h}{2} = 0$$

$f(0) = 0$

∴ $f(x)$ is continuous at $x = 0$.

(ii) $$f'(0^-) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sin\frac{\pi h}{2} - 0}{-h} = \lim_{h \to 0} \frac{\pi}{2} \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} = \frac{\pi}{2}$$

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\pi}{2} \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} = \frac{\pi}{2}$$

$f'(0^-) = f'(0^+)$

∴ $f(x)$ is differentiable at $x = 0$.

(iii) $$f'(1^-) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{\sin\left\{\frac{\pi(1-h)}{-h}\right\} - 1}{-h} = \lim_{h \to 0} \frac{\cos\frac{\pi h}{2} - 1}{-h}$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{\pi h}{4}}{\left(\frac{\pi h}{4}\right)^2} \cdot \frac{\pi^2 h}{16} = \lim_{h \to 0} \left(\frac{\pi^2 h}{8}\right) = 0$$

$$f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{|-1+h| - 1}{h}$$

$$= \lim_{h \to 0} \frac{|-1+h| - 1}{h} = \lim_{h \to 0} \frac{1 - h - 1}{h} = -1$$

$f'(1^-) \neq f'(1^+)$

Hence $f(x)$ is not differentiable at $x = 1$ but it is continuous there.

(iv) $$f'(3/2^-) = \lim_{h \to 0} \frac{f\left(\frac{3}{2} - h\right) - f\left(\frac{3}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{|3 - 2h - 3|\left[\frac{3}{2} - h\right] - 0}{-h}$$

$$= \lim_{h \to 0} \frac{2h \cdot 1}{-h} = -2$$

$$f'(3/2^+) = \lim_{h \to 0} \frac{f\left(\frac{3}{2} + h\right) - f\left(\frac{3}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{|2h|\left[\frac{3}{2} + h\right] - 0}{h} = \lim_{h \to 0} \frac{2h \cdot 1}{h} = 2$$

∴ $f(x)$ is not differentiable at $x = \frac{3}{2}$ but it is continuous there.

Problem 5:

Let $$f(x) = \begin{cases} \left(\dfrac{(1+\{x\})^{\frac{1}{\{x\}}}}{e}\right)^{\frac{1}{\{x\}}}, & x \neq \text{Integer} \\ \dfrac{2}{e}, & x = \text{Integer} \end{cases}.$$

Discuss the continuity and differentiability of $f(x)$ at any integral point, where $\{.\}$ denotes the fractional part.

Solution: Let $x = I_0$ be any arbitrary integers.

$$f(I_0) = \frac{2}{e}.$$

$$f(I_0^-) = \lim_{h \to 0} f(I_0 - h)$$

$$= \lim_{h \to 0} \left(\frac{(1+\{I_0 - h\})^{\frac{1}{\{I_0 - h\}}}}{e}\right) = \left(\frac{(1+1)^1}{e}\right)^1 = \frac{2}{e}$$

$$f(I_0 + h) = \lim_{h \to 0} (I_0 + h) = \lim_{h \to 0} \left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}$$

$$= e^{\lim\limits_{h \to 0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}} = e^{\lim\limits_{h \to 0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e} - 1\right)}$$

$$= e^{\lim\limits_{h \to 0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e} - 1\right)} = e^{\lim\limits_{h \to 0}\frac{(1+h)^{1/h} - e}{eh}}$$

Now $(1+h)^{1/h}$

$$= e^{\frac{1}{h}\ln(1+h)} = e^{\frac{1}{h}\left(h - \frac{h^2}{2} + \frac{h^3}{3} - ....\right)} = e.e^{-h\left(\frac{1}{2} - \frac{h}{3} + \frac{h^2}{4} - .....\right)}$$

$$\Rightarrow \quad f(I_0^+) = e^{\lim\limits_{h \to 0}\frac{e^{-h\left(\frac{1}{2} - \frac{h}{3} + \frac{h^2}{4} - ....\right)} - 1}{h}}$$

$$= e^{\lim\limits_{h \to 0}\frac{\left(e^{-h\left(\frac{1}{2} - \frac{h}{3} + \frac{h^2}{4} - ....\right)} - 1\right)\left(\frac{1}{2} - \frac{h}{3} + \frac{h^2}{4} - ....\right)(-1)}{-h\left(\frac{1}{2} - \frac{h}{3} + \frac{h^2}{4} - ....\right)}} = e^{-\frac{1}{2}}$$

Since $f(I_0^+) \neq f(I_0^-)$

$\Rightarrow$ $f(x)$ is discontinuous at all integral points and hence non-differentiable.

Problem 6: Let

$$f(x) = \begin{cases} \left(\ell n\left(e\left|[x]+[-x]\right|\right)^{x}\right)\cdot\left(\dfrac{2e^{\left(\frac{\{x\}+\{-x\}}{|x|}\right)}-5}{3+e^{\frac{1}{|x|}}}\right) & \text{for } x<0 \\ 0 & \text{for } x=0 \\ x\cdot\dfrac{1-e^{|x|+\{x\}}}{|x|+\{x\}} & \text{for } x>0 \end{cases}$$

where [ ], { } represents integral and fractional part functions respectively. Compute the one-sided derivatives at x = 0 and comment on the continuity and differentiability at x = 0.

Solution: $f'(0^+) = \lim\limits_{h\to0}\dfrac{h\left(\dfrac{1-e^{h+h}}{h+h}\right)-0}{h}$

$$= \lim_{h\to0}\frac{(1-e^{2h})}{(2h)h}h = -1$$

$$f'(0^-) = \lim_{h\to0}\frac{-h\cdot\ell n\left(e\left|-1+0\right|\right)\cdot\left(\dfrac{2e^{\frac{1-h+h}{|-h|}}-5}{3+e^{-\frac{1}{h}}}\right)}{-h}$$

$$= \lim_{h\to0}\frac{-h\cdot\left(\dfrac{2e^{\frac{1}{h}}-5}{3+e^{\frac{1}{h}}}\right)}{-h}$$

$$= \lim_{h\to0}\frac{2-5e^{-\frac{1}{h}}}{3e^{-\frac{1}{h}}+1} = 2$$

Hence f is continuous but not derivable at x = 0.

Problem 7: Let f be a continuous function from R to R such that f is differentiable at 0. Suppose that f(1/n) = 0 for all n ∈ N.

(i) Prove that f(0) = 0. (This fact does not requires the differentiability of f)

(ii) Prove that f ′(0) = 0.

Solution:

(i) Since f is continuous at 0 and $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right)=0$,

we must have $f(0)=\lim\limits_{n\to\infty}f\left(\dfrac{1}{n}\right)=0$,

since we are given that $f\left(\dfrac{1}{n}\right)=0$ for all n ∈ N.

(ii) By part (i) we have f(0) = 0 and so

$$f'(0) = \lim_{x\to0}\left(\frac{f(x)-f(0)}{x-0}\right) = \lim_{x\to0}\left(\frac{f(x)}{x}\right)$$

Thus, since $\dfrac{1}{n}\neq0$ and $\dfrac{1}{n}\to0$ as $n\to\infty$,

$$f'(0) = \lim_{n\to\infty}\left(\frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) = 0,$$

because $f\left(\dfrac{1}{n}\right)=0$ for all n ∈ N.

Problem 8: If $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$ then find the value of $\lim\limits_{x\to1^+}\dfrac{f(1-h)-f(1)}{h^3+3h}$.

Solution: $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$

$$l = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h}\cdot\frac{-h}{h(h^2+3)}$$

$$l = \lim_{h\to0} -f'(1)\cdot\frac{1}{h^2+3} = -\frac{1}{3}f'(1) \qquad ...(1)$$

Now, $f'(x) = 30x^9 - 56x^7 + 30x^5 - 63x^2 + 6x$

$f'(1) = 30 - 56 + 30 - 63 + 6 = -53$

∴ From (1), $l = -\dfrac{1}{3}(-53) = \dfrac{53}{3}$.

Problem 9: If $f(x + y) = f(x) + f(y) + |x|y + xy^2$, ∀ x, y ∈ R and f′(0) = 0, then prove that f is differentiable for all x ∈ R, but it is not twice differentiable at x = 0.

Solution:

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{f(h)+|x|h+xh^2}{h}$$

∵ f(0) = 0, we have

$$f'(x) = \lim_{h\to0}\left(\frac{f(h)-f(0)}{h}+|x|+xh\right)$$

f′(x) = f ′(0) + | x | = | x |.

Hence, f is differentiable for all x ∈ R.

It is easily seen that f ′(x) = | x | is not differentiable at x = 0. So, f is not twice differentiable at x = 0.

Problem 10: If the function f(x) = |x – a|.g(x), where g(x) is a continuous function, is differentiable at x = a then find g(a).

Solution: We have

$$f'(a^+) = \lim_{h\to0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to0}\frac{hg(a+h)-0}{h} = g(a).$$

[∵ g is continuous at x = a we have

$\lim\limits_{h\to0}g(a+h) = \lim\limits_{h\to0}g(a-h) = g(a)$]

$$f'(a^-) = \lim_{h\to0}\frac{f(a)-f(a-h)}{h} = \lim_{h\to0}\frac{0-hg(a-h)}{h} = -g(a)$$

Since, f is differentiable at x = a, therefore we have

g(a) = – g(a)

⇒ g(a) = 0.

Problem 11: If $f(x) = \begin{cases} \frac{2}{3}(a^2 - b^2), \ 0 \le x \le b \\ \frac{2}{3}a^2 - \frac{4}{9}x^2 - \frac{2b^3}{9x}, \ b < x \le a, \\ \frac{2}{9x}(a^3 - b^3), \ x > a \end{cases}$

then find whether $f'(a)$ exists.

Solution: We have,

$$f'(x) = \begin{cases} 0, \ 0 \le x < b \\ \frac{-8}{9}x + \frac{2b^3}{9x^2}, \ b < x < a, \\ \frac{-2}{9x^2}(a^3 - b^3), \ x > a \end{cases}$$

$$\therefore \quad f'(a^+) = \lim_{h \to 0} f'(a+h) = \lim_{h \to 0}\left[\frac{-2}{9}\cdot\frac{(a^3 - b^3)}{(a+h)^2}\right]$$

$$= \frac{-2a^3 + 2b^3}{9a^2}.$$

and $f'(a^-) = \lim\limits_{h \to 0} f'(a-h)$

$$= \lim_{h \to 0}\left[\frac{-8}{9}(a-h) + \frac{2b^3}{9(a-h)^2}\right]$$

$$= \frac{-8a}{9} + \frac{2b^3}{9a^2} = \frac{-8a^3 + 2b^3}{9a^2}.$$

$\therefore$ $f'(a^+) \ne f'(a^-)$ and hence $f'(a)$ does not exist.

Problem 12: Let $f : [0, 1] \to R$ be defined as

$$f(x) = \begin{cases} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then prove that (i) f(x) is differentiable in [0, 1]

(i) f(x) is bounded in [0, 1]

(ii) $f'(x)$ is bounded in [0, 1].

Solution: $f'(0^+) = \lim\limits_{h \to 0} \dfrac{h^3(1-h)\sin\frac{1}{h^2} - 0}{h} = 0$

$$f'(1^-) = \lim_{h \to 0} \frac{(1-h)^3(+h)\sin\frac{1}{(1-h)^2} - 0}{-h}$$

$$= \lim_{h \to 0} -(1-h)^3 \sin\frac{1}{(1-h)^2} = -\sin 1$$

Hence f is derivable in [0, 1].

Obviously f is continuous in a closed interval [0, 1] and hence f is bounded.

Now $f'(x)$

$$= \begin{cases} (x^3 - x^4)\cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right) + \sin\frac{1}{x^2}(3x^2 - 4x^3) & x \ne 0 \\ 0 \quad \text{if } x = 0 \end{cases}$$

and $\lim\limits_{x \to 1^-} f'(x) = (0) + \sin 1(3 - 4) = -\sin 1.$

Thus, $f'(x)$ is continuous in [0, 1] and hence it is also bounded.

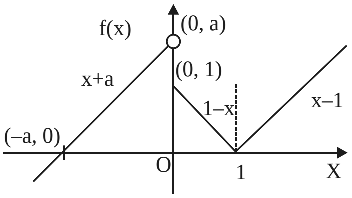
Problem 13: $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ x - 1| & \text{if } x \ge 0 \end{cases}$ and

$$g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$$

where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0 ? Justify your answer.

Solution: $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ x - 1| & \text{if } x \ge 0 \end{cases}$ and

$$g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$$



$$G(x) = g(f(x)) = \begin{cases} f(x) + 1 & f(x) < 0 \\ (f(x) - 1)^2 + b & f(x) \ge 0 \end{cases}$$

$$= \begin{cases} x + a + 1 & \text{if } f(x) < 0 \text{ or } x < -a \\ \begin{cases} (x + a - 1)^2 + b & \text{if } f(x) \ge 0 \text{ or } -a < x < 0 \\ (1 - x - 1)^2 + b & \text{if } f(x) \ge 0 \text{ or } 0 \le x < 1 \\ \quad = x^2 + b & \\ (x - 1 - 1)^2 + b & \text{if } f(x) \ge 0 \text{ or } x \ge 1 \\ = (x-2)^2 + b & \end{cases} \end{cases}$$

(gof) (x) is continuous :

Continuity at x = – a

$$\lim_{h \to 0} G(-a-h) = \lim_{h \to 0} (-a-h) + a + 1 = 1$$

$$\lim_{h \to 0} G(-a+h) = \lim_{h \to 0} (-a + h + a - 1)^2 + b$$

$$= b + 1$$

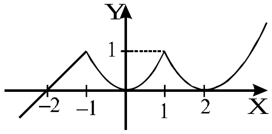
$\Rightarrow$ b + 1 = 1 $\Rightarrow$ b = 0.

Continuity at x = 0

$$\lim_{h \to 0} G(0-h) = (a-1)^2 \qquad (\text{as } b = 0)$$

$$\lim_{h \to 0} G(0+h) = 0 \Rightarrow a - 1 = 0 \Rightarrow a = 1$$

$$\therefore \quad G(x) = \begin{cases} x+2 & x<-1 \\ x^2 & -1 \le x < 1 \\ (x-2)^2 & x \ge 1 \end{cases}$$



$\Rightarrow$ G is differentiable at x = 0.

Problem 14: Let f : R → R be a real valued function ∀ x, y ∈ R such that $|f(x) - f(y)| < \sin^4(x-y)$. Prove that f(x) is a constant function.

Solution: Since $|f(x) - f(y)| < \sin^4(x-y)$,

$x \neq y \, |f(x) - f(y)| < |\sin^4(x-y)|$

Dividing by | x – y | on both sides,

$$\therefore \quad \left|\frac{f(x)-f(y)}{x-y}\right| < \left|\frac{\sin^4(x-y)}{x-y}\right|$$

Taking lim y → x, we get using domination law of limits that

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}\left|\frac{\sin^4(x-y)}{x-y}\right|$$

$$\Rightarrow \quad \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{y\to x}\frac{\sin^4(x-y)}{x-y}\right|$$

$\Rightarrow \quad |f'(x)| \le 0$

$\Rightarrow \quad |f'(x)| = 0 \quad (\because |f'(x)| \ge 0$, in general)

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad f(x) = c$ (constant)

Problem 15: A function f : R → [1, ∞) satisfies the equation f(xy) = f(x)f(y) – f(x) – f(y) + 2. If f is differentiable on R and f(2) = 5, then show that

$$f'(x) = \frac{f(x)-1}{x}.f'(1)$$. Hence, determine f(x).

Solution: We have

$f(xy) = f(x)f(y) - f(x) - f(y) + 2$ ...(1)

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)f\left(1+\frac{h}{x}\right)-f(x)-f\left(1+\frac{h}{x}\right)+2-f(x)}{h}$$

[using (1)]

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}}.\frac{f(x)-1}{x} \quad ...(2)$$

Putting x = 1 and y = 2 in equation (1), we have

$f(2) = f(1)\, f(2) - f(1) - f(2) + 2$

$\Rightarrow 5 = 5(f(1) - f(1) - 5 + 2$ [$\because$ f(2) = 5]

$\Rightarrow f(1) = 2.$

Now (2) reduces to

$$f'(x) = \frac{f(x)-1}{x}.\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}} = \frac{f(x)-1}{x}.f'(1)$$

To find f(x), write the above equation as

$$\frac{df}{f-1} = f'(1)\frac{dx}{x}$$

Integrating both sides w.r.t. x

we get ln |f–1| = f′(1).ln|x| + lnc.

Let f′(1) = a

ln|f–1| = a.ln|x|c

Now, using the condition f(1) = 2, we have

ln |2 – 1| = a.ln c ⇒ c = 1

Now, $f - 1 = \pm |x|^a$

$\Rightarrow \quad f(x) = 1 \pm |x|^a$

Using the condition f(2) = 5, we have

$5 = 1 \pm 2^a$

This gives a = 2, using + sign.

Hence, we have $f(x) = 1 + |x|^2$

which can also be written as $f(x) = 1 + x^2$.

Problem 16: Let $f\left(\frac{x_1+x_2+x_3+....+x_n}{n}\right)$

$$= \left(\frac{f(x_1)+f(x_2)+f(x_3)+....+f(x_n)}{n}\right)$$

where x_i are any real numbers and n ∈ N. If f(x) is differentiable and f ′(0) = a and f(0) = b, find f(x).

Solution: We have, $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

{taking $x_1 = x$, $x_2 = y$, n = 2}

This holds for any real x, y. So y is independent of x, i.e.,

$\frac{dy}{dx} = 0$.

Differentiating w.r.t x,

$$f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2}\left(1+\frac{dy}{dx}\right) = \frac{1}{2}\left\{f'(x)+f'(y)\frac{dy}{dx}\right\}$$

$$\therefore \quad \frac{1}{2}f'\left(\frac{x+y}{2}\right) = \frac{1}{2}f'(x)$$

Taking x = 0 and x in place of y we get

$$\frac{1}{2}f'\left(\frac{0+x}{2}\right) = \frac{1}{2}f'(0)$$

$$\therefore \quad f'\left(\frac{x}{2}\right) = f'(0) = a\,(\text{given}).$$

Integrating w.r.t. $\frac{x}{2}$, $f\left(\frac{x}{2}\right) = a \cdot \frac{x}{2} + c$

When x = 0, f(0) = 0 + c, i.e., b = c

[∵ f(0) = b (given)]

Thus, $f\left(\frac{x}{2}\right) = \frac{ax}{2} + b$ ...(1)

Again, putting y = 0 in $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ we get

$$f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} = \frac{f(x)+b}{2}$$

∴ From (1), $\frac{f(x)+b}{2} = \frac{ax}{2} + b$

∴ f(x) = ax + b.

Problem 17: Find a function continuous and derivable for all x and satisfying the functional relation, $f(x+y) \cdot f(x-y) = f^2(x)$, where x and y are independent variables and $f(0) \neq 0$.

Solution: Put y = x and x = 0 to get

$f(x) . f(-x) = f^2(0)$ ...(1)

Now $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) - \frac{f^2(0)}{f(-x)}}{h}$$

$$= \frac{1}{f(-x)} \cdot \lim_{h \to 0} \frac{f(x+h) . f(-x) - f^2(0)}{h}$$

$$= \frac{1}{f(-x)} \lim_{h \to 0} \frac{f\left(\frac{h}{2} + \left(\frac{h}{2} + x\right)\right) . f\left(\frac{h}{2} - \left(\frac{h}{2} + x\right)\right) - f^2(0)}{h}$$

$$= \frac{1}{f(-x)} \lim_{h \to 0} \frac{f^2\left(\frac{h}{2}\right) - f^2(0)}{h}$$

$$= \frac{1}{2f(-x)} \lim_{h \to 0} \frac{\left[f\left(\frac{h}{2}\right) + f(0)\right]\left[f\left(\frac{h}{2}\right) - f(0)\right]}{\frac{h}{2}}$$

$$= \frac{f(x)}{2f^2(0)} . 2f(0) . f'(0) = \frac{f'(0)}{f(0)} f(x)$$

$\Rightarrow \frac{f'(x)}{f(x)} = \frac{f'(0)}{f(0)} = k$ (say)

$\Rightarrow \ln|f(x)| = kx + c$

$\Rightarrow |f(x)| = e^{kx+c} = e^{kx} \cdot e^c$

$\Rightarrow f(x) = \pm e^{kx} \cdot e^c$

$\Rightarrow f(x) = a.e^{kx}$ where a and k are arbitrary constants.

Problem 18: Let f be a function such that

$f(x + f(y)) = f(f(x)) + f(y) \quad \forall\ x, y \in R$

and f(h) = h for $0 < h < \varepsilon$ where ε is a small positive quantity. Determine f''(x) and f(x).

Solution: Given f(x + f(y)) = f(f(x) + f(y))...(1)

Putting x = y = 0 in (1),

$f(0 + f(0)) = f(f(0)) + f(0) \Rightarrow f(f(0)) = f(f(0)) + f(0)$

∴ f(0) = 0 ...(2)

Now $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (for $0 < h < \varepsilon$)

putting x as h and y as x in (1)

$= \lim_{h \to 0} \frac{f(f(h))}{h} = \lim_{h \to 0} \frac{f(h)}{h}$ (∵ f(h) = h)

$= \lim_{h \to 0} \frac{h}{h} = 1.$

Integrating both sides with limits 0 to x

f(x) = x

∴ f''(x) = 0.

Problem 19: Let f(x) be a thrice differentiable function satisfying f(x + y) = f(x – y) + y[f ′(x + y) + f ′(x – y)] where f(0) = 0, f ′(0) = 1, and f(1) = 2. Find f(x).

Solution: Put y = x

f(2x) = f(0) + x[f ′(2x) + f ′(0)]

f(2x) = xf ′(2x) + xf ′(0) + f(0)

Differentiate both sides w.r.t. x

f′(2x). 2 = f′(2x) + xf ″(2x).2 + f′(0)

f′(2x) = 2xf ″(2x) + f′(0)

Replace 2x by x,

f′(x) = xf ″(x) + f′(0)

Again differentiate both sides w.r.t. x

f ″(x) = xf ‴(x) + f ″(x)

∴ xf ‴(x) = 0

⇒ f ‴(x) = 0

∴ $f(x) = ax^2 + bx + c$

Since f(0) = 0, c = 0

f′(0) = 1 ⇒ b = 1 and f(1) = 2 ⇒ a = 1.

Hence $f(x) = x^2 + x$.

Problem 20: Let f(x) be a positive differentiable function satisfying

$f\left(\frac{x+y}{2}\right) = \sqrt{f(x).f(y)}\ \forall\ x, y \in R$, where f ′(0) = 2.

Find f(x).

Solution: Put x = y = 0, $f(0) = \sqrt{f(0)^2}$

⇒ f(0) = 0, 1

Since f is a positive function f(0) = 1.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \frac{\sqrt{f(2x).f(2h)} - f(x)}{h}$$

Replace x by 2x and y = 0 in the rule :

$$f(x) = \sqrt{f(2x).f(0)}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x)\sqrt{\frac{f(2h)}{f(0)}} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)(\sqrt{f(2h)} - 1)}{h}$$

$$= f(x)\cdot \lim_{h\to 0} \frac{(f(2h) - 1)}{h.(\sqrt{f(2h)} + 1)}$$

$$= f(x)\cdot \lim_{h\to 0} \frac{f(2h) - 1}{2h}$$

$f'(x) = f(x) . f'(0)$

$f'(x) = 2f(x)$

$\ell n\ f(x) = 2x + c$

$f(x) = e^{2x + c}$

$f(0) = 1 \Rightarrow c = 0$

$\therefore$ $f(x) = e^{2x}$

Things to Remember

1. Right Hand Derivative (R.H.D.)

$$= f'(a^+) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Left Hand Derivative (L.H.D.)

$$= f'(a^-) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

f is differentiable at x = a if and only if L.H.D. = R.H.D.

2. A function will fail to have a derivative at a point where the graph has

(i) a corner, where the one-sided derivatives differ.

(ii) an oscillation point, where the one-sided derivative(s) does (do) not exist.

(iii) a vertical tangent, where the absolute value of slope of the secant approaches ∞.

(iv) a discontinuity.

3. The curve y = f(x) has a vertical tangent line at the point $(x_0, f(x_0))$ provided that f is continuous at x_0 and $|f'(x)| \to \infty$ as $x \to x_0$

4. The graph of a continuous function f has a cusp at x_0 if $f'(x)$ approaches ∞ from one side and –∞ from the other side.

5. If a function f is differentiable at x = a then it must be continuous at x = a.

If f is continuous at x = a, then f may or may not be differentiable at x = a.

6. If a function f is discontinuous at x = a then it is non-differentiable at x = a.

7. If a function f is non-differentiable at x = a but both the one-sided derivatives exist (though being unequal), then f is continuous at x = a.

8. If a function f is defined only in the left neighbourhood of a point x = a (for x ≤ a), it is said to be differentiable at a if

$$\text{L.H.D.} = f'(a^-) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$ exists. In such a case $f'(a) = f'(a^-)$.

Similarly, if a function f is defined only in the right neighbourhood of a point x = a (for x ≥ a), it is said to be differentiable at a if

$$\text{R.H.D.} = f'(a^+) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$ exists.

In such a case $f'(a) = f'(a^+)$.

9. A function f(x) is said to be differentiable in an open interval (a, b) if it is differentiable at all interior points in (a, b).

10. A function f(x) is said to be differentiable in a closed interval [a, b] if

(i) it is differentiable at all interior points in (a, b).

(ii) R.H.D. = $f'(a^+)$ exists at the left endpoint a.

(ii) L.H.D. = $f'(a^-)$ exists at the right endpoint b.

11. (i) All polynomial, exponential, logarithmic and trigonometric functions (inverse trigonometric not included) are differentiable at each point in their domain.

(ii) Modulus function and signum function are non differentiable at x = 0. Hence, y = | f(x) | and y = sgn(f(x)) should be checked at points where f(x) = 0.

(iii) Power function $y = x^p$, $0 < p < 1$ is non-differentiable at x = 0. Hence, $y = (f(x))^p$ should be checked at points where f(x) = 0.

(iv) The inverse trigonometric functions $y = \sin^{-1}x, \cos^{-1}x, \text{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points x = ± 1. Hence, $y = \sin^{-1}(f(x)), \cos^{-1}(f(x)), \text{cosec}^{-1}(f(x))$, and $\sec^{-1}(f(x))$ should be checked at points where f(x) = ± 1.

(v) Greatest integer function and fractional part functions are non differentiable at all integral x.

Hence, $y = [f(x)]$ and $y = \{f(x)\}$ should be checked at points where $f(x) = n,\ n \in I$.

(vi) Further, a function should be checked at all those points where discontinuity may arise.

12. An alternative limit form of the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

13. If $f'(a)$ exists, then $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

14. (i) If $f'(a)$ exists and $\psi(h) \to 0$ as $h \to 0$, then

$$\lim_{h\to 0} \frac{f(a+\psi(h))-f(a)}{\psi(h)} = f'(a)$$

(ii) If $f'(a)$ exists, and $\psi(h) \to 0$ and $\phi(h) \to 0$ as $h \to 0$,

then $\lim_{h\to 0} \frac{f(a+\psi(h))-f(a+\phi(h))}{\psi(h)-\phi(h)} = f'(a)$

15. Let the function $y = f(x)$ be defined by $x= x(t)$ and $y= y(t)$, where t is the parameter.

Then, $$\frac{dy}{dx} = \lim_{\delta\to 0} \frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}} = \lim_{\delta\to 0} \frac{y(t+\delta)-y(t)}{x(t+\delta)-x(t)}$$

16. In general, the limit of the derivative may not exist, but when it exists then it is equal to the value of the derivative. In such a case, we say that the derived function $f'(x)$ is continuous, or the function $f(x)$ is continuously differentiable.

17. (i) If $\lim_{x\to a^-} f'(x)$ and $\lim_{x\to a^+} f'(x)$ both exist or are infinite

then $\lim_{x\to a^-} f'(x) = f'(a^-)$

i.e L.H.L. of $f'(x)$ = L.H.D. of $f(x)$ at $x = a$

and $\lim_{x\to a^+} f'(x) = f'(a^+)$

i.e R.H.L. of $f'(x)$ = R.H.D. of $f(x)$ at $x = a$.

(a) If $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$, then $f(x)$ is differentiable and its derivative $f'(x)$ is continuous at $x = a$.

(b) If $\lim_{x\to a^-} f'(x) \neq \lim_{x\to a^+} f'(x)$, then $f(x)$ is non-differentiable and its derivative $f'(x)$ is discontinuous at $x = a$.

(c) If $\lim_{x\to a^-} f'(x)$ and $\lim_{x\to a^+} f'(x)$ are infinite, then $f(x)$ has an infinite derivative at $x = a$ and hence it is non-differentiable there and $f'(x)$ is discontinuous at $x = a$.

(ii) If any of the limits $\lim_{x\to a^-} f'(x)$ or $\lim_{x\to a^+} f'(x)$ does not exist, then we cannot conclude anything about the differentiability of the function. In such a case, we should try to find the derivative using its basic definition (i.e. first principles).

18. A function $f(x)$ is twice differentiable at $x = a$ if its derivative $f'(x)$ is differentiable at $x = a$ i.e.

the limit $f''(a) = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h}$ exists.

19. If $f(x)$ and $g(x)$ are differentiable at $x = a$, then the following functions are also differentiable at $x = a$.

(i) $cf(x)$ is differentiable at $x = a$, where c is any constant.

(ii) $f(x) \pm g(x)$ is differentiable at $x = a$.

(iii) $f(x).\ g(x)$ is differentiable at $x = a$.

(iv) $f(x)/g(x)$ is differentiable at $x = a$, provided $g(a) \neq 0$.

20. If $f(x)$ is differentiable at $x = a$ and $g(x)$ is non-differentiable at $x = a$, then we have the following results :

(i) Both the functions $f(x) + g(x)$ and $f(x) - g(x)$ are non-differentiable at $x = a$.

(ii) $f(x).\ g(x)$ is not necessarily non-differentiable at $x = a$. We need to find the result by first principles.

21. If $f(x)$ and $g(x)$ both are non-differentiable at $x = a$, then we have the following results.

(i) The functions $f(x) + g(x)$ and $f(x) - g(x)$ are not necessarily non-differentiable at $x = a$. However, atmost one of $f(x) + g(x)$ or $f(x) - g(x)$ can be differentiable at $x = a$. That is, both of them cannot be differentiable simultaneously at $x = a$.

(ii) $f(x).\ g(x)$ and $f(x)/g(x)$ are not necessarily non-differentiable at $x = a$. We need to find the result by applying first principles.

22. If $f(x)$ is differentiable at $x = a$ and $g(x)$ is differentiable at $x = f(a)$ then the composite function $(gof)(x)$ is differentiable at $x = a$.

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. If $f(x) = \frac{x}{1+e^{1/x}}$, $x \neq 0$ and $f(0) = 0$ then,

(A) $f(x)$ is continuous at $x = 0$ and $f'(x) = 1$

(B) $f(x)$ is discontinuous at $x = 0$

(C) $f(x)$ is continuous at $x = 0$ and $f'(x)$ does not exists

(D) $f(x)$ is continuous at $x = 0$ and $f'(x) = 0$

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \le x_0 \\ ax+b & \text{if } x > x_0 \end{cases}$

The values of the coefficients a and b for which the function is continuous and has a derivative at x_0, are

(A) $a = x_0,\ b = -x_0$ (B) $a = 2x_0,\ b = -x_0^2$

(C) $a = a = x_0^2,\ b = -x_0$ (D) $a = x_0,\ b = -x_0^2$

3. The number of points where function f(x) = minimum $\{x^3 - 1, -x + 1, \text{sgn}(-x)\}$ is continuous but not differentiable is

(A) One (B) Two
(C) Zero (D) None of these

4. Total number of the points where the function $f(x) = \min\{|x| - 1, |x - 2| - 1\}$ is not differentiable

(A) 3 points (B) 4 points
(C) 5 points (D) None of these

5. The set of values of x for which the function defined as

$$f(x) = \begin{cases} 1-x & x<1 \\ (1-x)(2-x) & 1 \le x \le 2 \\ 3-x & x>2 \end{cases}$$

fails to be continuous or differentiable, is

(A) {1} (B) {2}
(C) {1, 2} (D) ϕ

6. The number of points where $f(x) = (x + 1)^{2/3} + |x - 1|^{\sqrt{3}}$, is non-differentiable is

(A) 1 (B) 2
(C) 3 (D) none

7. The set of all points where $f(x) = \sqrt[3]{x^2|x|} - |x| - 1$ is not differentiable is

(A) {0} (B) {–1, 0, 1}
(C) {0, 1} (D) none of these

8. It f(x) is a differentiable function from R → Q then $\sum_{r=0}^{100} (-1)^r f(r)$ is

(A) 0 (B) –1
(C) 1 (D) none

9. The function $f(x) = \dfrac{|x| - x(3^{1/x} + 1)}{3^{1/x} - 1}$, $x \ne 0$, $f(0) = 0$ is

(A) discontinuous at x = 0
(B) continuous at x = 0 but not differentiable there
(C) both continuous and differentiable at x = 0
(D) differentiable but not continuous at x = 0

10. Let $f(x) = \begin{cases} g(x).\cos\dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ where g(x) is an even function differentiable at x = 0, passing through the origin. Then f ′(0)

(A) is equal to 1 (B) is equal to 0
(C) is equal to 2 (D) does not exist

11. Let $f(x) = \lim\limits_{n\to\infty} \dfrac{(x^2 + 2x + 3 + \sin \pi x)^n - 1}{(x^2 + 2x + 3 + \sin \pi x)^n + 1}$, then

(A) f(x) is continuous and differentiable for all x ∈ R.
(B) f(x) is continuous but not differentiable for all x ∈ R.
(C) f(x) is discontinuous at infinite number of points.
(D) f(x) is discontinuous at finite number of points.

12. Let f (x) is a function continuous for all x ∈ R except at x = 0 such that f ′(x) < 0 ∀ x ∈ (–∞, 0) and f(x) > 0 ∀ x ∈ (0, ∞). If $\lim\limits_{x\to 0^+} f(x) = 3$, $\lim\limits_{x\to 0^-} f(x) = 4$ and f (0) = 5, then the image of the point (0,1) about the line $y \lim\limits_{x\to 0} f(\cos^3 x - \cos^2 x)$

$= x \lim\limits_{x\to 0} f(\sin^2 x - \sin^3 x)$, is

(A) $\left(\dfrac{12}{25}, \dfrac{-9}{25}\right)$ (B) $\left(\dfrac{12}{25}, \dfrac{9}{25}\right)$
(C) $\left(\dfrac{16}{25}, \dfrac{-8}{25}\right)$ (D) $\left(\dfrac{24}{25}, \dfrac{-7}{25}\right)$

13. Let f be an injective and differentiable function such that f(x). f(y) + 2 = f(x) + f(y) + f(xy) for all non negative real x and y with f ′(0) = 0, f ′(1) = 2 ≠ f(0), then

(A) x f ′(x) – 2f(x) + 2 = 0
(B) x f ′(x) + 2 f(x) – 2 = 0
(C) x f ′(x) – f(x) + 1 = 0
(D) 2 f(x) = f’(x) + 2

14. Let f be a function such that f(x + y) = f(x) + f(y) for all x and y and $f(x) = (2x^2 + 3x)\, g(x)$ for all x where g(x) is continuous and g(0) = 3. Then f ′(x) is equal to

(A) 9 (B) 3
(C) 6 (D) none

15. Let f(x) be a function such that f(x + y) = f(x) + f(y) and f(x) = sin x g(x) for all x, y ∈ R. If g(x) is a continuous function such that g(0) = K, then f’(x) is equal to

(A) K (B) Kx
(C) Kg(x) (D) none

16. Let $f(x) = \begin{cases} \dfrac{3x^2 + 2x - 1}{6x^2 - 5x + 1} & \text{for } x \ne \dfrac{1}{3} \\ -4 & \text{for } x = \dfrac{1}{3} \end{cases}$ then $f'\left(\dfrac{1}{3}\right)$

(A) is equal to –9 (B) is equal to –27
(C) is equal to 27 (D) does not exist

17. Given f(x) is a differentiable function of x, satisfying f(x) . f(y) = f(x) + f(y) + f(xy) – 2 and that f(2) = 5. Then f(3) is equal to

(A) 10 (B) 24
(C) 15 (D) none

18. The function $f(x) = \begin{cases} 2x + 1 & , x \in Q \\ x^2 - 2x + 5 & , x \notin Q \end{cases}$ is

(A) continuous no where
(B) differentiable no where
(C) continuous but not differentiable exactly at one point
(D) differentiable and continuous only at one point and discontinuous elsewhere

19. Let f(x) be differentiable at x = h then

$$\lim_{x\to h}\frac{(x+h)\,f(x)-2h\ f(h)}{x-h}$$ is equal to

(A) f(h) + 2hf′ (h)
(B) 2 f(h) + hf′ (h)
(C) hf(h) + 2f′ (h)
(D) hf(h) – 2f′ (h)

20. If f(x) . f(y) = f(x) + f(y) + f(xy) – 2 ∀ x, y ∈ R and if f(x) is not a constant function, then the value of f(1) is

(A) 1
(B) 2
(C) 0
(D) – 1

21. If f(x) = |1 – x|, then the points where $\sin^{-1}$(f |x|) is non-differentiable are

(A) {0, 1}
(B) {0, –1}
(C) {0, 1, –1}
(D) none of these

22. Let f(x) be defined for all x ∈ R and the continuous. Let f(x + y) – f(x– y) = 4xy ∀ x, y = ∈ R and f(0) = 0 then

(A) f(x)is bounded

(B) $f(x) + f\left(\frac{1}{x}\right) = f\left(x+\frac{1}{x}\right)+2$

(C) $f(x)+f\left(\frac{1}{x}\right)=f\left(x-\frac{1}{x}\right)+2$

(D) none of these

23. If g′(x) exists for all x, g′ (0) = 2 and g(x + y) = e^yg(x) + e^xg(y) ∀ x, y. Then

(A) g(2x) = –2e^xg(x)
(B) g’(x) = g(x) + 2e^x
(C) $\lim_{h\to 0}$ g(h)/h = 3
(D) None of these

24. If y = |1–|2–|3–|4–x|||| ; then number of points where y is not differentiable; is

(A) 1
(B) 3
(C) 5
(D) > 5

25. $f(x) = \begin{cases} \dfrac{x}{2x^2+|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ then f(x) is

(A) Continuous but non-differentiable at x = 0
(B) Differentiable at x = 0
(C) Discontinuous at x = 0
(D) None of these

26. Let f(x) = cos x and g(x) =

$$g(x) = \begin{cases} \text{minimum}\ \{f(t) : 0 \le t \le x\}, & x \in [0,\ \pi] \\ \sin x - 1 \ , & x > \pi \end{cases}$$

then

(A) g(x) is discontinuous at x = π
(B) g(x) is continuous for x ∈ [0, ∞)
(C) g(x) is differentiable at x = π
(D) g(x) is differentiable for x [0, ∞)

27. If f(x + y + z) = f(x). f(y) . f(z) for all x, y, z and f(2) = 4, f′(0) = 3, then f’(2) equals

(A) 12
(B) 9
(C) 16
(D) 6

28. If f(x) = $\lim_{k\to\infty}$ $(\cos x)^{2k}$ + $\cos^{-1}$ (sin x); then the points of non-differentiability of f(x) is

(A) x = 1
(B) x = – 1
(C) x = nπ(n ∈ 1)
(D) none of these

29. If f(x) = Max. {1, (cos x + sin x), (sin x – cos x)} 0 ≤ x ≤ 5π/4, then

(A) f(x) is not differentiable at x = π/6
(B) f(x) is not differentiable at x = 5π/6
(C) f(x) is continuous for x ∈ [0, 5π/4]
(D) None of these

30. Let f : R ⟶ R satisfying |f(x)| ≤ x^2 ∀ x ∈ R, then

(A) ‘f’ is continuous but non-differentiable at x = 0
(B) ‘f’ is discontinuous at x = 0
(C) ‘f’ is differentiable at x = 0
(D) None of these

31. Choose the incorrect statement given that f is differentiable

(A) If f is odd and f′(c) = 3 then f′(–c) = 3
(B) If f is even and f′(c) = 3 then f′(–c) – 3
(C) If f is even then f′(0) = 0
(D) If f is odd then f′(0) = 0

32. Consider the function f(x)

$$= \begin{cases} x^3 & \text{if } \ x < 0 \\ x^2 & \text{if } \ 0 \le x < 1 \\ 2x - 1 & \text{if } \ 1 \le x < 2, \\ x^2 - 2x + 3 & \text{if } \ x \ge 2 \end{cases}$$

then f is continuous and differentiable for

(A) x ∈ R
(B) x ∈ R – {0, 2}
(C) x ∈ R – {2}
(D) x ∈ R – {1, 2}

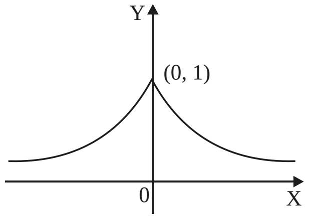
33. The number of points on [0, 2] where

$$f(x) = \begin{cases} x\{x\}+1 & 0 \le x < 1 \\ 2-\{x\} & 1 \le x \le 2 \end{cases}$$

fails to be continuous or derivable is

(A) 0
(B) 1
(C) 2
(D) 3

34. Which one of the following functions best represent the graph as shown adjacent ?



(A) $f(x) = \dfrac{1}{1+x^2}$
(B) $f(x) = \dfrac{1}{1+\sqrt{|x|}}$
(C) $f(x) = e^{-|x|}$
(D) $f(x) = a^{|x|}$ $(a > 1)$

35. The number of points where the function
$f(x) = (x^2 - 1)\,|x^2 - x - 2| + \sin(|x|)$
is not differentiable is
(A) 0
(B) 1
(C) 2
(D) 3

36. Let f(x) be differentiable at x = h then

$$\lim_{x\to h} \frac{(x+h)\,f(x) - 2h\,f(h)}{x-h} \text{ is equal to}$$

(A) $f(h) + 2hf'(h)$
(B) $2f(h) + hf'(h)$
(C) $hf(h) + 2f'(h)$
(D) $hf(h) - 2f'(h)$

37. The function $f(x) = \text{maximum}\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to
(A) 1
(B) 0, 2
(C) 0, 1
(D) 1, 2

38. Let $f(x) = \begin{cases} \dfrac{\sin|x^2-5x+6|}{x^2-5x+6}, & x \neq 2,3 \\ 1, & x = 2 \text{ or } 3 \end{cases}$.

The set of all points where f is differentiable is
(A) (∞, ∞)
(B) $(-\infty, \infty) - \{2\}$
(C) $(-\infty, \infty) - \{3\}$
(D) $(-\infty, \infty) - \{2, 3\}$

39. If $f'(a) = \dfrac{1}{4}$, then $\displaystyle\lim_{h\to 0} \frac{f(a+2h^2) - f(a-2h^2)}{f(a+h^3-h^2) - f(a-h^3+h^2)} =$
(A) 0
(B) 1
(C) –2
(D) none

40. If $f(x) = \begin{cases} [x] + \sqrt{\{x\}} & x < 1 \\ \dfrac{1}{[x]+\{x\}^2} & x \geq 1 \end{cases}$, then [where [.] and { . } represents greatest integer part and fractional part respectively.]
(A) f(x) is continuous at x = 1 but not differentiable
(B) f(x) is not continuous at x = 1
(C) f(x) is differentiable at x = 1
(D) $\displaystyle\lim_{x\to 1} f(x)$ does not exist

41. If f(x) has isolated point discontinuity at x = a such that |f(x)| is continuous at x = a then
(A) |f(x)| must be differentiable at x = a
(B) $\displaystyle\lim_{x\to a} f(x)$ does not exist
(C) $\displaystyle\lim_{x\to a} f(x) + f(a) = 0$
(D) f(a) = 0

42. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?
I. f is continuous on the closed interval [a, b]
II. f is bounded on the open interval (a, b)
III. If $a < a_1 < b_1 < b$, and $f(a_1) < 0 < f(b_1)$, then there is a number c such that $a_1 < c < b_1$ and f(c) = 0
(A) I and II only
(B) I and III only
(C) II and III only
(D) only III

43. Let f : R → R, $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1\ \forall\ x, y \in R$, if f(0) = 1 and f′(0) = 0, then
(A) $f(x) = 1 - \dfrac{x^2}{2}$
(B) $f(x) = x^2 + 1$
(C) $f(x) = \left(\dfrac{2x+1}{x+1}\right)$
(D) none of these

44. If f : [–2a, 2a] → R is an odd function such that f(x) = f(2a – x) for x ∈ (a, 2a). if the left hand derivative of f(x) at x = a is zero, then the left hand derivative of f(x) at x = –a is
(A) 1
(B) –1
(C) 0
(D) none

45. Suppose that the differentiable functions u, v, f, g : R ⟶ R satisfy $\displaystyle\lim_{x\to\infty} u(x) = 2,\ \lim_{x\to\infty} v(x) = 3,$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \text{ and } \frac{f'(x)}{g'(x)} + u(x)\frac{f(x)}{g(x)} = v(x)$$

then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is equal to (given that it exists)
(A) 1
(B) 1/2
(C) 2
(D) None

46. Suppose f is a differentiable function such that f(x + y) = f (x) + f (y) + 5xy for all x, y and f′(0) = 3. The minimum value of f (x) is
(A) – 1
(B) – 9/10
(C) – 9/25
(D) None

47. Let h(x) be differentiable for all x and let $f(x) = (kx + e^x)\,h(x)$ where k is some constant. If h(0) = 5, h’(0) = –2 and f’(0) = 18 then the value of k is equal to
(A) 5
(B) 4
(C) 3
(D) 2.2

48. If for a function f(x) : f(2) = 3, f′(2) = 4, then $\displaystyle\lim_{x\to 2} [f(x)]$, where [.] denotes the greatest integer function, is
(A) 2
(B) 3
(C) 4
(D) dne

49. If $f(x) = \begin{cases} [x]+\sqrt{\{x\}} & x<1 \\ \dfrac{1}{[x]+\{x\}^2} & x \ge 1 \end{cases}$, then

[where [.] and { . } represent greatest integer and fractional part functions respectively]

(A) f(x) = is continuous at x = 1 but not differentiable

(B) f(x) is not continuous at x = 1

(C) f(x) is differentiable at x = 1

(D) $\lim\limits_{x \to 1}$ f(x) does not exits

50. Let f″(x) be continuous at x = 0 and f″(0) = 4 then value of $\lim\limits_{x \to 0} \dfrac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is

(A) 11 (B) 2

(C) 12 (D) none

Multiple Correct Answer Type for JEE Advanced

51. Let f (x)= |x − 1| ([x] − [−x]) , then which of the following statement(s) is/are correct?

(where [x] denotes greatest integer function)

(A) f (x) is continuous at x = 1

(B) f (x) is derivable at x = 1

(C) f (x) is non-derivable at x = 1

(D) f (x) is discontinuous at x = 1.

52. Which of the following function will have tangent at indicated point

(A) $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$ at x = 0

(B) f(x) = sgn (sin x) at x = π

(C) f(x) = sgn (x^2) at x = 0

(D) f(x) = $|x - 1|^{11/10}$ at x = 1

53. Let f (x) = sin πx, g (x) = sgn (x) and h (x) = gof(x) then

(A) h(x) is discontinuous at infinite number of points

(B) h′(x) = 0 for all x ∈ R – I

(C) $\lim\limits_{x \to I} h(x)$ does not exist

(D) h(x) is periodic with period 1.

54. If f(x) = $\dfrac{\tan[x]\pi}{[1+|\log(\sin^2 x+1)|]}$, where [.] denotes the greatest integer function then f(x) is

(A) discontinuous ∀ x ∈ I

(B) continuous ∀ x ∈ R

(C) non differentiable ∀ x ∈ I

(D) a periodic function with no fundamental period

55. Let f and g be two functions defined as follows :

$f(x) = \dfrac{x+|x|}{2}$ for all x & $g(x) = \begin{bmatrix} x & \text{for } x<0 \\ x^2 & \text{for } x \ge 0 \end{bmatrix}$ then

(A) (gof)(x) & (fog)(x) are both continuous for all x ∈ R

(B) (gof)(x) & (fog)(x) are unequal functions

(C) (gof) is differentiable at x = 0

(D) (fog)(x) is not differentiable at x = 0 .

56. If a function f : R ⟶ R satisfies

$|\Sigma\, 3^k (f(x + ky) - f(x - ky))| \le 1$ for every n ∈ N and for all x y ∈ R, then

(A) $|3^n (f(x + ny) - f(x - ny))| \le 2$

(B) $|f(u) - f(v)| \le \dfrac{2}{3^n}$

(C) f(x) is a constant function

(D) f′(x) = 1 ∀ x ∈ R

57. If we define new derivative of function f to be

$\Delta(f(x)) = \lim\limits_{\Delta x \to 0} \dfrac{f^3(x+\Delta x) - f^3(x)}{\Delta x}$, then

(A) Δ (x) = $3x^3$

(B) Δ (tan x) at x = $\dfrac{\pi}{4}$ equals 6

(C) Δ (sin x) at x = 0 equals 1

(D) Δ ($\cos^{-1}$ x) at x = 0 equal – $\dfrac{3\pi^2}{4}$

58. if f(x) = minimum $\left(\cos x, \dfrac{1}{2}, \{\sin x\}\right)$, 0 ≤ x ≤ 2π, where { . } represents fractional part function, then f(x) is differentiable at

(A) $\dfrac{3\pi}{2}$ (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

59. If f(x) = minimum {x^3 – 1, – x + 1, sgn (–x)}, then f(x) is

(A) continuous at x = 0

(B) differentiable at x = 0

(C) continuous at x = 2

(D) differentiable at x = 2

60. If [.] denotes greatest integer function, then for the function y = sin π (|x| + [x]), which of the following is correct

(A) not continuous at x = –1/2

(B) continuous at x = 0

(C) differentiable in [–1, 0)

(D) differentiable in (0,1]

61. If f(x) is differentiable everywhere and f(a) = 0, then at x = a, f′(a) is non-zero
(A) | f | is differentiable
(B) $|f|^2$ is differentiable
(C) f | f | is differentiable
(D) $f + |f|^2$ is differentiable

62. Let $f(x) = x^2 \sin\frac{1}{x}$ for 0 < x ≤ 1 and f (0) = 0. If $g(x) = x^2$ for x ∈ [0, 1] then which of the following statement(s) is/are correct?
(A) f (x) is differentiable in [0, 1]
(B) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}$ does not exist.
(C) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}$ exist
(D) $\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist.

63. If $f(x) = \begin{cases} x+1 & , \quad x \ge 1 \\ 2-x & , \quad x < 1 \end{cases}$ and g(x) = f(x) + f(|x|), then g(x) is
(A) not continuous at x = –1
(B) not continuous at x = 1
(C) continuous at x = 0
(D) not differentiable at x = 0

64. A function f(x) satisfies the relation
f(x + y) = f(x) + f(y) + xy (x + y) ∀ x, y ∈ R.
If f′(0) = –1, then
(A) f(x) is a polynomial function
(B) f(x) is an exponential function
(C) f(x) is twice differentiable for all x ∈ R
(D) f′(3) = 8

65. Function f(x) = max (|tan x|, cos|x|) is
(A) non-differentiable at 4 points in (–π, π)
(B) discontinuous at 2 points in (–π, π)
(C) non-differentiable at only 2 points in (–π, π)
(D) in (–π, π) there are only 2 points where f(x) is continuous but not differentiable

66. Let a differentiable function f(x) be such that $|f(y) - f(y)| \le \frac{1}{2}|x - y|$ ∀ x, y ∈R and $f'(x) \ge \frac{1}{2}$.
Then the number of points of intersection of the graph of y = f(x) with
(A) the line y = x is one
(B) the curve $y = -x^3$ is one
(C) the curve 2y = |x| is three
(D) None of these

67. Given that f′(x) is continuous at x = a. State which of the following statements are true
(A) $f'(a) = \lim\limits_{h\to 0}\frac{f(h)-f(a)}{h-a}$
(B) $f'(a) = \lim\limits_{h\to 0}\frac{f(a)-f(a-h)}{h}$
(C) $f'(a) = \lim\limits_{t\to 0}\frac{f(a+2t)-f(a)}{t}$
(D) $f'(a) = \lim\limits_{t\to 0}\frac{f(a+2t)-f(a+t)}{2t}$

68. Let $f(x) = \begin{cases} \sin^2 x.\cos\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$
(A) f is continuous at x = 0
(B) f′(0) = 1
(C) f′(0) does not exist
(D) f′ is not continuous at x = 0.

69. Let $f(x) = g'(x)\,\frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}}$ where g′ is the derivative of g and is a continuous function and a > 0 then $\lim\limits_{x\to 0} f(x)$ exists if
(A) g(x) is polynomial
(B) g(x) = x
(C) $g(x) = x^2$
(D) $g(x) = x^3\,h(x)$ where h(x) is a polynomial

70. Let f : R ⟶ R be a function such that,
$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, f(0)=3 and f′(0)=3, then
(A) $\frac{f(x)}{x}$ is differentiable in R
(B) f′ (x) = 3
(C) f(x) is continuous in R
(D) f(x) is bounded in R

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

71. Consider the function $f(x) = x^2 - 2x$ and g(x) = –|x|
Assertion (A) : The composite function F(x) = f(g(x)) is not derivable at x = 0.
Reason (R) : $F'(0^+) = 2$ and $F'(0^-) = -2$.

72. **Assertion (A) :** There is no polynomial function f such that $f(x + y) = f(x) + yf(f(x))$ for all $x, y \in R$

Reason (R) : $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = f(f(x))$.

If f is of degree n then the equation $n - 1 = n^2$ has no positive integral solution.

73. **Assertion (A) :** There are exactly three functions $f : R \to R$ such that

$f(x) f(y) = f(x) + f(y) - f(xy)$ ...(1)

$1 + f(x + y) = f(xy) + f(x) f(y)$ ...(2)

Reason (R) : Adding (1) and (2), we get

$$1 + f(x + y) = f(x) + f(y).$$

Now $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{f(h)-1}{h} = f'(0)$.

$\Rightarrow$ $f(x) = ax + b$ where a & b can be determined by using function's value at $x = 0, 1$.

74. **Assertion (A) :** The function $y = \sin^{-1}(\cos x)$ is not differentiable at $x = n\pi$, $n \in I$ is particular at $= \pi$.

Reason (R) : $\frac{dy}{dx} = \frac{-\sin x}{|\sin x|}$ so the function is not differentiable at the points where $\sin x = 0$.

75. Consider two functions

$f(x) = \sin x$ and $g(x) = |f(x)|$.

Assertion (A) : The function $h(x) = f(x)\, g(x)$ is not differentiable in $[0, 2\pi]$.

Reason (R) : f (x) is differentiable and g (x) is not differentiable in $[0, 2\pi]$

76. **Assertion (A) :** The function

$f(x) = c_1 e^{|x|} + c_2 |x|^3$, where c_1, c_2 are constants is differentiable at $x = 0$, provided $c_1 = 0$.

Reason (R) : $e^{|x|}$ is not differentiable at $x = 0$.

77. Consider two functions $f(x) = \sin x$ and $g(x) = |f(x)|$.

Assertion (A) : The function $h(x) = f(x)\, g(x)$ is not differentiable in $[0, 2\pi]$

Reason (R) : f (x) is differentiable and g (x) is not differentiable in $[0, 2\pi]$

78. Consider the function $f(x) = \cot^{-1}\left(\text{sgn}\left(\frac{[x]}{2x-[x]}\right)\right)$ where [ ] denotes the greatest integer function

Assertion (A) : f(x) is discontinuous at x = 1.

Reason (R) : f(x) is non derivable at x = 1.

79. Let $f(x) = \text{sgn}\, x$ and $g(x) = \begin{cases} x-1 & 0 < x \le 2 \\ 1+x^2 & 2 < x \le 4 \end{cases}$ then

Assertion (A): The function (fog)(x) is differentiable at x = 2.

Reason (R) : If f(x) is differentiable at x= g(x) is differentiable at x = a then g(x) is differentiable at x = a and f(x) is differentiable at x = g(a).

80. **Assertion (A) :**

$f(x) = \left|(x-1)^{2p_1+1}(x-2)^{2p_2+1}....(x-n)^{2p_n+1}\right|$ is differentiable everywhere where $p_1, p_2, ...p_n \in N$.

Reason (R) : $f(x) = x|x|$ is differentiable everywhere.

Comprehension – 1

Let $f(x) = \begin{cases} e^{\{x^2\}} - 1, & x > 0 \\ \dfrac{\sin x - \tan x + \cos x - 1}{2x^2 + \ln(2+x) + \tan x}, & x < 0 \\ 0, & x = 0 \end{cases}$

where { } represents fractional part function. Lines L_1 and L_2 represent tangent and normal to curve y = f (x) at x = 0. Consider the family of circles touching both the lines L_1 and L_2.

81. Ratio of radii of two circles belonging to this family cutting each other orthogonally is

(A) $2 + \sqrt{3}$ (B) $\sqrt{3}$

(C) $2 + \sqrt{2}$ (D) $2 - \sqrt{2}$

82. A circle having radius unity is inscribed in the triangle formed by L_1 and L_2 and a tangent to it. Then the minimum area of the triangle possible is

(A) $3 + \sqrt{2}$ (B) $3 - \sqrt{2}$

(C) $3 + 2\sqrt{2}$ (D) $3 - 2\sqrt{2}$

83. If centres of circles belonging to family having equal radii 'r' are joined, the area of figure formed is

(A) $2r^2$ (B) $4r^2$

(C) $8r^2$ (D) r^2

Comprehension – 2

Let f(x) be a real valued function not identically zero, which satisfies the following conditions.

I. $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$, $n \in N$, x, y are any real numbers

II. $f'(0) \ge 0$.

84. The value of $f'(10)$ is

(A) 10 (B) 0

(C) 2n + 1 (D) 1

85. The function f(x) is

(A) odd

(B) even

(C) neither even nor odd

(D) even as well as odd

86. The function f(x) is

(A) Continuous and differentiable everywhere

(B) Continuous everywhere but not differentiable at some points

(C) Discontinuous at exactly one point

(D) Discontinuous at infinitely many points

Comprehension – 3

Suppose f, g and h be three real valued function defined on R.

Let $f(x) = 2x + |x|$, $g(x) = \frac{1}{3}(2x - |x|)$ and $h(x) = f(g(x))$

87. The range of the function

$k(x) = 1 + \frac{1}{\pi}(\cos^{-1}(h(x)) + \cot^{-1}(h(x)))$ is equal to

(A) $\left[\frac{1}{4}, \frac{7}{4}\right]$ (B) $\left[\frac{5}{4}, \frac{11}{4}\right]$

(C) $\left[\frac{1}{4}, \frac{5}{4}\right]$ (D) $\left[\frac{7}{4}, \frac{11}{4}\right]$

88. The domain of definition of the function $l(x) = \sin^{-1}(f(x)) - g(x))$ is equal to

(A) $\left[\frac{3}{8}, \infty\right)$ (B) $(-\infty, 1]$

(C) $[-1, 1]$ (D) $\left(-\infty, \frac{3}{8}\right]$

89. The function $T(x) = f(g(f(x))) + g(f(g(x)))$, is

(A) continuous and differentiable in $(-\infty, \infty)$.

(B) continuous but not derivable $\forall\, x \in R$.

(C) neither continuous nor derivable $\forall\, x \in R$.

(D) an odd function.

Comprehension – 4

A function f (x) is said to be differentiable at x = c, if $f'(c^+) = f'(c^-)$ = a finite quantity. Also f (x) is said to be bounded if $|f(x)| < n\ \forall\, x$ for which f (x) is defined (where n is finite)

90. Let $f: R \to R$ and $c \in R$ such that $\lim_{n\to\infty} n\left(f\left(c + \frac{1}{n}\right) - f(c)\right)$ = a ($a \in R$) then

(A) $f'(c)$ exists and is equal to a

(B) $f'(c)$ does not exists

(C) $f'(c)$ exists but is not equal to a

(D) $f'(c)$ may or may not exist.

91. If $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$ and f(x) be continuous on R. Then

(A) f (x) is bounded on R and attains both maximum and minimum on R.

(B) f (x) is unbounded on R but attains minimum on R.

(C) f (x) is bounded on R and attains either maximum or minimum on R.

(D) f is unbounded.

92. f (x) is defined on [0, 1] and

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$ where a, c ∈ R and c > 0

(A) $f'(0)$ exists if $0 < a < 1$

(B) f (x) is continuous if $a < 0$

(C) $f'(x)$ is bounded and continuous if $a > c$

(D) f (x) and $f'(x)$ are continuous if $a > 1 + c$.

Comprehension – 5

Let $f: R \to R$ be a differential function satisfying $f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3}$ for all real x and y and $f'(2) = 2$.

93. The range of $f(|x|)$ is

(A) $[0, \infty)$ (B) $[1, \infty)$

(C) $[2, \infty)$ (D) None of these

94. If $f(x) = |f(|x|) - 3|$ for all $x \in R$, then for g(x)

(A) one non-differentiable point

(B) two non-differentiable point

(C) three non-differentiable point

(D) four non-differentiable point

95. The number of solutions of the equation $x^2 + (f(|x|)^2) = 9$ are

(A) 0 (B) 2

(C) 3 (D) 5

MATCH THE COLUMNS FOR JEE ADVANCED

96.

	Column-I		Column-II
(A)	Let $f(x) = \begin{cases} \tan^{-1} x, & \lvert x\rvert \geq 1 \\ \frac{x^2-1}{4}, & \lvert x\rvert < 1 \end{cases}$ $\frac{x^2-1}{4}$, $\lvert x\rvert < 1$, then f(x) is not differentiable at x equal to	(P)	–1
(B)	$f(x) = (x^2 - 4)\lvert x^2 - 5x + 6\rvert + \cos\lvert x\rvert$ is non derivable at x equal to	(Q)	1
(C)	If $\sin(x + y) = e^{x+y} - 2$, then $\frac{dy}{dx}$ is equal to	(R)	2

(D) Let f: R ⟶ R is defined by the equation $f(x + y) = f(x)\, f(y)\ \ \forall x, y \in R$, $f(0) \neq 0$ and $f'(0) = 2$ then $\dfrac{f'(x)}{f(x)}$ is equal to	(S) 3 (T) None of the above values

97.

Column-I	Column-II
(A) Let $g(x) = \begin{cases} a\sqrt{x+1} & \text{if } 0 < x < 3 \\ bx + 2 & \text{if } 3 \le x < 5 \end{cases}$, if g(x) is differentiable on (0, 5) then (a + b) equals	(P) 0
(B) Let a and b be real numbers and let $f(x) = a \sin x + b\sqrt[3]{x} + 4, \forall x \in R$ If $f(\log_{10}(\log_3 10)) = 5$ then the value of $f(\log_{10}(\log_{10} 3))$ is equal to	(Q) 1
(C) $\displaystyle\lim_{x \to 0} \frac{\sin^4 x + \sin^4 2x + \sin^4 3x}{\sin^3 x + \sin^3 2x + \sin^3 3x}$ equals	(R) 2
(D) Number of solution(s) of the equation $2^{\log_{10} x} + 8 = (x-8)^{\frac{1}{\log_{10} 2}}$ is	(S) 3 (T) 4

98.

Column-I	Column-II
(A) If f(xy) = f(x) . f(y) and f is differentiable at x = 1 such that $f'(1) = 1$ also $f(1) \neq 0$, then $f'(7)$ equals	(P) 1
(B) If [.] denotes greatest integer function, then number of points at which the function $f(x) = \lvert x^2 - 3x + 2\rvert + \lvert\sin x\rvert - [x - 1/2]$, $-\pi \le x \le \pi$, is non differentiable, is	(Q) 13
(C) Let $f(x) = [a + 7 \sin x]$, $x \in (0, \pi)$, $a \in I$, [.] denotes greatest integer function. Then number of points at which f(x) is not differentiable is	(R) 5
(D) If for a continuous function, f, f(0) = f(1) = 0, f'(1) = 2 and $y(x) = f(e^x)\, e^{f(x)}$, then y'(0) is equal to	(S) 3 (T) 2

99.

Column-I	Column-II
(A) If $f(x) = \begin{cases} \dfrac{\left(\dfrac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\,(\{x\}-\{x\}^3)}, & x > 0 \\ k, & x = 0 \\ \dfrac{A\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,(1-\{x\})}, & x < 0 \end{cases}$ is continuous at x = 0, then the value of $\sin^2 k + \cos^2\left(\dfrac{A\pi}{\sqrt{2}}\right)$, is (where {x} denotes fractional part function)	(P) 2
(B) If $f(x) = [2 + 5\lvert n\rvert \sin x]$ where $n \in I$ has exactly 19 points of non-differentiability in (0, π), then possible value(s) of 'n' are (where [x] denotes greatest integer function)	(Q) – 2
(C) Let $f(x) = \begin{cases} x\,\dfrac{\left(\frac{3}{4}\right)^{\frac{1}{x}} - \left(\frac{3}{4}\right)^{\frac{-1}{x}}}{\left(\frac{3}{4}\right)^{\frac{1}{x}} + \left(\frac{3}{4}\right)^{\frac{-1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.	(R) 3

If $P = f'(0^-) - f'(0^+)$, then $\lim\limits_{x\to P^-} \dfrac{\left(\exp\left((x+2)\ln 4\right)\right)^{\frac{[x+1]}{4}} - 16}{4^x - 16}$ is less than (where [x] denotes greatest integer function)

(D)	The number of points at which $g(x) = \dfrac{1}{1+\dfrac{2}{f(x)}}$ is not differentiable where $f(x) = \dfrac{1}{1+\dfrac{1}{x}}$, is	(S) -3
		(T) 5

100.

	Column-I	Column-II
(A)	If f : [0, 1] ⟶ [0, 1] is continuous, then the number of roots of the equation $f(x) = x^3$ is	(P) one
(B)	If f is derivable, then between the consecutive roots of $f'(x) = 0$, the number of roots of $f(x) = 0$ is	(Q) at least one
(C)	The number of values of x at which $f(x) = \lvert 1 - \lvert x-1\rvert\rvert$ is not differentiable	(R) at most one
(D)	Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 2-x, & \text{if } x \text{ is irrational}\end{cases}$ The number of values of x at which f(x) is continuous is	(S) more than one

Review Exercises for JEE Advanced

1. Let $f(x) = \dfrac{\sqrt{x+1}-1}{\sqrt{x}}$ for $x \neq 0$, $f(0) = 0$. Is the function f(x) continuous and differentiable at x = 0 ?

2. Let g(x) be polynomial of degree atmost two and f(x) be continuous at x = 0 and differentiable at x = 1. Find

$$g(x) \text{ if } f(x) = \begin{cases} \dfrac{5xe^{1/x}+2x}{3-e^{1/x}} & , \ x<0 \\ g(x) & , \ 0 \le x \le 1. \\ (x-1)^2 \sin\dfrac{1}{(1-x)^2} + x & , \ x>1 \end{cases}$$

3. Let $f(x) = \begin{cases} -x-1 & , \ -2 \le x < 0 \\ x^2 - x & , \ 0 \le x < 2 \\ 2-x & , \ 2 \le x \le 3 \end{cases}$ and

$g(x) = f(|x|) + |f(x)|$ check continuity and differentiability of g(x) over [–2, 3].

4. Check the continuity and differentiability of

$$f(x) = \sin^{-1}\left(\frac{2\sqrt{x^2-1}}{x^2}\right).$$

5. Let $f(x) = (x^2-4)\,|(x^3 - 6x^2 + 11x - 6)| + \dfrac{x}{1+|x|}$.

Find the set of points at which the function f(x) is not differentiable.

6. Examine the differentiability of

$f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$ over its domain.

7. Let $f(x) = x^3 - 3x$ and

$$g(x) = \begin{cases} \min(f(t) : 0 \le t \le x) & , \ 0 \le x \le 2 \\ 2x-5 & , \ 2 < x \le 3. \\ (x-2)^2 & , \ x > 3 \end{cases}$$

Draw g(x) and discuss continuity and differentiability of g in (0, 4).

8. Let $f(x) = \begin{cases} x^2 + a & , \ 0 \le x < 1 \\ 2x+b & , \ 1 \le x \le 2 \end{cases}$

and $g(x) = \begin{cases} 3x+b & , \ 0 \le x < 1 \\ x^3 & , \ 1 \le x \le 2 \end{cases}$. If $\dfrac{d(f(x))}{d(g(x))}$ exists at x = 1, find the values of a, b and also its value.

9. If f and g be two functions having the same domain D, if f and g be derivable at $x_0 \in D$, and if $f(x_0) \neq g(x_0)$, then prove that each of the function max. {f, g} and min. {f, g} is derivable at x_0. What happens if $f(x_0) = g(x_0)$?

10. (i) $f : [0, 1] \to [1, 2]$ be a differentiable function, then prove that the number of solution of $f(x) - e^{x(x-1)} - \log_2(1+x) = 0$ will be at least one.

(ii) Let $f : [1, e] \to [0, 1]$ be continuous then prove that $f(x) = \ell n\, x$ has atleast one solution in [1, e].

11. If a differentiable function f(x) satisfies the relation $\frac{x}{y}\,f\left(\frac{x}{y}\right) = x\,f(x) - yf(y)\ \forall\ x, y \in R\ \&\ f(e) = 1/e$, then prove that $f(x) = \frac{\log x}{x}$.

12. Given a real valued function f(x) as follows :

$f(x) = \frac{x^2 + 2\cos x - 2}{x^4}$ for $x < 0$; $f(0) = \frac{1}{12}$ &

$f(x) = \frac{\sin x - \ell n(e^x \cos x)}{6x^2}$ for $x > 0$. Test the continuity and differentiability of f(x) at x = 0.

13. Let f be given by

$$f(x) = \begin{cases} x^2 \sin(x^{-4/3}) & \text{when } x \le 0 \\ 0 & \text{when } x = 0 \end{cases}.$$

Show that f has a derivative for all values of x, and that $f'(0) = 0$. Prove that f′ is not, however, continuous at x = 0

14. Draw the graph of the function $f(x) = x - |x - x^2|, -1 \le x \le 1$ and discuss its continuity in [–1, 1] and differentiability in (–1, 1).

15. If $f(x) = \begin{cases} \frac{\tan[x^2]\pi}{ax^2} + ax^3 + b & , \quad 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x & , \quad 1 < x \le 2 \end{cases}$ is differentiable in [0, 2], then $a = \frac{1}{k_1}$ and $b = \frac{\pi}{4} - \frac{26}{k_2}$.

Find $k_1^{\,2} + k_2^{\,2}$ { where [ ] denotes greatest integer function}.

16. If $f(x) = \begin{cases} x^a \,.\, \cos\left(\frac{1}{x^3}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is continuous but non-differentiable at x = 0, then all possible values of a.

17. Examine the function, $f(x) = x \,.\, \frac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \ne 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin .

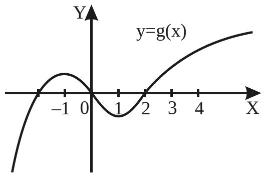
18. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24$ and

$$g(x) = \begin{cases} \min f(t)\ ;\ x \le t \le x+1, -1 \le x \le 1 \\ x - 10\ ;\ x > 1 \end{cases}$$

Discuss the differentiability of g(x) in [–1, 2].

19. For the function g whose graph is given, arrange the following numbers in increasing order.

0, g ′(–2), g ′(0), g ′(2), g ′(4)



20. Let f (x) = max. {|x² – 2 |x| |, |x|} and g (x) = min. {|x² – 2 |x| |, |x|} then show that f(x) is not differentiable at 5 points and g (x) is non differentiable at 7 points.

21. Consider function f : R – {–1, 1} ⟶ R.

$f(x) = \frac{x}{1 - |x|}$. Find f ′(x), draw the graph of f′ and prove that it is not derivable at the origin.

22. Suppose the function f satisfies the conditions :

(i) f(x + y) = f(x)f(y) for all x and y.

(ii) f(x) = 1 + x.g(x) where $\lim_{x \to 0} g(x) = 1$

Show that the derivative f ′(x) exists and f ′(x) = f(x) for all x.

23. If f(x) = |x – 1| . ([x] – [–x]), then find $f'(1^+)$ and $f'(1^-)$ where [x] denotes greatest integer function.

24. Let f : R ⟶ R be continuous function $|f(x)| \le x^{4n}$, $n \in N$. Prove that f(x) is differentiable at x = 0.

25. Find the values of

(a) $\lim_{h \to 0} \frac{f(3+h)^2 - f(3-h^2)}{2h^2}$ if f ′(3) = 2 ;

(b) $\lim_{h \to 0} \frac{f(a+2h^2) - f(a-2h^2)}{h^2}$ if $f'(a) = \frac{1}{4}$.

26. If $f(x) = \begin{cases} \frac{5e^{1/x} + 2}{3 - e^{1/x}} & , \ x \ne 0 \\ 0 & , \ x \ne 0 \end{cases}$ Examine f(x), xf(x) and $x^2 f(x)$ for continuity and differentiability at x= 0

27. $f(x) = \begin{bmatrix} 1 - x\ , (0 \le x \le 1) \\ x + 2, (1 < x < 2) \\ 4 - x\ , (2 \le x \le 4) \end{bmatrix}$. Discuss the continuity & differentiability of y = f (f(x)) for $0 \le x \le 4$.

28. The function $f(x) = \begin{cases} ax(x-1) + b\ , & x < 1 \\ x - 1 & , \ 1 \le x \le 3 \\ px^2 + qx + 2\ , & x > 3 \end{cases}$ find the values of the constant a, b p and q so that

(i) f(x) is continuous for all x.

(ii) f ′(1) does not exist.

(iii) f ′(x) is continuous at x = 3.

29. Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function f(x) = x.

$\sin \frac{1}{x}\ \sin\left(\frac{1}{x.\sin {}^{1}\!/_{x}}\right)$ where $x \ne 0$, $x \ne 1/r\pi$ &

f(0) = f(1/rπ) = 0, r = 1, 2, 3, ...

30. Suppose that instead of the usual definition of the derivative Df(x), we define a new of derivative, D*f(x), by the formula,

$$D^*(x) = \lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h},$$

where $f^2(x)$ means $[f(x)]^2$. Derive the formula for computing the derivative D* of a product & quotient.

31. Let $f(x + y) - 2f(x - y) + f(x) - 2f(y) = y - 2$ for all real x and y. If f'(x) exist, find f(x).

32. Let $f(xy) = xf(x) + yf(y)$ for all real x and y. If f is a differentiable function and $f(0) = 0$, find f(x).

33. A differentiable function f satisfies the relation $f(xyz) = f(x) + f(y) + f(z)\ \forall\ x, y, z \in R^+$ and $f'(1) = 1$. Find f(x).

34. A differentiable function f satisfies the relation $xf(y) + yf(x) = (x + y)\ f(x)\ .\ f(y)\ \ \forall\ x, y \in R$. If $f'(0) = 0$, find f(x).

35. A differentiable function f satisfies the relation $f(x + y) = f(x) + f(y) + f(x).f(y)\ \forall\ x, y \in R$. If $f(0) \neq -1$ and $f'(0) = -1$, find f(x).

Target Exercises for JEE Advanced

1. Let 'f' be a function whose graph is obtained by summing the ordinate values of the graph of another function 'g' after shifting its graph by 1 unit leftward and rightward.

If $g(x) = \begin{cases} 1-|x| & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$ discuss the continuity and differentiability of f(x).

2. Check the differentiability of

$$f(x) = \begin{cases} (x-e)2^{-2^{\left(\frac{1}{e-x}\right)}}, & x \neq e \\ 0 \quad , & x = e \end{cases} \quad \text{at } x = e.$$

3. Let $f(x) = \begin{cases} \sqrt{x}(1+x\sin\frac{1}{x}) & , x > 0 \\ -\sqrt{|x|}(1+x\cos\frac{1}{x}) & , x < 0 \\ 0 & , x = 0 \end{cases}$.

Show that f(x) is not differentiable at x = 0.

4. A differentiable function f satisfies the relation $f(x^2 + y^2) = (f(x))^2 + (f(y))^2\ \forall\ x, y \in R$. Find f(x).

5. If $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\ \forall\ x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\frac{dy}{dx}\ \frac{f(x)}{x} = 1$, find f(x).

6. Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = 3 - x$

We define a function

$$h(x) = \begin{cases} \max\{f(u) & , 0 \le u \le x\}\ , 0 \le x \le 1 \\ \min\{g(v) & , 1 < v \le x\}\ , 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of h(x) in the interval [0, 2].

7. If $g(x) = \begin{cases} 0 & \text{for } -e \le x < 1 \\ \left\{1+\frac{1}{3}\sin(\ln x^{2\pi})\right\} & \text{for } 1 \le x \le e \end{cases}$. where { } denotes the fractional part function.

$$f(x) = \begin{cases} xg(x) & \text{for } 1 \le x \le e \\ x(g(x)+1) & \text{otherwise} \end{cases}.$$

Discuss continuity and differentiability of f(x) over its domain.

8. Suppose f is differentiable on the interval (a, ∞) and $\lim\limits_{x\to\infty}(f(x)+f'(x)) = \ell\ (\ell \in R)$.

Prove that $\lim\limits_{x\to\infty} f(x) = \ell$ and $\lim\limits_{x\to\infty} f'(x) = 0$.

9. Let f be a continuous and differentiable function in (a, b), $\lim\limits_{x\to a^+} f(x) = \infty$

and $\lim\limits_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \ge -1$ for $a < x < b$.

Prove that $b - a > \pi$.

10. If

$$f(n) = \lim_{X\to 0} \frac{1-\cos^2 x\cos^2\frac{x}{2}.\cos^2\frac{x}{8}.......\cos^2\frac{x}{2^{n-1}}}{x^2}$$

and $\lambda = \lim\limits_{n\to\infty} f(n)$. Find the absolute value of 27abc for which the function

$$g(x) = \begin{cases} \frac{a}{\pi}\cot^{-1}x & x < 0 \\ \lambda & x = 0 \\ \frac{b}{\pi}x + c & x > 0 \end{cases}$$

is differentiable at x = 0.

11. Let $f(x) = \begin{cases} ax(x-1)+b & ; & x < 1 \\ x+2 & ; & 1 \le x \le 3 \\ px^2+qx+2 & ; & x > 3 \end{cases}$ is continuous for all x except x = 1 but |f(x) | is differentiable every where and f'(x) is continuous at x = 3 and $|a + b + p + q| = \frac{k}{18}$, then find the value of k.

12. Two real valued, continuous and differentiable functions f(x) and g(x) defined on R satisfy the conditions

$f(x + y) = f(x)g(y) + g(x)f(y)$

$g(x + y) = g(x)g(y) - f(x)f(y)$

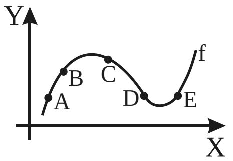
If $g'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$.

13. Sketch the graph of the function

$$f(x) = \left[\begin{array}{ll} \dfrac{x}{1+|x|} & \text{if } |x| \ge 1 \\ \dfrac{x}{1-|x|} & \text{if } |x| < 1 \end{array}\right.$$

and find the domain of $f'(x)$.

14. Use the graph of f to answer each question.



(i) Between which two consecutive points is the average rate of change of the function greatest ?

(ii) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B ?

(iii) Sketch a tangent line to the graph between the points C and D so that the slope of the tangent line is the same as the average rate of change of the function between C and D.

(iv) Give any sets of consecutive points for which the average rates of change of the function are approximately equal.

15. Given f(x + y + z) = f(x) f(y) f(z) for all real x, y, z if f(2) = 4 and $f'(0) = 3$, then show that f(0) = 1 and $f'(2) = 12$.

16. Let $f(x) = x^3 - 3x^2 + 6$ ∀ $x \in R$ and

$$g(x) = \begin{cases} \max.\{f(t),\ x+1 \le t \le x+2\}, & -3 \le x \le 0 \\ 1 - x \text{ for } x \ge 0 \end{cases}.$$

Test continuity of g(x) for $x \in [-3, 1]$.

17. If $f(x) = x^2 - 2|x|$ and

$$g(x) = \begin{cases} \min.\{f(t) : -2 \le t \le x\}, & -2 \le x \le 0 \\ \max.\{f(t) : 0 \le t \le x\}, & 0 \le x \le 3 \end{cases}.$$

(a) Draw the graph of f(x) and discuss its continuity and differentiability.

(b) Find and draw the graph of g(x). Also, discuss the continuity.

18. Suppose that f(x) is continuous at a point 'a'. Prove that it is differentiable at 'a' if and only if there exists some linear function L(x) such that $\dfrac{|f(x) - L(x)|}{|x - a|}$ tends to '0' as x tends to 'a'.

19. Suppose that f(x) is a differentiable function such that $f'(x) = g(x)$, $g''(x)$ exists and $|f(x)| \le 1$ for all real x. If $\{f(0)\}^2 + \{g(0)\}^2 = 9$, then show that for some $c \in (-3, 3)$, $g(c)g''(c) < 0$.

20. If $f(x) = \cos x . \cos\left(\dfrac{1}{\cos x}\right)$ when $x \in (0, 2\pi) - \{\pi/2, 3\pi/2\}$, and f(x) = 0 when $x \in \{\pi/2, 3\pi/2\}$. Show that f is continuous but not differentiable at $x = \pi/2, 3\pi/2$.

21. Prove that the function $f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ is infinitely differentiable at the point x = 0.

22. A function $f : R \to [1, \infty)$ satisfies the equation
f(xy) = f(x) f(y) – f(x) – f(y) + 2
If f is differentiable on R and f(2) = 5, then show that $f'(x) = \dfrac{f(x) - 1}{x} . f'(1)$ Hence, determine f(x).

23. Let f : R → R satisfy $|f(x)| \le x^2 e^x$ ∀ $x \in R$, then show that f(x) is differentiable at x = 0.

24. Let $f(x) = \dfrac{|x-1|}{1+1/x}$ and $g(x) = \dfrac{1}{1 + \dfrac{1}{f(x)}}$, then find the points where g(x) is not differentiable.

25. If f(x + y) = f(x) + f(y) ∀ x, y ∈ R, f(1) = 1 then find

$$\lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$$

26. Let $f(x) = \begin{cases} x - 1, & -1 \le x < 0 \\ x^2, & 0 \le x \le 1 \end{cases}$. Discuss the differentiability of h(x) = f(|sin x|) + |f(sin x)| in [0, 2π].

27. Let f : R → R satisfy f(xy) = f(x). f(y) – f(x + y) + 1 for all x, y ∈ R. If f(x) is differentiable and $f'(0) = 1$ then determine all functions 'f'.

28. Suppose that function g(t) and h(t) are defined for all values of t and that g(0) = h(0) = 0. Can $\lim\limits_{t \to 0} g(t)/(h(t)$ exist? If it does exist, must it equal zero? Give reason for your answer.

29. Given a function f, we say that f is differentiable at $x_0 \in D_f$ if there exists a real number A such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - A(x - x_0)|}{|x - x_0|}.$$

Prove that if f is differentiable at x_0 then the value of A in the above definition is the derivative $f'(x_0)$. (Note that the magnification function $A(x - x_0)$ is called the differential of f at x_0, and A is called the differential coefficient.)

30. A differentiable function f satisfies
$(x - y)\, f(x + y) - (x + y)\, f(x - y)$
$= 4xy(x^2 - y^2)\ \forall\ x, y \in R$, where $f(1) = 1$.
Find f(x).

31. A differentiable function f satisfies the relation $f(x+y) - f(x - y) = f(x) . f(y)\ \forall\ x, y \in R$.
If $f'(0) = 0$, find f(x).

32. Let f be defined on (a, b) and differentiable at c, where $a < c < b$. Suppose $x_n \to c$ and $z_n \to c$, where $x_n < c < z_n$ for all n.

Prove that $\dfrac{f(x_n) - f(z_n)}{x_n - z_n} \to f'(c)$.

33. A differentiable function f satisfies the relation $f\left(\dfrac{x+y}{1+xy}\right) = f(x) . f(y)\ \forall\ x, y \in R - \{-1\}$, where $f(0) \neq 0$ and $f'(0) = 1$. Find f(x).

34. A differentiable function f satisfies the relation
$[1 + f(x).f(y)]\, f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$.
If $f(0) = 0$ and $f'(0) = 1$, find f(x).

35. Given a differentiable function f(x) defined for all real x, and is such that $f(x + h) - f(x) \le 6h^2$ for all real h and x. Show that f(x) is constant.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. Let $f(x) = \begin{cases} (x-1)^2 \sin\dfrac{1}{(x-1)} - |x| & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$

be a real-valued functions. Then the set of points where f(x) is not differentiable is........... **[IIT - 1981]**

2. Let $f(x) = x\,|x|$. The set of points where f(x) is twice differentiable is............... **[IIT - 1992]**

B. Multiple Choice Questions with ONE correct answer :

3. For a real number y, let [y] denotes the greatest integer less than or equal to y : Then the fuction
$f(x) = \dfrac{\tan(\pi[x - \pi])}{1 + [x]^2}$ is **[IIT - 1981]**
(A) discontinuous at some x
(B) continuous at all x, but the derivative $f'(x)$ does not exist for some x
(C) $f'(x)$ exist for all x, but the derivative f''(x does not exist for some x
(D) $f'(x)$ exits for all x

4. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then **[IIT - 1985]**
(A) f(x) is continuous but not differentiable at x = 0
(B) f(x) is differentiable at x = 0
(C) f(x) is not differentiable at x = 0
(D) none of these

5. Let [x] denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then f(x) is **[IIT - 1986]**
(A) continuous at x = 0
(B) continuous in (–1, 0)
(C) differentiable at x = 1
(D) differentiable in (–1, 1)
(E) none of these

6. The set of all points where the function
$f(x) = \dfrac{x}{(1 + |x|)}$ is differentiable, is **[IIT - 1987]**
(A) $(-\infty, \infty)$ (B) $[0, \infty)$
(C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$
(E) none of these

7. Let $f : R \to R$ be a differentiable function and f(1) = 4. Then the value of $\lim\limits_{x\to 1} \displaystyle\int_4^{f(x)} \frac{2t}{x-1}\, dt$ is **[IIT - 1990]**
(A) $8f'(1)$ (B) $4f'(1)$
(C) $2f'(1)$ (D) $f'(1)$

8. Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then : **[IIT - 1993]**
(A) $\lim\limits_{x\to 0} f(x)$ does not exist
(B) f(x) is continuous at x = 0
(C) f(x) is not differentiable at x = 0
(D) $f'(0) = 1$

9. The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is Not differentiable at - **[IIT - 1999]**
(A) – 1 (B) 0
(C) 1 (D) 2

10. Find left hand derivative of $f(x) = [x]\sin(\pi x)$ at k, k an integer if **[IIT - 2001]**
(A) $(-1)^k (k - 1)\pi$ (B) $(-1)^{k-1} (k - 1)\pi$
(C) $(-1)^k (k - 1)k\pi$ (D) $(-1)^{k-1} (k - 1)k\pi$

11. Let $f : R \to R$ be a function which is defined by $f(x) = \max\{x, x^3\}$. The set of all points where f(x) is not differentiable is - **[IIT - 2001]**

(A) {–1, 1} (B) {–1, 0}
(C) {0, 1} (D) {–1, 0, 1}

12. Which of the following functions is differentiable at x = 0 ? **[IIT - 2001]**
(A) cos (|x|) + |x| (B) cos (|x|) – |x|
(C) sin (|x|) + |x| (D) sin (|x|) – |x|

13. The domain of the derivative of the function

$$f(x)=\begin{cases}\tan^{-1}x, \text{ if } |x|\le 1\\ \frac{1}{2}(|x|-1), \text{ if } |x|>1\end{cases} \text{ is}$$ **[IIT - 2002]**

(A) R – {0} (B) R – {1}
(C) R – {–1} (D) R – {–1, 1}

14. If f(x) is differentiable and strictly increasing function, then the value of $\lim\limits_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is **[IIT - 2004]**
(A) 1 (B) 0
(C) –1 (D) 2

15. The function given by y = ||x| – 1| is differentiable for all real number except the points **[IIT - 2004]**
(A) {0, 1, –1} (B) ± 1
(C) 1 (D) –1

16. If f is a differentiable function and f(1/n) = 0 ∀ n ≥ 1 and n ∈ I, then **[IIT - 2005]**
(A) f(x) = 0, x ∈ (0, 1]
(B) f(0) = 0, f ′(0) = 0
(C) f(0) = 0 = f ′(x), x ∈ (0, 1]
(D) f(0) = 0 and f ′(0) need not to be zero

17. Let f(x) be differentiable on the interval (0, ∞) such f(1) = 1, and $\lim\limits_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each x > 0.
Then f(x) is **[IIT - 2007]**
(A) $\frac{1}{3x}+\frac{2x^2}{3}$ (B) $\frac{-1}{3x}+\frac{4x^2}{3}$
(C) $\frac{-1}{x}+\frac{2}{x^2}$ (D) $\frac{1}{x}$

18. Let $g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)}$; 0 < 2, m and n are integers, m ≠ 0, n > 0 and let p be the left hand derivative of |x – 1| at x = 1. If $\lim\limits_{x\to 1}g(x)=p$, then : **[IIT - 2008]**
(A) n = 1 n = 1 (B) n = 1, m = –1
(C) n = 2, m = 2 (D) n > 2, m = n

19. Let $f(x)=\begin{cases}x^2\left|\cos\frac{\pi}{x}\right|, x\ne 0\\ 0, x=0\end{cases}, x\in,$ then f is
(A) differentiable both at x = 0 and at x = 2
(B) differentiable at x = 0 but not differentiable at x = 2
(C) not differentiable at x = 0 but differentiable at x = 2
(D) differentiable neither at x = 0 nor at x = 2 **[IIT - 2012]**

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answer :

20. If x + |y| = 2y, then y as a function of x is **[IIT - 1984]**
(A) defined for all real x
(B) continuous at x – 0
(C) differentiable for all x
(D) such that $\frac{dy}{dx}=\frac{1}{3}$ for x < 0

21. The function f(x) = 1 + |sin x| is **[IIT - 1986]**
(A) continuous now here
(B) continuous everywhere
(C) differentiable nowhere
(D) not differentiable at x = 0
(E) not differentiable at infinite number of points.

22. The function $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$ is **[IIT - 1988]**
(A) continuous at x = 1
(B) differentiable at x = 1
(C) continuous at x = 3
(D) differentiable at x = 3

23. Let h(x) = min {x, x^2}, for every real number of x. Then **[IIT - 1998]**
(A) h is continuous for all x
(B) h is differentiable for all x
(C) h’(x) = 1 , for all x > 1
(D) h is not differentiable at two values of x’

24. Let $f(x)=\begin{cases}0, & x<0\\ x^2, & x\ge 0\end{cases}$ then for all x **[IIT - 1994]**
(A) f ′ is differentiable (B) f is differentiable
(C) f ′ is continuous (D) f is continuous

25. Let g(x) = x f(x), where $f(x)=\begin{cases}x\sin(1/x) & , x\ne 0\\ 0 & , x=0\end{cases}$ At x = 0 **[IIT - 1994]**
(A) g is differentiable but g’ is not continuous
(B) g is differentiable while f is not
(C) both f and g are differentiable
(D) g is differentiable and g’ is continuous

26. The function f(x) = max {(1 – x), (1 + x), 2}, x ∈ (–∞, ∞) is **[IIT - 1995]**
(A) continuous at all points
(B) differentiable at all points

(C) differentiable at all points except at $x = 1$ and $x = -1$

(D) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous

27. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is - **[IIT - 2000]**

(A) onto if f is onto

(B) one-one if f is one-one

(C) continuous if f is continuous

(D) differentiable if f is differentiable

28. If $f(x) = \min. \{1, x^2, x^3\}$, then **[IIT - 2006]**

(A) f(x) is continuous $\forall\ x \in R$

(B) f(x) is continuous & differentiable everywhere

(C) f(x) is continuous but not differentiable $\forall\ x \in R$

(D) f(x) is not differentiable at two points

29. If $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x - 1 & 0 < x \le 1 \\ \ell n\ x, & x > 1 \end{cases}$, then **[IIT - 2011]**

(A) f(x) is continuous at $x = -\dfrac{\pi}{2}$

(B) f(x) is not differentiable at $x = 0$

(C) f(x) is differentiable at $x = 1$

(D) f(x) is differentiable at $x = -\dfrac{3}{2}$

30. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y)$, $\forall\ x, y \in R$. If f(x) is differentiable at $x = 0$, then **[IIT - 2011]**

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous $\forall\ x \in R$

(C) $f'(x)$ is constant $\forall\ x \in R$

(D) f(x) is differentiable except at finitely many points

31. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x + y) = f(x) + f(y),\ \forall\ x, y \in \mathbb{R}$$

If f(x) is differentialbe at $x = 0$, then **[IIT - 2011]**

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous $\forall\ x \in \mathbb{R}$

(C) f'(x) is constant $\forall\ x \in \mathbb{R}$

(D) f(X) is differentiable except at finitely many points.

32. If $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, x \le -\dfrac{\pi}{2} \\ -\cos x, -\dfrac{\pi}{2} < x \le 0 \\ x - 1, 0 < x \le 1 \\ \ln x, x > 1 \end{cases}$, then **[IIT - 2011]**

(A) f(x) is continuous at $x = -\dfrac{\pi}{2}$

(B) f(x) is not differentiable at $x = 0$

(C) f(x) is differentiable at $x = 1$

(D) f(x) is differentiable at $x = -\dfrac{3}{2}$

33. Let $g : \mathbf{R} \to \mathbf{R}$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \ne 0$. Let

$$f(x) = \begin{cases} \dfrac{x}{|x|} g(x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

and $h(x) = e^{|x|}$ for all $x \in \mathbf{R}$. Let (foh) (x) denote f(h(x)) and (hof)(x) denote h(f(x)), then which of the following is (are) true ?

(A) f is differentiable at $x = 0$

(B) h is differentiable at $x = 0$

(C) foh is differentiable at $x = 0$

(D) hof is differentiable at $x = 0$ **[2015]**

34. Consider the hyperbola $H : x_2 - y_2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

(A) $\dfrac{dl}{dx_1} = 1 - \dfrac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\dfrac{dm}{dx_1} = \dfrac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\dfrac{dl}{dx_1} = 1 + \dfrac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\dfrac{dm}{dy_1} = \dfrac{1}{3}$ for $y_1 > 0$ **[2015]**

35. Let $f : \left[-\dfrac{1}{2}, 2\right] \to \mathbf{R}$ and $g : \left[-\dfrac{1}{2}, 2\right] \to \mathbf{R}$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\, f(x) + |4x - 7|\, f(x)$, where [y] denotes the greatest integer less than or equal to y for $y \in \mathbf{R}$. Then

(A) f is discontinuous exactly at three points in $\left[-\dfrac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points is $\left[-\dfrac{1}{2}, 2\right]$

(C) g is not differentiable exactly at four points is $\left(-\dfrac{1}{2}, 2\right)$

(D) g is not differentiable exactly at five points in $\left(-\dfrac{1}{2}, 2\right)$ **[2016]**

36. Let a, b $\in$ **R** and f : **R** $\to$ **R** be defined by f (x) = a cos ($|x^3 - x|$) + b $|x|$ sin ($|x^3 + x|$). Then f is

(A) differentiable at x = 0, if a = 0 and b = 1

(B) differentiable at x = 1, if a = 1 and b = 0

(C) not differentiable at x = 0, if a = 1 and b = 0

(D) not differentiable at x = 1, if a = 1 and b = 1 **[2016]**

37. Let f: $(0, \infty) \to$ **R** be differentiable function such that $f'(x) = 2 - \dfrac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$ then

(A) $\lim\limits_{x \to 0+} f'\left(\dfrac{1}{x}\right) = 1$

(B) $\lim\limits_{x \to 0+} xf\left(\dfrac{1}{x}\right) = 2$

(C) $\lim\limits_{x \to 0+} x^2 f'(x) = 2$

(D) $|f(x)| \le 2$ for all $x \in (0, 2)$ **[2016]**

38. Let f : **R** $\to$ **R**, g : **R** $\to$ **R** and h : **R** $\to$ **R** be differentiable function such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \dfrac{1}{15}$ (B) $h'(1) = 666$

(C) $h(0) = 16$ (D) $h(g(3)) = 36$ **[2016]**

39. For every twice differentiable function f : **R** $\to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE?

(a) there exist r, s $\in$ **R**, where r < s such that f is one-one on the open interval (r, s).

(b) there exits $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(c) $\lim\limits_{x \to 0} f(x) = 1$

(d) there exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$ **[2018]**

D. Assertion & Reasoning :

40. Let f and g be real valued functions defined on interval (–1, 1) such that $g''(x)$ is continuous, $g(0) \neq 0$, $f''(0) = 0$ and $f(x) = g(x) \sin x$.

Statement–1 : $\lim\limits_{x \to 0} [g(x) \cot x - g(0) \operatorname{cosec} x] = f''(0)$

Statement–2 : $f'(0) = g(0)$ **[IIT - 2008]**

E. Integer Answer Type :

41. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is : **[IIT - 2009]**

42. Let $y'(x) + y(x)\, g'(x) = g(x)$, $y(0) = 0$, $x \in \mathbb{R}$, where $f'(x)$ denotes $\dfrac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on $\mathbb{R}$ with $g(0) = g(2) = 0$. Then the value of y(2) is **[IIT - 2011]**

F. Subjective Problems :

43. Let $f(x) = \dfrac{x^2}{2}$, $0 \le x < 1$ **[IIT - 1983]**

$= 2x^2 - 3x + \dfrac{3}{2}$, $1 \le x \le 2$

Discuss the continuity of f, f′ and f″ on [0, 2].

44. Let $f(x) = x^3 - x^2 + x + 1$ and **[IIT - 1985]**

$g(x) = \max \{f(t);\ 0 \le t \le x\}$, $0 \le x \le 1$

$= 3 - x$ $1 < x \le 2$

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

45. Let f(x) be defined in the interval [–2, 2] such that

$f(x) = 1$, $-2 \le x \le 0$

$= x - 1$, $0 < x \le 2$

and $g(x) = f(|x|) + |f(x)|$

Test the differentiability of g(x) on $x \in [-2, 2]$

[IIT - 1986]

46. Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If f’(0) exists, find its value. **[IIT - 1987]**

47. Draw a graph of the function $y = [x] + |1 - x|$, $-1 \le x \le 3$. Determine the points, if any, where this function is not differentiable. **[IIT - 1989]**

48. A function R $\to$ R satisfies the equation $f(x + y) = f(x)\, f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at x = 0 and f’(0) = 2. Show that f’(x) = 2 f(x) for all x in R . Hence, determine f(x). **[IIT - 1990]**

49. Let $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$ for all real x and y. If f′(0) exists and equals – 1 and f(0) = 1. Find f(2)

[IIT - 1995]

50. Let $f(x) = \begin{cases} xe^{\left(-\frac{1}{|x|} + \frac{1}{x}\right)} & ;\ x \neq 0 \\ 0 & ;\ x \neq 0 \end{cases}$ test whether

[IIT - 1997]

(a) f(x) is continuous at x = 0

(b) f(x) is differentiable at x = 0

51. Determine the values of x for which the following function fails to be continuous or differentiable

$$f(x) = \begin{cases} 1 - x & ;\ x < 1 \\ (1 - x)(2 - x) & ;\ 1 \le x \le 2 \\ 3 - x & ;\ x > 2 \end{cases}$$

Justify your answer. **[IIT - 1997]**

52. Discuss the continuity and differentiability of the function

[IIT - 1998]

$$f(x) = \begin{cases} 2 + \sqrt{1 - x^2} & ;\ |x| \le 1 \\ 2e^{(1-x)^2} & ;\ |x| > 1 \end{cases}$$

53. Let $\alpha \in R$. Prove that a function $f : R \rightarrow R$ is differentiable at α if and only if there is a function $g: R \rightarrow R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. **[IIT - 2001]**

54. Let $f(x) = \begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x \geq 0 \end{cases}$, and

$g(x) = \begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x \geq 0 \end{cases}$, where a and b are non-negative real numbers Determine the composite function g of. If (g o f) (x) is continuous for all real x, determine the values of a and b. Further, for these value of a and b, is gof differentiable at x = 0 ? Justify your answer. **[IIT - 2002]**

55. If a function $f : [-2a, 2a] \rightarrow R$, is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and left hand derivative at $x = a$ is 0, then find the left hand derivative at $x = -a$. **[IIT - 2003]**

56. Let $f(x) = \begin{cases} b\sin^{-1}\dfrac{x+c}{2} & ; \quad -1/2<x<0 \\ 1/2 & ; \quad x=0 \\ \dfrac{e^{ax/2}-1}{x} & ; \quad 0<x<\dfrac{1}{2} \end{cases}$

If f(x) be a differentiable function at x = 0, & $|c| < 1/2$, then find the value of 'a' and prove that $64\,b^2 = (4 - c^2)$. **[IIT - 2004]**

57. If $f(x - y) = f(x) \cdot g(y) - f(y) \cdot g(x)$ and $g(x - y) = g(x) \cdot g(y) + f(x) \cdot f(y)$ for all $x, y \in R$. If right hand derivative at x = 0 exists for f(x). Find derivative of g(x) at x = 0 **[IIT - 2005]**

G. Match the Columns

58.

Column-I	**Column-II**
(A) $x\,\lvert x\rvert$	**(P)** continuous in (–1, 1)
(B) $\sqrt{\lvert x\rvert}$	**(Q)** differentiable in (–1, 1)
(C) $x + [x]$	**(R)** strictly increasing in (–1, 1)
(D) $\lvert x - 1\rvert + \lvert x + 1\rvert$	**(S)** not differentiable at least at one point in (–1, 1)

59. Let $f_1: \mathbf{R} \rightarrow \mathbf{R}$, $f_2: [0, \infty) \rightarrow \mathbf{R}$, $f_3: \mathbf{R} \rightarrow \mathbf{R}$ and $f_4: \mathbf{R} \rightarrow [0, \infty)$ be defined by

$f_1(x) = \begin{cases} |x| & \text{if } \quad x<0, \\ e^x & \text{if } \quad x \geq 0; \end{cases}$

$f_2(x) = x^2$;

$f_3(x) = \begin{cases} \sin x & \text{if } \quad x<0, \\ x & \text{if } \quad x \geq 0 \end{cases}$

and

$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } \quad x<0, \\ f_2(f_1(x))-1 & \text{if } \quad x \geq 0. \end{cases}$

List I	List II
P. f_4 is	1. onto but not one-one
Q. f_3 is	2. neither continuous nor one-one
R. f_2 o f_1 is	3. differentiable but not one-one
S. f_2 is	4. continuous and one-one

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

[2014]

Previous Year's Questions (JEE Main Papers)

1. Let f be a differentiable function such that $8f(x) + 6f\left(\dfrac{1}{x}\right) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$, then $\dfrac{dy}{dx}$ at $x = -1$ is

(a) $\dfrac{15}{14}$ (b) $-\dfrac{15}{14}$

(c) $-\dfrac{1}{14}$ (d) $\dfrac{1}{14}$ **[2013, online]**

2. For $a > 0$, $t \in \left(0, \dfrac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$. Then $1+\left(\dfrac{dy}{dx}\right)^2$ equals

(a) $\dfrac{x^2}{y^2}$ (b) $\dfrac{y^2}{x^2}$

(c) $\dfrac{x^2+y^2}{y^2}$ (d) $\dfrac{x^2+y^2}{x^2}$ **[2013, online]**

3. If $y = \sec(\tan^{-1} x)$, then $\left.\dfrac{dy}{dx}\right|_{x=1}$ is equal to

(a) $\dfrac{1}{2}$ (b) 1

(c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{2}}$ **[2013]**

4. Let $f(x) = \dfrac{x^2-x}{x^2+2x}$, $x \neq 0, -2$. Then $\dfrac{d}{dx}(f^{-1}(x))$ (wherever defined) is equal to

(a) $\dfrac{-1}{(1-x)^2}$ (b) $\dfrac{3}{(1-x)^2}$

(c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$ **[2013, online]**

5. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then g′(x) is equal to
(a) $1+x^5$ (b) $5x^4$
(c) $\frac{1}{1+(g(x))^5}$ (d) $1+(g(x))^5$ **[2014]**

6. Let f(x) = x|x|, g(x) = sin x and h(x) = go f(x). Then
(a) h(x) is not differentiable at x = 0
(b) h(x) is differentiable at x = 0, but h′(x) is not continuous at x = 0
(c) h′(x) continuous at x = 0, but it is not differentiable at x = 0
(d) h′(x) is differentiable at x = 0 **[2014, online]**

7. If $f(x) = x^2 - x + 5$, $x > \frac{1}{2}$, and g(x) is its inverse function, then g′(7) equals
(a) $-\frac{1}{3}$ (b) $\frac{1}{13}$
(c) $\frac{1}{3}$ (d) $-\frac{1}{13}$ **[2014, online]**

8. Let f : **R** → **R** be function such that $|f(x)| \le x^2$ for all x ∈ **R**. Then at x = 0, f is
(a) continuous but not differentiable
(b) continuous as well as differentiable
(c) neither continuous nor differentiable
(d) differentiable but not continuous **[2014, online]**

9. If $y = e^{nx}$, then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is equal to
(a) ne^{nx} (b) ne^{-nx}
(c) 1 (d) $-ne^{-nx}$ **[2014]**

10. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of k + m is
(a) 2 (b) $\frac{16}{5}$
(c) $\frac{10}{3}$ (d) 4 **[2015]**

11. For x ∈ **R**, f(x) = |log2 – sin x| and g(x) = f(f(x)), then
(a) g is not differentiable at x = 0
(b) g′(0) = cos(log 2)
(c) g′(0) = –cos(log 2)
(d) g is differentiable at x = 0 and g′(0) = –sin(log2) **[2016]**

12. If f(x) is differentiable function in the internal (0, ∞) such that f(1) = 1 and $\lim\limits_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t-x} = 1$ for each x > 0, then $f\left(\frac{3}{2}\right)$ is equal to
(a) $\frac{23}{18}$ (b) $\frac{13}{6}$
(c) $\frac{25}{9}$ (d) $\frac{31}{8}$ **[2016, online]**

13. If the function $f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x+b) & 1 \le x \le 2 \end{cases}$ is differentiable at x = 1, then $\frac{a}{b}$ is equal to
(a) $\frac{\pi+2}{2}$ (b) $\frac{\pi-2}{2}$
(c) $\frac{-\pi-2}{2}$ (d) $-1-\cos^{-1}(2)$ **[2016]**

14. If $y = \left[x+\sqrt{x^2-1}\right]^{15} + \left[x-\sqrt{x^2-1}\right]^{15}$ then $(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to
(a) 125 y (b) $224\ y^2$
(c) $225\ y^2$ (d) 225 y **[2017 online]**

15. If $2x = y^{1/5} + y^{-1/5}$ and $(x^2-1)\frac{d^2y}{dx^2} + \lambda x\frac{dy}{dx} + ky = 0$ then λ + k is equal to
(a) –23 (b) –24
(c) 26 (d) –26 **[2017 online]**

16. If for x ∈(0, 1/4), derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$ then g(x) is equal to
(a) $\frac{3}{1+9x^3}$ (b) $\frac{9}{1+9x^3}$
(c) $\frac{3x\sqrt{x}}{1-9x^3}$ (d) $\frac{3x}{1-9x^3}$ **[2017]**

17. If $x^2 + y^2 + \sin y = 4$, then the value of $\frac{d^2y}{dx^2}$ at the point (–2, 0) is
(a) –34 (b) –32
(c) –2 (d) 4 **[2018 online]**

18. If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim\limits_{x\to 0}\frac{f'(x)}{x}$
(a) exists and is equal to 0
(b) exists and is equal to –2
(c) exists and is equal to 2
(d) does not exist **[2018 online]**

19. If $f(x) = \sin^{-1}\left(\frac{2\times 3^x}{1+9^x}\right)$, then $f'\left(-\frac{1}{2}\right)$ is equal to

(a) $\sqrt{2}\log\sqrt{3}$ (b) $-\sqrt{3}\log 3$
(c) $-\sqrt{3}\log\sqrt{3}$ (d) $\sqrt{3}\log 3$ **[2018 online]**

20. For each x ∈**R**, let f(x) = |x – 1|, g(x) = cos x and ϕ(x) = f(g(2 sin x) – g(f(x)), then ϕ is

(a) differentiable at each point of **R**
(b) not differentiable at 0
(c) not differentiable at 1
(d) differentiable only in (–π/2, π/2) **[2018]**

21. Let S = {t ∈R : f(x) = |x – π|. $(e^{|x|} - 1)$ sin |x| is not differentiable at t}. Then the set S is equal to

(a) {0} (b) {π}
(c) {0, π} (d) ϕ (empty set) **[2018]**

22. If x = 3 tan t and y = 3 sec t, then the value of $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$ is

(a) $\frac{3}{2\sqrt{2}}$ (b) $\frac{1}{3\sqrt{2}}$
(c) $\frac{1}{6}$ (d) $\frac{1}{6\sqrt{2}}$ **[2019]**

23. Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8-2|x|, & 2<|x|\le 4 \end{cases}$

Let S be the set of points in the interval (–4, 4) at which f is not differentiable. Then S:

(a) is an empty set
(b) equals {–2, –1, 1, 2}
(c) equals {–2, –1, 0, 1, 2}
(d) equals {–2, 2}

24. Let f: (–1, 1) → **R** be a function defined by $\max\{-|x|, -\sqrt{1-x^2}\}$. If k be the set of all points at which f is not differentiable then k has exactly

(a) three elements (b) one elements
(c) five elements (d) two elements **[2019]**

25. If x log(log x) – $x^2 + y^2$ = 4 (y > 0), then $\frac{dy}{dx}$ at x = e is equal to

(a) $\frac{e}{\sqrt{4+e^2}}$ (b) $\frac{1+2e}{2\sqrt{4+e^2}}$
(c) $\frac{2e-1}{2\sqrt{4+e^2}}$ (d) $\frac{1+2e}{\sqrt{4+e^2}}$ **[2019]**

26. Let $f(x) = \begin{cases} -1, & -2\le x<0 \\ x^2-1, & 0\le x\le 2 \end{cases}$ and g(x) = |f(x)| + f(|x|). Then in the interval (–2, 2), g is

(a) differentiable at all points
(b) not differentiable at two points
(c) not continuous
(d) not differentiable at one point **[2019]**

27. Let f be a differentiable function such that f(1) = 2 and f′(x) = f(x) for all x ∈ R. If h(x) = f(f(x)) then h′(1) is equal to

(a) 4e (b) $4e^2$
(c) 2e (d) $2e^2$ **[2019]**

28. For x > 1, if $(2x)2^y = 4e^{2x-2y}$, then $(1 + \log 2x)^2\frac{dy}{dx}$ is equal to

(a) log 2x (b) $\frac{x\log 2x+\log 2}{x}$
(a) x log 2x (b) $\frac{x\log 2x-\log 2}{x}$ **[2019]**

29. Let K be the set of all real values of x where the function f(x) = sin |x| – |x| + 2(x – p) cos |x| is not differentiable, then the set K is equal to

(a) {π} (b) {0}
(c) ϕ (an empty set) (d) {0, π} **[2019]**

ANSWERS

CONCEPT PROBLEMS—A

1. f ′(0⁺) = – 1 ; f ′(0⁻) = 1; non-differentiable

2. m = – 1, n = π.

3. f ′(1+) = – 1; f ′(1–) = 3 ln 3 hence f is not differentiable at x = 1

4. (b) $(2a^{-1/3})/3$

5. yes; y = 1.

6. non-differentiable at x = 0

PRACTICE PROBLEMS—A

7. – h; 0.

8. (a) –1,1 (b) $\frac{2}{a}, \frac{-2}{a}$
(c) 1, 0 (d) 0, 0.

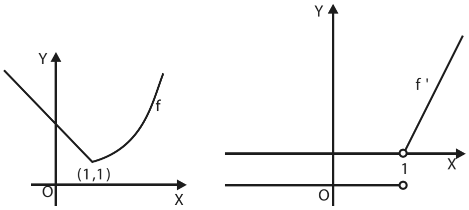
9. f(x) is not differentiable at x = 0

14. 0

15. 0

Concept Problems—B

1. (a) –1, 2 (b) 1, 3
2. (a) 0 (b) 1, 3
3. (a) 5 (b) 2, 3
4. – 4 (discontinuity), – 1 (corner), 2 (discontinuity), 5 (vertical tangent).
5. No



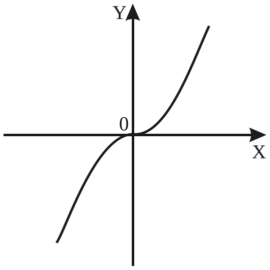
7. a = 2, b = 0
8. continuous and differentiable.
9. cont. but not diff.at x = 0 ; diff. & cont. at x = π/2
10. a = 1, b = 1
11. discontinuous and non-differentiable at x = 1.

Practice Problems—B

12. continuous; diff at x = 1, non diff. at x = 0
13. continuous but non diff at x = 0
14. a = 2
15. diff.
16. discontinuous & not derivable at x = 1, continuous but not derivable at x = 2
18. discontinuous at x = 1 and hence non-differentiable.

Concept Problems—C

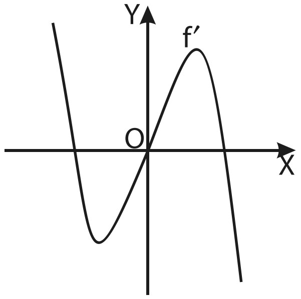
1. (a) x = – 3, 1 (b) x = ± 1.
2. yes
3. (a) –1, 1
   (b) x = 10/3, L.H.D. = –3, R.H.D. = 3
5. $f'(o)$ does not $f'(o^-) = 0$ $f'(o^+)$
6. f(x) is cont. but not diff. at x = 0
7. (a)



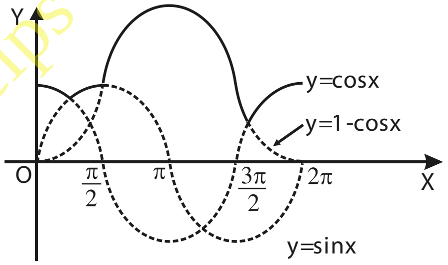
   (b) All x (c) $f'(x) = 2|x|$
8. discontinuous at x = 0, 1, 2, 3 hence non-differentiable
9. The Dirichlet function is nowhere continuous, and hence it is nowhere derivable.
10. diff every where.

Practice Problems—C

11. continuous but non-differentiable.
12. differentiable.
13. x = 0, ± 1
14. a = 35/9, b = 10/3
15. Non differentiable.
16.



17. f is cont. but not diff. at x = 1, discont. at x = 2 & x = 3. cont.& diff. at all other points
18. 3 ; The bold line represents the graph of y = f(x).



It is non-differentiable at three points namely

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{3}.$$

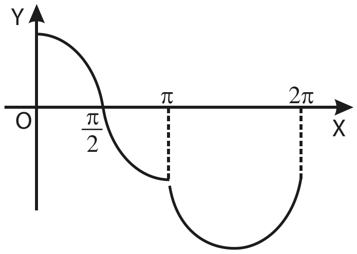
19. g o f (x) = 0 for all x ∈ R and

$$f\,o\,g = \begin{cases} 0 \text{ if } x \text{ is an integer} \\ n \text{ if } x \in R^{+} \text{and } \sqrt{n} < x < \sqrt{n+1} \text{ or} \\ x \in R^{-} \text{and } -\sqrt{n+1} < x < -\sqrt{n} \end{cases}$$

g o f is differentiable everywhere and f o g is differentiable on R – I.

20. $g(x) = \begin{cases} \cos x\,, & 0 \le x \le \pi \\ \sin x - 1\,, & x > \pi \end{cases}$

Adjacent figure represents the graph of g(x). Clearly, g(x) is continuous but non-differentiable at x = π.



Concept Problems—D

1. $6 \cos 9$
2. $f(x) = x^6$, $a = 2$
3. (i) $f(x) = 2^x$, $a = 5$
   (ii) $f(x) = \cos x$, $a = \pi$ or $f(x) = \cos(\pi + x)$, $a = 0$
4. The only doubtful point is 0. But $|f(x)/x| = |x| \to 0$ as $x \to 0$, so $f'(0) = 0$. For $x > 0$, $f'(x) = 2x$ and for $x < 0$, $f'(x) = -2x$. therefore $f'(x) = 2|x|$ is continuous.
5. 1
6. 2/3
7. non diff. at $x = 0$
8. $a = 6$, $b = -3$.

Concept Problems—E

2. (a) x for $x \geq 0$, $-x$ for $x < 0$
   (b) Yes
   (c) 1 for $x > 0$, -1 for $x < 0$
3. (a) $\frac{4}{3}x^{1/3}$ (b) Yes
   (c) $\frac{4}{9x^{2/3}}$
4. (a) $g'(x) = 2x$ for $x \leq 1$, $g'(x) = 2$ for $x > 1$
   $g''(x) = 2$ for $x \leq 1$, $g''(x) = 0$ for $x > 1$
   (b) g is differentiable ; g' is not
5. $-\sin a$

Practice Problems—D

6. (a) $y'' = 6|x|$; $y''(0) = 0$.
   (b) $y'' = 2 \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} - \frac{1}{x^2} \sin \frac{1}{x}$ at $x \neq 0$, $y''(0)$ does not exist, since $y'(x)$ is discontinuous at $x = 0$.
8. $2\sec^x x \tan x$
9. $n - 1 < k < n$

Concept Problems—F

1. (a) Yes (b) No
2. No
3. (a) No (b) No
4. If $g(x_0) \neq 0$ then f is non-diff. as $x = x_0$. If $g(x_0) = 0$ then f is differentiable is $x = x_0$.

Practice Problems—E

6. Yes
7. No
8. (b) f is twice but not thrice differentiable at $x = x_0$ and the derivatives are 0.
9. They are of opposite signs.
10. non-differentiable.
11. Cont. but non diff. at $x = 1$, discont. and non diff. at $x = 2, 3$
12. non diff. at $x = \pi, 2\pi$
13. $p > 1$
14. non-diff at $x = 2, 3$ if $a \neq 4, 9$; non diff at $x = 3$ if $a = 4$; non-diff at $x = 2$ if $a = 9$.

Practice Problems—F

2. $x, |x|$
3. -1
4. 0
5. $2x$
7. $\frac{1}{2}, \ell n2$
8. $x^2 + x e^x$
10. $f(x) = 2x$

OBJECTIVE EXERCISES

1. C
2. B
3. A
4. A
5. B
6. A
7. A
8. D
9. B
10. B
11. A
12. D
13. A
14. A
15. A
16. B
17. A
18. D
19. A
20. B
21. D
22. C
23. B
24. D
25. C
26. B
27. A
28. C
29. C
30. C
31. D
32. A
33. C
34. C
35. C
36. A
37. D
38. D
39. C
40. A
41. C
42. D
43. A
44. C
45. A
46. B
47. C
48. D
49. A
50. C
51. A,C
52. A,D
53. A,B,C
54. B,D
55. A,C
56. A,B,C
57. D
58. B,D
59. A,B,C
60. B,C,D
61. B,D
62. B,D
63. A,B,C,D
64. A,C,D
65. A,B,D
66. A,B
67. A, B, C
68. A,C,D
69. C,D
70. B,C
71. A
72. A
73. D
74. A
75. D
76. A
77. D
78. B
79. C
80. B
81. A
82. C
83. B
84. D
85. A
86. A
87. B
88. D
89. B
90. D

91. C **92.** D **93.** C

94. C **95.** B

96. (A)–(P,Q), (B)–(S), (C)–(P), (D)–(R)

97. (A)–R,(B)–S, (C)–P, (D)–Q

98. (A)–(P), (B)–(R), (C)–(Q), (D)–(T)

99. (A)–P; (B)–(P, Q), C–PRT, D–R

100. (A)–(Q), B–(R), (C)–(S), (D)–(P)

REVIEW EXERCISES for JEE ADVANCED

1. continuous but non-differentiable.

2. g(x) = x

3. discont and non-diff at $x = 0, 2$; cont but non-diff at $x = \pm 1$.

4. discont and non-diff at $x = 0$; cont but non-diff at $x = \pm\sqrt{2}$ and –1,1

5. {1, 3}

6. non-diff at x = 4

7. discontinuous and non-differentiable at x = 2.

8. a = – 1, b = – 2, value = 2/3.

9. If $f(x_0) = g(x_0)$, then the two functions may or may not be differentiable at $x = x_0$.

12. f is cont. but not derivable at x = 0.

14. continuous but non-differentiable at x = 0.

15. 180

16. $a \in (0, 1]$

17. If $a \in (0, 1)$, $f'(0^+) = -1$, $f'(0^-) = 1$

$\Rightarrow$ continuous but not derivable ; If a = 1, f(x) = 0 which is constant $\Rightarrow$ continuous and derivable;

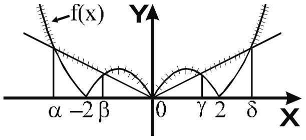
If a > 1, $f'(0^-) = -1$, $f'(0^+) = 1$

$\Rightarrow$ continuous but not derivable.

18. non diff. at x = 1

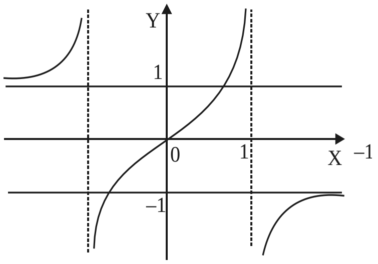
19. $g'(0)$, 0, $g'(4)$, $g'(2)$, $g'(-2)$

20. f(x) is non differentiable at $x = \alpha, \beta, 0, \gamma, \delta$ and g(x) is non differentiable at $x = \alpha, \beta, 0, -2, 2$



21. $$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \ge 0,\ x \ne 1 \\ \dfrac{x}{1+x} & \text{if } x < 0,\ x \ne -1 \end{cases}$$

and $$f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} & \text{if } x > 0,\ x \ne 1 \\ \dfrac{1}{(1+x)^2} & \text{if } x < 0,\ x \ne -1 \end{cases}$$



23. $f'(1^+) = 3$, $f'(1^-) = -1$

25. (a) 2 (b) 1

26. f(x) is not cont. not diff, xf(x) is cont. and non diff. $x^2 f(x)$ is cont and diff.

27. cont but not diff at x = 1, discont. at x = 2, 3

28. $a \ne 1$, b = 0, p = 1/3, q = – 1

29. cont in $0 \le x \le 1$, but not diff. at x = 0

30. $D^*[f(x).\,g(x)] = f^2(x)\,D^*g(x) + g^2(x).\,D^*f(x)$;

$$D^*\left(\frac{f(x)}{g(x)}\right) = \frac{g^2(x).D^*f(x) - f^2(x).D^*g(x)}{[g(x)]^4}$$

31. x + 1

32. f(x) = 0

33. f(x) = ln x

34. f(x) = 1, f(x) = 0.

35. $f(x) = e^{-x} - 1$

TARGET EXERCISES for JEE ADVANCED

1. Continuous but non-differentiable at x = –2, –1, 0, 1, 2

2. not differentiable at x = e; $f'(e^+) = 1$, $f'(e^-) = 0$.

4. f(x) = 0, f(x) = 1/2, f(x) = x. **5.** $f(x) = \sin^{-1} x)$

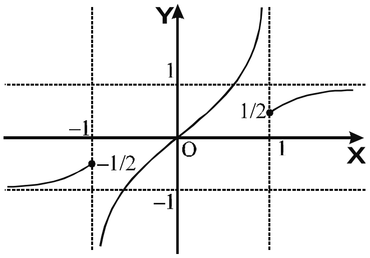
6. continuous but non-differentiable at x = 1.

7. f is continuous and differentiable for all $x \in$ domain , except non-differentiable at x = 1

10. 256

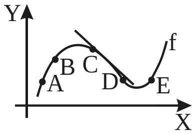
11. 54

13. R – {–1, 1}



14. (i) A and B (ii) Greater

(iii)



(iv) B and C, D and E

16. g(x) is continuous in [–3, 1] except x = 0

17. (a) continuous for all $x \in R$ not differentiable for all $x \in R - \{0\}$

(b) g(x) is not continuous at x = 0

22. $1 + x^2$ **24.** $\{\pm 1, 0, 1 - \sqrt{2}\,\})$

25. $\frac{\ell n2}{2}$ **26.** Non - diff. at $x = \pi, 2\pi$

27. x + 1

28. For g(t) = mt and h(t) = t, $\text{limit}_{t \to 0}$ g(t)/(h(t) = m, which need not be zero.

30. $f(x) = x^3$ **31.** f(x) = 0

33. $f(x) = \frac{1-x}{1+x}$. **34.** $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. {0} **2.** R – {0}
3. D **4.** B
5. ABD **6.** A
7. A **8.** B
9. D **10.** A
11. D **12.** D
13. D **14.** C
15. A **16.** B
17. A **18.** C
19. B **20.** ABD
21. BDE **22.** ABC
23. ACD **24.** BCD
25. AB **26.** AC
27. C **28.** AC
29. ABCD **30.** BC
31. BC **32.** ABCD
33. AD **34.** ABD
35. BC **36.** AB
37. A **38.** BC **39.** ABD
40. B **41.** 2 **42.** 0

43. f and f′ are continuous and f″ is discontinuous on [0, 2]

44. cont. on (0, 2) and differentiable on (0, 2) = {1}

45. not differentiable at x = 0, 1

46. f′(0) = 0 **47.** x = 0, 1, 2, 3

48. $f(x) = e^{2x}$ **49.** – 1

50. cont. but not diff.

51. f(x) is not continuous and thus not differentiable at x = 2

52. continuous but not differentiable at discontinuous and not differentiable at x = –1.

54. $g(f(x)) = \begin{cases} x + a + 1 & \text{if } x < -a \\ (x + a - 1)^2 + b & \text{if } -a \le x < 0 \\ x^2 + b & \text{if } 0 \le x < 1 \\ (x - 2)^2 + b & \text{if } x \ge 1 \end{cases}$ a = 1, b = 0,

gof differentiable at x = 0

55. 0

56. a = 1

57. 0

58. (A)→(P, Q, R), (B)→(P, S), (C)→(R, S), (D)→(P, Q)

59. D

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. (c) **2.** (d) **3.** (d) **4.** (b)
5. (d) **6.** (c) **7.** (c) **8.** (b)
9. (d) **10.** (a) **11.** (b) **12.** (c)
13. (a) **14.** (d) **15.** (b) **16.** (b)
17. (a) **18.** (b) **19.** (a) **20.** (a)
21. (d) **22.** (d) **23.** (c) **24.** (a)
25. (c) **26.** (d) **27.** (a)
28. (d) **29.** (c)

HINTS & SOLUTIONS

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. $$f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$$

LHD =

$$\lim_{h\to 0}\frac{\dfrac{-h}{1+e^{-1/h}}-0}{-h}=\lim_{h\to 0}\frac{1}{1+e^{-1/h}}=1$$

RHD → $$\lim_{h\to 0}\frac{\dfrac{h}{1+e^{1/h}}-0}{h}=0$$

Since LHD and RHD exist, f(x) is continuous but not differentiable at $x=0$

Hence, C is correct.

2. For f(x) to be continuous at $x = x_0$.

$ax_0+b=x_0^2$ (1)

For $f(x)$ to be differentiable

LHD ⇒

$$\lim_{h\to 0}\frac{(x_0-h)^2-x_0^2}{-h}=\lim_{h\to 0}\frac{h^2-2x_0h}{-h}=2x_0$$

RHD ⇒

$$\lim_{h\to 0}\frac{a(x_0+h)+b-(x_0)^2}{h}=a$$

For differentiable, LHD = RHD.

$\Rightarrow a=2x_0\ \&\ b=-x_0^2$

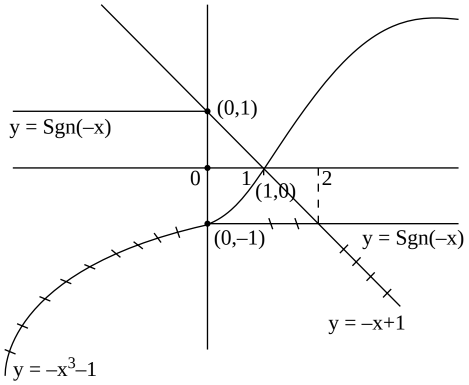
Hence, B is correct.

3. $f(x)=\min\left\{x^3-1,\ -x+1,\ \text{sgn}(-x)\right\}$

At x = 0, f(x) is differentiable as LHD = RHD = 0

At x = 2, f(x) is continuous but not differentiable.

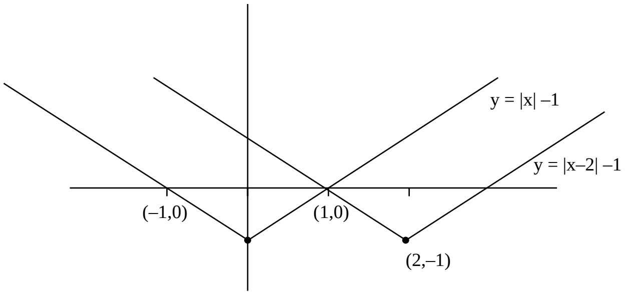
Hence, A is correct.



4. $f(x)=\min\{|x|-1,\ |x-2|-1\}$

f(x) is not differentiable at x = 0, 1, 2

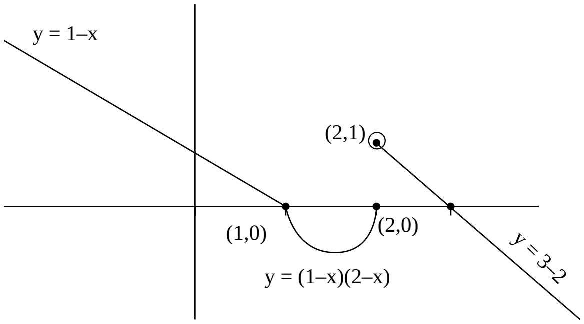
Hence, A is correct.



5. $$f(x)=\begin{cases}1-x, & x<1\\ (1-x)(2-x), & 1\le x\le 2\\ 3-x, & x>2.\end{cases}$$

f(x) is discontinuous at $x = 2$ & non-differentiable at $x = 2$. However, it is differentiable at $x = 1$ since LHD = RHD = −1

Hence, B is correct.



6. $y=x^p, p<1$ is non differentiable

at $x=0$

$y=|x|^p, p>1$ is

differentiable at $x=0$

f(x) is non differentiable at $x=-1$ only.

Hence, A is correct.

7. $f(x)=\sqrt[3]{x^2|x|}-|x|-1$

$=|x|^{5/3}-|x|-1$

is non differentiable at $x=0$ because of $|x|$

Hence, A is correct.

8. If $f(x)$ is differentiable function from $R \to Q$.

$\Rightarrow f(x)$ is constant function.

$$\sum_{r=0}^{100}(-1)^r f(r) = f(0) - f(1) + f(2) - f(3) + \ldots + f(100)$$

$= f(100)$ which is some constant

Hence, D is correct.

9. $$f(x) = \begin{cases} \dfrac{-x3^{1/x}}{3^{1/x}-1}, & x > 0 \\ \dfrac{-x\left(3^{1/x}+2\right)}{3^{1/x}-1} & x < 0 \\ 0 & x = 0 \end{cases}$$

LHD at $x = 0$

$$\lim_{h\to 0}\frac{h\left(3^{-1/h}+2\right)}{3^{-1/h}-1}\times\frac{1}{-h} = 2$$

RHD at $x = 0$

$$\lim_{h\to 0}\frac{-h3^{1/h}}{3^{1/h}-1}\times\frac{1}{h} = 1$$

So, $f(x)$ is not derivable at $x = 0$

but it is continuous

Hence, B is correct.

10. Since *g*(*x*) is even and differentiable at $x = 0$,

$g'(0) = 0$

$$\Rightarrow \lim_{h\to 0}\frac{g(h)}{h} = 0$$

$$f'(0) = \lim_{h\to 0}\frac{g(h)\cos\left(\frac{1}{h}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{g(h)}{h} = 0$$

$\Rightarrow \quad f'(0) = 0$

Hence, B is correct.

11. $$f(x) = \lim_{n\to\infty}\left(1 - \frac{2}{\left(x^2+2x+3+\sin\pi x\right)^n+1}\right)$$

As $x^2+2x+3+\sin\pi x > 2 \quad \forall x \in R$.

$f(x) = 1$ for all *x*.

Hence, A is correct.

12. $$y\lim_{x\to 0} f\left(\cos^3 x - \cos^2 x\right) = x\lim_{x\to 0} f\left(\sin^2 x - \sin^3 x\right)$$

$\cos^3 x < \cos^2 x$

$\sin^2 x > \sin^3 x$

$\Rightarrow 4y = 3x \Rightarrow$ line eqn

Let image be (h, k)

$\Rightarrow$ Pt $\left(\frac{h}{2}, \frac{k+1}{2}\right)$ will fulfill the equation of line

$$4\left(\frac{k+1}{2}\right) = 3\left(\frac{h}{2}\right)$$

$\Rightarrow 4k + 4 = 3h$ ....(I)

$$-\frac{4}{3} = \frac{k-1/2}{h/2}$$

$\Rightarrow -4h = 3k - 3$ ....(II)

Solving (I) & (II)

$$h = \frac{24}{25}, \quad k = \frac{-7}{25}$$

Hence, D is correct.

13. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x(1+h/x)) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(1+h/x) + 2 - 2f(x) - f(1+h/x)}{h}$$

$$= \lim_{h\to 0}(f(x)-1)\left(\frac{f(1+h/x)-2}{h}\right)$$

$$= (f(x)-1)\lim_{h\to 0}\frac{f(1+h/x)-2}{h/x}\times\frac{1}{x} \quad (f(1) = 2)$$

$$= (f(x)-1)\frac{f'(1)}{x}$$

$xf'(x) = 2f(x) - 2$

Hence, A is correct

14. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h} = f'(0)$$

$$f'(0) = \lim_{h\to 0}\frac{(2h^2+3h)g(h)}{h}$$

$$= \lim_{h\to 0}(2h+3)g(h)$$

$= (3)g(0)$

$= 3\times 3 = 9$

$f'(x) = f'(0) = g$

Hence, A is correct.

15. $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{f(h)}{h} = f'(0)$

$f'(0) = \lim\limits_{h\to 0}\dfrac{\sin h\, g(h)}{h}$

$g(0) = k$

Hence, A is correct.

16. $f(x) = \dfrac{3x^2+2x-1}{6x^2-5x+1} = \dfrac{(3x-1)(x+1)}{(3x-1)(2x-1)} = \dfrac{x+1}{2x-1}$

$$f'\left(\frac{1}{3}\right) = \lim_{h\to 0}\frac{\dfrac{1/3+h+1}{2(1/3+h)-1}+4}{h}$$

$$= \lim_{h\to 0}\frac{4/3+8/3+9h-4}{h(2(1/3+h)-1)}$$

$= -27.$

Hence, B is correct.

17. $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(1+h/x)-2f(x)+2-f\left(1+h/x\right)}{h}$$

$$= \frac{(f(x)-1)}{x}f'(1)$$

Now, $\dfrac{df}{f(x)-1} = f'(1)\dfrac{dx}{x}.$

Integrating both sides w.r.t x, $\ln|f(x)-1| = n\ln|x| + \ln C$

$\Rightarrow f(x) = 1 + cx^n$

Put x = 1, y = 2 to get f(1)=2

Use $f(1) = 2$ and $f(2) = 5$ to get

$f(x) = x^2+1$

Hence $f(3) = 10$

Hence, A is correct.

18. For $f(x)$ to be continuous,

$2x+1 = x^2-2x+5$

$\Rightarrow x^2-4x+4 = 0$

$\Rightarrow (x-2)^2 = 0$

$\Rightarrow x = 2$

For $f(x)$ to be differentiable at $x = 2$

$$= \lim_{h\to 0}\frac{2(2+h)+1-5}{h} = \lim_{h\to 0}\frac{(2+h)^2-2(2+h)+5-5}{h}$$

(h rational) (h irrational)

Both are equal to 2.

Hence, D is correct.

19. $\lim\limits_{x\to h}\dfrac{(x+h)f(x)-2hf(h)}{x-h}$

$\lim\limits_{x\to h}\dfrac{xf(x)-hf(h)}{x-h} + \lim\limits_{x\to h}\dfrac{hf(x)-hf(h)}{x-h}$

$= f(h)+hf'(h) + hf'(h)$

$= f(h)+2hf'(h)$

Hence, A is correct.

20. Put $x = 1, y = 1 \Rightarrow f^2(1)-3f(1)+2 = 0$

$\Rightarrow f(1) = 1, 2$

If $f(1) = 1$, put $y = 1$ to get $f(x) = 1$

which is a constant function

Hence $f(1) = 2$

Hence, B is correct.

21. $y = \sin^{-1}(|1-|x||)$

This function is non differentiable at

$x = 0, \pm 1$ and $\left|1-|x|\right| = 1 \Rightarrow x = \pm 2$

$\Rightarrow x = \pm 2$

Hence, D is correct .

22. $f(x+y)-f(x-y) = 4xy$

Put $x = 0$

$f(y) = f(-y) \Rightarrow f(x)$ is even function

Put $y = x$

$f(2x) = 4x^2 = (2x)^2$

$f(x) = x^2$. It is unbounded.

$$f(x)+f\left(\frac{1}{x}\right) = x^2+\frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2+2$$

$$= f\left(x-\frac{1}{x}\right)+2$$

Hence, C is correct.

23. $g(x+y)=e^y g(x)+e^x g(y)$

Put $x=0,\ y=0 \Rightarrow g(0)=0$

$$g'(x)=\lim_{h\to 0}\frac{e^h g(x)+e^x g(h)-g(x)-e^x g(0)}{h}$$

$$g'(x)=g(x)+2e^x$$

Hence, B is correct

24. $y=|1-|2-|3-|4-x||||$

$f(x)$ is not differentiable when

$x=4$, $|3-|4-x||=0, |2-|3-|4-x|||=0, |1-|2-|3-|4-x|||=0$

y is not differentiable at more than 5 points

Hence, D is correct.

25. $f(x)=\begin{cases}\dfrac{1}{2x+1}, & x>0\\ \dfrac{1}{2x-1} & x<0\\ 1 & x=0\end{cases}$

$\lim\limits_{x\to 0^+} f(x)=1$, $\lim\limits_{x\to 0^-} f(x)=-1$, $\quad f(0)=1$

Hence, C is correct.

26. $y=\cos x$ is decreasing in $[0, \pi]$

$$g(x)=\begin{cases}\cos x, & x\in[0,\pi]\\ \sin x-1, & x>\pi\end{cases}$$

$\lim\limits_{x\to\pi^-}\cos x=-1$, $\lim\limits_{x\to\pi^+} f(x)=-1$, $f(\pi)=-1$

At $x=\pi$, LHD = 0 RHD = 1. Not differentiable

Hence, B is correct.

27. Put $x=y=z=0 \Rightarrow f(0)=1$

Put $z=0$

$\Rightarrow f(x+y)=f(x)\cdot f(y).$

$$f'(x)=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$$f'(x)=3f(x)$$

Put $x=2$

$f'(2)=12$

Hence, A is correct.

28. $f(x)=\lim\limits_{k\to\infty}(\cos x)^{2k}+\cos^{-1}(\sin x)$

$$=\begin{cases}\cos^{-1}\sin x, & x\neq n\pi\\ 1+\cos^{-1}\sin x & x=n\pi\end{cases}$$

$f(x)$ is discontinuous at $x=n\pi$

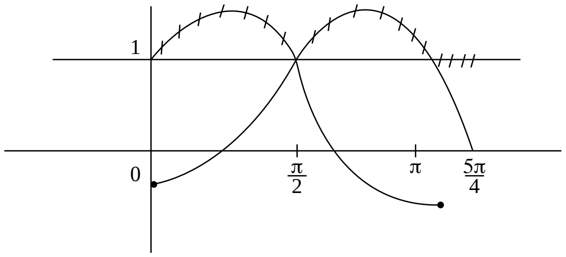
Hence, C is correct.

29. $f(x)=\max\{1, \cos x+\sin x,\ \sin x-\cos x\}$

$$=\max\left\{1, \sqrt{2}\sin\left(\frac{\pi}{4}+x\right), \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\right\}$$

f(x) is not differentiable at $x=\dfrac{\pi}{2},\pi$

Hence, C is correct.



30. $|f(x)|\le x^2$

$|f(0)|\le 0 \Rightarrow f(0)=0$

$|f(h)|\le h^2$

$$\Rightarrow \lim_{h\to 0}\left|\frac{f(h)-f(0)}{h}\right|\le \lim_{h\to 0}|h|$$

$$|f'(0)|\le 0$$

$$\Rightarrow f'(0)=0$$

Hence, C is correct.

31. If f(x) is even and differentiable for all x

$\Rightarrow f'(0)=0$

Hence, option D is correct.

32. $f(x)=\begin{cases}x^3, & x<0\\ x^2, & 0\le x<1\\ 2x-1, & 1\le x<2\\ x^2-2x+3, & x\ge 2\end{cases}$

$$f'(x)=\begin{cases}3x^2, & x<0\\ 2x, & 0\le x<1\\ 2, & 1\le x<2\\ 2x-2, & x\ge 2\end{cases}$$

$f(x)$ is continuous and differentiable for $x\in R$.

Hence, A is correct.

33. $f(x)=\begin{cases}x^2+1, & 0\le x<1\\ 3-x, & 1\le x<2\\ 2, & x=2\end{cases}$

$\Rightarrow f(x)$ is discontinuous at $x=2$ and non-derivable at $x=1$

Hence, C is correct.

34. As $x \to \pm\infty,\ f(x) \to 0$

As $x > 0,\quad f''(x) > 0$

As $x < 0,\quad f''(x) > 0$

& at $x = 0,\ f'(x)$ is non differentiable.

Hence, C is correct.

35. $f(x) = \left(x^2 - 1\right)|(x-2)(x+1)| + \sin|x|$

$y = (x+1)|(x+1)|$ is differentiable at $x = -1$

$f(x)$ is non differentiable at $x = 0$ & 2

Hence, C is correct.

36. Put $x = h + t$ where $t \to 0$

$$\lim_{t\to 0} \frac{(2h+t)f(h+t) - 2hf(h)}{t}$$

$$= 2h \lim_{t\to 0} \frac{f(h+t) - f(h)}{t} + \lim_{t\to 0} f(h+t)$$

$$= f(h) + 2hf'(h)$$

Hence, A is correct

37. $f(x) = \max\{\sqrt{x(2-x)}, 2-x\}$

Now as $y = \sqrt{x(2-x)}$

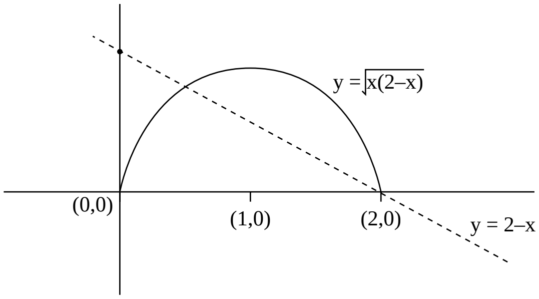
$\Rightarrow y^2 = 2x - x^2 + 1 - 1$

$y^2 + (x-1)^2 = 1$

$f(x)$ has corner at $x = 1$ and has infinite derivative at $x = 2$

$f(x)$ is non differentiable at $x = 1, 2$.

Hence, D is correct.



38. $$f(x) = \begin{cases} \dfrac{\sin\left(x^2 - 5x + 6\right)}{x^2 - 5x + 6}, & x < 2, \text{ or } x > 3 \\ -\dfrac{\sin\left(x^2 - 5x + 6\right)}{x^2 - 5x + 6}, & 2 < x < 3 \\ 1 & x = 2 \text{ or } 3 \end{cases}$$

$f(x)$ is non differentiable at $x = 2$ & 3 as it is discontinuous there.

Hence, D is correct.

39. $$\lim_{h\to 0} \frac{f\left(a+2h^2\right) - f\left(a-2h^2\right)}{4h^2}$$

$$\times \frac{2h^3 - 2h^2}{f\left(a+h^3-h^2\right) - f\left(a-h^3+h^2\right)} \times \frac{4h^2}{2h^3 - 2h^2}$$

$$f'(a) \times \frac{1}{f'(a)} \times \frac{4}{-2} = -2$$

Hence, C is correct.

40. $$f(x) = \begin{cases} [x] + \sqrt{\{x\}}, & x < 1 \\ \dfrac{1}{[x] + \{x\}^2}, & x \geq 1 \end{cases}$$

$\lim_{x\to 1^-} f(x) = 1,\ \lim_{x\to 1^+} f(x) = 1,\ f(1) = 1$

LHD → $\lim_{h\to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \dfrac{\sqrt{\{1-h\}}}{-h} =$ does not exist

Hence, A is correct.

41. $-f(a) = \lim_{x\to a} f(x)$

Hence, C is correct.

42. Statement III is true using Intermediate Value Theorem.

Hence, D is correct.

43. $f(x - f(y)) = f(f(y)) + x\,f(y) + f(x) - 1$

Put $y = 0$

$f(x-1) = f(1) + x + f(x) - 1$

Put $x = 1$

$f(1) = 1/2$.

$f(x-1) = x + f(x) - \dfrac{1}{2}$

Now,

$f(0) = 1 + f(1) - 1/2$

$f(1) = 2 + f(2) - 1/2$.

$f(x-2) = x - 1 + f(x-1) - 1/2$

$f(x-1) = x + f(x) - \dfrac{1}{2}$

Add all equations,

$f(0) = \dfrac{x(x+1)}{2} - \dfrac{x}{2} + f(x)$

$f(x) = 1 - \dfrac{x^2}{2}$

Hence, A is correct.

44. Conceptual Problem.

C is Correct.

45. $\lim\limits_{x\to\infty}\mu(x)=2,\ \lim\limits_{x\to\infty}v(x)=3,\ \lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}g(x)=\infty$

$$\frac{f'(x)}{g'(x)}+\mu(x)\frac{f(x)}{g(x)}=v(x)$$

Let $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=a$

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}+\mu(x)\frac{f(x)}{g(x)}=\lim_{x\to\infty}v(x)$$

$\Rightarrow\ a+2a=3$ using LH Rule

$a=1$

Hence, A is correct.

46. $f'(x)=\lim\limits_{h\to0}\dfrac{f(x)+f(h)+5xh-f(x)-f(0)}{h}$

$=\lim\limits_{h\to0}\dfrac{f(h)}{h}+5x\qquad(\because f(0)=0)$

$f'(x)=f'(0)+5x$

$=5x+3$

$f(x)=\dfrac{5}{2}x^2+3x$

Minimum value of $f(x)=\dfrac{-D}{4a}=-\dfrac{9}{10}$

Hence, B is correct.

47. $f(x)=\left(kx+e^x\right)h(x)$

Put $x=0,\quad f(0)=5$

Differentiate w.r.t x

$f'(x)=\left(k+e^x\right)h(x)+\left(kx+e^x\right)h'(x)$

Put $x=0$

$f'(0)=(k+1)h(0)+h'(0)$

$18=5k+5-2$

$k=3$

Hence, C is correct.

48. If $f'(2)=4\Rightarrow f(x)$ is differentiable and continuous

at $x=2$

as $f(2)=3$

$\lim\limits_{x\to2^+}f(x)=3+h$

$\lim\limits_{x\to2^-}f(x)=3-h$

So, $\lim\limits_{x\to2}[f(x)]$ will not exist.

Hence, D is correct.

49. Similar to Q . 40.

50. $\lim\limits_{x\to0}\dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$

Applying L-Hospital Rule,

$$\lim_{x\to0}\frac{2f'(x)-6f'(2x)+4f'(4x)}{2x}$$

$$\lim_{x\to0}\frac{2f''(x)-12f''(2x)+16f''(4x)}{2}$$

= 12

Hence, C is correct.

MULTIPLE CORRECT ANSWER TYPE

51. $f(x)=|x-1|\ ([x]-[-x])$

$\lim\limits_{x\to1^+}f(x)=0,\ \lim\limits_{x\to1^-}f(x)=0,\ f(1)=0$

LHD $\to\lim\limits_{h\to0}\dfrac{h(1)}{-h}=-1$

RHD $\to\lim\limits_{h\to0}\dfrac{h(3)}{h}=3$

Hence, A, C is correct.

52. A) LHD $\to\lim\limits_{h\to0}\dfrac{-h^2\sin 1/h}{-h}=0$

RHD $\to\lim\limits_{h\to0}\dfrac{h^2\sin(1/h)}{h}=0$

B) LHD $\to$ Not defined & $f(x)$ is discontinuous at $x=\pi$.

RHD $\to$ Not defined

C) $f(x)=\begin{cases}1, & x\neq0\\ 0, & x=0\end{cases}$

D) LHD = 0

RHD = 0

Hence, A, D are correct.

53. $h(x)=g\circ f(x)=\text{sgn}(\sin\pi x)$

$$=\begin{cases}0, & x=I\\ 1 & x\in(0,1),\ (2,3),\ldots\\ -1 & x\in(1,2),\ (3,4),\ldots\end{cases}$$

Hence, A, B, C are correct.

54. $f(x)=\dfrac{\tan[x]\pi}{\left[1+\left|\log\left(\sin^2 x+1\right)\right|\right]}$

$=0,\ x\in R$

Hence, B, D are correct.

55. $f(x)=\begin{cases} x, & x\ge 0\\ 0 & x<0\end{cases},\qquad g(x)=\begin{cases} x, & x<0\\ x^2, & x\ge 0\end{cases}$

$f\circ g(x)=\begin{cases} 0, & x<0\\ x^2 & x\ge 0\end{cases}$

$g\circ f(x)=\begin{cases} x^2, & x\ge 0\\ 0, & x<0\end{cases}$

Hence, A, C are correct.

56. Replacing n by (n – 1), in $\left|\sum_{k=1}^{n} 3^k f(x+ky)-f(x-ky)\right|\le 1$ 1

we get $\left|\sum_{k=1}^{n-1} 3^k\left(f(x+ky)-f(x-ky)\right)\right|\le 1$ 2

Subtract 2 from 1,

$\left|3^n\left(f(x+ny)-f(x-ny)\right)\right|\le 2$

$\Rightarrow |f(x+ny)-f(x-ny)|\le \dfrac{2}{3^n}$ 3

Put x + xy = u, x – xy = v

eq 3 becomes $|f(u)-f(v)|\le \dfrac{2}{3^n}$.

As $n\to\infty$,

$|f(u)-f(v)|\ \le 0$

$\Rightarrow\ f(u)=f(v)\ \Rightarrow f(x)$ is constant function

Hence, A, B, C are correct.

57. A) $\Delta(x)=\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}=\lim\limits_{h\to 0}(x+h)^2+x^2+x(x+h)$

$=3x^2$

B) $\Delta(\tan x)=\lim\limits_{h\to 0}\dfrac{\tan^3(x+h)-\tan^3 x}{h}$

$=3\tan^2(x)\sec^2 x$

at $x=\dfrac{\pi}{4}$, $\Delta(\tan x)=6$

C) $\Delta(\sin x)=\lim\limits_{h\to 0}\dfrac{\sin^3(x+h)-\sin^3(x)}{h}$

$=3\sin^2 x\cos x$

at $x=0,\quad \Delta(\sin x)=0$.

D) $\Delta\left(\cos^{-1}x\right)=\lim\limits_{h\to 0}\dfrac{\cos^{-1}(x+h)^3-\left(\cos^{-1}x\right)^3}{h}$

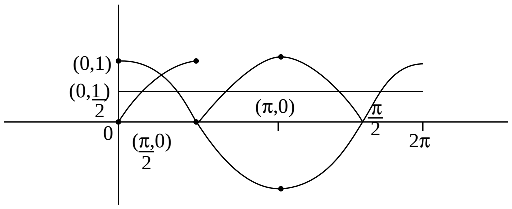
$=3\left(\cos^{-1}x\right)^2\times\dfrac{-1}{\sqrt{1-x^2}}$

at $x=0,\quad \Delta\left(\cos^{-1}x\right)=-3\dfrac{\pi^2}{4}$

Hence, D is correct.

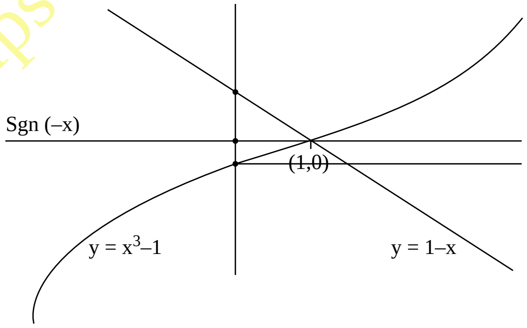
58. $f(x)=\min\left(\cos x,\dfrac{1}{2},\{\sin x\}\right)$

Hence, B & D are correct.



59. $f(x)=\min\left\{x^3-1,\,1-x,\,\text{sgn}(-x)\right\}$

Hence, A, B, C are correct



60. $y=\sin\pi(|x|+[x])$

$y=\begin{cases}\sin\pi(x+[x]), & x\ge 0\\ \sin\pi(-x+[x]) & x<0\end{cases}$

at $x=-\dfrac{1}{2}$

LHL = –1, RHL = –1, $f\left(-\dfrac{1}{2}\right)=-1$

at $x=0$

LHL = 0, RHL = 0, $f(0)=0$

$\text{LHD}=\lim\limits_{h\to 0}\dfrac{\sin\pi(-1+h)}{-h}=\pi$

$\text{RHD}=\lim\limits_{h\to 0}\dfrac{\sin\pi(h)}{h}=\pi$

Hence, B, C, D are correct.

61. For | f|

$\text{LHD}=\lim\limits_{h\to 0}\dfrac{|f(a-h)|-|f(a)|}{-h}$

$$= \lim_{h\to 0}\frac{|f(a-h)|}{-h}$$

$$\text{RHD} = \lim_{h\to 0}\frac{|f(a+h)|}{h}$$

So, $|f(x)|$ is not differentiable at $x = a$. as LHD & RHD will be of opposite sign.

But $|f(x)|^2$ will be differentiable.

Hence, B & D is correct.

62. (A) $\text{RHD} = \lim_{h\to 0}\dfrac{h^2\sin\frac{1}{h}}{h} = 0$

$f(x)$ is differentiable only in $(0,1)$.

(B) $\lim_{x\to 0}\dfrac{x^2\sin 1/x}{x^2} =$ Does not exist

(D) $\lim_{x\to 0}\dfrac{-\cos 1/x + 2x\sin(1/x)}{2x} =$ Does not exist

Hence, B, D are correct.

63. $f(x) = \begin{cases} x+1, & x \ge 1 \\ 2-x, & x < 1 \end{cases}$

$$f(|x|) = \begin{cases} |x|+1, & |x| \ge 1 \\ 2-|x| & |x| < 1 \end{cases}$$

$$= \begin{cases} x+1 & x \ge 1 \\ 1-x & x \le -1 \\ 2-x & 0 < x < 1 \\ 2+x & -1 < x \le 0 \end{cases}$$

$$g(x) = f(x) + f(|x|)$$

$$= \begin{cases} 2x+2, & x \ge 1 \\ 3-2x, & x \le -1 \\ 4-2x, & 0 < x < 1 \\ 4, & -1 < x \le 0 \end{cases}$$

Hence, A, B, C, D are correct.

64. $f(x+y) = f(x) + f(y) + xy(x+y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x)+f(h)+xh(x+h)-f(x)}{h}$$

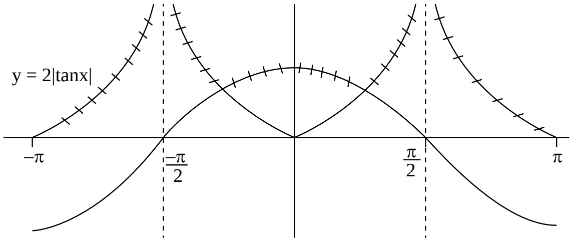
$$= x^2 + f'(0) \quad (\because f(0) = 0)$$

$$f'(x) = x^2 - 1$$

Hence, A, C, D are correct.

65. $f(x) = \max(|\tan x|, \cos|x|)$

Hence, A, B, D are correct.



66. $\left|\dfrac{f(x)-f(y)}{x-y}\right| \le \dfrac{1}{2}$

Put $x \to y+h$

$$|f'(x)| \le \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \le f'(x) \le \frac{1}{2} \quad \& \quad f'(x) \ge \frac{1}{2}$$

$$f'(x) = \frac{1}{2}$$

Integrating, we get $f(x) = \dfrac{x}{2} + c$

Hence, A , B is correct.

67. Conceptual.

Hence, A , B is correct.

68. $f(x) = \begin{cases} \sin^2 x\cos\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x)$$

$$f'(x) = \begin{cases} 2\sin x\cos x\cos\frac{1}{x} + \frac{1}{x^2}\sin^2 x\sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Hence, A, C, D is correct.

69. $\lim_{x\to 0} f(x) = \lim_{x\to 0} g'(x)\dfrac{e^{a/x}-e^{-a/x}}{e^{a/x}+e^{-a/x}}$

$\text{RHL} \to \lim_{x\to 0} g'(x)$

$\text{LHL} \to \lim_{x\to 0} -g'(x)$

for $\lim_{x\to 0} f(x)$ to exist, LHL = RHL.

$$\Rightarrow g'(0) = 0$$

Hence, C, D is correct.

70. $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)}{3}$, $f(0) = 3$, $f'(0) = 3$

$$f(x+y) = \frac{f(3x)+f(3y)}{3}$$

$$f'(x) = \lim_{h\to 0}\frac{1}{3}\left[\frac{f(3x)+f(3h)-f(3x)-f(0)]}{h}\right]$$

$$= \frac{1}{3}\lim_{h\to 0}\frac{f(3h)-f(0)}{h}$$

$$= 3$$

$$f(x) = 3x+3$$

Hence B, C is correct.

Assertion & Reason

71. A: $F(x) = |x|^2 + 2|x|$

$$= \begin{cases} x^2+2x, & x \ge 0 \\ x^2-2x, & x < 0 \end{cases}$$

$F(x)$ is not derivable at $x = 0$.

R : $F'\left(0^+\right) = 2$ & $F'\left(0^-\right) = -2$.

Hence, A is correct.

72. $f'(x) = \lim_{h\to 0}\frac{f(x)+hf(f(x)-f(x)}{h}$

$$f'(x) = f\left(f(x)\right)$$

Now If $f(x)$ has degree ' n ', $f'(x)$ has degree $n-1$ & $f\circ f(x)$ has degree n^2

Hence, there is no polynomial possible.

Hence, A is correct.

73. Hence, D is correct.

74. $y = \sin^{-1}\cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 x}}x - \sin x = \frac{-\sin x}{|\sin x|}$$

It is not differentiable at $x = n\pi$

Hence, A is correct.

75. A : $h(x) = \sin x\,|\sin x|$

$$= \begin{cases} \sin^2 x, & x \in [0,\pi] \\ -\sin^2 x, & x \in (\pi, 2\pi] \end{cases}$$

$h(x)$ is differentiable in $x \in [0, 2\pi]$

R: It is correct.

Hence, D is correct.

76. A : $f(x) = c_1 e^{|x|} + c_2|x|^3$

For $f(x)$ to be differentiable, $c_1 = 0$

$$f(x) = c_2|x|^3 = c_2 x^3 \quad \forall x$$

R: It is correct.

Hence, A is correct.

77. Same as Q. 75.

78. $f(x) = \cot^{-1}\left(\operatorname{sgn}\left(\frac{[x]}{2x-[x]}\right)\right)$

$$\lim_{x\to 1^+} f(x) = \frac{\pi}{4}$$

$$\lim_{x\to 1^-} f(x) = \frac{\pi}{2}$$

$f(x)$ is discontinuous & hence, non derivable at $x = 1$

Hence, B is correct.

79. $f\circ g(x) = \begin{cases} -1, & 0 < x < 1 \\ 1 & 1 < x \le 4 \\ 0 & x = 1 \end{cases}$

Hence, C is correct.

80. $f(x) = |(x-1)^{2p_1+1}(x-2)^{2P_2+1}\ldots(x-n)^{2p_n+1}|$

Hence, B is correct.

Comprehension-1:

LHD at $x = 0$

$$\lim_{h\to 0}\frac{-\sin h + \tan h + \cos h - 1}{\left(2h^2 + \ln(2-h) - \tan h\right)^{-h}}$$

$$= \lim_{h\to 0}\frac{(1-\cos h)(\tan h - 1)\cdot h}{(h) - h\left[2h^2 + \ln(2-h) - \tan h\right]}$$

$$\Rightarrow = 0$$

RHD at $x = 0$

$$\lim_{h\to 0}\frac{e^{h^2}-1}{h} = 0$$

$L_1 \Rightarrow y = 0$

$L_2 \Rightarrow x = 0$

81. Circle touching both lines

$$(x \pm r)^2 + (y \pm r)^2 = r^2$$

$$x^2 + y^2 \pm 2rx \pm 2ry + r^2 = 0$$

Two circles are orthogonal

$$2r_1r_2 + 2r_1r_2 = r_1^2 + r_2^2$$

$$r_1^2 + r_2^2 - 4r_1r_2 = 0$$

$$\frac{r_1}{r_2} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Hence A is correct.

82. $x^2 + y^2 + 2x + 2y + 1 = 0$

$$(a+b)^2 + (a+c)^2 = (b+c)^2$$

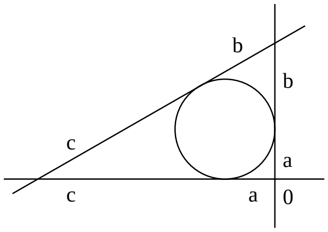
$$2a^2 + 2a(b+c) = 2bc$$

$$a^2 + ab + ac = bc$$

As $a = 1$

$$1 + b + c = bc$$

$$b = \frac{c+1}{c-1}$$



$$\text{Area of } \Delta = \frac{1}{2}(1+b)(1+c)$$

$$= \frac{1}{2}(1+b+c+bc)$$

$$= bc$$

$$A = \frac{c(c+1)}{c-1}$$

$$\frac{dA}{dc} = 0 \Rightarrow (c-1)(2c+1) = \left(c^2+c\right)$$

$$\Rightarrow \quad 2c^2 - 2c + \cancel{c} - 1 = c^2 + \cancel{c}$$

$$\Rightarrow c^2 - 2c - 1 = 0$$

$$c = \frac{2 \pm \sqrt{4+4}}{2} = 1 + \sqrt{2}$$

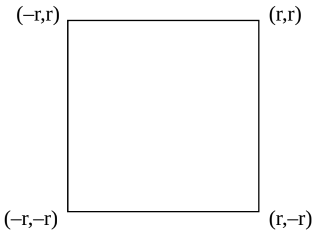
$$\text{Min value of } A = \frac{\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)}{\sqrt{2}}$$

$$= \left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right) = \left(3+2\sqrt{2}\right)$$

Hence, C is correct.

83. Area of figure $= 4r^2$

Hence, B is correct.



Comprehension 2.

$$f(x) = \begin{cases} 3x, & x \ge 0 \\ x & x < 0 \end{cases}, \quad g(x) = \begin{cases} x/3, & x \ge 0 \\ x & x < 0 \end{cases}$$

$$h(x) = f(g(x))$$

$$= \begin{cases} x, & x \ge 0 \\ x & x < 0 \end{cases} = x \quad \forall x \in R$$

84. $k(x) = 1 + \frac{1}{\pi}\left(\cos^{-1} x + \cot^{-1} x\right)$

$$\text{Min value at } x = 1 \qquad \Rightarrow \frac{5}{4}$$

$$\text{Max value at } x = -1 \qquad \Rightarrow \frac{11}{4}$$

Hence, B is correct.

85. $l(x) = \begin{cases} \sin^{-1}\frac{8x}{3}, & x \ge 0 \\ 0 & x < 0 \end{cases}$

Hence, D is correct.

86. $g(f(x)) = x, \; x \in R$

$$T(x) = f(x) + g(x)$$

$$= \begin{cases} \frac{10x}{3}, & x \ge 0 \\ 2x, & x < 0 \end{cases}$$

Hence, B is correct.

Comprehension 3:

87. $\lim\limits_{n \to \infty} \frac{f(c+1/n) - f(c)}{1/n} = f'(c) = a$

This holds true only if $f'(c)$ exist

Hence, D is correct.

88. $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to -\infty} f(x) = 0$ & $f(x)$ is continuous

Hence, C is correct.

89. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 0$

$$f'(x) = \begin{cases} ax^{a-1}\sin\left(x^{-c}\right)^{-c} x^a \cos x^{-c} x^{-c-1}, & x \ne 0 \\ 0 & x = 0 \end{cases}$$

$f'(x)$ is continuous if $a - c - 1 > 0$

Hence, D is correct.

Comprehension 4

$$f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3}, \quad f'(2) = 2$$

$$f(x+y) = \frac{2 + f(3x) + f(3y)}{3}$$

$$f'(x) = \lim_{h \to 0} \frac{f(3h) - f(0)}{h} = 3f'(3h) = 3f'(3)$$

Put $x = 0, \; y = 0, \quad f(0) = 2$

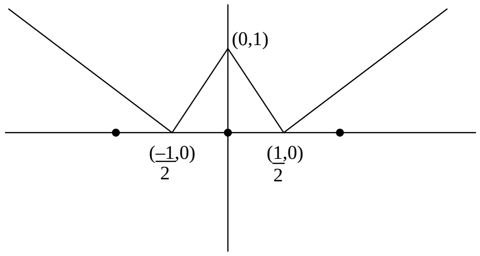
$$f(x) = 2x + 2$$

90. $\begin{cases} g(x) & x > 0 \end{cases}$

Hence, C is correct.

91. $g(x) = |2|x| - 1|$

Hence, C is correct



92. $x^2 + 4(|x|+1)^2 = 9$

$5x^2 + 8|x| - 5 = 0$

$$|x| = \frac{-8 \pm \sqrt{64+100}}{10}$$

$$= \frac{-8 \pm \sqrt{164}}{10}$$

$$|x| = \frac{-8 \pm \sqrt{164}}{10}$$

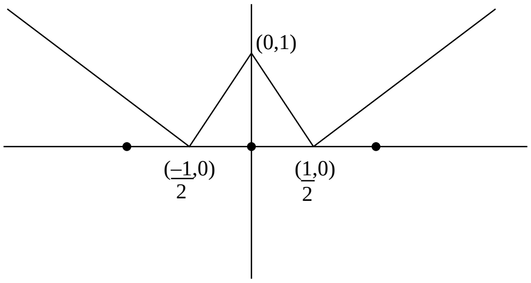
So, two values of *x* are the soln of eqn.
Hence, B is correct

93. $\begin{cases} g(x) & x > 0 \end{cases}$

Hence, C is correct.

94. $g(x) = |2|x| - 1|$

Hence, C is correct



95. $x^2 + 4(|x|+1)^2 = 9$

$5x^2 + 8|x| - 5 = 0$

$$|x| = \frac{-8 \pm \sqrt{64+100}}{10}$$

$$= \frac{-8 \pm \sqrt{164}}{10}$$

$$|x| = \frac{-8 \pm \sqrt{164}}{10}$$

So, two values of *x* are the soln of eqn.
Hence, B is correct

MATCH THE FOLLOWING

96.

(A) $f(x) = \begin{cases} \tan^{-1} x, & |x| \ge 1 \\ \dfrac{x^2 - 1}{4}, & |x| < 1 \end{cases}$

At $x = 1$

$$\text{RHD} \to \lim_{h \to 0} \frac{\tan^{-1}(1+h) - \pi/4}{h} = \frac{1}{2}$$

$$\text{LHD} \to \lim_{h \to 0} \left(\frac{(1-h)^2 - 1)}{4} \right) - \pi/4 \Big/ -h$$

$$\Rightarrow \lim_{h \to 0} \frac{(1-h)^2 - 1 - \pi}{-4h} = \text{Not defined}$$

At $x = -1$

$$\text{RHD} \to \frac{\lim\limits_{h \to 0} \dfrac{(-1+h)^2 - 1}{4} + \dfrac{\pi}{4}}{-h} = \lim_{h \to 0} \frac{(-1+h)^2 - 1 + \pi}{-4h}$$

$$= \text{Not defined}$$

$f(x)$ is non differentiable at $x = +1, -1$.

(B) $f(x) = (x^2 - 4)|x^2 - 5x + 6| + \cos|x|$

$f(x) = (x^2 - 4)|(x-2)(x-3)| + \cos x$

$f(x)$ is non differentiable at $x = 3$

(C) $\sin(x+y) = e^{x+y} - 2$

$\Rightarrow \cos(x+y)(1+y') = e^{x+y}(1+y')$

$\Rightarrow y' = -1$

(D) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x) \cdot f(h) - f(x)}{h} = 2f(x)$

Hence $\to$ A $\to$ (PQ), B $\to$ (S), C $\to$ (P), D $\to$ (R)

97.

(A) $g(x) = \begin{cases} a\sqrt{x+1}, & 0 < x < 3 \\ bx + 2, & 3 \le x < 5 \end{cases}$

As $g(x)$ is continuous at $x = 3$.

$\Rightarrow 2a = 3b + 2$ (I)

$$g'(x)=\begin{cases}\dfrac{a}{2\sqrt{x+1}}, & 0<x<3\\ b & 3\le x<5\end{cases}$$

As $g(x)$ is differentiable at $x=3$,

$$\frac{a}{4}=b \qquad \text{(II)}$$

Solving (I) & (II)

$$a=\frac{8}{5},\quad b=\frac{2}{5}\quad a+b=2.$$

(B) $\log_{10}\log_3 10=\log_{10}\left(\dfrac{1}{\log 3}\right)=-\log\log 3$

& $\log_{10}\log_{10}3=\log\log 3$

$f(-x)=-a\sin x-b\sqrt[3]{x}+4$

$f(x)=a\sin x+b\sqrt[3]{x}+4$

$\Rightarrow f(x)+f(-x)=8$

$\Rightarrow f(-x)=8-f(x)$

$$\Rightarrow f\left(\log_{10}\log_{10}3\right)=8-f\left(\log_{10}\log_3 10\right)$$
$$=8-5=3$$

(C) $$\lim_{x\to 0}\frac{\sin^4 x+(2\sin x\cos x)^4+\left(3\sin x-4\sin^3 x\right)^4}{\sin^3 x+(2\sin x\cos x)^3+\left(3\sin x-4\sin^3 x\right)^3}$$

$$\lim_{x\to 0}\frac{\sin x(1+(2\cos x)^4+(3-4\sin^2 x)^4)}{\left(1+(2\cos x)^3+\left(3-4\sin^2 x\right)^3\right)}$$

$=0$

(D) $2^{\log_{10}x}+8=(x-8)^{\log_2 10}$

By drawing a rough sketch, it can be observed that there is only one point of intersection of

$y=2^{\log_{10}x}+8$

& $y=(x-8)^{\log_2 10}$

Hence $D\to Q$.

$A\to R,\ B\to S,\ C\to P,\ D\to Q$.

98.

(A) $$f'(x)=\lim_{h\to 0}\frac{f(x)f(1+h/x)-f(x)}{h}$$

$$=\frac{f(x)}{x}f'(1)=\frac{f(x)}{x}$$

$f(x)=x$

$\Rightarrow f'(x)=1$

(B) $f(x)=|(x-1)(x-2)|+|\sin x|+\left[x-\dfrac{1}{2}\right]$

$f(x)$ is non differentiable at $x=-\dfrac{1}{2},\ 0,\ \dfrac{1}{2},\ 1,\ 2$.

(C) $f(x)=[a+7\sin x]$, $\quad x\in(0,\pi)$, $\quad a\in I$

$=a+[7\sin x]$

$f(x)$ is non differentiable when

$\sin x=\dfrac{1}{7},\dfrac{2}{7},\dfrac{3}{7},\dfrac{4}{5},\dfrac{5}{7},\dfrac{6}{7},1$

There are 13 points where $f(x)$ is not differentiable

(D) $y(x)=f(e^x).\ e^{f(x)}$

$y'(x)=f'\left(e^x\right)\cdot e^x\cdot e^{f(x)}+e^{f(x)}f'(x)f\left(e^x\right)$

$y'(0)=2$ Hence,

$A\to(P),\ B\to(R),\ C\to(\theta),\ D\to(\tau)$

99.

(A) $$\lim_{x\to 0^+}\frac{\left(\frac{\pi}{2}-\sin^{-1}\left(1-\{x\}^2\right)\right)\sin^{-1}\left(1-\{x\}\right)}{\sqrt{2}\left(\{x\}-\{x\}^3\right)}$$

$$\lim_{x\to 0^+}\frac{\cos^{-1}\left(1-x^2\right)\sin^{-1}(1-x)}{\sqrt{2}\left(x-x^2\right)}=\frac{\pi}{2}$$

$$\lim_{x\to 0^-}\frac{A\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2}\sqrt{\{x\}}(1-\{x\})}$$

$$=\frac{A\pi}{2\sqrt{2}}$$

$\Rightarrow k=\dfrac{\pi}{2}\ \&\ A=\sqrt{2}$

Then, $\sin^2 k+\cos^2\dfrac{A\pi}{\sqrt{2}}=1+1=2$

(B) $f(x)=[2+5|n|\sin x]=2+[5|n|\sin x]$

is non - differentiable at

$\sin x=\dfrac{1}{5|n|},\dfrac{2}{5|n|},\dfrac{3}{5|n|},\dfrac{4}{5|n|},\cdots\dfrac{5|n|}{5|n|}$

$\Rightarrow$ For 19 points, $|n|=2$

$\Rightarrow n=\pm 2.$

(C) $$f'\left(0^+\right)=\lim_{h\to 0}\frac{h\left((3/4)^{1/h}-(3/4)^{-1/h}\right)}{h\left((3/4)^{1/h}+(3/4)^{-1/h}\right)}=-1$$

$$f'\left(0^-\right)=\lim_{h\to 0}\frac{(3/4)^{-1/h}-(3/4)^{1/h}}{(3/4)^{-1/h}+(3/4)^{1/h}}=1$$

$$P = f'(0^-) - f'(0^+) = 2$$

$$\lim_{x\to 2^-} \frac{(e^{(x+2)\ln 4})^{\frac{[x+1]}{4}} - 16}{4^x - 16}$$

$$\lim_{x\to 2^-} \frac{4^{\frac{(x+2)}{2}} - 16}{4^x - 16} = \lim_{x\to 2^-} \frac{\left(4^{x/2-1} - 1\right)}{4^{x-2} - 1}$$

$$\Rightarrow \lim_{x\to 2^-} \frac{4^{\frac{(x-2)}{2}} - 1}{x - 2} \times \frac{x-2}{4^{x-2} - 1} = 1/2$$

(D) $f(x) = \dfrac{x}{x+1}, \quad x \neq -1$

$$g(x) = \frac{1}{1 + \frac{2x}{1+x}} = \frac{1+x}{1+3x}, \quad x \neq \frac{-1}{3}$$

g(x) is not differentiable at $x = -1, \dfrac{-1}{3}$ & 0.

Hence, $A \to P$, $B \to (P, Q)$, $C \to$, $D \to (R)$

100.

(A) $f(x) = x^3$

If $f(x)$ is continuous in $[0, 1]$, no of roots can be exactly one if range of $f(x) = [0, 1]$.

(B) between any two consecutive roots of $f'(x) = 0$, $f(x)$ can have atmost one root.

(C) $f(x) = |1 - |x - 1||$ is not differentiable at $x = 2, 0, 1$

(D) For $f(x)$ to be continuous,

$$x = 2 - x$$

$$x = 1$$

Hence,

$A \to (P)$, $B \to (R)$, $C \to (S)$, $D \to (P)$

Review Exercises

1. $f(x) = \begin{cases} \dfrac{\sqrt{x+1} - 1}{\sqrt{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x+1} - 1}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x+1} + 1} = 0$$

At $x = 0$

RHD

$$\lim_{h\to 0} \frac{\sqrt{h+1} - 1}{\sqrt{h}} = 0$$

LHD

$$\lim_{h\to 0} \frac{\sqrt{1-h} - 1}{\sqrt{-h}} = \lim_{h\to 0} \frac{\sqrt{-h}}{\sqrt{1-h} + 1} = \text{Not defined}$$

Hence, $f(x)$ is continuous but not derivable at $x = 0$.

2. Let $g(x) = ax^2 + bx + c$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{5xe^{1/x} + 2x}{3 - e^{1/x}} = 0$$

$$\lim_{x\to 0^+} f(x) = c$$

$$\Rightarrow c = 0$$

$$\lim_{x\to 1^-} f(x) = a + b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 \cdot \sin\left(\frac{1}{(1-x)^2}\right) + x$$

$$= 1$$

$\Rightarrow a + b = 1 \quad (1)$

LHD at $x = 1 \Rightarrow 2a + b$

RHD at $x = 1a$

$$\lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h^2}\right) + 1 + h - 2a - b}{h}$$

$\Rightarrow \quad 2a + b = 1 \qquad (II)$

upon solving

$a = 0, \; b = 1$

$\Rightarrow g(x) = x$

3. $f(|x|) = \begin{cases} -|x| - 1, & -2 \leq |x| < 0 \\ x^2 - |x|, & 0 \leq |x| < 2 \\ 2 - |x|, & 2 \leq |x| \leq 3 \end{cases}$

$$= \begin{cases} x^2 + x & -2 < x \leq 0 \\ x^2 - x & 0 < x < 2 \\ 2 - x & 2 \leq x \leq 3 \\ 2 + x & -3 \leq x \leq -2 \end{cases}$$

$$|f(x)|=\begin{cases}-x-1, & -2\le x\le -1\\ x+1, & -1<x<0\\ x-x^2 & 0\le x<1\\ x^2-x & 1\le x<2\\ x-2 & 2\le x\le 3\end{cases}$$

$$g(x)=\begin{cases}x^2-1, & -2\le x\le -1\\ x^2+2x+1, & -1<x<0\\ 0, & 0\le x<1\\ 2\left(x^2-x\right), & 1\le x<2\\ 0 & 2\le x\le 3\end{cases}$$

$g(x)$ is discontinuous at $x=0$ & 2
& $g(x)$ is non derivable at $x=0,\ \pm1, 2$

4. $f(x)=\sin^{-1}\left(\dfrac{2\sqrt{x^2-1}}{x^2}\right)$

For $f(x)$ to be defined, $\dfrac{2\sqrt{x^2-1}}{x^2}\le 1$

$$\Rightarrow 4\left(x^2-1\right)\le x^4$$

$$\Rightarrow \left(x^4-4x^2+4\right)\ge 0$$

$$\Rightarrow \left(x^2-2\right)^2\ge 0$$

$f(x)$ to be checked at $x=0,\ \pm\sqrt{2}$.

$$f'(x)=\frac{1}{\sqrt{1-\dfrac{4\left(x^2-1\right)}{x^4}}}\times\frac{x^2\times\dfrac{2x}{\sqrt{x^2-1}}-4x\sqrt{x^2-1}}{x^4}$$

$$\Rightarrow \frac{x^2}{\sqrt{\left(x^2-2\right)^2}}\times\frac{2x^3-4x\left(x^2-1\right)}{x^4\sqrt{x^2-1}}$$

$$=\frac{4-2x^2}{x\sqrt{x^2-1}\left|x^2-2\right|}$$

$f(x)$ is not derivable at $x=0,\pm\sqrt{2}$
but continuous at $x=\pm\sqrt{2}$.

5. $f(x)=\left(x^2-4\right)\left|x^3-6x^2+11x-6\right|+\dfrac{x}{1+|x|}$

$$=\left(x^2-4\right)|(x-1)(x-2)(x-3)|+\frac{x}{1+|x|}$$

$f(x)$ is not derivable at $x=1$ & 3.

6. $f(x)=\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}$

$$=\left|\sqrt{x-2}+\sqrt{2}\right|+\left|\sqrt{x-2}-\sqrt{2}\right|$$

$$f(x)=\begin{cases}2\sqrt{2}, & 2\le x<4\\ 2\sqrt{x-2}, & 4\le x\end{cases}$$

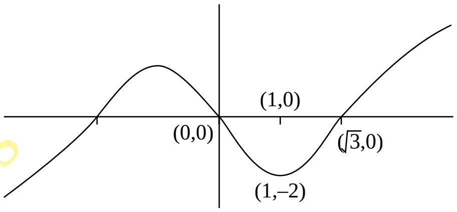
$f(x)$ is not derivable at $x=2$ & 4.

7. $f(x)=x^3-3x$

$$f'(x)=3x^2-3$$

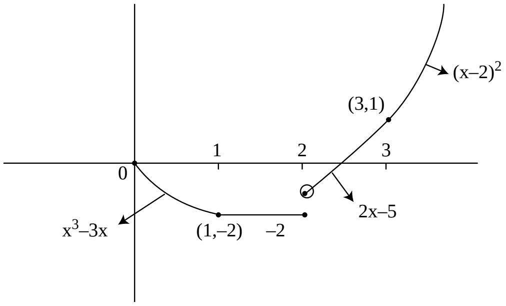
$$=3\left(x^2-1\right)=0$$

$$\Rightarrow x=\pm1$$



$$g(x)=\begin{cases}x^3-3x, & 0\le x\le 1\\ -2, & 1<x\le 2\\ 2x-5, & 2<x\le 3\\ (x-2)^2, & x>3\end{cases}$$

$g(x)\Rightarrow$



$g(x)$ is discontinuous & non derivable at $x=2$.

8. $f(x)=\begin{cases}x^2+a, & 0\le x<1\\ 2x+b, & 1\le x\le 2\end{cases}$

$$g(x)=\begin{cases}3x+b, & 0\le x<1\\ x^3, & 1\le x\le 2\end{cases}$$

RHD at $x=1$ for $f(x)$

$$\lim_{h\to0}\frac{2+2h+b-(2+b)}{h}=\lim_{h\to0}\frac{2h}{h}=2$$

LHD at $x=1$ for $f(x)$

$$\lim_{h\to0}\frac{(1-h)^2+a-2-b}{-h}=\lim_{h\to0}\frac{h^2-2h+a-1-b}{-h}=2$$

if $a = 1 + b$ (I)

RHD at $x = 1$ for $g(x)$

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h} = \lim_{h\to 0}\frac{(1+h)^2+1+1+h}{} = 3$$

LHD at $x = 1$ for $g(x)$.

$$\lim_{h\to 0}\frac{3-3h+b-1}{-h} = 3 \quad \text{if } b = -2$$

$\Rightarrow a = -1$

Value of $\frac{d\,(f(x))}{d\,(g(x))} = \frac{2}{3}$

9. If f & g are derivable & $f(x_0) \neq g(x_0)$

$\Rightarrow$ f, g will not intersect.

So, max $\{f, g\}$ or min $\{f, g\}$ will be either f or g respectively for all $x \in D$.

So, they are derivable.

If $f(x_0) = g(x_0) \Rightarrow$ they will intersect

so, we shall have a sharp pt at x_0 for

max $\{f, g\}$ or min $\{f, g\}$

so, they are not derivable at $x = x_0$ (May or May not)

10. Let $g(x) = e^{x(x-1)} + \log(1+x)$

(i) $g'(x) = (2x-1)e^{x(x-1)} + \frac{1}{1+x} \geq 0$

$g(0) = 1$

$g(1) = 1 + 1 = 2.$

As $g(x)$ is continuous & differentiable in [0, 1] & will vary from [1, 2]

So, $f(x) = g(x)$ will have atleast one solution

(ii) $f(x) = \ln x$

$f'(x) = \frac{1}{x} > 0$

$f(1) = 0, \quad f(e) = 1$

As $f(x)$ has atleast one root in [1, e]

11. $\frac{x}{y} f\left(\frac{x}{y}\right) = xf(x) - yf(y)$

$f\left(\frac{x}{y}\right) = yf(x) - \frac{y^2}{x} f(y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f((x)/(1+h/x)^{-1})-f(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x)}{1+h/x} - \frac{1}{x(1+h/x)^2} f\left(1+\frac{h}{x}\right) - f(x)\right)\times\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{\left(x\left(1+\frac{h}{x}\right)f(x) - f\left(1+\frac{h}{x}\right) - f(x)x\left(1+\frac{h}{x}\right)^2\right)}{x\left(1+\frac{h}{x}\right)^2 h}$$

$$= \lim_{h\to 0}\frac{(xf(x)+hf(x)-f(1+h/x)-xf(x)-f(x)h^2/x-2\,hf(x))}{x(1+h/x)^2 h}$$

$$f'(x) = -\frac{f(x)}{x} - \frac{f'(1)}{x^2}$$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{f'(0)}{x^2}$$

$$\text{I.F} = e^{\int 1/x\,dx} = x$$

$$xy = \int -\frac{f'(1)}{x}dx$$

$$xy = -f'(1)\ln x + c$$

$$y = \frac{\ln x}{x}$$

Hence proved.

12. $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\sin x - \ln\left(e^x\cos x\right)}{6x^2}$$

$$= \lim_{x\to 0^+}\frac{\cos x - \frac{1}{e^x\cos x}\left[-e^x\sin x + e^x\cos x\right]}{12x}$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x - 1}{12x} + \frac{e^x\tan x}{12x}\right) = \frac{1}{12}a$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{x^2+2\cos x-1}{x^4} = \text{Not defined}$$

Hence, $f(x)$ is not continuous at $x = 0$.

13. $$f(x) = \begin{cases} x^2\sin\left(x^{-4/3}\right), & x \leq 0 \\ 0, & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x\sin x^{-4/3} - \frac{4}{3}x^{-1/3}\left(\cos x^{-4/3}\right), & x \leq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x\to 0^-} f'(x) = \text{Not defined} \ \& \ f'(0)=0$$

LHD at $x = 0$

$$\lim_{h\to 0}\frac{h^2\sin(h)^{-4/3}}{-h}=0$$

Hence, *f* has a derivative for all x & $f'(x)$ is not continuous at *x* = 0.

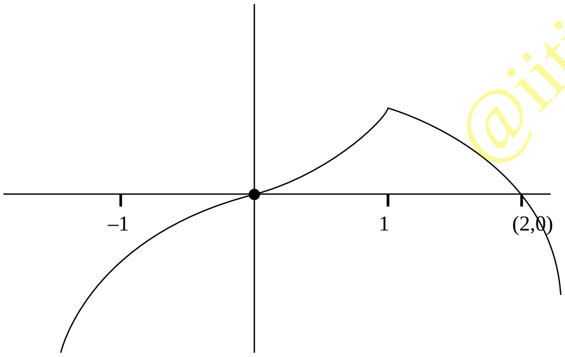
14. $f(x)=x-\left|x-x^2\right|=x-|x(1-x)|$

$$=\begin{cases} x+x-x^2, & x\ge 1\\ x-x+x^2, & 0<x<1\\ x+x-x^2 & x\le 0\end{cases}$$

$$=\begin{cases} 2x-x^2, & x\ge 1\\ x^2 & 0<x<1\\ 2x-x^2, & x\le 0\end{cases}$$

$f(x)$ is continuous in $x\in[-1,1]$

& not differentiable at $x = 0$



15. LHD at $x=1$

$$f'(1^-)=\lim_{h\to0}\frac{a(1-h)^3+b-a-b}{-h}$$

$$=a\lim_{h\to0}\left(\frac{(1-h)^3-1}{-h}\right)=3a$$

RHD at $x=1$

$$f'\left(1^+\right)=\lim_{h\to0}\frac{2\cos\pi(1+h)+\tan^{-1}(1+h)-a-b}{h}$$

if $a+b+2+\pi/4=0$

$$f'\left(1^+\right)=+\frac{1}{2}$$

$$\Rightarrow a=+\frac{1}{6},\ b=-\frac{\pi}{4}-\frac{13}{6}$$

$\Rightarrow k_1=6,\ \ k_2=12$

$k_1^2+k_2^2=180.$

16. $\lim\limits_{x\to0^+} x^a\cos\left(\dfrac{1}{x^3}\right)=0$ if $a>0$

LHD at x = 0

$$\lim_{h\to0}-(-h)^{a-1}\cos\frac{1}{h^3}$$

RHD at x = 0

$$\lim_{h\to0}h^{a-1}\cos\frac{1}{h^3}$$

For *f(x)* to be non differentiable, $a-1\le 0$

$\Rightarrow\ a\le 1$

17. $f(x)=\begin{cases} x\cdot\dfrac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}, & x\ne 0\\ 0, & x=0\end{cases}$

For $a>0,\ a\ne 1$

$$\lim_{x\to0^+}x\cdot\frac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}=0$$

$$\lim_{x\to0^-}x\cdot\frac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}=0$$

LHD at $x = 0$

$$\lim_{h\to0}\frac{(-h)a^{-1/h}-a^{1/h}}{(-h)a^{-1/h}+a^{1/h}}=-1$$

RHD at $x = 0$

$$\lim_{h\to0}\frac{h\cdot\left(a^{1/h}-a^{-1/h}\right)}{h\cdot\left(a^{1/h}+a^{1/h}\right)}=1$$

Hence, $f(x)$ is continuous at $x=0$ but not derivable.

If $a=1$

$f(x)=0$ Hence, f(x) is continuous & derivable at $x=0$.

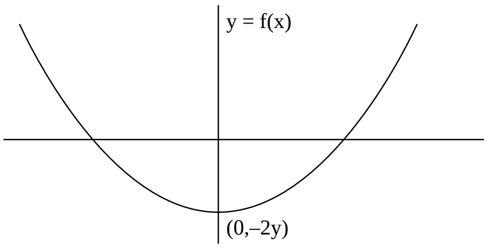
18. $f(x)=x^4-8x^3+22x^2-24.$

$f'(x)=4x^3-24x^2+44x$

$=4x\left(x^2-6x+11\right)$

$$g(x)=\begin{cases} -24, & -1\le x\le 0\\ x^4-8x^3+22x^2-24, & 0<x\le 1\\ x-10, & x>1\end{cases}$$

Hence, $g(x)$ is non differentiable at $x = 1$.



19. From the graph,

$g'(x) =$ slope of tangent at pt x.

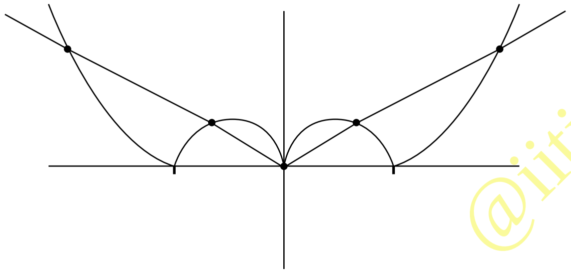
Hence, $g'(0),\ 0,\ g'(4),\ g'(2),\ g'(-2)$

20. $f(x) = \max\left\{\left|x^2 - 2|x|\right|, |x|\right\}$.

From graph,

$f(x)$ is non differentiable at 5 points (marked - $\circ$)

& $g(x)$ is non differentiable at 7 points (marked - //)



21. $f(x) = \dfrac{x}{1-|x|} = \begin{cases} \dfrac{x}{1-x}, & x \ge 0 \\ \dfrac{x}{1+x}, & x < 0 \end{cases}$

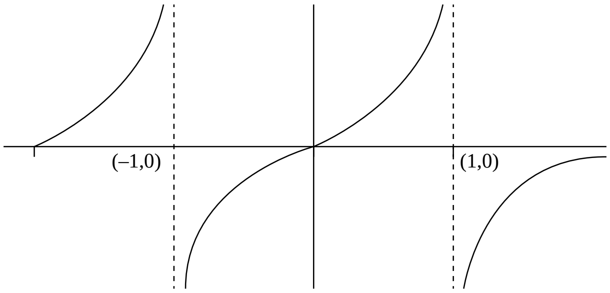
For $f(x)$

LHD at $x = 0$

$$\lim_{h\to 0} \frac{-h}{1+h} \times \frac{1}{-h} = 1$$

RHD at $x = 0$

$$\lim_{h\to 0} \frac{h}{1-h} \times \frac{1}{h} = 1$$



Now $f'(x) = \begin{cases} \dfrac{1}{(1-x)^2}, & x > 0,\ x \ne 1 \\ \dfrac{1}{(1+x)^2}, & x < 0,\ x \ne -1 \end{cases}$

For $f'(x)$

LHD at x = 0

$$\lim_{h\to 0} \frac{\frac{1}{(1-h)^2} - 1}{-h} = \lim_{h\to 0} \frac{1-(1-h)^2}{(1-h)^2(-h)}$$

$$= \lim_{h\to 0} \frac{2h - h^2}{(1-h)^2(-h)} = -2$$

RHD at x = 0

$$\lim_{h\to 0} \frac{\frac{1}{(1-h)^2} - 1}{h} = 2$$

Hence, $f'(x)$ is not derivable at x = 0.

22. $f(x+y) = f(x)\cdot f(y)$

$\Rightarrow f(x) = a^x$

$\Rightarrow g(x) = \dfrac{a^x - 1}{x}$, Now $\lim\limits_{x\to 0} g(x) = 1$

$\Rightarrow a = e.$

Hence, $g(x) = \dfrac{e^x - 1}{x}$

Now, $f(x)$ is derivable for $\forall x$ & $f'(x) = f(x)$ for all x.

23. $f(x) = |x-1| \cdot ([x] - [-x])$

$$f'\left(1^+\right) = \lim_{h\to 0} \frac{h(1-(-2))}{h} = 3$$

$$f'(1^-) = \lim_{h\to 0} \frac{h\cdot 1}{-h} = -1$$

24. $|f(x)| \le x^{4n} \Rightarrow -x^{4n} \le f(x) \le x^{4n}$

LHD at x = 0

$$f'(x) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)}{-h} = f'(0^-) = 0$$

RHD at

$$f'(x) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = f'(0^+) = 0$$

Hence, $f(x)$ is differentiable at $x = 0$.

25. (a) $\lim\limits_{h\to 0}\dfrac{f\left(3+h^2\right)-f\left(3-h^2\right)}{2h^2}$

$\Rightarrow f'(3)=2$

(b) $\lim\limits_{h\to 0}\dfrac{f\left(a+2h^2\right)-f\left(a-2h^2\right)}{h^2}$

$\Rightarrow 4f'(a)=1$

26. $f(x)=\begin{cases}\dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$

$\lim\limits_{x\to 0^+} f(x)=-5,\quad \lim\limits_{x\to 0^-} f(x)=\dfrac{2}{3}$

$f(x)$ is discontinuous & non derivable at $x=0$.

Now, $xf(x)$

$$x f(x)=\begin{cases}x\cdot\left(\dfrac{5e^{1/x}+2}{3-e^{1/x}}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

$\lim\limits_{x\to 0^+} f(x)=0,\quad \lim\limits_{x\to 0^-} f(x)=0,\quad f(0)=0$

LHD at $x=0$

$$\lim_{h\to 0}\frac{-h\left(5e^{-1/h}+2\right)}{-h\left(3-e^{-1/h}\right)}=\frac{2}{3}$$

RHD at $x=0$

$$\lim_{h\to 0}\frac{h\left(5e^{1/h}+2\right)}{h\left(3-e^{1/h}\right)}=-5$$

Hence, $x\,f(x)$ is continuous but not differentiable

Now, $x^2 f(x)$

$$x^2 f(x)=\begin{cases}x^2\dfrac{\left(5e^{1/x}+2\right)}{3-e^{1/x}}, & x\neq 0\\ 0 & x=0\end{cases}$$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=f(0)=0$

LHD at $x=0$

$$\lim_{h\to 0}\frac{h^2}{-h}\frac{(5e^{-1/h}+2)}{\left(3-e^{-1/h}\right)}=0$$

RHD at $x=0$

$$\lim_{h\to 0}\frac{h^2}{h}\frac{\left(5e^{1/h}+2\right)}{\left(3-e^{1/h}\right)}=0$$

Hence, $x^2 f(x)$ is continuous and derivable at $x=0$.

27. $f\circ f(x)=\begin{cases}1-f(x), & 0\le f(x)\le 1\\ f(x)+2, & 1<f(x)<2\\ 4-f(x), & 2\le f(x)\le 4\end{cases}$

$$=\begin{cases}x, & 0\le x\le 1\\ 2-x, & 1<x<2\\ 6-x, & 2<x<3\\ 2, & x=2\\ 0, & x=3\\ x-3, & 3<x\le 4\end{cases}$$

$f\circ f(x)$ is discontinuous at $x=2,\ 3$ and non derivable at $x=1,\ 2,\ 3$.

28. $f(x)=\begin{cases}ax^2-ax+b, & x<1\\ x-1, & 1\le x\le 3\\ px^2+qx+2, & x>3\end{cases}$

If $f(x)$ is continuous for all $x\in R$

$b=0$ (I)

$2=9p+3q+2$

$\Rightarrow\ 3p+q=0$ (II)

$$f'(x)=\begin{cases}2ax-a, & x<1\\ 1, & 1\le x\le 3\\ 2px+q, & x>3\end{cases}$$

LHD at $x=1=a\neq 1$ (III)

$6p+q=1$ (IV)

From (II) & (IV)

$p=\dfrac{1}{3}$

$q=-1$

29. $f(x)=x\sin\left(\dfrac{1}{x}\right)\sin\left(\dfrac{1}{x\sin 1/x}\right),\qquad x\neq 0,\quad x\neq\dfrac{1}{r\pi}$

$\lim\limits_{x\to 0^+} f(x)=0,\quad f(0)=0$

at $x\neq\dfrac{1}{r\pi}$

$\lim\limits_{x\to\frac{1^+}{r\pi}} f(x)=0,\ \lim\limits_{x\to\frac{1^-}{r\pi}} f(x)=0,\ f\left(\dfrac{1}{r\pi}\right)=0$

Hence, $f(x)$ is continuous for $x\in[0,1]$.

LHD at $x=0$

$$\lim_{h\to 0}\frac{-h\sin\left(\dfrac{1}{-h}\right)\sin\left(\dfrac{1}{h\sin 1/h}\right)}{-h}=\text{Not defined}$$

Hence, $f(x)$ is not derivable at $x=0$.

30. $$D^*(f\cdot g)=\lim_{h\to 0}\frac{f^2(x+h)g^2(x+h)-f^2(x)g^2(x)}{h}$$

$$f^2(x+h)\left(g^2(x+h)-g^2(x)\right)+$$

$$g^2(x+h)\left(f^2(x+h)-f^2(x)\right)+$$

$$=\frac{1}{2}\lim_{h\to 0}\frac{g^2(x)f^2(x+h)+f^2(x)g^2(x+h)-2f^2(x)g^2x}{h}$$

$$=\left[f^2(x)D^*g(x)+g^2(x)D^*f(x)\right]$$

$$D^*\frac{f(x)}{g(x)}=\lim_{h\to 0}\left(\frac{f^2(x+h)}{g^2(x+h)}-\frac{f^2(x)}{g^2(x)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{\begin{array}{c}f^2(x+h)g^2(x)-f^2(x)g^2(x)-\\ f^2(x)g^2(x+h)+f^2(x)g^2(x)\end{array}}{g^2(x)g^2(x+h)h}$$

$$=\frac{g^2(x)D^*f(x)-f^2(x)D^*g(x)}{(g(x))^4}$$

31. $f(x+y)-2f(x-y)+f(x)-2f(y)=y-2$

Put $x=y=0$

$f(0)=1$

Put $x=0$

$f(y)-2f(-y)+1-2f(y)=y-2$

$f(y)+2f(-y)=3-y$ (I)

Put $y=-y$

$f(-y)+2f(y)=3+y$ (II)

Solving (I) & (II)

$-3f(y)=-3-3y$

$f(y)=1+y \quad \Rightarrow f(x)=1+x$

32. $f(xy)=x\,f(x)+y\,f(y)$

Put $x=y=0\Rightarrow f(0)=0,\quad f(1)=0$ as $x=y=1$

$$f'(x)=\lim_{h\to 0}\frac{xf(x)+\left(1+\frac{h}{x}\right)f\left(1+\frac{h}{x}\right)-xf(x)}{h}$$

$$f'(x)=\frac{f'(1)}{x} \qquad \text{(I)}$$

Differentiating wrt x.

$y\,f'(xy)=x\,f'(x)+f(x)$

Put $y=0$ & $x=1$

$f'(0)=0$

Hence, $f'(x)=0$

$f(x)=0$ as $f(0)=0$.

33. $f(x\,y\,z)=f(x)+f(y)+f(z)$

Put $x=y=z=1 \quad \Rightarrow f(1)=0$

$f(xy)=f(x)+f(y)$

$\Rightarrow f(x)=\ln x$.

34. $xf(y)+yf(x)=(x+y)f(x)\cdot f(y)$

Put $y=0$

$xf(0)=xf(x)f(0)$

$\Rightarrow xf(0)[1-f(x)]=0$

either $f(x)=1$ or $f(0)=0$

Put $x=-1$

$f(y)+yf(1)=(1+y)f(1)f(y)$ (II)

Put $x=1,\ y=1$

$2f(1)=2f^2(1)$

$\Rightarrow 2f(1)(f(1)-1)=0$

$\Rightarrow$ either $f(1)=0$ or $f(1)=1$

From (II)

If $f(1)=1$

$f(y)+y=(1+y)\,f(y)$

$\Rightarrow \quad f(y)=1$

or If $f(1)=0$

$f(y)=0$

$\Rightarrow \quad f(x)=0$

$\Rightarrow \quad f(x)=0$ or $f(x)=1$

35. $f(x+y)=f(x)+f(y)+f(x)\,f(y)$

Put $y=0,\ f(0)=0$

$$f'(x)=\lim_{h\to 0}\frac{f(x)+f(h)+f(x)\,f(h)-f(x)}{h}$$

$=(1+f(x))f'(0)$

$f'(x)=-1-f(x)$

$$\frac{dy}{dx}=-1-y.$$

$\ln(1+y)=-x+c.$

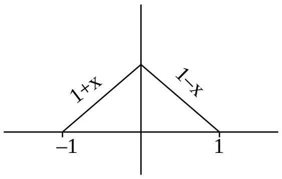
$1+y=e^{-x+c}$

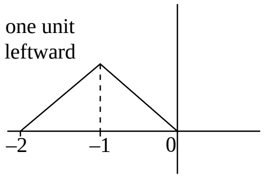
$y=e^{-x+c}-1$

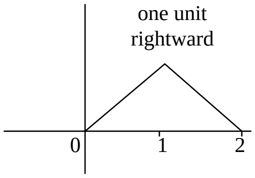
$y=e^{-x}-1$

Target Exercises

1. $$g(x)=\begin{cases}1-|x|, & |x|\le 1\\ 0, & |x|>1\end{cases}$$







$$f(x)=\begin{cases}0, & x<-2,\ x>2\\ x+2, & -2<x\le -1\\ -x, & -1<x\le 0\\ x, & 0<x\le 1\\ -x+2, & 1<x<2\end{cases}$$

Hence, $f(x)$ is continuous but not derivable at $x=-2,-1,0,1,2.$

2. LHD at $x=e$

$$\lim_{h\to 0}\frac{(-h)\,2^{-2^{\frac{1}{h}}}}{-h}=0$$

RHD at $x=e$

$$\lim_{h\to 0}\frac{h\ 2^{-2\frac{1}{-h}}}{h}=1.$$

$f(x)$ is not differentiable at $x=e$

3. LHD at $x=0$

$$\lim_{h\to 0}\frac{-\sqrt{h}\left(1-h\cos\frac{1}{h}\right)}{-h}=\lim_{h\to 0}\frac{\left(1-h\cos\frac{1}{h}\right)}{-\sqrt{h}}=\infty$$

RHD at $x=0$

$$\lim_{h\to 0}\frac{\sqrt{h}(1+h\sin 1/h)}{h}=\lim_{h\to 0}\frac{(1+h\sin 1/h)}{\sqrt{h}}=\infty$$

$f(x)$ is not derivable at $x=0.$

4. $f\left(x^2+y^2\right)=(f(x))^2+(f(y))^2$

Put $y=h$

$$\frac{f\left(x^2+h^2\right)-f\left(x^2\right)}{h^2}=\frac{f(x)^2-f\left(x^2\right)+f(h)^2}{h^2}$$

Put $x=0,\ y=0$

$f(0)=2f(0)^2$

$\Rightarrow$ either $f(0)=0$ or $f(0)=\frac{1}{2}$

If $f(0)\ =\ 0$

$f(x)^2=f\left(x^2\right)$

$2f(x)f'(x)=2xf'\left(x^2\right)$

Put

Differentiating wrt x.

$2xf'\left(x^2+y^2\right)=2f(x)f'(x)$

Put $y=0$

$xf'\left(x^2\right)=f(x)f'(x)$

Integrating

$$\frac{1}{2}f\left(x^2\right)=\frac{f^2(x)}{2}$$

5. $f(x)+f(y)=f\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

As $f'(0)=1\cdot$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}\qquad \text{as } f(x) \text{ is odd}$$

$$=\lim_{h\to 0}f\frac{\left(x+h\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

Integrating both sides.

$f(x)=\sin^{-1}x+c$

$f(x)=\sin^{-1}x\qquad$ as $\ f(0)=0$

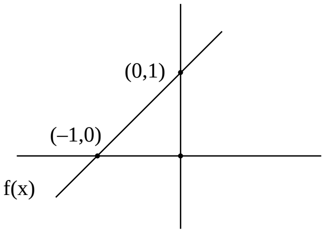
6. $f(x) = x^3 - x^2 + x + 1$

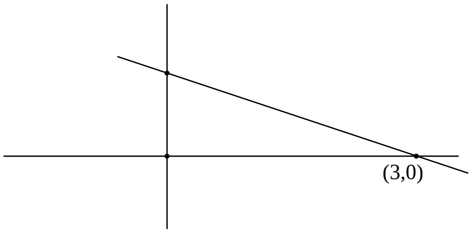
$f'(x) = 3x^2 - 2x + 1 > 0$

$g(x) = 3 - x$

$$h(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x & 1 < x \le 2 \end{cases}$$

$h(x)$ is continuous but not derivable at $x = 1$





7. $$g(x) = \begin{cases} 0, & -e \le x < 1 \\ \left\{1 + \frac{1}{3}\sin \ln x^{2\pi}\right\}, & 1 \le x \le e \end{cases}$$

$$= \begin{cases} 0 & -e \le x < 1 \\ 0 & x = 1,\ x = e \\ \dfrac{\sin 2\pi \ln x}{3} & 1 < x < e^{1/2} \\ \dfrac{-\sin 2\pi \ln x}{3} & e^{1/2} < x < e \end{cases}$$

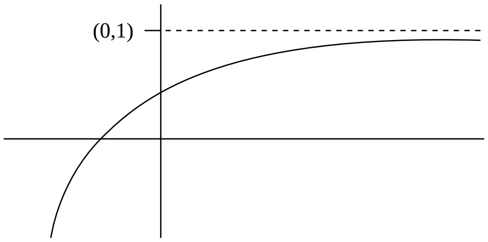
$$f(x) = \begin{cases} x, & -e \le x < 1 \\ 0, & x = 1 \text{ or } e \\ x\dfrac{\sin 2\pi \ln x}{3}, & 1 < x < e^{1/2} \\ -x\left(\dfrac{\sin 2\pi \ln x}{3}\right), & e^{1/2} < x < e \end{cases}$$

Hence, $f(x)$ is discontinuous at $x = 1$ & non derivable also.

8. $\lim\limits_{x \to \infty} \left(f(x) + f'(x)\right) = 1$

If $f(x)$ is differentiable & limit of $f(x) + f'(x)$ exist, it is possible only when

$f(x) \to 1$ & $f'(x) \to 0$ for $x \to \infty$



9. $\dfrac{f'(x)}{1 + f^2(x)} \ge -1$

Integrating

$$\int_{\infty}^{\infty} \tan^{-1} f(x) \ge -\int_a^b x$$

$-\pi \quad \ge -(b - a)$

$\pi < b - a$

10. $$f(x) = \lim_{x \to 0} \frac{\left(1 - \cos^2 x \cos^2 x/2 \cos^2 x/8 \cdots \cos^2 x/2^{n-1}\right)}{x^2}$$

$f(n) = -1 - \dfrac{1}{4} - \dfrac{1}{16} \cdots$

$d = \lim\limits_{n \to \infty} f(n) = \dfrac{-1}{1 - 1/4} = -\dfrac{4}{3}$

RHD at $x = 0 \quad = \dfrac{b}{\pi}$

LHD at $x = 0 \Rightarrow -\dfrac{9}{\pi}$

$\lim\limits_{x \to 0^-} f(x) = \dfrac{a}{2}$

$\lim\limits_{x \to 0^+} f(x) = c$

$a = -\dfrac{8}{3}$

$b = \dfrac{8}{3}$

$c = \dfrac{-4}{3}$

$\Rightarrow 27abc = 256$.

11. $\lim\limits_{x \to 3^-} f(x) = 5, \quad \lim\limits_{x \to 3^+} f(x) = 9p + 3q + 2$.

$\Rightarrow 9p + 3q = 3$

$\Rightarrow \quad 3p + q = 1$ (I)

$\lim\limits_{x\to1^+} f(x) = 3, \quad \lim\limits_{x\to1^-} f(x) = b$

$\Rightarrow b \neq 3$ (II)

$\lim\limits_{x\to3^+} f'(x) = 6p+q, \qquad \lim\limits_{x\to3^-} f'(x) = 1$

$\Rightarrow 6p+q=1$ (III)

From (I) & (III)

$p=0,\ q=1$

LHD for $|f(x)|$ at $x=1$

$$\lim_{h\to0}\frac{|-ah(h+1)+b|-3}{-h} = -a$$

So, $b=-3$ for LHD to exist

RHD for $|f(x)|$ at $x=1 \quad =1$.

$\Rightarrow a \quad = -1$

$$|a+b+p+q| = |-1-3+0+1| = 3 = \frac{k}{18}$$

$\Rightarrow k=54$

12. $f(x-y) = f(x)\,g(y) - f(y)\,g(x)$

Put $x=y \Rightarrow \quad f(0)=0$

Put $y=0 \Rightarrow g(0)=1$

$$f\left(0^+\right) = \lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to0^+}\frac{f(0)g(-h)-g(0)f(-h)-f(0)}{h} = f'\left(0^-\right)$$

Put $y=x$ in $g(x-y) = g(x)g(y)+f(x)f(y)$

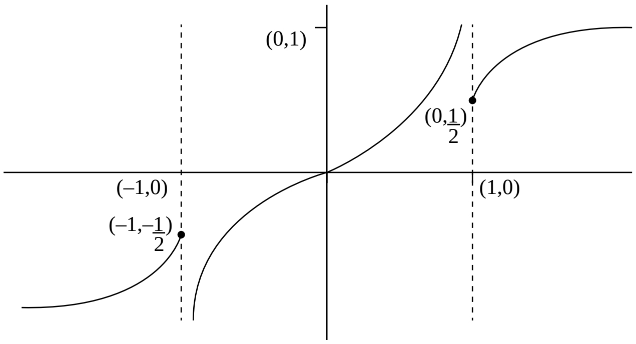
$\Rightarrow g^2(x)+f^2(x)=1$

13. $$f(x) = \begin{cases} \dfrac{x}{1+x}, & x\ge1,\ -1<x<0 \\ \dfrac{x}{1-x} & x\le-1,\ 0\le x<1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x\ge1,\ \ -1<x<0 \\ \dfrac{1}{(1-x)^2}, & x\le-1,\ 0\le x<1 \end{cases}$$

Graph of f(x)

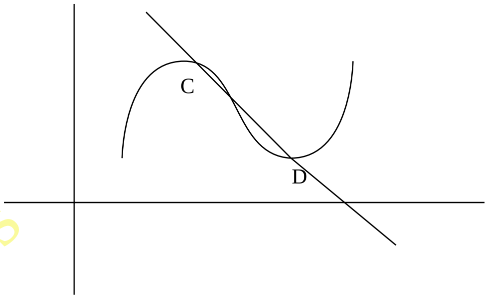
Domain of $f'(x) = R \sim \{-1,1\}$



14. (i) Between points A & B

(ii) Greater

(iii)



(iv) B, C & D, E Points

15. $f(x+y+z) = f(x)\cdot f(y)\cdot f(z)$

Put $z=0$

$f(x+y) = f(x)f(y)f(0)$

$$f'(x) = \lim_{h\to0}\frac{f(x)f(h)f(0)-f(x)f(0)f(0)}{h}$$

$= 3\,f(x)\,f(0)$

$\dfrac{f'(x)}{f(x)} = 3f(0) \quad \Rightarrow$ Put $x=0 \Rightarrow f^2(0)=1$

$\Rightarrow f(0)=\pm1$

Put $\quad x=2$

$\Rightarrow f'(2) = 3f(0)f(2)$

$= 12f(0)$

Integrating

$f(x) = e^{3f(0)x}+c.$

As $c=0$

$f(x) = e^{3f(0)x}$

As $f(x)$ is always +ve $\Rightarrow f(0)$ is +ve

$\Rightarrow f(0)=1$

& Hence, $f'(2)=12$

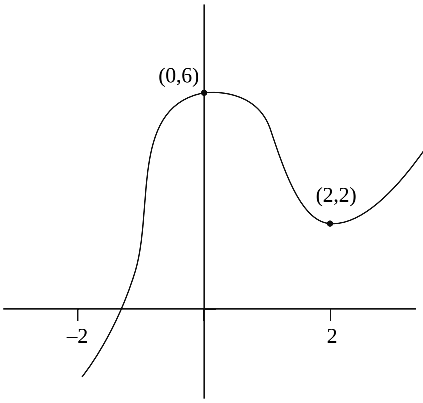
16. $f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x$

$= 3x(x-2)$

$$g(x) = \begin{cases} (x+2)^3 - 3(x+2)^2 + 6, & -3 \le x \le -2 \\ 6 & -2 < x \le -1 \\ 1-x & x \ge 0 \\ (x+1)^3 - 3(x+1)^2 + 6 & -1 < x < 0 \end{cases}$$

g(x) is continuous for $x \in [-3, 1]$ expect at $x = 0$



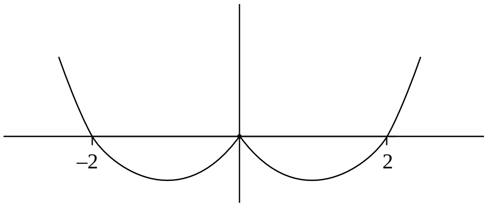
17. $$f(x) = \begin{cases} x^2 - 2x, & x \ge 0 \\ x^2 + 2x & x < 0 \end{cases}$$

a) Graph of $f(x)$

$f(x)$ is continuous for $x \in R$ but not derivable at $x = 0$

b) $$g(x) = \begin{cases} x^2 + 2x, & -2 \le x \le -1 \\ -1, & -1 < x \le 0 \\ 0 & 0 < x \le 2 \\ x^2 - 2x & 2 < x \le 3 \end{cases}$$

g(x) is continuous for $x \in [-2, 3] \sim \{0\}$



18. $$\lim_{x \to a} \left| \frac{f(x) - L(x)}{x - a} \right| = \left| \lim_{x \to a} \frac{f(x) - L(x)}{x - a} \right| =$$

As $f(x)$ is continuous $\Rightarrow \lim_{x \to a} L(x) \to f(a)$

For differentiability $\Rightarrow |f'(x)| = 0$

$\Rightarrow f'(x) = 0$

19. Let assume that g(x) & $g''(x)$ are +ve for $x \in (-3, 3)$

$\Rightarrow -1 \le f(x) \le 1$ & $2\sqrt{2} \le g(x) \le 3$

Now, $f'(x) = g(x)$

Integrating x

$$f(x) = \int_{-3}^{x} g(x)dx + f(-3)$$

If g(x) is decreasing

$$\int_{-3}^{x} g(x)dx > 6\sqrt{2} \Rightarrow f(x) > 2\sqrt{2}(3-1) > 1$$

which is a contradiction

If g(x) is increasing

$$\int_{-3}^{x} g(x)dx > 6\sqrt{2} \Rightarrow f(x) > 4\sqrt{2} \ge f(x) > 1$$

which is a contradiction

Hence, g(x) & $g''(x)$ can't be of same sign

$\Rightarrow g(x)g''(x) < 0$

20. $\lim_{x \to \frac{\pi^+}{2}} f(x) = 0, \quad \lim_{x \to \frac{\pi^-}{2}} f(x) = 0$

$\lim_{x \to \frac{3\pi^+}{2}} f(x) = 0, \quad \lim_{x \to \frac{3\pi^-}{2}} f(x) = 0$

So, f(x) is continuous at $x = \pi/2 \diagup 3\pi/2$.

LHD at x = $\pi/2$

$$\lim_{h \to 0} \frac{\cos(\pi/2 - h)\left(\cos \dfrac{1}{\cos\left(\dfrac{\pi}{2} - h\right)}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\sin h \cos\left(\dfrac{1}{\sin h}\right)}{-h}$$

= does not exist

LHD at x = $3\pi/2$

$$\lim_{h \to 0} \frac{\cos(3\pi/2 - h)\left(\cos \dfrac{1}{\cos\left(\dfrac{3\pi}{2} - h\right)}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cos\left(\frac{1}{\sin h}\right)$$

$$= \text{Not defined}$$

Hence, f(x) is non derivable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

21. LHD at x = 0

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{-h} = \lim_{h\to 0} \frac{-2}{h^3} e^{-1/h^2} = 0$$

RHD at *x* = 0

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{2}{h^3} e^{-1/h^2} = 0$$

Hence, f(x) is infinitely differentiable at x = 0

22. $f(xy) = f(x)\cdot f(y) - f(x) - f(y) + 2$

As f(0) = 1

$$f'(x) = \lim_{h\to 0} \frac{6(x)f\left(1+\frac{h}{x}\right) - f(x) - f(1+h/x) + 2 + f(x)}{h}$$

$$f'(x) = (f(x)-1)\lim \frac{f(1+h/x)-2}{h}$$

$$= \frac{(f(x)-1)}{x} f'(1)$$

$$\frac{f'(x)}{f(x)-1} = \frac{f'(1)}{x}$$

As Put x =1, y = 2

$\Rightarrow f(1) = 2$

Integrating both sides,

$$f(x) - 1 = cx^{f'(1)}$$

$$f(x) = cx^{f'(1)} + 1$$

$$f(x) = 1 + x^2$$

23. $|f(x)| \le x^2 e^x$

Put $x = 0 \Rightarrow f(0) = 0$

Divide by |x| both sides

$$\left|\frac{f(x)-f(0)}{x}\right| \le |x| e^x$$

Now $\lim_{x\to 0}\left|\frac{f(x)-f(0)}{x}\right| \le \lim_{x\to 0} |x| e^x = 0$

$\Rightarrow$ f(x) is differentiable at x = 0.

24. $$f(x) = \begin{cases} \dfrac{x(x-1)}{x+1}, & x \ge 1 \\ \dfrac{-x(x-1)}{x+1} & x < 1 \end{cases}$$

$$g(x) = \frac{1}{1+\frac{1}{f(x)}}$$

$$= \begin{cases} \dfrac{x(x-1)}{x^2+1}, & x \ge 1 \\ \dfrac{-x(1-x)}{-x^2+2x+1}, & x < 1 \end{cases}$$

g(x) is non differentiable at $x = \pm 1, 0, 1-\sqrt{2}$

25. $f'(x) = \lim_{h\to 0} \frac{f(x)+f(h)-f(x)}{h} = f'(1)$

Integrating

$$f(x) = f'(1)x + c$$

$$f(x) = x \quad \text{as } f(0) = 0 \ \& \ f(1) = 1$$

$$\lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \sin x} = \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{(\tan x - \sin x)}{x^2 \sin x}$$

$$= \frac{\ln}{2}$$

26. $$f(\sin x) = \begin{cases} \sin x - 1, & -1 \le \sin x < 0 \\ \sin^2 x, & 0 \le x \le 1 \end{cases}$$

$$f(|\sin x|) = \begin{cases} |\sin x| - 1, & -1 \le |\sin x| < 0 \\ \left|\sin^2 x\right| & 0 \le |\sin x| \le 1 \end{cases}$$

$$= \left\{\sin^2 x, \qquad x \in R\right.$$

$$|f(\sin x)| = \begin{cases} |\sin x - 1|, & -1 \le \sin x < 0 \\ \sin^2 x, & 0 \le \sin x < 1 \end{cases}$$

$$= \begin{cases} 1 - \sin x, & -1 \le \sin x < 0 \\ \sin^2 x, & 0 \le \sin x < 1 \end{cases}$$

$$h(x) = \begin{cases} 1, & -1 \le \sin x < 0 \\ 2\sin^2 x & 0 \le \sin x < 1 \end{cases}$$

Hence, h(x) is not differentiable at $x = \pi \ \& \ 2\pi$ in $x \in [0, 2\pi]$.

27. $f'(x) = \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(xh) + 1 - f(x)}{h}$

$$f'(x) = f(x)f'(0) - xf'(0)$$

$$f'(x) = f(x) - x$$

Put $y = f(x)$

$$\frac{dy}{dx} = y - x$$

$$\frac{dy}{dx} - 1(y) = -x$$

Integrating factor $= e^{-\int 1dx} = e^{-x}$

Integrating both sides.

$$\Rightarrow ye^{-x} = -\int xe^{-x}dx$$

$$= -[-xe^{-x} - e^{-x}] + c$$

So,

$$y = x + 1$$

28. $\lim\limits_{t\to 0}\frac{g(t)}{h(t)}$. It may or may not exist at $t = 0$ and if it exist, it may not always be zero.

e.g

$$g(t) = \begin{cases} mx + 4, & t > 0 \\ 0 & t = 0 \\ mx^2 + 2 & t < 0 \end{cases}$$

$$f(t) = \begin{cases} x + 4, & t > 0 \\ 0, & t = 0 \\ mx^2 + 4, & t < 0 \end{cases}$$

or

$$g(t) = \begin{cases} mt, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

$$f(t) = \begin{cases} t, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

29. $(x-y)f(x+y) - (x+y)f(x-y) = 4xy\left(x^2 - y^2\right)$

$$\Rightarrow (x-y)\ \frac{f(x+y)}{y}\ -(x+y)\frac{f(x-y)}{y} = 4x\left(x^2 - y^2\right)$$

$$\Rightarrow\ (x-y)\left(\frac{f(x+y) - f(x)}{y}\right) -(x+y)\left(\frac{f(x-y) - f(x)}{y}\right)$$

$$+\frac{f(x)}{y}[(x-y) - x - y] = 4x\left(x^2 - y^2\right)$$

Take limit $y \to 0$

$$\Rightarrow xf'(x) + xf'(x) - 2f(x) = 4x^3$$

$$xf'(x) - f(x) = 2x^3$$

Put $t = f(x)$

$$\frac{dt}{dx} - \frac{t}{x} = 2x^2$$

Integrating both sides

$$\frac{t}{x} = x^2 + c$$

$$f(x) = x^3 + cx$$

$$\Rightarrow f(x) = x^3 \quad \text{as } f(1) = 1$$

30. $\frac{f(x+y)}{y} - \frac{f(x-y)}{y} = f(x)\frac{f(y)}{y}$

$$\Rightarrow \frac{f(x+y) - f(x)}{y} - \frac{f(x-y) - f(x)}{y} = f(x)\frac{f(y)}{y}$$

Taking lim $y \to 0$

$$\Rightarrow\quad f'(x) + f'(x) = f(x)f'(0)$$

$$\Rightarrow 2f'(x) = f(x)f'(0) = 0$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow f(x) = d = 0 \quad \text{as } f(0) = 0$$

31. $I = \frac{f\left(x_n\right) - f\left(z_n\right)}{x_n - z_n}$

as $x_n = c - h$

$z_n = c + h.$

$$I = \lim_{h\to 0}\frac{f(c-h) - f(c+h)}{-2h}$$

$= f'(c)$ if $f(x)$ is differentiable at $x = c$.

32. $f\left(\frac{x+y}{1+xy}\right) = f(x)\cdot f(y)$

Put $x = 0,\ y = 0$

$f(0) = 1$

Put $y = -x$.

$1 = f(x)\cdot f(-x)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \frac{1}{f(-x)} \lim_{h\to 0} \frac{(f(-x)f(x+h)-1)}{h}$$

$$\Rightarrow f'(x) = \frac{1}{f(-x)}\left(f(-x)f'(x) - f(x)f'(-x)\right)$$

$$f'(x) = f'(x) - \frac{f(x)}{f(-x)}f'(-x)$$

$$\Rightarrow f(x) = 0 \text{ or } f'(-x) = 0$$

33. $f\left(\dfrac{x+y}{1+xy}\right) = f(x)\cdot f(y)$

Put $x = 0, y = 0 \Rightarrow f(0) = 1$ as $f(0) \neq 0$

Put $y = -x$.

$$f(x) = \frac{1}{f(-x)}$$

Put $y = 1$

$$\Rightarrow f(1) = 0$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(-x)f(x+h)-1}{hf(-x)}$$

$$= \lim_{h\to 0} \frac{f\left(\dfrac{h}{1-x(x+h)}\right)-1}{hf+(-x)}$$

$$= \lim_{h\to 0} \frac{f'\left(\dfrac{h}{1-x(x+h)}\right)}{f(-x)} \times \frac{1-x(x+h)-h(-x)}{(1-x(x+h))^2}$$

$$f'(x) \Rightarrow \frac{1}{f(-x)} \frac{\left(1-x^2\right)}{\left(1-x^2\right)^2}$$

$$f'(x) = \frac{f(x)}{\left(1-x^2\right)}$$

$$\frac{f'(x)}{f(x)} = \frac{1}{\left|1-x^2\right|}$$

$$\ln f(x) = \frac{1}{2}\ln\frac{(1-x)}{1+x} + \frac{1}{2}\ln x$$

$$f^2(x) = c\left(\frac{1-x}{1+x}\right)$$

$$\Rightarrow f(x) = \sqrt{\frac{1-x}{1+x}}$$

34. $f(x+h) - f(x) \leq 6h^2$

If $h > 0$

$$\frac{f(x+h)-f(x)}{h} \leq 6h$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \leq 0 \qquad (1)$$

If $h < 0$

$$\frac{f(x+h)-f(x)}{h} \geq 6h$$

$$\lim \frac{f(x+h)-f(x)}{h} \geq 0 \qquad (2)$$

For differentiability of f(x), (1) & (2) should be satisfied together.

$$\Rightarrow f^1(x) = 0$$

$\Rightarrow f(x)$ is a constant function.

35. $f(x+h) - f(x) \leq 6h^2$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} \leq 6h$$

$$\Rightarrow \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \leq \lim_{h\to 0} 6h$$

$$\Rightarrow \frac{dy}{dx} \leq 0$$

Replace h by $-h$.

$$f(x-h) - f(x) \leq 6h^2$$

$$\frac{f(x-h)-f(x)}{-h} \geq -6h$$

$$\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \geqslant 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 0 = \frac{dy}{dx}$$

So, $f(x)$ is a constant function.

Previous Year's Questions (JEE Advanced)

1. $f(x)=\begin{cases}(x-1)^2\sin\dfrac{1}{(x-1)}-|x|, & x\neq 1\\ -1, & x=1\end{cases}$

As $|x|$ is not derivable at x = 0

f(x) is non derivable at x = 0 only

2. $f(x)=x|x|=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$

$f'(x)=\begin{cases}2x, & x\geq 0\\ -2x, & x<0\end{cases}$

$f''(x)=\begin{cases}2, & x\geq 0\\ -2, & x<0\end{cases}$

Hence, $f''(x)$ is derivable for $x\in R\sim\{0\}$

3. $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}=0 \quad \forall\ x\in R$

Hence, D is correct.

4. $f(x)=x(\sqrt{x}-\sqrt{x+1})$

$\lim\limits_{x\to 0^+} f(x)=f(x)=0$

RHD at x = 0

$f'(x)=\lim\limits_{h\to 0}\dfrac{h(\sqrt{h}-\sqrt{h+1})}{h}=-1$

LHD at x = 0 not defined

Hence, f(x) is not differentiable at x = 0

5. $f(x)=[x\sin\pi x]$

A) $\lim\limits_{x\to 0^+} f(x)=0,\ \lim\limits_{x\to 0^-} f(x)=0, f(0)=0$

B) For $x\in(-1,0),\ x\sin\pi x\in(0,1)$

So, f(x) is continuous.

C) RHD at x = 1

$\lim\limits_{h\to 0}\dfrac{[-(1+h)\sin\pi h]}{h}$ = not defined

So, f(x) is not derivable at x = 1

D) f(x) is differentiable $\forall\, x\in(-1,1)$

Hence, A B D are correct.

6. $f(x)=\dfrac{x}{1+|x|}$

$=\begin{cases}\dfrac{x}{1+x} & ,x\geq 0\\[2mm] \dfrac{x}{1-x} & ,x<0\end{cases}$

$f'(x)=\begin{cases}\dfrac{1}{(1+x)^2} & ,x\geq 0\\[2mm] \dfrac{1}{(1-x)^2} & ,x<0\end{cases}$

f(x) is differentiable $\forall\ x\in R$

Hence, A is correct.

7. $\lim\limits_{x\to 1}\displaystyle\int_4^{f(x)}\dfrac{2t}{x-1}dt$

$=\lim\limits_{x\to 1}\dfrac{\int_4^{f(x)} 2t\,dt}{x-1}$

$=\lim\limits_{x\to 1} 2f(x)f'(x)$ (Applying L-Hospital's rule)

$=8f'(1)$

Hence, A is correct.

8. $f(x)=\left[\tan^2 x\right]$

$\lim\limits_{x\to 0^+} f(x)=0=\lim\limits_{x\to 0^-} f(x)=f(0)$

Hence, B is correct.

9. $f(x)=\left(x^2-1\right)\left|x^2-3x+2\right|+\cos|x|$

f(x) is non differentiable at x = 2.

Hence, D is correct.

10. LHD at x = k

$I=\lim\limits_{h\to 0}\dfrac{[k-h]\sin\pi(k-h)-k\sin k\pi}{-h}$

$I=\lim\limits_{h\to 0}\dfrac{(k-1)\sin(k\pi-h\pi)}{-h}$

If k is odd

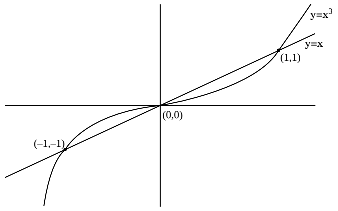
$I=\lim\limits_{h\to 0}\dfrac{(k-1)\sin h\pi}{-h}=(-1)(k-1)\pi$

If K is even,

$I=\lim\limits_{h\to 0}\dfrac{(k-1)\sinh\pi}{h}=(k-1)\pi$

Hence, A is correct

11. $f(x)=\max\left\{x,x^3\right\}$



f(x) is non differentiable at three points {−1, 0, 1}

Hence, D is correct.

12. For $f(x) = \sin|x| - |x|$

RHD at x = 0

$$\lim_{h\to 0}\frac{\sin h - h}{h} = 0$$

LHD at x = 0

$$\lim_{h\to 0}\frac{-\sin h - h}{-h} = 0$$

So, $\sin|x| - |x|$ is differentiable at x = 0.

Hence, D is correct.

13. $$f'(x) = \begin{cases} \dfrac{1}{1+x^2} & ,|x| \le 1 \\ \dfrac{1}{2} & ,x > 1 \\ -\dfrac{1}{2} & ,x < -1 \end{cases}$$

f(x) is not derivable at x = 1 & −1

Hence, D is correct.

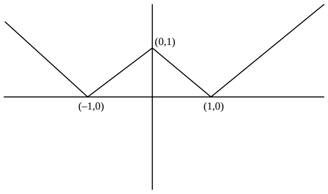
14. $$\lim_{x\to 0}\frac{f\left(x^2\right) - f(x)}{f(x) - f(0)}$$

$$= \lim_{x\to 0}\frac{2xf'\left(x^2\right) - f'(x)}{f'(x)}$$

$= -1$

Hence, C is correct.

15. $y = \left||x| - 1\right|$



||x| − 1| is non differentiable at x = −1, 0, 1.

Hence, A is correct.

16. $\lim\limits_{n\to\infty} f\left(\dfrac{1}{n}\right) = 0$ & f(x) is continuous for x = 0

So, f(0) = 0

Now, $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x}$

$$= \lim_{x\to 0}\frac{f(x)}{x}$$

Put $x = \dfrac{1}{n}$

$$f'(0) = \lim_{n\to\infty}\frac{f(1/n)}{1/n} = 0$$

Hence, B is correct

17. $$\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t - x}$$

$$\lim_{t\to x}\ 2tf(x) - x^2 f'(t)$$

$$2xf(x) - x^2 f'(x) = 1$$

$$\Rightarrow 2xy - x^2\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = -\frac{1}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\text{I.F} = e^{-\int \frac{4}{x}dx} = e^{-2\ln x} = x^{-2}$$

Integrating both sides,

$$yx^{-2} = -\int x^{-4}dx = \frac{x^{-3}}{3} + c$$

Put x = 1, y = 1

$$c = \frac{2}{3}$$

$$y = \frac{1}{3x} + \frac{2x^2}{3}$$

Hence, A is correct.

18. $$g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$$

$$\lim_{x\to 1} g(x) = \lim_{h\to 0}\frac{(h)^n}{\log\cos^m(h)}$$

$$= \lim_{h\to 0}\frac{-(n)h^{n-1}}{m\sin h}\cos(h) = -1 \quad (\text{asp} = -1)$$

$\Rightarrow n = 2, m = 2$

Hence, C is correct.

19. $$f(x) = \begin{cases} x^2\left|\cos\left(\dfrac{\pi}{x}\right)\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

RHD at x = 0

$$\lim_{h\to 0}\frac{h^2\left|\cos\left(\dfrac{\pi}{h}\right)\right|}{h} = 0$$

LHD at x = 0

$$\lim_{h\to 0}\frac{h^2\left|\cos\left(\dfrac{\pi}{-h}\right)\right|}{-h} = 0$$

f (x) is derivable at x = 0

RHD at x = 2

$$\lim_{h\to 0}\frac{(2+h)^2\left|\cos\left(\dfrac{\pi}{2+h}\right)\right|}{h} = \lim_{h\to 0}\frac{4\cos\left(\dfrac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h\to 0} 4\sin\left(\frac{\pi}{2+h}\right)\times\frac{-\pi}{(2+h)^2}$$

$$= -\pi$$

LHD at x = 2

$$\lim_{h\to 0}\frac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right|}{-h} = \pi$$

f(x) is not derivable at x = 2 only

Hence, B is correct.

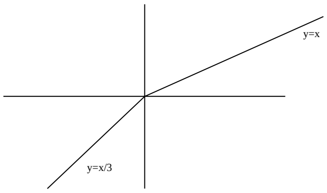
20. $x+|y| = 2y$

If $y \ge 0$

$y = x$

& If $y < 0$

$x = 3y$



Hence, A, B, D is correct.

21. $f(x) = 1+|\sin x|$

$$= \begin{cases} 1+\sin x, & x\in[0+2n\pi, \pi+2n\pi] \\ 1-\sin x & x\in[2n\pi+\pi, 2n\pi+2\pi] \end{cases}$$

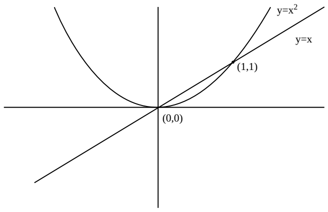
Hence, B, D, E is correct.

22. $$f(x) = \begin{cases} 3-x, & 1\le x\le 3 \\ x-3, & x>3 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1 \end{cases}$$

$$f'(x) = \begin{cases} -1, & 1\le x\le 3 \\ 1, & x>3 \\ \frac{x}{2}-\frac{3}{2}, & x<1 \end{cases}$$

Hence, A, B, C is correct.

23. $h(x) = \min\{x, x^2\}$



h (x) is continuous for all x & not differentiable at x = 0 & 1

Hence, A, C, D is correct.

24. $$f(x) = \begin{cases} 0 & ,x<0 \\ x^2 & ,x\ge 0 \end{cases}$$

$$f'(x) = \begin{cases} 0 & ,x<0 \\ 2x & ,x\ge 0 \end{cases}$$

Hence, B, C, D are correct.

25. $$g(x) = \begin{cases} x^2\sin(1/x), & x\ne 0 \\ 0, & x=0 \end{cases}$$

RHD at x = 0

$$\lim_{h\to 0}\frac{h^2\sin(1/h)}{h} = 0$$

LHD at x = 0

$$\lim_{h\to 0}\frac{h^2\sin(1/h)}{h} = 0$$

$$g'(x) = \begin{pmatrix} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\ne 0 \\ 0, & x=0 \end{pmatrix}$$

$$\lim_{x\to 0^+} g'(x) = \text{Not defined}$$

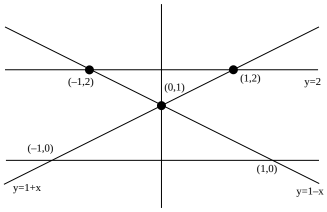
$$f(x) = \begin{cases} x\sin(1/x), & x\ne 0 \\ 0, & x=0 \end{cases}$$

LHD at x = 0

$$\lim_{h\to 0}\frac{-h\sin(-1/h)}{-h} = \text{Not defined}$$

Hence, A, B is correct.

26. $f(x) = \max\{(1-x), 1+x, 2\}$

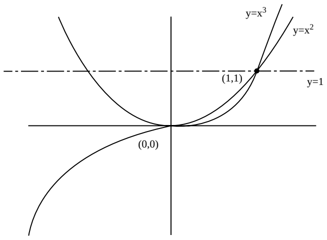


Hence, A, C are correct.

27. Conceptual

Hence, C is correct.

28. $f(x) = \min\{1, x^2, x^3\}$



Hence, A, D is correct.

29. $$\lim_{x\to -\frac{\pi^-}{2}} f(x) = 0,\ \lim_{x\to -\frac{\pi^+}{2}} f(x) = 0$$

LHD at x = 0

$$\lim_{h\to 0}\frac{-\cos h-(-1)}{-h} = 0$$

RHD at x = 0

$$\lim_{h\to 0}\frac{h-1-(-1)}{h}=1$$

LHD at x = 1

$$\lim_{h\to 0}\frac{(1-h)-1}{-h}=1$$

RHD at x = 1

$$\lim_{h\to 0}\frac{\ln(1+h)}{h}=1$$

Hence, A, B, C, D is correct.

30. $f(x+y)=f(x)+f(y)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)}{h}=f'(0)$$

$$f(x)=f'(0)x+c$$

$$f(x)=f'(0)x$$

Hence B, C is correct.

31. $f(x+y)=f(x)+f(y)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)}{h}=f'(0)$$

$$f(x)=f'(0)x+c$$

$$f(x)=f'(0)x$$

Hence, B, C is correct.

32. $\lim\limits_{x\to -\frac{\pi^-}{2}} f(x)=0,\ \lim\limits_{x\to -\frac{\pi^+}{2}} f(x)=0$

LHD at x = 0

$$\lim_{h\to 0}\frac{-\cos h-(-1)}{-h}=0$$

RHD at x = 0

$$\lim_{h\to 0}\frac{h-1-(-1)}{h}=1$$

LHD at x = 1

$$\lim_{h\to 0}\frac{(1-h)-1}{-h}=1$$

RHD at x = 1

$$\lim_{h\to 0}\frac{\ln(1+h)}{h}=1$$

Hence, A, B, C, D is correct.

33. Differentiability of f(x) at x = 0

$$\text{LHD} \Rightarrow \lim_{h\to 0}\frac{f(0)-f(0-h)}{h}=\lim_{h\to 0}\frac{0+g(-h)}{h}=0$$

$$\text{RHD} \Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{g(h)}{h}=0$$

⇒ f(x) is differentiable at x = 0

Differentiability of h(x) at x = 0

$h'(0^+) = 1$ & h(x) is even function

⇒ h(x) is non differentiable at x = 0

Differentiability of f(h(x)) at x = 0

$$f(h(x))=g\left(e^{|x|}\right)$$

$$\text{LHD} \Rightarrow \lim_{h\to 0}\frac{f(h(0))-f\left(h(0-h)\right)}{h}$$

$$=\lim_{h\to 0}\frac{g(1)-g\left(e^{h}\right)}{h}=-g'(1)$$

$$\text{RHD} \to \lim_{h\to 0}\frac{f(0+h)-f\left(h(0)\right)}{h}$$

$$=\lim_{h\to 0}\frac{g\left(e^{h}\right)-g(1)}{h}=g'(1)$$

So, f(h(x)) is non diff at x = 0

Differentiability of h(f(x)) at x = 0

$$h(f(x))=\begin{cases} e^{|f(x)|}, & x\neq 0\\ 1, & x=0\end{cases}$$

$$\text{LHD}=\lim_{h\to 0}\frac{h(f(0))-h(f(0-h))}{h}$$

$$=\lim_{h\to 0}\frac{1-e^{|g(-h)|}}{|g(-h)|}\times\frac{|g(-h)|}{h}=0$$

$$\text{RHD}=\lim_{h\to 0}\frac{h\left(f(0+h)\right)-h\left(f(0)\right)}{h}$$

$$=\lim_{h\to 0}\frac{e^{|g(h)|}-1}{|g(h)|}-\frac{|g(h)|}{h}=0$$

Hence, A, D is correct.

34. Tangent at P → $xx_1-yy_1=1$ intersects X-axis at

$$M\Rightarrow\left(\frac{1}{\lambda_1},0\right)$$

Slope of Normal $=\dfrac{-y_1}{x_1}=\dfrac{y_1-0}{x_1-x_2}$

$$\Rightarrow x_2=2x_1 \Rightarrow N\equiv(2x_1,0)$$

For centroid $l=\dfrac{3x_1+\dfrac{1}{x_1}}{3},\ m=\dfrac{y_1}{3}$

$$\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}\quad \&\quad \frac{dm}{dy_1}=\frac{1}{3}\frac{dy}{dx_1}=\frac{x_1}{3\sqrt{x_1^2-1}}$$

Hence, A, B, D is correct.

35. $f(x)=[x^2-3]$ which is discontinuous at $x=1,\sqrt{2},\sqrt{3},2$

Now, $g(x)=f(x)[|x|+|4x-7|]$ is non differentiable at x=1,√2,√3

& $|x|+|4x-7|$ is non differentiable at $x=0,\frac{7}{4}$

But $f(x)=0\ \forall\ x\in[\sqrt{3},2)$

Hence, g(x) is non differentiable at $0,1,\sqrt{2},\sqrt{3}$

Hence, B, C is correct.

36. $f(x)=a\cos\left(x^3-x\right)+bx\sin\left(x\left(x^2+1\right)\right)$

It is differentiable ∀ x ∈ R

Hence, A, B is correct.

37. $f'(x)+\frac{1}{x}f(x)=2 \quad \forall x\in(0,\infty)$

$\Rightarrow f(x)=x+\frac{c}{x}$ as $f(1)\neq 1\Rightarrow c\neq 0$

A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}\left(2-1-cx^2\right)=1$

B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}\left(1+cx^2\right)=1$

C) $\lim\limits_{x\to 0^+} x^2f'(x)=\lim\limits_{x\to 0^+}\left(x^2-c\right)=-c$

D) for c ≠ 0, f(x) is unbounded function

Hence, A is correct.

38. As $g'(f(x))f'(x)=1$

$\Rightarrow g'(2)=\frac{1}{3}$

As $h(x)\ =f\circ f(x)$

$\Rightarrow h(0)=16$

$h(x)\ =f\circ f(x)$

$\Rightarrow h'(x)\ =f'\circ f(x)\times f'(x)$

$h'(1)=f'(f(1))\times f'(1)=111\times 6=666$

Hence, B, C is correct.

39. Apply LMVT in [−4, 0)

$\frac{f(0)-f(-4)}{4}=f'\left(x_0\right)$

$\Rightarrow\left|f'\left(x_0\right)\right|\leq\frac{|f(0)+|f(-4)|}{4}\leq 1$

If f(x)is periodic, $\Rightarrow \lim\limits_{x\to\infty} f(x)\neq 1$

Similarly, $\left|f'\left(x_1\right)\right|\leq 1$ for some $x,\in(0,4)$

$g(x)=\left(f(x)\right)^2+\left(f'(x)\right)^2$

$g\left(x_0\right)\leq 5$ & $g(x_1)\leq 5$

g(0) = 85, it has local maximum value ≥ 85, Say α

$\Rightarrow g'(\alpha)=0, g''(\alpha)<0$

$\Rightarrow 2f(\alpha)f'(\alpha)+2f'(\alpha)f''(\alpha)=0$

$\Rightarrow\left(f(\alpha)+f''(\alpha)\right)f'(\alpha)=0$

$\Rightarrow f(\alpha)+f''(\alpha)=0$ as $f'(\alpha)\neq 0$

Hence, A, B, D is correct.

40. Statement-1

$\lim\limits_{x\to 0}(g(x)\cot x-g(0)\operatorname{cosec} x)$

$\lim\limits_{x\to 0}\frac{g(x)\cos x-g(0)}{\sin x}=g'(0)$

& $f(x)=g(x)\sin x$

$f'(x)=g'(x)\sin x+g(x)\cos x$

$f''(x)=g'(x)\cos x+g''(x)\sin x+g'(x)\cos x-g(x)\sin x$

$f''(0)=2g'(0)$

Hence, B is correct.

41. $f(x)=x^3+e^{x/2}$

$g'(x)=\frac{1}{f'(x)}$

$=\frac{1}{3x^2+\frac{1}{2}e^{x/2}}$

$g'(1)=\frac{1}{1/2}=2$

42. $y'(x)+y(x)g'(x)=g(x)$

Integrating factor $e^{\int g'(x)dx}$

$y(x)e^{g(x)}=\int e^{g(x)}g(x)dx$

43. $f(x)=\begin{cases}\frac{x^2}{2}, & 0\leq x<1\\ 2x^2-3x+\frac{3}{2}, & 1\leq x\leq 2\end{cases}$

$\lim\limits_{x\to 1^-} f(x)=\frac{1}{2},\ \lim\limits_{x\to 1^+} f(x)=\frac{1}{2}=f(1)$

$f'(x)=\begin{cases}x, & 0\leq x<1\\ 4x-3, & 1\leq x\leq 2\end{cases}$

$\lim\limits_{x\to 1^-} f'(x)=1,\ \lim\limits_{x\to 1^+} f'(x)=1,\quad f'(1)=1$

$f''(x)=\begin{cases}1, & 0\leq x<1\\ 4, & 1\leq x\leq 2\end{cases}$

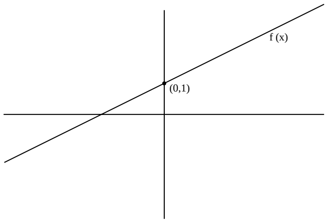
f(x) & f′(x) are continuous everywhere

While f″(x) is discontinuous at x = 1.

44. $f(x)=x^3-x^2+x+1$

$g(x)=\max\begin{cases}f(t); 0\leq t\leq x\}, & 0\leq x\leq 1\\ 3-x, & 1<x\leq 2\end{cases}$

$f'(x)=3x^2-2x+1>0$



So, f(x) is strictly increasing $\forall x \in R$

$$g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = 2 \quad = \lim_{x \to 1^+} f(x) = f(1)$$

$$g'(x) = \begin{cases} 3x^2 - 2x + 1, & 0 \le x \le 1 \\ -1, & 1 < x \le 2 \end{cases}$$

g(x) is continuous but not differentiable at x=1

45. $$f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$$

$$g(x) = f(|x|) + |f(x)|$$

$$f(|x|) = \begin{cases} 1, & x = 0 \\ -x - 1, & -2 \le x < 0 \\ x - 1, & 0 < x \le 2 \end{cases}$$

$$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ 1 - x, & 0 < x \le 1 \\ x - 1, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} -x, & -2 \le x < 0 \\ 2, & x = 0 \\ 0, & 0 < x \le 1 \\ 2x - 2, & 1 < x \le 2 \end{cases}$$

g(x) is not differentiable at x = 0 & 1

46. $$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

For f(x) to be differentiable, f′(0) = 0

47. $y = [x] + |1 - x|$

$$y = \begin{cases} -x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ x + 1, & 2 \le x < 3 \\ 5, & x = 3 \end{cases}$$

f(x) is not differentiable at x = 0, 1, 2, 3

48. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} f(x) \frac{(f(h) - 1)}{h} = f'(0) f(x) = 2f(x)$$

Integrating both sides,

$$f(x) = e^{2x}$$

49. $$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(2x) + f(2h) - 2f(x)}{2h}$$

$$= \lim_{h \to 0} \frac{f(2h) - f(0)}{2h}$$

$$f'(x) = f'(0)$$

$$f(x) = f'(0)x + c$$

$$f(x) = -x + 1$$

50. a) $$f(x) = \begin{cases} xe^{\left(-\frac{1}{|x|} + \frac{1}{x}\right)}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ xe^{2/x}, & x < 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 0 = f(0)$$

$$\lim_{x \to 0^-} xe^{2/x} = 0$$

f(x) is continuous at x = 0

b) $$f'(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ e^{2/x}, & x < 0 \end{cases}$$

So, f(x) is derivable at x = 0.

51. $$f(x) = \begin{cases} 1 - x, & x < 1 \\ 2 - 3x + x^2, & 1 \le x \le 2 \\ 3 - x, & x > 2 \end{cases}$$

$$f'(x) = \begin{cases} -1 & x < 1 \\ 2x - 3 & 1 \le x \le 2 \\ -1 & x > 2 \end{cases}$$

So, f(x) is not continuous at x = 2 & not derivable at x = 2

52. $$f(x) = \begin{cases} 2 + \sqrt{1 - x^2}, & |x| \le 1 \\ 2e^{(1-x)^2}, & |x| > 1 \end{cases}$$

$$\lim_{x \to 1^+} f(x) = 2, \qquad \lim_{x \to 1^-} f(x) = 2, \quad f(1) = 2$$

$$\lim_{x \to -1^+} f(x) = 2, \quad \lim_{x \to -1^-} f(x) = 2e^2, \quad f(-1) = 2$$

For x = 1

LHD

$$\lim_{h \to 0} \frac{2 + \sqrt{1 - (1-h)^2} - 2}{-h} = \lim_{h \to 0} \frac{\sqrt{2h - h^2}}{-h} = \text{not defined}$$

For x = −1

RHD

$$\lim_{h \to 0} \frac{2 + \sqrt{1 - (-1+h)^2} - 2}{h} = \lim_{h \to 0} \frac{\sqrt{2h - h^2}}{h} = \text{not defined}$$

f (x) is continuous but not derivable at x = 1

& it is discontinuous and not derivable at x = −1

53. $f'(\alpha) = \lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{h}$

$= \lim\limits_{h\to 0} \dfrac{g(\alpha+h)h}{h}$

$= g(\alpha)$

For f(x) is differentiable, g(x) needs to be continuous at $x = \alpha$

54. $f(x) = \begin{cases} x+a, & x<0 \\ x-1, & x\ge 1 \\ 1-x, & 0\le x<1 \end{cases}$

$g(x) = \begin{cases} x+1, & x<0 \\ (x-1)^2+b, & x\ge 0 \end{cases}$

$g\circ f(x) = \begin{cases} f(x)+1, & f(x)<0 \\ (f(x)-1)^2+b, & f(x)\ge 0 \end{cases}$

$= \begin{cases} x+a+1, & x<-a \\ (x+a-1)^2+b, & -a\le x<0 \\ x^2+b, & 0\le x\le 1 \\ (x-2)^2+b, & x>1 \end{cases}$

For continuity at

$x=-a \Rightarrow 1 = 1+b \Rightarrow b=0$

$x=0 \Rightarrow a=1$

For differentiability at x = 0

RHD = 0

LHD = 0

g o f(x) is differentiable at x = 0

55. LHD at x = –a

$\lim\limits_{h\to 0} \dfrac{f(-a-h)-f(-a)}{-h}$

$\lim\limits_{h\to 0} \dfrac{-f(a+h)+f(a)}{-h}$

$\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{h} = 0$

56. $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2}<x<0 \\ \dfrac{1}{2}, & x=0 \\ \dfrac{e^{ax/2}-1}{x}, & 0<x<\dfrac{1}{2} \end{cases}$

For continuity at x = 0

$\lim\limits_{x\to 0^-} f(x) = b\sin^{-1}\dfrac{c}{2}, \quad f(0) = \dfrac{1}{2}$

$\lim\limits_{x\to 0^+} f(x) = \dfrac{a}{2}$

$\Rightarrow a = 1 \quad \& \quad b\sin^{-1}\dfrac{c}{2} = \dfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

For differentiability at x = 0

RHD

$\lim\limits_{h\to 0}\left(\dfrac{e^{ah/2}-1}{h} - \dfrac{1}{2}\right)\times\dfrac{1}{h}$

$\lim\limits_{h\to 0} \dfrac{2e^{ah/2}-h-2}{2h^2}$

$\lim\limits_{h\to 0} \dfrac{ae^{ah/2}-1}{4h} = \dfrac{a}{8} = \dfrac{1}{8}$

LHD

$\lim\limits_{h\to 0} \dfrac{b\sin^{-1}\left(\dfrac{c-h}{2}\right) - \dfrac{1}{2}}{-h}$

$\lim\limits_{h\to 0} \dfrac{b}{\sqrt{1-\dfrac{(c-b)^2}{4}}} \times \dfrac{1}{2} = \dfrac{1}{8}$

$\Rightarrow \quad 16b^2 = 1 - \dfrac{c^2}{4}$

$\Rightarrow \quad 64b^2 = 4 - c^2$

57. $f(x-y) = f(x)g(y) - f(y)g(x)$

Put x = y $\Rightarrow f(0) = 0$

Put y = 0 $\Rightarrow g(0) = 1$

$f'\left(0^+\right) = \lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{f(0)g(-h)-g(0)f(-h)-f(0)}{h}$

$= f'(0)$

Hence, f(x) is differentiable at x = 0

Put y = x in $g(x-y) = g(x)g(y) + f(x)f(y)$

$\Rightarrow f^2(x) + g^2(x) = 1$

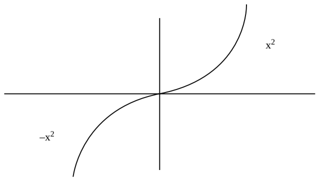
Differentiating,

$2f(x)f'(x) + 2g(x)g'(x) = 0$

$\Rightarrow g'(0) = 0$

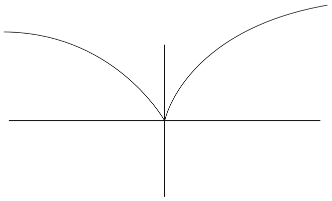
58.

A) $x|x| = \begin{Bmatrix} x^2, x\ge 0 \\ -x^2, \ x<0 \end{Bmatrix}$



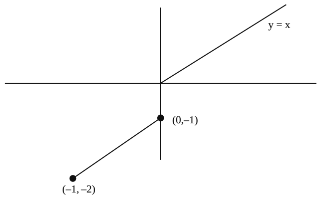
Hence, A → (P, Q ,R)

B) $y=\sqrt{|x|}$



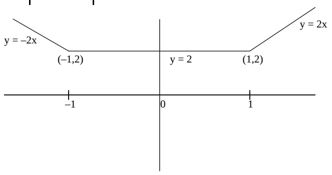
Hence, B $\to$ (P, S)

C) $x+[x]=\begin{cases} x-1 & -1<x<0 \\ x & 0\le x<1 \end{cases}$



Hence, C $\to$ (R, S)

D) $|x-1|+|x+1|$



Hence, D $\to$ (P, Q)

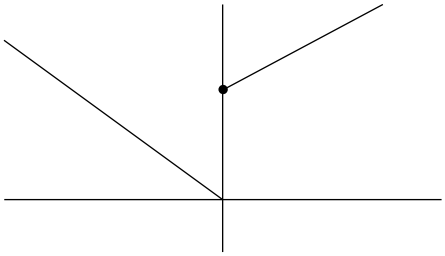
59. $f_2(f_1)=\begin{cases} x^2, & x<0 \\ e^{2x}, & x\ge 0 \end{cases}$

$f_4: R \to [0,\infty)$

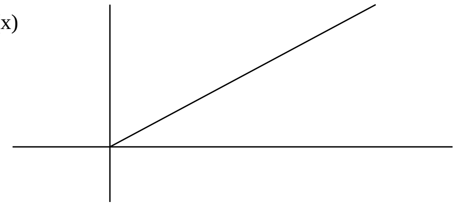
$f_4(x)=\begin{cases} f_2f_1(x), & x<0 \\ f_2f_1(x)-1, & x\ge 0 \end{cases}$

$=\begin{cases} x^2, & x<0 \\ e^{2x}-1, & x\ge 0 \end{cases}$

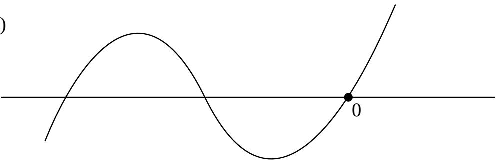
$y = f_1(x)$



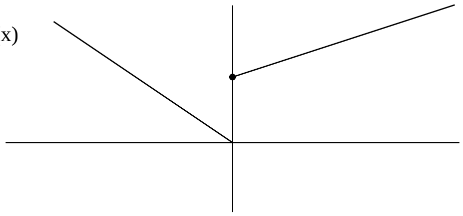
$y = f_2(x)$



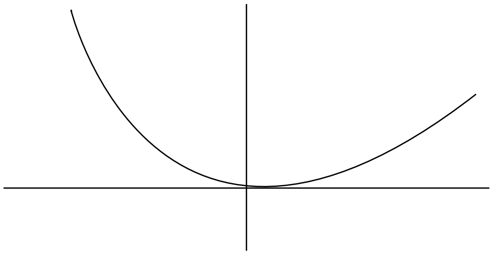
$y = f_3(x)$



$y = f_2f_1(x)$



$y = f_4(x)$



Hence, D is correct

Previous Year's Questions (JEE Main Papers)

1. $8f(x)+6f\left(\frac{1}{x}\right)-x=5$ ....(I) $(x\ne 0)$

$y=x^2f(x)$

Put $x=\frac{1}{x}$

$8f\left(\frac{1}{x}\right)+6f(x)=5+\frac{1}{x}$ ....(II)

Solving (I) & (II)

$\Rightarrow 28f(x)=40+8x-30-\frac{6}{x}=10+8x-\frac{6}{x}$

$\Rightarrow f(x)=\frac{5}{14}+\frac{2x}{7}-\frac{3}{14x}$

Now, $y=\frac{5}{14}x^2+\frac{2}{7}x^3-\frac{3x}{14}$

$\frac{dy}{dx}=\frac{5}{7}x+\frac{6}{7}x^2-\frac{3}{14}$

For $x=-1, \frac{dy}{dx}=\frac{-1}{14}$

Hence, C is correct.

2. $x=\sqrt{a^{\sin^{-1}t}}$,

$$\frac{dx}{dt}=\frac{1}{2\sqrt{a^{\sin^{-1}t}}}\times\frac{a^{\sin^{-1}t}\ln a}{\sqrt{1-t^2}} \qquad \text{...............................(I)}$$

$$y=\sqrt{a^{\cos^{-1}t}}$$

$$\frac{dy}{dt}=\frac{-1}{2\sqrt{a^{\cos^{-1}t}}}\times\frac{a^{\cos^{-1}t}\ln a}{\sqrt{1-t^2}} \qquad \text{...............................(II)}$$

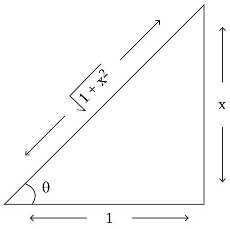
Now, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}=\frac{-y}{x}$

So, $1+\left(\frac{dy}{dx}\right)^2=\frac{x^2+y^2}{x^2}$. Hence, D is correct

3. $y=\sec\left(\tan^{-1}x\right)$

Let $\tan^{-1}x=\theta$

$\therefore\ x=\tan\theta$



$\therefore y=\sec\theta=\sqrt{1+x^2}$

On differenciating w.r.t x, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1+x^2}}\times 2x$$

$\therefore$ At $x=1$

$$\frac{dy}{dx}=\frac{1}{\sqrt{2}}$$

Hence D is correct.

4. $f(x)=\frac{x^2-x}{x^2+2x},\ x\neq 0,-2$

$$f(x)=\frac{x(x-1)}{x(x+2)}=\frac{x-1}{x+2}$$

Now, $y=\frac{x-1}{x+2}$

$$\Rightarrow xy+2y=x-1\ \Rightarrow x=\frac{2y+1}{1-y}$$

$$\Rightarrow f^{-1}(x)=\frac{2x+1}{1-x}$$

$$f^{-1}(x)=\frac{3}{1-x}-2$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{3}{(1-x)^2}$$

Hence, B is correct

5. $f'(x)=\frac{1}{1+x^5}$

Here, g is the inverse of x

$\Rightarrow f\circ g(x)\ =x$

On, differenciating w.r.t x we get

$f'\{g(x)\}\times g'(x)=1$

$$\Rightarrow g'(x)=\frac{1}{f'(g(x))}$$

$$g'(x)=\frac{1}{\frac{1}{1+\{g(x)\}^5}}$$

$\Rightarrow g'(x)=1+\{g(x)\}^5$

Hence the correct option is D.

6. $f(x)=x|x|$

$g(x)=\sin x\ \ \&\ \ h(x)=g\circ f(x)$

As $f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$

$$h(x)=g\circ f(x)=\begin{cases}\sin x^2, & x\geq 0\\ -\sin x^2, & x<0\end{cases}$$

Now, $h'(x)=\begin{cases}2x\cos x^2, & x\geq 0\\ -2x\cos x^2, & x<0\end{cases}$

LHL = RHL = $h'(0)$ at $x=0$

$\Rightarrow h'(x)$ is continuous at $x=0$

$$\text{RHD}=\lim_{h\to 0}\frac{2h\cos h^2}{h}=2,\quad \text{LHD}=\lim_{h\to 0}\frac{2h\cos h^2}{-h}=-2$$

Hence, option C is correct

7. $f(x)=x^2-x+5,\ x>\frac{1}{2}$

As $x^2-x+5=7$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$\Rightarrow x=-1$ or 2

$\Rightarrow x=2$ at $x>\frac{1}{2}$

$$g'(7)=\frac{1}{f'(2)}=\frac{1}{2\times 2-1}=\frac{1}{3}$$

Hence, C is correct

8. $|f(x)|\leq x^2$

Put $x=0$, $|f(0)|\leq 0\ \Rightarrow f(0)=0$

$$\lim_{x\to 0}\left|\frac{f(x)-f(0)}{x}\right|\leq\lim_{x\to 0}|x|$$

$\Rightarrow |f'(0)|\leq 0\ \Rightarrow f'(0)=0$

Hence, B is correct

9. $y=e^{nx};\ \left(\frac{d^2y}{dx^2}\right)\frac{d^2x}{dy^2}=?$

As $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{1}{dy/dx}\right)=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\times\frac{1}{dy/dx}$

$$=\frac{-1}{(dy/dx)^2}\times\frac{d^2y}{dx^2}\times\frac{1}{(dy/dx)}$$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)=-(y')^3\left(\frac{d^2x}{dy^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

Now, $y' = ne^{nx}$

$y'' = n^2e^{nx}$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)=\frac{-\left(n^2e^{nx}\right)^2}{n^3\left(e^{nx^3}\right)^3}=-ne^{-nx}$$

Hence, D is correct

10. $g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$

$$g'(x)=\begin{cases}\dfrac{k}{2\sqrt{x+1}}, & 0\le x\le 3\\ m, & 3<x\le 5\end{cases}$$

As $g'(x)$ is differentiable

$$\Rightarrow \frac{k}{4}=m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

As g(x) is continuous,

$$\Rightarrow 2k=3m+2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

Solving (I) & (II)

$m=\frac{+2}{5},\ k=\frac{8}{5}$

$\Rightarrow k+m=2$

Hence, A is correct

11. $f(x)=|\log 2-\sin x| \quad \& \quad g(x)=f(f(x))$

Note that,

For $x\to 0, \quad \log 2>\sin x$

$\therefore f(x)=\log 2-\sin x$

$\Rightarrow g(x)=\log 2-\sin(f(x))$

$=\log 2-\sin(\log 2-\sin x)$

Clearly, g(x) is differentiable at x=0 as sin x is differentiable.

Now,

$g'(x)=-\cos(\log 2-\sin x)(-\cos x)$

$=\cos x\cdot\cos(\log 2-\sin x)$

$\Rightarrow g'(0)=1\cdot\cos(\log 2)$

Option B is correct.

12. $\lim\limits_{t\to x}\dfrac{t^2f(x)-x^2f(t)}{t-x}=1$

$\lim\limits_{t\to x}\dfrac{2tf(x)-x^2f'(t)}{t}=1$

$\Rightarrow 2xf(x)-x^2f'(x)=1$

$\Rightarrow f'(x)-\dfrac{2}{x}f(x)=\dfrac{-1}{x^2}$

Integrating

$\Rightarrow x^{-2}f(x)=+\dfrac{x^{-3}}{3}+c$

$\Rightarrow f(x)=\dfrac{1}{3x}+\dfrac{2}{3}x^2$

$f\left(\dfrac{3}{2}\right)=\dfrac{2}{9}+\dfrac{18}{6\times 2}$

$\Rightarrow\dfrac{8+18\times 3}{36}$

$\Rightarrow\dfrac{4+27}{18}=\dfrac{31}{18}$

13. $f(x)=\begin{cases}-x, & x<1\\ a+\cos^{-1}(x+b), & 1\le x\le 2\end{cases}$

$$f'(x)=\begin{cases}-1, & x<1\\ \dfrac{-1}{\sqrt{1-(x+b)^2}}, & 1\le x\le 2\end{cases}$$

$\Rightarrow\sqrt{1-(1+b)^2}=1$

$\Rightarrow b=-1$

As f(x) is continous at x = 1

$\Rightarrow -1=a+\dfrac{\pi}{2}$

$\Rightarrow a=-\left(\dfrac{\pi}{2}+1\right)$

$\Rightarrow\dfrac{a}{b}=\dfrac{\pi+2}{2}$ Hence, A is correct.

14. $y=[x+\sqrt{x^2-1}]^{15}+[x-\sqrt{x^2-1}]^{15}$

$$y'=\frac{15[x-\sqrt{x^2-1}]^{15}}{\sqrt{x^2-1}}-\frac{15}{\sqrt{x^2-1}}[x-\sqrt{x^2-1}]^{15}$$

$$y'=\frac{15}{\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{15}-(x-\sqrt{x^2-1})^{15}]$$

$$(\sqrt{x^2-1})y'=15\left[(x+\sqrt{x^2-1})^{15}-(x-\sqrt{x^2-1})^{15}\right]$$

Again differentiate,

$$(\sqrt{x^2-1})y''+\frac{x}{\sqrt{x^2-1}}y''=\frac{225[(x+\sqrt{x^2-1})^{15}+(x-\sqrt{x^2-1})^{15}}{\sqrt{x^2-1}}$$

$$\left(x^2-1\right)y''+xy'=225y$$

Hence, D is correct

15. $2x=y^{1/5}+y^{-1/5} \quad \& \quad \left(x^2-1\right)\dfrac{d^2y}{dx^2}+\lambda x\dfrac{dy}{dx}+ky=0$

$$\Rightarrow 2=\frac{1}{5}y^{-4/5}y_1-\frac{1}{5}y^{-6/5}y,$$

$$\Rightarrow 10y=y_1\left(y^{1/5}-y^{-1/5}\right)$$

$$\Rightarrow 10y_1=y_{11}\left(y^{1/5}-y^{-1/5}\right)+(y)^2\left(\frac{1}{5}y^{-4/5}+\frac{1}{5}y^{-\frac{6}{5}}\right)$$

$$\Rightarrow 10y_1=y_1\left(y^{1/5}-y^{-1/5}\right)+\left(y_1\right)^2\frac{2x}{5y}$$

$$\Rightarrow 10y_1 = y_{11}\frac{(10y)}{y_1} + \left(y_1\right)^2 \frac{2x}{5y}$$

$$\Rightarrow 10\left(y_1\right)^2 = (10y)y_{11} + \left(y_1\right)^3 \frac{2x}{5y}$$

$$\Rightarrow 50yy_1^2 = 50y^2 y_{11} + 2xy_1^3$$

$$\Rightarrow 50y = 50\left(\frac{y}{y_1}\right)^2 y_{11} + 2xy_1^3$$

$$\Rightarrow 50y = \frac{50}{100}((y^{1/5} + y^{-1/5})^2 - 4)y_{11} + 2xy_1$$

$$\Rightarrow \left(x^2 - 1\right)y_{11} + xy_1 - 25y = 0 \quad \Rightarrow \lambda + k = -24$$

Hence, B is correct

16. Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left[\frac{2\cdot\left(3x^{3/2}\right)}{1-\left(3x^{3/2}\right)^2}\right]$

$$= 2\tan^{-1}\left(3x^{3/2}\right)\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{1+\left(3x^{3/2}\right)^2}\cdot 3\times\frac{3}{2}(x)^{1/2}$$

$$= \frac{9}{1+9x^3}\cdot\sqrt{x}$$

$$g(x) = \frac{9}{1+9x^3}$$

Option B is correct

17. $x^2 + y^2 + \sin y = 4$

$2x + 2yy + y_1\cos y = 0 \quad \Rightarrow \quad$ at $(-2,0)$, $\quad y_1 = 4$

$$2 + 2\left(y_1\right)^2 + 2y_1y_{11} - y_1\sin y + y_{11}\cos y = 0$$

At $(2,0) \Rightarrow 2 + 32 + y_{11} = 0$

$$y_{11} = -34$$

Hence, A is correct

18. $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

As $\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & 1 & 1 \\ \frac{2\sin x}{x} & 1 & 2 \\ \tan x & 1 & 1 \end{vmatrix}$

$$\lim_{x\to 0}\frac{f(x)}{x^2} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

As $\lim\limits_{x\to 0}\frac{f(x)}{x^2} = \lim\limits_{x\to 0}\frac{f'(x)}{2x}$

$$\Rightarrow \lim_{x\to 0}\frac{f'(x)}{x} = -2$$

Hence, B is correct.

19. $f(x) = \sin^{-1}\left(\frac{2\times 3^x}{1+9x}\right) \quad$ Let $3^x = \tan t$

$$f(x) = 2t = 2\tan^{-1}3^x$$

$$f'(x) = \frac{2}{1+9^x}\times 3^x\ln 3$$

$$f'\left(-\frac{1}{2}\right) = \frac{2}{1+\frac{1}{3}}\times\frac{1}{\sqrt{3}}\ln 3$$

$$\Rightarrow \frac{\sqrt{3}}{2}\ln 3$$

$$= \sqrt{3}\ln\sqrt{3}$$

Hence, A is correct

20. $f(x) = |x-1|;\ g(x) = \cos x$

$$\phi(x) = f(g(2\sin x)) - g(f(x))$$

As $f\circ g(x) = |\cos x - 1| = 2\sin^2\frac{x}{2}$

& $g\circ f(x) = \cos|x-1| = \cos(x-1)$

So, $\phi(x) = 2\sin^2(\sin x) - \cos(x-1)$

$\phi(x)$ is differentiable $\forall\ x \in R$

Hence, A is correct.

21. $f(x) = |x-\pi|\cdot(e^{|x|}-1)\sin|x|$

$$f(x) = \begin{cases} (\pi-x)\left(e^x-1\right)\sin x, & 0 < x < \pi \\ (\pi-x)\left(e^x-1\right)\sin x, & x \ge \pi \\ (\pi-x)\left(e^{-x}-1\right)\sin(-x), & x \le 0 \end{cases}$$

RHD at $x = 0 \Rightarrow \lim\limits_{h\to 0}(\pi-h)\frac{\left(e^n-1\right)}{n}\sin h = 0$

LHD at $x = 0 \Rightarrow \lim\limits_{h\to 0}(\pi+h)\frac{\left(e^n-1\right)}{-n}\sin h = 0$

RHD at $x = \pi \Rightarrow \lim\limits_{h\to 0}\frac{(h)\left(e^{\pi+h}-1\right)(-\sin h) = 0}{n}$

LHD at $x = \pi \Rightarrow \lim\limits_{h\to 0}\frac{h\left(e^{\pi-h}-1\right)\sin h = 0}{-n}$

Hence, D is correct

22. $x = 3\tan t;\ y = 3\sec t$

$$\frac{dx}{dx} = 3\sec^2 t, \quad \frac{dy}{dt} = 3\sec t\tan t$$

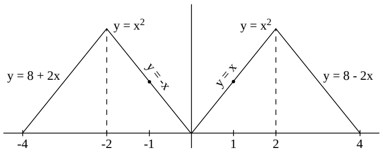
$$\frac{dy}{dx} = \sin t$$

$$\frac{d^2y}{dx^2}=\frac{\cos t}{3\sec^2 t}=\frac{\cos^3 t}{3}=\frac{1}{6\sqrt{2}}$$

Hence, D is correct

23. $f(x)=\begin{cases}\max\ [|x|, x^2], & |x|\le 2\\ 8-2|x|, & 2<|x|\le 4\end{cases}$

$$f(x)=\begin{cases}8+2x, & -4\le x<-2\\ x^2, & -2\le x\le -1\\ |x|, & -1<x<1\\ x^2, & 1\le x\le 2\\ 8-2x, & 2<x\le 4\end{cases}$$



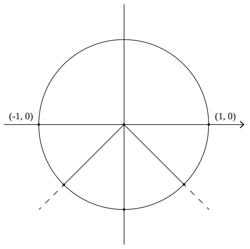
f(x) is not differenciable at x= −2, −1, 0, 1, 2

S = {−2, −1, 0, 1, 2}

Hence, C is correct

24. $\max\{-|x|, -\sqrt{1-x^2}\}$

Graph of the given function is shown as



Clearly, f(x) is not differential at 3 points in (−1, 1).

Hence, A is correct.

25. $x\log(\log x)-x^2+y^2=4$ ....(I)

Differenciating both sides w. r. t x, we get

$$x\frac{1}{\log_e x}\times\frac{1}{x}+\ln(\ln x)-2x+2y\frac{dy}{dx}=0$$

At $x=e,\ 1-2e+2y\cdot\frac{dy}{dx}=0$

$$\Rightarrow\frac{dy}{dx}=\frac{2e-1}{2y}$$ ....(II)

Also from (I), $-e^2+y^2=4$

$y=\sqrt{4+e^2}$

$\therefore$ from (II), $\frac{dy}{dx}=\frac{2e-1}{2\sqrt{4+e^2}}$

Hence, C is correct

26. $f(x)=\begin{cases}-1, & -2\le x<0\\ x^2-1, & 0\le x\le 2\end{cases}$

$g(x)=|f(x)|+f(|x|)$

$$|f(x)|=\begin{cases}1, & -2\le x<0\\ 1-x^2, & 0\le x\le 1\\ x^2-1, & 1<x\le 2\end{cases}$$

$f(|x|)=\{x^2-1\quad -2\le x\le 2$

$$g(x)=\begin{cases}x^2-2, & -2\le x<0\\ 0, & 0\le x\le 1\\ 2x^2-2, & 1<x\le 2\end{cases}$$

g(x) is not differentiable at x = 0.

Hence, D is correct.

27. $f'(x)=f(x)\Rightarrow\ln f(x)=x+c$

$\Rightarrow\ln f(x)=x+\ln 2-1$

$\Rightarrow f(x)=e^x\cdot e^{\ln 2-1}$

$f(x)=2e^{x-1}$

Now, $h(x)=2e^{2\left(e^{x-1}\right)^{-1}}=\frac{2}{e}e^{2e^{x-1}}$

$h'(x)=\frac{2}{e}e^{2\left(e^{x-1}\right)}\times 2\left(e^{x-1}\right)$

$h'(1)=\frac{2}{e}e^2\times 2=4e$

Hence, A is correct.

28. $(2x)^{2y}=4e^{2x-2y}$

$2y\ln 2x=\ln 4+2x-2y$

$\Rightarrow 2y_1\ln 2x+2\frac{y}{x}=2-2y_1$

$\Rightarrow y_1(1+\ln 2x)+\frac{y}{x}=1$

$\Rightarrow y_1(1+\ln 2x)+\frac{2x+\ln 4}{2(\ln 2x+1)}\times\frac{1}{x}=1$

$y_1(1+\ln 2x)+\frac{x+\ln 2}{(1+\ln 2x)x}=1$

$y_1(1+\ln 2x)^2=(1+\ln 2x)-\left(\frac{x+\ln 2}{x}\right)$

$=\frac{x\ln 2x-\ln 2}{x}$

Hence, D is correct.

29. $f(x)=\sin|x|-|x|+2(x-p)\cos|x|$

$$f(x)=\begin{cases}\sin x-x+2(x-p)\cos x, & x\ge 0\\ -\sin x+x+2(x-p)\cos x, & x<0\end{cases}$$

$$f'(x)=\begin{cases}\cos x-1+2\cos x-2(x-p)\sin x, & x\ge 0\\ -\cos x+1+2\cos x-2(x-p)\sin x, & x<0\end{cases}$$

f′(x) is continuous at x = 0

⇒ f(x) is always differentiable .

Hence, C is correct.

4

METHODS OF DIFFERENTIATION

4.1 Introduction

In the previous chapter, we have seen how to interpret derivatives as slopes and rates of change. We used the following definition of a derivative to calculate the derivatives of functions defined by formulae.

Definition The derivative of the function y = f(x) with respect to the variable x is the function f′(x) is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

i.e. $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

or, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{...(1)}$$

provided the limit exists.

Given any number x for which this limit exists, we assign to x the number f′(x). So we can regard f′ as a new function, called the derivative of f and defined by above equation. We know that the value of f′(x), can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, f(x)) or the instantaneous rate of change of f(x) with respect to x.

The function f′ is called the derivative of f because it has been "derived" from f by the limiting operation. The domain of f′ is the set {x | f′(x) exists} and may be smaller than the domain of f.

We have a variety of notations for the derivative:

$$\frac{df(x)}{dx}, \frac{df}{dx}, Df(x), f'(x), f', \dot{f}, f_x, y', \dot{y}, y_1.$$

The process of finding the derivative is called differentiation.

Note:

1. In the study of motion, Newton's dot notation is often used. If y is a function of time t, then $\dot{y}$ denotes the derivative dy/dt.
2. The symbols D and d/dx are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.
3. There is another notation which omits reference to y and f altogether. For example, we can write $\frac{d}{dx}(x^2) = 2x$ which is read "the derivative of x^2 with respect to x is 2x."
4. When we wish to denote the value of the derivative at c, in the Leibniz notation, we shall write $\left.\frac{dy}{dx}\right|_{x=c}$. For instance, we would evaluate $\frac{dy}{dx} = 2x$ at x = 3 by writing $\left.\frac{dy}{dx}\right|_{x=3} = \left.2x\right|_{x=3} = 2(t3) = 6.$

Although, by definition, a derivative is a limit, in practice the computation of derivatives does not directly involve the evaluation of limits, and instead involves the differentiation rules and tabulated derivatives. In this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions.

Note that all the difficulties encountered in evaluating the limits were surmounted when the tabulated derivatives were found and when proving the differentiation rules.

4.2 Derivative Using First Principles (*AB Initio*) Method

Before investigating general rules for calculating the derivatives of given analytical functions, we may discuss a few examples independently from first principles.

If the derivative is found using the limit (1), we call this as finding the derivative using first principles.

To differentiate one variable with respect to another we merely carry out the steps suggested by the definition.

(i) Let y = f(x). Give x an increment Δx and by substituting x + Δx in the equation defining y, calculate y + Δy, the new value of y.

(ii) Subtract y from y + Δy, thus obtaining Δy in terms of x and Δx.

(iii) Divide Δy by Δx, thus obtaining the value of Δy/Δx

(iv) Find $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$, this being the value of dy/dx.

This procedure is called differentiation using first principles. It should be clearly understood that during the process of differentiation x and y are regarded as fixed values of the variables x and y, and that Δx and Δy are variables.

Example 1: Using first principles, find $\dfrac{dy}{dx}$, if $y = x^3 - 2x + 4$.

Solution: Substituting x + Δx for x, we have

$$y + \Delta y = (x + \Delta x)^3 - 2(x + \Delta x) + 4$$
$$= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 2x - 2\Delta x + 4.$$

Substracting y from y + Δy,

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 2\Delta x$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 - 2$$

Let $\Delta x \to 0$.

The second and third terms of $\dfrac{\Delta y}{\Delta x}$ will evidently approach 0, therefore $\dfrac{dy}{dx} = 3x^2 - 2$.

Example 2: Find the derivative of $f(x) = \dfrac{2x-3}{3x+4}$.

Solution: $f(x+\Delta x) = \dfrac{2(x+\Delta x)-3}{3(x+\Delta x)+4}$

$$f(x+\Delta x) - f(x) = \frac{2x+2\Delta x-3}{3x+3\Delta x+4} - \frac{2x-3}{3x+4}$$

$$= \frac{(3x+4)[(2x-3)+2\Delta x]-(2x-3)[(3x+4)+3+\Delta x]}{(3x+4)(3x+3\Delta x+4)}$$

$$= \frac{(6x+8-6x+9)\Delta x}{(3x+4)(3x+3\Delta x+4)} = \frac{17\Delta x}{(3x+4)(3x+3\Delta x+4)}$$

$$\Rightarrow \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{17\Delta x}{(3x+4)(3x+3\Delta x+4)}$$

$$f'(x) = \lim_{\Delta x\to 0}\frac{17}{(3x+4)(3x+3\Delta x+4)} = \frac{17}{(3x+4)^2}.$$

Example 3: Using the definition of the derivative, find the derivative of the function cos ax at x.

Solution: Let y = cos ax, we have

$$\Delta y = \cos a(x + \Delta x) - \cos ax$$

$$= -2\sin\left(ax + \frac{a}{2}\Delta x\right)\sin\frac{a\Delta x}{2}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{-2\sin\left(ax+\frac{a}{2}\Delta x\right)\sin\frac{a\Delta x}{2}}{\Delta x}$$

Hence

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -2\lim_{\Delta x\to 0}\sin\left(ax+\frac{a}{2}\Delta x\right)\lim_{\Delta x\to 0}\frac{\sin\frac{a}{2}\Delta x}{\Delta x}$$

$$= -a\sin ax.$$

In particular, if a = 1, then y = cos x and $y' = -\sin x$.

Example 4: Find the derivative of y = tan x using first principles.

Solution: $\dfrac{dy}{dx} = \lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan x}{h}$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\cdot\frac{1}{\cos(x+h)\cos x} = \sec^2 x.$$

Example 5: Find the derivative of $\tan^{-1}x$ with respect to x by using first principles.

Solution: Let $\tan^{-1}x = \theta$, $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$$\Rightarrow x = \tan\theta \quad \text{and } \tan^{-1}(x+h) = \theta + \Delta\theta \qquad \text{...(1)}$$

$$\Rightarrow x + h = \tan(\theta + \Delta\theta) \qquad \text{...(2)}$$

Let $\lim\limits_{h\to 0}\dfrac{\tan^{-1}(x+h)-\tan^{-1}x}{h} = L$

$$L = \lim_{h\to 0}\frac{\theta+\Delta\theta-\theta}{h} = \lim_{h\to 0}\frac{\Delta\theta}{h} \quad \text{(from (1) and (2))}$$

$$= \lim_{h\to 0}\frac{\Delta\theta}{\tan(\theta+\Delta\theta)-\tan\theta}$$

$$= \lim_{\Delta\theta\to 0}\frac{\Delta\theta}{\tan(\theta+\Delta\theta)-\tan\theta}$$

$$= \lim_{\Delta\theta\to 0}\frac{\Delta\theta\cos(\theta+\Delta\theta)\cos\theta}{\sin\Delta\theta} = \cos^2\theta$$

$$= \frac{1}{\sec^2\theta} = \frac{1}{1+x^2}.$$

Example 6: Using first principles, find the derivative of the function $y = -\cot x - x$.

Solution: $\Delta y = -\cot(x + \Delta x) - (x + \Delta x) + \cot x + x$

$$= \cot x - \cot(x + \Delta x) - \Delta x$$

Using the formula $\cot\alpha - \cot\beta = \dfrac{\sin(\beta - \alpha)}{\sin\alpha \sin\beta}$, we get

$$\Delta y = \frac{\sin(x + \Delta x - x)}{\sin x \sin(x + \Delta x)} - \Delta x = \frac{\sin \Delta x}{\sin x \sin(x + \Delta x)} - \Delta x$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\dfrac{\sin \Delta x}{\Delta x}}{\sin x \sin(x + \Delta x)} - 1$$

and, consequently,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{\sin \Delta x}{\Delta x}}{\sin x \sin(x + \Delta x)} - 1$$

$$= \frac{1}{\sin^2 x} - 1.$$

Thus, we have $y' = \dfrac{1}{\sin^2 x} - 1 = \cot^2 x$.

Example 7: Find by first principles the derivative of $\dfrac{e^x}{x}$ w.r.t. x.

Solution: $y = \dfrac{e^x}{x}$, $\quad y + \Delta y = \dfrac{e^{x+\Delta x}}{x + \Delta x}$

$$\Delta y = \frac{e^{x+\Delta x}}{x + \Delta x} - \frac{e^x}{x}$$

$$= \frac{x \,.\, e^x \,.\, e^{\Delta x} - e^x (x + \Delta x)}{x(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = \frac{x e^x \dfrac{(e^{\Delta x} - 1)}{\Delta x} - e^x \,.\, \dfrac{\Delta x}{\Delta x}}{x(x + \Delta x)}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{x e^x \left(\dfrac{e^{\Delta x} - 1}{\Delta x} \right) - e^x}{x(x + \Delta x)} \right]$$

$$\therefore \quad \frac{d}{dx}(x e^x) = \frac{x e^x - e^x}{x^2}.$$

Example 8: Find derivative of $f(x) = e^{\sin x}$ using first principles.

Solution: $f'(x) = \lim\limits_{h \to 0} \dfrac{e^{\sin(x+h)} - e^{\sin x}}{h}$

$$= \lim_{h \to 0} e^{\sin x} \frac{\left[e^{\sin(x+h) - \sin x} - 1 \right]}{\sin(x+h) - \sin x} \left(\frac{\sin(x+h) - \sin x}{h} \right)$$

$$= e^{\sin x} \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= e^{\sin x} \cos x.$$

Concept Problems A

1. If $F(x) = \dfrac{3}{2x + 1}$, compute $F'(3)$ using first principles.

2. Using first principles, compute $f'(a)$ for each of the following functions:

(i) $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$ (ii) $f(x) = x + \dfrac{1}{x^2}$, $x \neq 0$

3. Differentiate the following functions w.r.t. x using first principles:

(i) $x^2 \cos x$ (ii) $\cos^{-1} x^2$

(iii) $\sqrt[3]{\sin x}$ (iv) e^{x^2}

(v) $\dfrac{\ln x}{x}$

4.3 Derivative of Standard Functions

(i) $\dfrac{d}{dx}(x^n) = n.x^{n-1}$; $n \in R$.

(ii) $\dfrac{d}{dx}(e^x) = e^x$

(iii) $\dfrac{d}{dx}(a^x) = a^x.\ln a, \quad a > 0$

(iv) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(v) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x.\ln a} = \dfrac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \,.\, \tan x$

(x) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \,.\, \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

For example, $\frac{d}{dx}(x^{-7}) = -7x^{-8}$.

$$\frac{d}{dx}(x^{-2/3}) = (-2/3)x^{-2/3-1} = (-2/3)x^{-5/3}.$$

$$\frac{d}{dx}(2^x) = 2^x. \ln 2.$$

Example 1: If $y = x^{-1/2} + \log_5 x + \frac{\sin x}{\cos x} + 2^x$, find $\frac{dy}{dx}$.

Solution: Here $y = x^{-1/2} + \log_5 x + \tan x + 2^x$
On differentiating w.r.t. x we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{-1/2} + \frac{d}{dx}(\log_5 x) + \frac{d}{dx}\tan x + \frac{d}{dx}(2^x)$$

$$= -\frac{1}{2}(x)^{-1/2-1} + \frac{1}{x\log_e 5} + \sec^2 x + 2^x \ln 2$$

$$= -\frac{1}{2}(x)^{-3/2} + \frac{1}{x\log_e 5} + \sec^2 x + 2^x \ln 2.$$

Example 2: Let $y = \frac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$ and $\frac{dy}{dx} = ax + b$, find the value of a and b.

Solution: $y = \frac{x^4 - (x^2 + 2x + 1)}{x^2 - x - 1} = x^2 + x + 1.$

$$\therefore \quad \frac{dy}{dx} = 2x + 1 = ax + b$$

Hence, a = 2 and b = 1.

Example 3: Find $f'(x)$ when $f(x) = |x|$.

Solution: For $x < 0$, $f(x) = -x$ and

$$f'(x) = \lim_{\Delta x \to 0} \frac{-(x+\Delta x) - (-x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -1 = -1.$$

Similarly, for $x > 0$, $f(x) = x$ and

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1.$$

At $x = 0$, $f(x) = 0$ and $\lim_{\Delta x \to 0} \frac{f(0) + (\Delta x) - f(0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}.$$

As $\Delta x \to 0^-$, $\frac{|\Delta x|}{\Delta x} = \frac{-\Delta x}{\Delta x} = -1 \to -1.$

But, as $\Delta x \to 0^+$, $\frac{|\Delta x|}{\Delta x} = \frac{\Delta x}{\Delta x} = 1 \to 1.$

Hence, the derivative does not exist at x = 0.

Finally, we have $\frac{d}{dx}(|x|) = \frac{x}{|x|}$.

Example 4: Prove that the derivative of [x] is equal to $1 - \frac{\sin \pi x}{\sin \pi x}$.

Solution: Since [x] is constant between every two integral values of x, its derivative, whenever it exists, has the value zero. Thus, the derivative of [x], is a function equal to zero for all values of x except integers and undefined for integers. The function $1 - \frac{\sin \pi x}{\sin \pi x}$ has exactly the same properties.

4.4 Rules of Differentiation

(i) The Constant Rule

$$\frac{d}{dx}(c) = 0, \text{ where c is a constant.}$$

(ii) The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cu(x)] = c\frac{d}{dx}u(x).$$

Proof Let $v(x) = cu(x)$. Then

$$v'(x) = \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \lim_{h \to 0} \frac{cu(x+h) - cu(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h}\right]$$

$$= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} = cu'(x).$$

(iii) The Sum Rule

If u and v are both differentiable, then

$$\frac{d}{dx}[u(x)+v(x)]=\frac{d}{dx}u(x)+\frac{d}{dx}v(x)$$

Proof Let F(x) = u(x) + v(x). Then

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$=\lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$

$$=\lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]$$

$$=\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$$

$$=u'(x)+v'(x)$$

The sum rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(u+v+w)'=[(u+v)+w]'$$
$$=(u+v)'+w'=u'+v'+w'.$$

When differentiating a sum of terms, we simply differentiate each term and then add the result.

By writing u – v as u + (–1) v and applying the sum rule and the constant multiple rule, we get the following formula.

(iv) The Difference Rule

If u and v are both differentiable, then

$$\frac{d}{dx}[u(x)-v(x)]=\frac{d}{dx}u(x)-\frac{d}{dx}v(x).$$

The Extended Linearity Rule

If u_1, u_2 ....., u_n are differentiable functions and c_1, c_2, ..., c_n are constants, then

$$\frac{d}{dx}[c_1u_1+c_2u_2+\ldots\ldots+c_n u_n]$$
$$=c_1\frac{du_1}{dx}+c_2\frac{du_2}{dx}+\ldots..+c_n\frac{du_n}{dx}.$$

(v) The Product Rule

If u and v are both differentiable, and y(x) = u(x).v(x) then

$$\frac{d(u\,v)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

i.e. $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$.

The derivative of the product of two functions is:
(First Function) × (Derivative of Second)
+ (Second Function) × (Derivative of First)

Proof $\frac{d}{dx}(uv)=\lim_{x\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$

$$=\lim_{x\to 0}\frac{v(x+h)\{u(x+h)-u(x)\}+u(x)\{v(x+h)-v(x)\}}{h}$$

$$=\lim_{x\to 0}v(x+h).\frac{u(x+h)-u(x)}{h}$$
$$+\lim_{x\to 0}u(x).\frac{v(x+h)-v(x)}{h}$$

= vu′ + uv′.

Dividing by uv the above result may be written as

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}.$$

The rule may be extended to product of more than two functions.

For example, if y = uvw; let vu = z, then y = uz.

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{z}\frac{dz}{dx}$$

but $\frac{1}{z}\frac{dz}{dx}=\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$,

by substitution we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}.$$

Generally, if y = uvwt.....

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}+\frac{1}{t}\frac{dt}{dx}+\ldots\ldots.,$$

and if we multiply by uvwt...... we obtain

$$\frac{dy}{dx}=(vwt\ldots)\frac{du}{dx}+(uwt\ldots)\frac{dv}{dx}+(uvt\ldots)\frac{dw}{dx}+\ldots$$

i.e. multiply the derivative of each separate function by the product of all the remaining functions and add up all the results. The sum will be the derivative of the product of all the functions.

In short, (uv)′= uv′ + vu′
(uvw)′= u′vw + uv′w + uvw′
(uvwt...)′= u′vwt... + uv′wt... + uvw′t... + ...

(vi) The Quotient Rule

If u and v are both differentiable, then

$$\frac{d}{d\,x}\left(\frac{u}{v}\right)=\frac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$$

Proof $\frac{d}{dx}\left(\frac{u}{v}\right)=\lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}}{h}$

$$=\lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{hv(x)v(x+h)}$$

$$=\lim_{h\to 0}\frac{v(x)\{u(x+h)-u(x)\}-u(x)\{v(x+h)-v(x)\}}{hv(x)v(x+h)}$$

$$=\frac{1}{\{v(x)^2\}}\lim_{h\to 0}\left[v(x).\frac{u(x+h)-u(x)}{h}\right.$$

$$-u(x).\frac{v(x+h)-v(x)}{h}\Bigg] = \frac{vu'-uv'}{v^2}$$

i.e. $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$.

Caution

It is important to realize that the symmetry present in the product rule is missing in the quotient rule. For the former, order is immaterial. The prime appears once on one factor and once on the other, and that is all there is to remember. This is not so for the quotient rule, however, where the wrong order will result in the wrong sign in the answer. There is no help for it but to memorize the formula precisely.

The derivative of a quotient of two functions is **:**

$$\frac{(\text{Derivative of Nr.})(\text{Dr.})-(\text{Derivative of Dr.})(\text{Nr.})}{\text{Square of Denominator}}$$

Note: If u is differentiable, the Reciprocal Rule says that

$$\frac{d}{dx}\left(\frac{1}{u(x)}\right) = -\frac{u'(x)}{(u'(x))^2}.$$

Example 1: If f(x) = $\sqrt{x}$ g(x), where g(4) = 2 and g′(4) = 3, find f′(4).

Solution: Applying the product rule, we get

$$f'(x) = \frac{d}{dx}[\sqrt{x}g(x)] = \sqrt{x}\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[\sqrt{x}]$$

$$= \sqrt{x}\,g'(x) + g(x)\cdot\frac{1}{2}x^{-1/2}$$

$$= \sqrt{x}\,g'(x) + \frac{g(x)}{2\sqrt{x}}.$$

So $f'(4) = \sqrt{4}g'(4) + \frac{g(x)}{2\sqrt{x}} = 2.3 + \frac{2}{2.2} = 6.5.$

Example 2: If y = $\sqrt{x}$ sin xcos x, then find y′.

Solution: $y' = (\sqrt{x})'\sin x\cos x + \sqrt{x}(\sin x)'\cos x + \sqrt{x}\sin x(\cos x)'$

$$= \frac{1}{2\sqrt{x}}\sin x\cos x + \sqrt{x}\cos x\cos x + \sqrt{x}\sin x(-\sin x)$$

$$= \frac{1}{2\sqrt{x}}\sin x\cos x + \sqrt{x}(\cos^2 x - \sin^2 x)$$

$$= \frac{\sin 2x}{4\sqrt{x}} + \sqrt{x}\cos 2x.$$

Example 3: Find the derivative of $y = \frac{x^2+x-2}{x^3+6}$.

Solution:

$$y' = \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^2+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1)-(x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{(2x^4+x^3+12x+6)-(3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}.$$

Example 4: Find $\frac{d}{dx}\left(\frac{5x-2}{x^2+1}\right)$

Solution: $\frac{d}{dx}\left(\frac{5x-2}{x^2+1}\right)$

$$= \frac{(x^2+1)\frac{d}{dx}[5x-2]-(5x-2)\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)^2}$$

$$= \frac{(5x^2+5)-(10x^2-4x)}{(x^2+1)^2} = \frac{-5x^2+4x+5}{(x^2+1)^2}.$$

Note: We do not use the quotient rule every time we see a quotient. Sometimes it's easier to rewrite a quotient in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2+2\sqrt{x}}{x}$$

using the quotient rule, it is much easier to perform the division first and write the function as

$F(x) = 3x + 2x^{-1/2}$ before differentiating.

Example 5: If $y = \frac{x^3+x^2+x}{1+x^2}$, find $\left.\frac{dy}{dx}\right]_{x=0}$.

Solution: $y = \frac{x(1+x^2)+x^2+1-1}{(1+x^2)}$

$$= x + 1 - \frac{1}{(1+x^2)}$$

$$y' = 1 + 0 - \frac{(1+x^2)\cdot 0 - 1\cdot 2x}{(1+x^2)^2} = 1 + \frac{2x}{(1+x^2)^2}.$$

Hence, $\left.\dfrac{dy}{dx}\right]_{x=0} = 1.$

Example 6: Find the derivative of

$$y = \frac{(x-1)\,(x^2-2x)}{x^4}.$$

Solution: Rather than using the quotient rule to find the derivative of $y = \dfrac{(x-1)\,(x^2-2x)}{x^4}$

we expand the numerator and divide by x^4 :

$$y = \frac{(x-1)\,(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4}$$

$= x^{-1} - 3x^{-2} + 2x^{-3}.$

Then we use the sum and power rules:

$$\frac{dy}{dx} = -x^{-2} - 3\,(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

Example 7: If $y = \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$,

then find $\dfrac{dy}{dx}$.

Solution:

$N^r = \cos 6x + (1+5)\cos 4x + (5+10)\cos 2x + 10$

$= \cos 6x + \cos 4x + 5(\cos 4x + \cos 2x)$

$\quad +10\,(1 + \cos 2x)$

$= 2\cos 5x\ \cos x + 10\cos 3x\ \cos x + 20\cos^2 x$

$= 2\cos x\,[\cos 5x + 5\cos 3x + 10\cos x]$

$\therefore\quad y = \dfrac{N^r}{D^r} = 2\cos x$

$\therefore\quad \dfrac{dy}{dx} = -2\sin x.$

Example 8: Find the derivative of the function

$$y = xe^x + \frac{\sin x}{\ln x}.$$

Solution: According to the rule for differentiating a sum, we have $y = \left(xe^x + \dfrac{\sin x}{\ln x}\right)' = (xe^x)' + \left(\dfrac{\sin x}{\ln x}\right)'$.

We then use the rule for differentiating a product and a quotient, respectively

$$y' = (xe^x)' + \left(\frac{\sin x}{\ln x}\right)'$$

$$= (x)'\,e^x + x(e^x)' + \frac{(\sin x)'\ln x - \sin x(\ln x)'}{\ln^2 x}$$

$$\Rightarrow\quad y' = 1\cdot e^x + xe^x + \frac{\cos x\ln x - \sin x\cdot\dfrac{1}{x}}{\ln^2 x}.$$

Example 9: Find the rate at which the function $y = x^2 \sin x$ is changing with respect to x when $x = \pi$.

Solution: For any x, the instantaneous rate of change is the derivative,

$$\frac{dy}{dx} = 2x\sin x + x^2\cos x.$$

Thus, the rate when $x = \pi$ is

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi\sin\pi + \pi^2\cos\pi$$

$= 2\pi(0) + \pi^2\,(-1) = -\pi^2.$

The negative sign indicates that when $x = \pi$, the function is decreasing at the rate of π^2 units of y for each one unit increase in x.

Note: Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when we are trying to match the answers to those given in the book.

Example 10: Differentiate both forms of

$$y = \frac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$$

Solution: First form $y = \dfrac{1-\cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x}.$$

Second form $\quad y = \operatorname{cosec} x - \cot x$

$y' = -\operatorname{cosec} x\cot x + \operatorname{cosec}^2 x$

To show that the two derivatives are equal, we write

$$\frac{1-\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$= \operatorname{cosec}^2 x - \operatorname{cosec} x\cot x.$

Example 11: A student made the mistake in writing quotient rule of differentiation and writes

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f(x).g'(x) - g(x)f'(x)}{(g(x))^2}$$

and still gets correct answer. Let $F(n) = \dfrac{f(n)}{g(n)}$, $n \in N$, then prove that

(i) $F(n) = F(n+1)$

(ii) $F(n) = F(n+2)$

(iii) $\sum_{n=1}^{20} F(n) = \sum_{n=1}^{20} F(2n-1)$

(iv) F(1), F(2), F(3) ....form a G.P.

Solution: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f(x).g'(x) - g(x)f'(x)}{(g(x))^2}$

$= -\frac{(f'(x).g(x) - f(x)g'(x))}{(g(x))^2}$

$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0$

$\Rightarrow \frac{f(x)}{g(x)} = c$ where c = constant

$\therefore$ $F(x) = c \ \forall \ n \in N$.

$\therefore$ $F(n) = F(n+1) = F(n+2)$

$\sum_{n=1}^{20} F(n) = \sum_{n=1}^{20} F(2n-1) = 20c$ and

F(1), F(2), F(3) ......are in G.P. with common ratio 1.

Concept Problems B

1. If $f(x) = 5x^3 + x^8 + 3 \tan x$, find $f'(x)$.

2. Check to see that for the function $f(x) = x^2$ the relationship $f'(a + b) = f'(a) + f'(b)$ is valid. Will this identity be valid for the function $f(x) = x^3$?

3. If $f(x) = 4 - 5x + 2x^3 - x^5$, show that $f'(a) = f'(-a)$.

4. At what points does the derivative of the function $f(x) = x^3$ coincide numerically with the value of the function itself?

5. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.

(i) $(fg)'(5)$

(ii) $(f/g)'(5)$

(iii) $(g/f)'(5)$

6. Differentiate the following expressions, and show that they are each positive for all positive values of x:

(i) $(x - 2)\,e^x + x + 2$,

(ii) $(x - e)\,e^x + \frac{x^2}{2} + 2x + 3$,

7. Find the derivative of the following functions at the point x = 1.

(i) $h(x) = \frac{x^3}{x^2 - 2x}$ (ii) $f(x) = \frac{\sqrt{x}}{3x - x^3}$

8. If $y = \frac{\sec x + \tan x - 1}{\tan x - \sec x + 1}$, find $\left.\frac{dy}{dx}\right]_{x=\pi/4}$

9. Prove the correctness of the formula

$$\left(\frac{k}{f(x)}\right)' = \frac{-kf'(x)}{f^2(x)}$$

(for the function f differentiable at the point x where $f(x) \neq 0$).

10. Criticize the following "proof" that 3 = 2. Begin by writing $x^3 = x \,.\, x^2 = x^2 + x^2 + ... + x^2$ (x summands) Differentiate to obtain $3x^2 = 2x + 2x + ...+ 2x$ (still x summands). Thus $3x^2 = 2x^2$, and "therefore" 3 = 2.

Practice Problems A

11. Differentiate the functions:

(i) $f(x) = (x^2 + 3x - 1)(x^2 - 3x + 1)$.

(ii) $f(t) = (t^2 + 1)(t^4 + 1)(t^8 + 1)$

(iii) $f(x) = \frac{x - 1}{x^2 - 2x + 2}$

(iv) $f(x) = (x^n + 1)(x^{2n} - x^n + 1)$, (n an integer).

12. Differentiate the following functions:

(i) $\frac{\sin x}{1 + \cos x}$ (ii) $x\,e^x \sin x$

(iii) $\frac{x^2 + \sec x}{1 + \tan x}$

13. If $y = \frac{x^4 + x^2 + 1}{x^2 + x + 1}$ and $\frac{dy}{dx} = ax + b$, find a and b.

14. If $y = \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x}$ then find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

15. Find the derivative of

$$f(x) = \begin{cases} x^2 + 5x + 4 & \text{for} \quad x \le 0 \\ 5x + 4 & \text{for } 0 < x < 6 \\ x^2 - 2 & \text{for} \quad x \ge 6 \end{cases}$$

16. If $f(x) = 1 + x + x^2 + ... + x^{100}$ then find $f'(1)$.

17. Find the slope of the curve whose equation is $y = (x - 1)(x - 2)(x + 1)$ at the points where it crosses the x–axis.

18. Find the slope of the curve whose equation is $y = \frac{(x-a)(x-b)}{2x-(a+b)}$ at each of the points where the curve cuts the x-axis. Calculate for the case when $a = 2$ and $b = 8$.

19. When $x = 2$ the following equation is true: $x^2 = 2x$. Does it follow from this that $(x^2)' = (2x)'$ when $x = 2$?

20. When a professor asked the calculus class to find the derivative of y^2 with respect to x^2 for the function $y = x^2 - x$, one student found $\frac{dy}{dx}\cdot\frac{y}{x}$.

Was this answer correct?

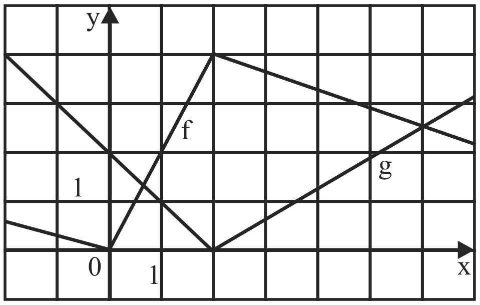
Suppose $y = x^2 - 3x^3 + \frac{7}{x}$. What is the derivative of y^2 with respect to x^2 for this function?

21. If $y = y_1 y_2 \ldots y_n$, then prove that

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \ldots + \frac{1}{y_n}\frac{dy_n}{dx}$$

22. If f and g are the function whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(i) Find $u'(1)$ (ii) Find $v'(5)$.



4.5 The Chain Rule

The above examples were solved using the differentiation rules and the tabulated formulas. Note that not any elementary function can be differentiated in such a way.

Let us consider, for instance, the function $y = \sqrt{x^2+1}$, which is neither a sum nor a product nor a quotient are not applicable to it. Since $y = \sqrt{x^2+1}$ is a composite function, namely, $y = \sqrt{u}$, $u = x^2 + 1$, its derivative can be found using the rule for differentiating a composite function. This rule together with the earlier rules and the tabulated derivatives enable us to find the derivative of any elementary function.

Derivative of Composite Function

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

In function notation, the chain rule is as follows:
if $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ then

$$\frac{dy}{dx} = (f(g(x)))' = f'(g(x)).g'(x)).$$

Proof We have

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{y\{u(x+h)\} - y\{u(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{y\{u(x+h)\} - y\{u(x)\}}{u(x+h) - u(x)}\cdot\frac{u(x+h) - u(x)}{h}$$

$$= \frac{dy}{du}\cdot\frac{du}{dx}$$

Note that this proof is inapplicable if $u(x+h) - u(x)$ turns into zero.

When deriving the chain rule above, we assumed that $\Delta u = u(x+h) - u(x) \neq 0$ in some neighbourhood of x. Sometimes, this assumption is not true, for instance, for the function $u = x^2 \sin(1/x)$ at $x = 0$. Therefore our reasoning only satisfies a particular case of the theorem. It can nevertheless be shown that the theorem remains valid in this case as well.

Alternative Proof

Let $y = \phi(x) = f(g(x))$, then we have to prove that

$$\phi'(x) = f'(g(x))g'(x).$$

We write $g(x) = u$, $g(x + h) = u + k$, so that $k \to 0$ when $h \to 0$ and $k/h \to g'(x)$ ...(1)

We must now distinguish two cases.

(a) Suppose that $g'(x) \neq 0$, and that h is small, but not zero. Then $k \neq 0$, because of (1), and

$$\frac{\varphi(x+h) - \varphi(x)}{h} = \frac{f(u+k) - f(u)}{k}\frac{k}{h} \to f'(u)g'(x).$$

(b) Suppose that $g'(x) = 0$, and that h is small, but not zero. There are now two possibilities.

If $k = 0$, then $\frac{\phi(x+h) - \phi(x)}{h} = 0$

If $k \neq 0$, then $\frac{\varphi(x+h) - \varphi(x)}{h} = \frac{f(u+k) - f(u)}{k}\frac{k}{h}$.

The first factor is nearly $f'(u)$, and the second is small, because $k/h \to 0$. hence $\{\phi(x + h) - \phi(x)\}/h$ is small in any case and

$$\frac{\phi(x+h) - \phi(x)}{h} \to 0 = f'(u)\, g'(x).$$

Example 1: If $F(x) = (x^2 + 2)^3$, compute $F'(x)$.

Solution: One way to do this problem is to expand $(x^2 + 2)^3$ and use the differentiation formulae.

$$F(x) = (x^2 + 2)^3 = x^6 + 6x^4 + 12x^2 + 8,$$

$$F'(x) = 6x^5 + 24x^3 + 24x.$$

Another method uses the chain rule. Let g and f be the functions defined, respectively, by $g(x) = x^2 + 2$ and $f(y) = y^3$. Then

$$f(g(x)) = (x^2 + 2)^3 = F(x),$$

and, according to the chain rule,

$$F'(x) = [f(g(x))]' = f'(g(x))\, g'(x).$$

Since $g'(x) = 2x$ and $f'(y) = 3y^2$, we get $f'(g(x)) = 3(x^2 + 2)^2$ and $F'(x) = 3(x^2 + 2)^2\,(2x)$

$$= 6x\,(x^4 + 4x^2 + 4),$$

which agrees with the alternative solution above.

Note: In using the chain rule we work from the outside to inside.

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

In practical terms, this means "differentiate the 'outside function,' plug in the 'inside function,' and then multiply by the derivative of the inside function."

For example, Let us differentiate $y = \sin(x^2)$

The outer function is the sine function and the inner function is the squaring function, so the chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Remark $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$. ...(1)

The idea that one can simply cancel out du in (1) is very appealing and accounts for the popularity of the notation. It is important to realize that the cancellation is valid because the Chain Rule is true, and not vice versa. Note also that (1) is incomplete in the sense that it does not say explicitly at what points to evaluate the derivatives. We can add this information by writing

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=u(a)}\cdot\left.\frac{du}{dx}\right|_{x=a}$$

Example 2: If $w = z^2 + 2z + 3$ and $z = \dfrac{1}{x}$, find $\dfrac{dw}{dx}$ (2).

Solution: By the chain rule, $\dfrac{dw}{dx} = \dfrac{dw}{dz}\,\dfrac{dz}{dx}$

$$= (2z + 2)\left(-\frac{1}{x^2}\right).$$

When $x = 2$, we have $z = \dfrac{1}{2}$.

Hence, $$\frac{dw}{dx}(2) = \left(2.\frac{1}{2} + 2\right)\left(-\frac{1}{4}\right) = -\frac{3}{4}.$$

Corollary. If $y = f(x)$, $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{du} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$$

i.e. $\quad (f(g(h)))' = f'(g(h))g'(h)h'$.

We apply the chain rule by working from the outside to the inside. At each step the derivative of the outside function is multiplied by the derivative of the inside function. We continue until no "inside function" remains undifferentiated. This process reminds of peeling an onion, one layer at a time, until its core is reached.

For example,

$$\frac{d(\ln\sin x^2)}{dx} = \frac{d(\ln\sin x^2)}{d(\sin x^2)}\cdot\frac{d\sin x^2}{dx^2}\cdot\frac{dx^2}{dx}$$

$$= \frac{1}{\sin x^2}\cdot\cos x^2\cdot 2x = 2x\cot x^2.$$

The general chain rule may be expressed as:

$$\frac{d\,(1^{st}\ \text{Func.})}{dx} = \frac{d\,(1^{st}\ \text{Func.})}{d\,(2^{nd}\,\text{Func.})}\cdot\frac{d(2^{nd}\,\text{Func.})}{d(3^{rd}\,\text{Func.})}\ \cdots\ \frac{d\,(\text{Last Func.})}{dx}$$

Example 3: If $y = \ln(\tan^{-1}\sqrt{1+x^2})$, find $\dfrac{dy}{dx}$.

Solution: $y = \ln(\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$\frac{dy}{dx} = \frac{d[\ln(\tan^{-1}\sqrt{1+x^2})]}{d(\tan^{-1}\sqrt{1+x^2})}\cdot\frac{d(\tan^{-1}\sqrt{1+x^2})}{d(\sqrt{1+x^2})}\cdot\frac{d(1+x^2)}{dx}$$

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}}\cdot\frac{1}{1+(\sqrt{1+x^2})^2}\cdot\frac{1}{2\sqrt{1+x^2}}.2x$$

$$= \frac{x}{(\tan^{-1}\sqrt{1+x^2})\{1+(\sqrt{1+x^2})^2\}\sqrt{1+x^2}}$$

$$= \frac{x}{(\tan^{-1}\sqrt{1+x^2})(2+x^2)\sqrt{1+x^2}}.$$

Example 4: Differentiate $g(x) = \sqrt[4]{\dfrac{x}{1-3x}}$.

Solution: We write $g(x) = \left(\dfrac{x}{1-3x}\right)^{1/4} = u^{1/4}$

where $u = \dfrac{x}{1-3x}$ is the inner function and $u^{1/4}$ is the outer function.

Then $g'(x) = (u^{1/4})'u'(x) = \dfrac{1}{4}\, u^{-3/4}u'(x)$

and we have $g'(x) = \dfrac{1}{4}\left(\dfrac{x}{1-3x}\right)^{-3/4}\left(\dfrac{x}{1-3x}\right)'$

$$= \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{(1-3x)(1)-x(-3)}{(1-3x)^2}\right]$$

$$= \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{1}{(1-3x)^2}\right] = \frac{1}{4x^{3/4}(1-3x)^{5/4}}.$$

Example 5: Differentiate $y = \sqrt{z}$, where z is a known function of x.

Solution: Here $y = z^{\frac{1}{2}}$,

and $\dfrac{dy}{dz} = \dfrac{1}{2}z^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{z}}$,

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} = \dfrac{1}{2\sqrt{z}}\cdot\dfrac{dz}{dx}$.

Note: This form occurs so often that it will be found convenient to commit it to memory.

Example 6: Find the derivative of $y = \sin\sqrt{x^2-4}$

Solution:

$$\frac{dy}{dx} = \frac{d(\sin\sqrt{x^2-4})}{d(\sqrt{x^2-4}}\cdot\frac{d(\sqrt{x^2-4})}{d(x^2-4)}\cdot\frac{d(x^2-4)}{dx}$$

$$= \cos\sqrt{x^2-4}\;.\;\frac{1}{2\sqrt{x^2-4}}\;.\;2x.$$

Example 7: Find $\dfrac{d}{dx}\left(\dfrac{3}{4}\cos x - \cos^3 x\right)$ when $x = 18°$.

Solution: Consider $y = \dfrac{3}{4}\cos x - \cos^3 x$

We write $y = \dfrac{1}{4}\cos 3x$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{4}\sin 3x$$

$$\left.\frac{dy}{dx}\right|_{x=18°} = \frac{3}{4}\sin 54° = \frac{3}{4}\left(\frac{(\sqrt{5}+1)}{4}\right)$$

$$= \frac{3}{16}(\sqrt{5}+1).$$

Example 8: Find the derivative w.r.t. x:

(i) $y = \log_{10}(2 + \sin x)$.

(ii) $y = \cos x^2 + 5\left(\dfrac{3}{x}+4\right)^6$

(iii) $y = \cos^4 (3x + 1)^2$

(iv) $y = \sin\sqrt{\cos x}$

Solution:

(i) Using formula, we have

$$\frac{dy}{dx} = \frac{d}{dx}\log_{10}(2+\sin x)$$

$$= \frac{1}{(2+\sin x)\ln 10}\cdot\frac{d}{dx}(2+\sin x)$$

$$= \frac{\cos x}{(2+\sin x)\ln 10}.$$

(ii) $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos x^2 + 5\dfrac{d}{dx}(3x^{-1}+4)^6$

$$= -\sin x^2\frac{d}{dx}(x^2) + 5\left[6(3x^{-1}+4)^5\frac{d}{dx}(3x^{-1}+4)\right]$$

$$= (-\sin x^2)(2x) + 30(3x^{-1}+4)^5(-3x^{-2})$$

$$= -2x\sin x^2 - 90x^{-2}(3x^{-1}+4)^5.$$

(iii) $\dfrac{dy}{dx} = 4\cos^3(3x+1)^2\dfrac{d}{dx}\cos(3x+1)^2$

$$= 4\cos^3(3x+1)^2\,.\,[-\sin(3x+1)^2]\,.\,\frac{d}{dx}(3x+1)^2$$

$$= -4\cos^3(3x+1)^2\sin(3x+1)^2\,.\,2(3x+1)(3)$$

$$= -24(3x+1)\cos^3(3x+1)^2\sin(3x+1)^2.$$

(iv) $\dfrac{dy}{dx} = \dfrac{d\sin\sqrt{\cos x}}{dx}$

$$= \frac{d\sin\sqrt{\cos x}}{d\sqrt{\cos x}}\cdot\frac{d\sqrt{\cos x}}{d\cos x}\cdot\frac{d(\cos x)}{dx}$$

$$= \cos\sqrt{\cos x}\;.\;\frac{1}{2\sqrt{\cos x}}\;.\;(-\sin x)$$

$$= -\frac{\sin x\cos\sqrt{\cos x}}{2\sqrt{\cos x}}.$$

Example 9: Find the derivative of

(i) $y = \ln^2\sin(3x+2)$

(ii) $y = e^{(x^3-1)^{-1/3}}$

Solution:

(i) $$\frac{dy}{dx}=\frac{d\{\ln^2\sin(3x+2)\}}{d\{\ln\sin(3x+2)\}}\cdot\frac{d\{\ln\sin(3x+2)\}}{d\{\sin(3x+2)\}}$$

$$\cdot\frac{d\{\sin(3x+2)\}}{d(3x+2)}\cdot\frac{d(3x+2)}{dx}$$

$$= \ln\sin(3x+2).\ \frac{1}{\sin(3x+2)}.\ \cos(3x+2)\ .\ 3$$

(ii) $$\frac{dy}{dx}=\frac{d\{e^{(x^3-1)^{-1/3}}\}}{d\{(x^3-1)^{-1/3}\}}\cdot\frac{d\{(x^3-1)^{-1/3}\}}{d(x^3-1)}\cdot\frac{d(x^3-1)}{dx}$$

$$= e^{(x^3-1)^{\frac{-1}{3}}}.\frac{-1}{3}(x^3-1)^{-\frac{1}{3}-1}.(-3x^2)$$

$$= \frac{-x^2e^{(x^3-1)^{-\frac{1}{3}}}}{(x^3-1)^{4/3}}.$$

Derivative of ln |x|

If f(x) = ln |x|, x ≠ 0, then $f'(x)=\frac{1}{x}$.

Proof : Using the definition of absolute value,

$$f(x)=\begin{cases}\ln x & \text{if } x>0\\ \ln(-x) & \text{if } x<0\end{cases}$$

So, $$f'(x)=\begin{cases}\frac{1}{x} & \text{if } x>0\\ \frac{1}{-x}(-1)=\frac{1}{x} & \text{if } x<0\end{cases}$$

Thus, $f'(x)=\frac{1}{x}$ for all x ≠ 0.

The derivative of the function ln|x| is $\frac{1}{x}$.

Also, if u is a differentiable function of x, then

$$\frac{d}{dx}\ln|u|=\frac{1}{u}\frac{du}{dx}.$$

Example 10: Differentiate $g(x)=\ln\left|\frac{x}{1-3x}\right|$.

Solution: $$g'(x)=\left(\frac{1-3x}{x}\right)\left(\frac{x}{1-3x}\right)'$$

$$=\left(\frac{1-3x}{x}\right)\left[\frac{(1-3x)(1)-x(-3)}{(1-3x)^2}\right]$$

$$=\frac{1}{x(1-3x)}.$$

Example 11: Differentiate w.r.t. x:

(i) $y=\ln\,(\sqrt{2\sin x+1}+\sqrt{2\sin x-1})$.

(ii) $y=e^x.\tan^{-1}e^x-\ln\sqrt{1+e^{2x}}$.

(iii) $y=\frac{\sin x}{\cos^2 x}+\ln\frac{1+\sin x}{\cos x}$

(iv) $y = x^2\cos(\log x)$

Solution:

(i) $$y'=\left(\frac{1}{\sqrt{2\sin x+1}+\sqrt{2\sin x-1}}\right).$$

$$(\sqrt{2\sin y+1}+\sqrt{2\sin x-1})'$$

$$=\frac{1}{\sqrt{2\sin x+1}+\sqrt{2\sin x-1}}.$$

$$\left(\frac{2\cos x}{2\sqrt{2\sin x+1}}+\frac{2\cos x}{2\sqrt{2\sin x-1}}\right)$$

$$=\frac{1}{\sqrt{2\sin x+1}+\sqrt{2\sin x-1}}.$$

$$\frac{\cos x\ (\sqrt{2\sin x+1}+\sqrt{2\sin x-1}}{\sqrt{4\sin^2 x-1}}$$

$$=\frac{\cos x}{\sqrt{4\sin^2 x-1}}.$$

(ii) Writing the given function in the form

$$y=e^x\tan^{-1}e^x-\frac{1}{2}\ln(1+e^{2x}).$$

We obtain

$$y'=e^x\cdot\frac{1}{1+e^{2x}}\cdot e^x+e^x\tan^{-1}e^x$$

$$-\frac{1}{2}\frac{1}{1+e^{2x}}\cdot e^{2x}\cdot 2$$

$$=\frac{e^{2x}}{1+e^{2x}}+e^x\tan^{-1}e^x-\frac{e^{2x}}{1+e^{2x}}=e^x\tan^{-1}e^x.$$

(iii) Let us transform the given function:

$$y=\frac{\sin x}{\cos^2 x}+\ln(1+\sin x)-\ln\cos x.$$

Then we have

$$y'=\frac{\cos^2 x\cos x-\sin x\cdot 2\cos x(-\sin x)}{\cos^4 x}$$

$$+\frac{1}{1+\sin x}\cos x-\frac{1}{\cos x}(-\sin x),$$

or $$y'=\frac{\cos^2 x+2\sin^2 x}{\cos^3 x}+\frac{\cos x(1-\sin x)}{1-\sin^2 x}+\frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} + \frac{1-\sin x}{\cos x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} + \frac{1}{\cos x} = \frac{2}{\cos^3 x} = 2\sec^3 x.$$

(iv) $\frac{dy}{dx} = \frac{d}{dx}\{x^2 \cos(\log x)\}$

$$= x^2 \frac{d\{\cos(\log x)\}}{dx} + \cos(\log x)\frac{d(x^2)}{dx}$$

$$= x^2 . \frac{d\cos(\log x)}{d\log x} + \frac{d(\log x)}{dx} + \cos(\log x) . 2x$$

$$= x^2 (-\sin(\log x)\} . \frac{1}{x} + 2x\cos(\log x)$$

$$= -x\sin(\log x) + 2x\cos(\log x).$$

Example 12: $y = \sin^{-1}\frac{2x^2}{1+x^4}$, $|x| < 1$

Solution: $y' = \frac{1}{\sqrt{1-\left(\frac{2x^2}{1+x^4}\right)^2}} \cdot \left(\frac{2x^2}{1+x^4}\right)'$

$$= \frac{1}{\sqrt{1-\left(\frac{2x^2}{1+x^4}\right)^2}} \times \frac{(1+x^4)\cdot 4x - 2x^2 \cdot 4x^3}{(1+x^4)^2}$$

$$= \frac{1}{\sqrt{1-2x^4+x^8}} \cdot \frac{4x(1-x^4)}{1+x^4} = \frac{4x}{1+x^4}.$$

Example 13: Differentiate $y = 2^{\ell n \sin e^{\tan^{-1}\sqrt{x^2-1}}}$.

Solution: $y = 2^{\ell n \sin e^{\tan^{-1}\sqrt{x^2-1}}}$

$$\frac{dy}{dx} = 2^{\ln \sin e^{\tan^{-1}\sqrt{x^2-1}}} \ln 2 . \frac{1}{\sin(e^{\tan^{-1}\sqrt{x^2-1}})}\cos(e^{\tan^{-1}\sqrt{x^2-1}})$$

$$\cdot e^{\tan^{-1}\sqrt{x^2-1}} . \frac{1}{1+(x^2-1)} \frac{1}{2\sqrt{x^2-1}}.2x$$

$$= 2^{\ell n \sin e^{\tan^{-1}\sqrt{x^2-1}}} .$$

$$\ell n2. \frac{1}{\sin\left(e^{\tan^{-1}\sqrt{x^2-1}}\right)}\cos\left(e^{\tan^{-1}\sqrt{x^2-1}}\right)$$

$$\cdot e^{\tan^{-1}\sqrt{x^2-1}} . \frac{1}{x\sqrt{x^2-1}} .$$

Example 14: Let g be a differentiable function of x. If $f(x) = \frac{g(x)}{x^2}$ for $x > 0$, $g(2) = 3$ and $g'(2) = -2$, find $f'(2)$.

Solution: $f'(x) = \frac{x^2 g'(x) - g(x)2x}{x^4}$

Hence, $f'(2) = \frac{4(-2) - 3\cdot 4}{16} = -\frac{5}{4}$.

Example 15: Suppose that f is a differentiable function such that $f(2) = 1$ and $f'(2) = 3$ and let $g(x) = f(xf(x))$. Find $g'(2)$.

Solution: $g(x) = f(x \cdot f(x))$

$$g'(x) = f'(x\,f(x)) \cdot \frac{d}{dx}(xf(x))$$

$$= f'(xf(x))\,(x\,f'(x) + f(x))$$

$$g'(x) = f'(2)\,(2\cdot 3 + 1) = 3 \cdot 7 = 21.$$

Example 16: Find dy/dx if

(a) $y = f(x^2 + 1)$ and $f'(x) = \sin x^2$

(b) $y = f(\sin x^2)$ and $f'(x) = x^2 + 1$.

Solution:

(a) Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{d\{f(x^2+1)\}}{d(x^2+1)} . \frac{d(x^2+1)}{dx} = f'(x^2+1).2x$$

$$= \sin(x^2+1)^2 . 2x \qquad [\because f'(x) = \sin x^2]$$

(b) Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{d\{f(\sin x^2)\}}{d(\sin x^2)} . \frac{d(\sin x^2)}{d(x^2)} . \frac{d(x^2)}{dx}$$

$$= f'(\sin x^2) . \cos x^2 . 2x \qquad [\because f'(x) = x^2 + 1].$$

Example 17: If $u = f(x^2)$, $v = g(x^3)$, $f'(x) = \sin x$ and $g(x) = \cos x$ then find $\frac{du}{dv}$.

Solution: Differentiating $u = f(x^2)$ and $v = g(x^3)$ w.r.t x we get, $\frac{du}{dx} = f'(x^2) \cdot 2x = \sin(x^2) \cdot 2x$.

$$\{\because f'(x) = \sin x \Rightarrow f'(x^2) = \sin(x^2)\}$$

$$\frac{dv}{dx} = g'(x^3) \cdot 3x^2 = \cos(x^3) \cdot 3x^2.$$

$$\{\because g'(x) = \cos x \Rightarrow g'(x^3) = \cos(x^3)\}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\sin(x^2)\cdot 2x}{\cos(x^3)\cdot 3x^2} = \frac{2}{3x} \cdot \frac{\sin x^2}{\cos x^3}.$$

Example 18: If $y = f\left(\frac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$ then find $\frac{dy}{dx}$.

Solution: $y = f\left(\frac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$

$$\frac{dy}{dx} = f'\left(\frac{3x+4}{5x+6}\right) \cdot \frac{3(5x+6) - 5(3x+4)}{(5x+6)^2}$$

$$\frac{dy}{dx} = \tan\left(\frac{3x+4}{5x+6}\right)^2 \cdot \frac{-2}{(5x+6)^2}.$$

Example 19: If $f(1) = \pi$, $g(1) = 3$, $f'(1) = 5$ and $g'(1) = 0$ then find the value of $h'(1)$ where $h(t) = \sin\left(\frac{f(t)}{g(t)}\right)$.

Solution: $h(t) = \sin\left(\frac{f(t)}{g(t)}\right)$,

$$h'(t) = \cos\left(\frac{f(t)}{g(t)}\right)\frac{g(t) \cdot f'(t) - f(t)g'(t)}{g^2(t)}$$

$$h'(1) = \cos\left(\frac{\pi}{3}\right) \cdot \frac{(3)(5) - (\pi)(0)}{9} = \frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}.$$

Concept Problems C

1. Find $f'(x)$ if $f(x) = (3x^2 + 6x + 5)^{7/2}$.
2. If $z = 5y^7 + 2y^2 + 1$ and $y = 2x^2 - 6$, find $\frac{dz}{dx}(2)$.
3. Let $y = x^2 + 3x + 2$ and $x = \frac{t-1}{t+1}$. Compute $\frac{dy}{dt}(2)$.
4. Let $w = z - \frac{1}{z}$, $z = \sqrt{y^3+1}$, and $y = 2x^3 - x + 1$. Find $\frac{dw}{dx}(1)$.
5. If f is a positive differentiable function and r is any real number, then prove that $(f^r)' = rf^{r-1} f'$.
6. If $y = \sec 4x$ and $t = \tan x$, prove that
$$\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}.$$
7. (i) Write $|x| = \sqrt{x^2}$ and use the chain rule to show that $\frac{d}{dx}|x| = \frac{x}{|x|}$.
   (ii) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?
   (iii) If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?
8. If f is differentiable, then $\frac{d}{dx}f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$. Is this true?
9. $\frac{d}{dx}|x^2 + x| = |2x + 1|$. Is this true?
10. Let f be a function for which $f'(x) = \frac{1}{x^2+1}$
    (i) If $g(x) = f(3x - 1)$, what is $g'(x)$?
    (ii) If $h(x) = f\left(\frac{1}{x}\right)$, what is $h'(x)$?
11. Given the functions $f(x) = \tan x$ and $\varphi(x) = \ln(1 - x)$, find $\frac{f'(0)}{\varphi'(0)}$.
12. Let f(x) be a differentiable function with the property that $f'(x) = \frac{1}{x}$. If g(x) is a differentiable function with the property that its composition with f is the identity function, i.e., $f(g(x)) = x$. Prove that $g' = g$.
13. If $u = \phi\,[\psi\,\{F\,(f(x)]$, prove that
$$\frac{du}{dx} = \phi'\,[\psi\,\{F(x)\}] \times \psi'\,\{F(f(x)\} \times F'(f(x)) \times f'(x)$$

Practice Problems B

14. Find the derivative of the following functions:
    (i) $\sqrt{\tan(1+x^2)}$ (ii) $\cos(\tan\sqrt{x+1})$
    (iii) $\frac{3x^2+2}{x(x^2+1)}$ (iv) $(3x^2 + 6x + 5)^{7/2}$
    (v) $\ell n(\sin x - \cos x)$
15. (i) If $F(x) = \ln|\cos x|$, show that $F'(x) = -\tan x$.
    (ii) If $F(x) = \ln|\sec x + \tan x|$, show that $F'(x) = \sec x$.
16. Find the derivative of each of the following functions and state the domain of each derivative.
    (i) $f(x) = |x^2 - 4|$ (ii) $f(t) = (|t| - |t - 1|)^2$
    (iii) $f(y) = y|y^3 - 1|$ (iv) $f(x) = \sqrt[3]{x+|x|}$

17. In each of the functions evaluate the derivative at the given points:

(i) $f(x) = |x - 1| + |x - 3|$, find $f'(2)$.

(ii) $f(x) = x\,[x] + \sqrt{x}$, find $f'\left(\frac{3}{2}\right)$.

(iii) $f(x) = \pi \ln |x| + \sqrt{2}\ |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

(iv) $f(x) = \frac{x-4}{2\sqrt{x}}$, find $f'(4)$.

18. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

19. Find $f'(x)$ if it is known that $\frac{d}{dx}[f(2x)] = x^2$.

20. Suppose that $f(0) = 0$ and that $f'(0)=1$. Calculate the derivative of $f(f(f(x))$ at $x = 0$.

21. Let $f(x) = \frac{1}{1+1/x}$ if $x \neq 0$, and let $g(x) = \frac{1}{1+1/f(x)}$.

Compute $f'(x)$ and $g'(x)$.

22. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$. Find $F'(3)$.

23. If $f(y) = y^{-2}$ and $g(x) = \frac{x^{1/2}}{\sqrt{5x^3+6x^2+4x}}$,

compute the derivative of the composite function f(g) in two ways: (a) by finding f(g(x)) first and then taking its derivative, (b) by the chain rule.

24. Suppose f is a differentiable function such that

$f(x) = x^2 + x$. Find $\frac{d}{dx} f(x^2+x)$.

25. Find $f'(x)$ if

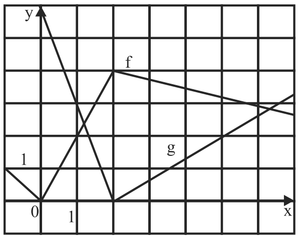
(i) $f(x) = \ln |x|\ (x \neq 0)$

(ii) $f(x)=\begin{cases}1-x & \text{for } x \le 0,\\ e^{-x} & \text{for } x > 0.\end{cases}$

26. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if exists. If it does not exist, explain why.

(i) $u'(1)$ (ii) $v'(1)$

(iii) $w'(1)$



27. The geometric mean of x and x + n is

$g = \sqrt{x(x+n)}$ and the arithmetic mean is

$a = [x + (x + n)]/2$. Show that $\frac{dg}{dx} = \frac{a}{g}$.

28. Find $f'(x)$ if $f(x) = x\,\frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}$.

29. If $f'(x) = \sin x + \sin 4x \cdot \cos x$, find

$\left(f\left(2x^2+\frac{\pi}{2}\right)\right)'$ at $x = \sqrt{\frac{\pi}{2}}$.

4.6 Logarithmic Differentiation

Logarithmic differentiation is a procedure in which logarithms are used to trade the task of differentiating products and quotients for that of differentiating sums and differences. It is especially valuable as a means for handling complicated product or quotient functions and power functions where variables appear in both the base and the exponent.

Logarithmic differentiation is used to find the derivative of:

(i) a function which is the product or quotient of a number of functions, or

(ii) a function of the form $[f(x)]^{g(x)}$ where f and g are both derivable.

Here, it will be found convenient to take the logarithm of the function first and then differentiate.

Thus, if

$$y = \frac{u_1u_2u_3\ldots}{v_1v_2v_3\ldots}, \text{ we have}$$

$$\ln y = \ln u_1 + \ln u_2 + \ln u_3 + \ldots - \ln v_1 - \ln v_2 - \ln v_3 - \ldots\ldots$$

and therefore,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \frac{1}{u_3}\frac{du_3}{dx} + \ldots - \frac{1}{v_1}\frac{dv_1}{dx} - \frac{1}{v_2}\frac{dv_2}{dx} - \frac{1}{v_3}\frac{dv_3}{dx} - \ldots.$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the laws of logarithm to simplify.
2. Differentiate both sides with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln(f(x))$ is not defined. If f(x) is not positive-valued everywhere,

then y should be replaced with $|y| = |f(x)|$ and

$\ln|y| = \ln |f(x)|$.

The differentiation then leads to the result
dy/dx = f(x).(ln|f(x)|)′,

using $\dfrac{d}{dx}\ \ln|x| = \dfrac{1}{x}$.

In practice we need not be overly concerned in advance with the sign of f(x), because the appearance of what seems to be the logarithm of a negative quantity will signal the fact that absolute values should be used.

For example,

$$\frac{d}{dx}\ \ln|x^2 - x| = \frac{1}{(x^2-x)}\frac{d}{dx}(x^2 - x) = \frac{2x-1}{x(x-1)},$$

$x \neq 0, x \neq 1$.

In contrast, $\dfrac{d}{dx}\ (\ln(x^2 - x) = \dfrac{2x-1}{x(x-1)}$

but only if x (x – 1) > 0, that is, for x > 1 or x < 0.

Example 1: Find the derivative of

$$y = \frac{(x+1)^2(x-1)}{(x-2)^3}$$

Solution: Taking modulus on both sides, we have

$$|y| = \left|\frac{(x+1)^2(x-1)}{(x-2)^3}\right|$$

Now we apply logarithm on both sides,
ln |y| = 2 ln |x + 1| + ln|x – 1| – 3 ln |x – 2|
Differentiating w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} + \frac{1}{x-1} - \frac{3}{x-2}$$

$$\frac{dy}{dx} = \frac{(x+1)^2(x-1)}{(x-2)^3}\left(\frac{2}{x+1} + \frac{1}{x-1} - \frac{3}{x-2}\right)$$

$$= \frac{2(x+1)(x-1)}{(x-2)^3} + \frac{(x+1)^2}{(x-2)^3} - \frac{3(x+1)(x-1)}{x-2}$$

Note that we cancelled factors like x + 1 and x – 1. But this does not prevent us from using this formula to evaluate the derivative at x = –1 or 1 because the same formula will be obtained when we use the product and quotient rules.

Also the application of modulus before we take logarithm on both sides of the function is unneccesary in practice.

Example 2: Differentiate $y = \dfrac{(2x-1)^3 \cdot \sqrt{3x+2}}{(5x+4)\sqrt[3]{1-x}}$.

Solution: Before seeking the derivative, it is useful to take the logarithm of the given function:

$$\ln|y| = 3\ln|2x-1| + \frac{1}{2}\ln|3x+2|$$
$$-2\ln|5x+4| - \frac{1}{3}|1-x|$$

$$\frac{y'}{y} = \frac{3}{2x-1}\cdot 2 + \frac{1}{2}\cdot\frac{3}{3x+2} - 2\cdot\frac{5}{5x+4} + \frac{1}{3(1-x)}$$

$$y' = \frac{(2x-1)^3\sqrt{3x+2}}{(5x+4)^2\sqrt[3]{1-x}}$$
$$\times\left[\frac{6}{2x-1} + \frac{3}{2(3x+2)} - \frac{10}{5x+4} + \frac{1}{3(1-x)}\right].$$

Example 3: Find the derivative w.r.t. x:

(a) $y = \sqrt[5]{\dfrac{x(x^2+1)}{\sqrt[3]{1-x}}}$ (b) $y = \sqrt{\dfrac{1-x}{1+x}}\dfrac{\sin^3 x}{\cos x}$

(c) $y = \sqrt[3]{\dfrac{1-\cos 3x}{1+\sin 3x}}$

Solution:

(a) Taking logarithm on both sides,

$$\ln|y| = \ln\sqrt[5]{\frac{|x|(x^2+1)}{\sqrt[3]{|1-x|}}}$$

$$= \frac{1}{5}\ln|x| + \frac{1}{5}\ln(x^2+1) - \frac{1}{15}\ln|x-1|$$

Differentiating w.r.t. x, we have

$$\frac{y'}{y} = \frac{1}{5x} + \frac{2x}{5(x^2+1)} - \frac{1}{15(x-1)}$$

$$\Rightarrow y' = \sqrt[5]{\frac{x(x^2+1)}{\sqrt[3]{1-x}}}\cdot\frac{8x^3-9x^2+2x-3}{15(x-1)(x^2+1)}$$

(b) Taking logarithm on both sides,

$$\ln|y| = \ln\left(\sqrt{\frac{1-x}{1+x}}\frac{|\sin^3 x|}{|\cos x|}\right)$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + 3\ln|\sin x| - \ln|\cos x|$$

Differentiating w.r.t. x, we have

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{3\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{1}{x^2-1} + 3\cot x + \tan x$$

$$\Rightarrow y' = \sqrt{\frac{1-x}{1+x}}\frac{\sin^3 x}{\cos x}\left[\frac{1}{x^2-1} + 3\cot x + \tan x\right].$$

(c) Taking logarithm on both sides,

$$\ln|y| = \ln\sqrt[3]{\frac{|1-\cos 3x|}{1+\sin 3x}}$$

$$= \frac{1}{3}[\ln|1-\cos 3x| - \ln(1+\sin 3x)]$$

Differentiating w.r.t. x, we have

$$\frac{y'}{y}=\frac{1}{3}\left[\frac{3\sin 3x}{1-\cos 3x}-\frac{3\cos 3x}{1+\sin 3x}\right]$$

$$=\frac{1+\sin 3x-\cos 3x}{(1-\cos 3x)(1+\sin 3x)}$$

$$\Rightarrow \quad y' = \sqrt[3]{\frac{|1-\cos 3x|}{1+\sin 3x}}\ \frac{1+\sin 3x-\cos 3x}{(1-\cos 3x)(1+\sin 3x)}.$$

Example 4: If f (x) = (x + 1)(x + 2)(x + 3)...(x + n) then find f′(0).

Solution: f(x) = (x + 1) (x + 2) (x + 3) .... (x + 100)

$\ln(f(x)) = \ln(x+1) + \ln(x+2) + \ldots\ldots + \ln(x+100)$

Differentiating w.r.t. x

$$\frac{f'(x)}{f(x)}=\frac{1}{x+1}+\frac{1}{x+2}+\ldots.+\frac{1}{x+100}$$

$$f'(0) = 100!.\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.+\frac{1}{100}\right).$$

Example 5: If $f(x)=\prod_{n=1}^{100}(x-n)^{n(101-n)}$

then find $\frac{f(101)}{f'(101)}$.

Solution: Applying log on both sides

$$\ell n\, f(x)=\sum_{n=1}^{100} n(101-n)\,\ell n(x-n)$$

Note that ∏ changes to ∑.

Differentiating w.r.t. x, we get,

$$\frac{f'(x)}{f(x)}=\sum_{n=1}^{100}\frac{n(101-n)}{(x-n)}$$

$$\frac{f'(101)}{f(101)}=\sum_{n=1}^{100}\frac{n(101-n)}{(101-n)} = \sum_{n=1}^{100} n = 5050$$

$$\therefore \quad \frac{f(101)}{f'(101)}=\frac{1}{5050}.$$

Differentiation of Power-Exponential Function

A function of the form $y = [f(x)]^{g(x)}$, f(x) > 0

where the base and the exponent are both functions of the independent variable x is referred to as a composite exponential (or power-exponential) function.

A simple example of such a function is $y = x^x$, x > 0.

To find a general formula for derivative of such functions, let us differentiate $y = \{u(x)\}^{v(x)}$.

Taking logarithm on both sides, we have

$\ln y = v(x) \ln u(x)$

Differentiating w.r.t. x, we have

$$\text{i.e.}\quad \frac{1}{y}\frac{dy}{dx}=v\left(\frac{1}{u}\frac{du}{dx}\right)+\ln u\left(\frac{dv}{dx}\right)$$

Multiplying throughout by $y = u^v$ we get

$$\frac{dy}{dx}=vu^{v-1}\frac{du}{dx}+u^v\ln u\frac{dv}{dx}$$

i.e. if $y = u^v$, we obtain $y' = vu^{v-1}u' + u^v v'.\ln u$

Thus, the derivative of a composite exponential function consists of two terms: the first term is obtained by assuming, when differentiating, that u is a function of x and v is a constant (that is to say, if we regard u^v as a power function); the second term is obtained on the assumption that v is a function of x, and u = constant (i.e., if we regard u^v as an exponential function).

For example, if $y = x^x$, then

$y' = xx^{x-1}(x') + x^x(x')\ln x$

or $\quad y' = x^x + x^x \ln x = x^x (1 + \ln x)$.

Further, if $y = (\sin x)^{x^2}$, then

$y' = x^2(\sin x)^{x^2-1}(\sin x)' + (\sin x)^{x^2}(x^2)' \ln x$

$= x^2(\sin x)^{x^2-1}\cos x + (\sin x)^{x^2}\, 2x \ln \sin x.$

Example 6: Find the derivative w.r.t. x:

(a) $y = x^{\sqrt{x}}$ (b) $y = (x + 1)^{2x}$

(c) $y = (\sqrt{\sin x})^x$ (d) $y = (\sin x)^{\tan x}$.

Solution:

(a) $y = x^{\sqrt{x}}$

Taking logarithm on both sides,

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ \ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\ \frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}\right)=x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right).$$

Note that we can also differentiate $y = x^{\sqrt{x}}$ by first expressing $y = e^{\sqrt{x}\ln x}$ and then using chain rule.

(b) $y = (x + 1)^{2x}$

$\ln y = \ln[(x + 1)^{2x}] = 2x \ln(x + 1)$

Differentiate both sides of this equation:

$$\frac{1}{y}\frac{dy}{dx}=2x\left\{\frac{d}{dx}[\ln(x+1)]\right\}+\left[\frac{d}{dx}(2x)\right]\ln(x+1)$$

$$=2x\left[\frac{1}{x+1}(1)\right]+2\ln(x+1)=\frac{2x}{x+1}+2\ln(x+1)$$

Finally, multiply both sides by $y = (x + 1)^{2x}$:

$$\frac{dy}{dx}=\left[\frac{2}{x+1}+2\ln(x+1)\right](x+1)^{2x}.$$

(c) $y = (\sqrt{\sin x})^x$

$$\ln y = \ln(\sqrt{\sin x})^x = \frac{1}{2} x \ln(\sin x)$$

Differentiating w.r.t. x, we have

$$\frac{y'}{y} = \frac{1}{2}\left[\ln \sin x + x\frac{\cos x}{\sin x}\right]$$

$$\Rightarrow \quad y' = \frac{1}{2}(\sqrt{\sin x})^x\ [\ln \sin x + x \cot x].$$

(d) $y = (\sin x)^{\tan x}$

$\ln y = \tan x \cdot \ln \sin x$

$$\frac{y'}{y} = \tan x \cdot \frac{1}{\sin x} \cdot \cos x + \sec^2 x \cdot \ln \sin x$$

$= 1 + \sec^2 x \cdot \ln \sin x$

$y' = y\,(1 + \sec^2 x) \cdot \ln \sin x)$

$= (\sin x)^{\tan x}\ (1 + \sec^2 x \cdot \ln \sin x).$

Example 7: If $y = e^x \sin x^3 + (\tan x)^x$, find $\dfrac{dy}{dx}$.

Solution: Let $u = e^x \sin x^2$ and $v = (\tan x)^x$

Now $u = e^x \sin x^3$

Differentiating w. r. t. x, we get

$$\frac{du}{dx} = e^x, \frac{d\{\sin(x)^3\}}{dx} + \sin x^3, \frac{d}{dx}(e^x)$$

$= e^x \,.\, \cos x^3 \,.\, 3x^2 + \sin x^3 \,.\, e^x$

Hence $\dfrac{du}{dx} = 3x^2 e^x \cos x^3 + e^x \sin x^3$ and

$v = (\tan x)^x$

$\therefore \quad \ln v = x \ln(\tan x)$

Differentiating w. r. t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1.\log(\tan x) + x.\frac{1}{\tan x}\sec^2 x$$

$$\therefore \quad \frac{dv}{dx} = v[\ln(\tan x) + x \cot x.\sec^2 x]$$

$= (\tan x)^x\ [\ln(\tan x) + x \cot x \sec^2 x]$

Now $y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = 3x^2 e^x \cos(x^3) + e^x \sin(x^3)$$

$$+ (\tan x)^x\ [\log(\tan x) + x\cot x \sec^2 x].$$

Example 8: Let f (x)

$$= \lim_{h\to 0} \frac{(\sin(x+h))^{\ln(x+h)} - (\sin x)^{\ln x}}{h},$$

then find $f\left(\dfrac{\pi}{2}\right)$.

Solution: Let $g(x) = (\sin x)^{\ln x} = e^{\ln x \cdot \ln(\sin x)}$

$$f(x) = g'(x) = (\sin x)^{\ln x}\left[\cot x(\ln x) + \frac{\ln(\sin x)}{x}\right]$$

Hence $f\left(\dfrac{\pi}{2}\right) = g'\left(\dfrac{\pi}{2}\right) = 1(0 + 0) = 0.$

Example 9: If $y = \sqrt{x}^{\sqrt{x}} \cdot e^{x^2}$ find $y'(1)$

Solution: $\dfrac{dy}{dx} = \sqrt{x}^{\sqrt{x}}\,(e^{x^2} \cdot 2x)$

$$+ e^{x^2} \cdot (\sqrt{x})^{\sqrt{x}} \cdot \left(\frac{\ln\sqrt{x}}{2\sqrt{x}} + \frac{\sqrt{x}}{2} \cdot \frac{1}{x}\right)$$

$y'(1) = 2e + e(0 + (1/2) = 2e + (e/2) = 5e/2.$

Example 10: If $y = 2^{\log_2 x^{2x}} + \left(\tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$ then find $\left.\dfrac{dy}{dx}\right]_{x=1}$.

Solution: $y = 2^{\log_2 x^{2x}} + \left(\tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$

$$= (x^2)^x + \left(\tan\frac{\pi x}{4}\right)^{\frac{4}{\pi x}}$$

$$= e^{x \ln x^2} + e^{\frac{4}{\pi x}\ln\left(\tan\frac{\pi x}{4}\right)}$$

$$\frac{dy}{dx} = (x^2)^x\left[\frac{x}{x^2}\cdot 2x + \ln x^2\right] + \left(\tan\frac{\pi x}{4}\right)^{\frac{4}{\pi x}}$$

$$\left[\frac{4}{\pi x}\cdot\frac{1}{\tan\frac{\pi x}{4}}\sec^2\frac{\pi x}{4}\frac{\pi}{4} + \ln\left(\tan\frac{\pi x}{4}\right)\left(-\frac{4}{\pi x^2}\right)\right]$$

$$\left.\frac{dy}{dx}\right|_{x=1} = (2 + 0) + \left[\frac{4}{\pi}\cdot\frac{2\pi}{4} - 0\right] = 2 + 2 = 4.$$

Example 11: Let $y = \left(1+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$ then find $y'(1)$

Solution: $y = e^{x\ln\left(1+\frac{1}{x}\right)} + e^{\left(1+\frac{1}{x}\right)\ln x}$

$$y' = \left(1+\frac{1}{x}\right)^x \cdot \left[\ln\frac{x+1}{x} + x\left(\frac{1}{x+1} - \frac{1}{x}\right) + x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\cdot\left[\left(\frac{x+1}{x}\right)\cdot\frac{1}{x} + \ln x\cdot\left(-\frac{1}{x^2}\right)\right]$$

$$y'(1) = 2\left[\ln 2 + \left(-\frac{1}{2}\right)\right] + 1[2]$$

$= 2\ ln\ 2 - 1 + 2 = 2\ ln\ 2 + 1 = ln\ (4e).$

Concept Problems D

1. If $y = \dfrac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\dfrac{dy}{dx}$.

2. If $y = (\sin x)^{\ell n\, x}$, find $\dfrac{dy}{dx}$.

3. Find the derivatives of the following functions:

   (i) $y = x^{x^2}$ (ii) $(\sin x)^{\cos x}$

4. Differentiate the following functions:

   (i) $y = \left(1+\dfrac{1}{x}\right)^x$ (ii) $y = e^{x^x}$

   (iii) $y = (\ln x)^x + (x)^{\sin x}$

5. Let $g(x) = f_1(x).f_2(x) \ldots f_n(x)$. Show that for those points x, where none of the function values $f_1(x), \ldots., f_n(x)$ are zero, we have

$$\frac{g'(x)}{g(x)} = \frac{f_1'(x)}{f_1(x)} + \ldots + \frac{f_n'(x)}{f_n(x)}.$$

6. Suppose that $y = uvw/pqr$, where u, v, w, p, q and r are non-zero differentiable functions of x. Show by logarithmic differentiation that

$$\frac{dy}{dx} = y.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} - \frac{1}{p}\frac{dp}{dx} - \frac{1}{q}\frac{dq}{dx} - \frac{1}{r}\frac{dr}{dx}\right).$$

Practice Problems C

7. If $f(x) = (1+x)(3+x^2)^{1/2}(9+x^3)^{1/3}$ then find $f'(-1)$.

8. If $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots. \sin nx$, find $y'(\pi)$.

9. Find $f'(x)$ if $f(x) = \dfrac{\sqrt[3]{x+\sqrt{2-x^2}}\,\sqrt[6]{1-x\sqrt{2-x^2}}}{\sqrt[3]{1-x^2}}$.

10. Find the derivative of $(\sin x)^{\tan x} + (\tan x)^{\sin x}$.

11. A differentiable function f(x) is so defined that for all real x, $\{f(x)\}^n = f(nx)$. Prove that $f(x) \cdot f'(nx) = f'(x) \cdot f(nx)$.

12. Differentiate $(\tan x)^{\sin x} + (\ln x)^{e^x}$.

13. Find the derivatives of the following functions:

   (i) $y = (\cos x)^{\ln x} + (\ln x)^{x}$

   (ii) $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$

   (iii) $y = \log_e(\tan^{-1}\sqrt{1-x^2})$

14. If $y = (\sin x)^{\ln x}\,\text{cosec}\,(e^x(a+bx))$ and $a + b = \dfrac{\pi}{2e}$ then the value of $\dfrac{dy}{dx}$ at $x = 1$.

4.7 Derivative of Inverse Functions

First of all, we consider the methods of differentiating inverse trigonometric functions.

The function $y = \sin^{-1}x,\ x \in [-1, 1]$, ...(1)

is the inverse of the function

$$x = \sin y, \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \ldots(2)$$

Let us find the derivative of function (1) on the interval (–1, 1).

$$\frac{dx}{dy} = \cos y$$

$$(\sin^{-1}x)' = \frac{dy}{dx} = \frac{1}{\cos y}$$

We express now cos y in terms of x.

Since $\cos y = \sqrt{1-\sin^2 y}$, taking into account equality (2), we have

$$\cos y = \sqrt{1-\sin^2 y}.$$

The radical should be taken with the plus sign, since

cos y on the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is positive.

Thus, $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}$.

We now find the derivative of $y = \sec^{-1}x$.

Here $x \in R - (-1,1)$ and $y \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$.

We have $y = \sec^{-1}x \Leftrightarrow x = \sec y$.

Thus, $\dfrac{dx}{dy} = \sec y \tan y$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}, \text{ if } \tan y \neq 0 \text{ i.e., } y \neq 0, \pi.$$

$$= \pm \frac{1}{\sec y\sqrt{(\sec^2 y - 1)}} = \pm \frac{1}{x\sqrt{(x^2-1)}}$$

where the sign in the same as that of tan y.

Now, $x \in (1, \infty) \Rightarrow y \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan y > 0$

$$x \in (-\infty, -1) \Rightarrow y \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \tan y < 0$$

Thus, $x \in (1, \infty) \Rightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{(x^2-1)}}$, and

$$x \in (-\infty, -1) \Rightarrow \frac{dy}{dx} = -\frac{1}{x\sqrt{(x^2-1)}}.$$

It follows that $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{(x^2-1)}}$ for all admissible values of x.

Note that slope of the tangent drawn to the graph of y = $\sec^{-1}x$ is always positive and hence the derivative should be positive for all x.

We have the following standard results:

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1$

(ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \ -1 < x < 1$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \ x \in R$

(iv) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}, \ |x| > 1$

(v) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x| \sqrt{x^2-1}}, \ |x| > 1$

(vi) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, \ x \in R$.

Example 1: Find the derivative of the function $y = \sin^{-1}\sqrt{x}$.

Solution: $(\sin^{-1}\sqrt{x})' = \frac{(\sqrt{x})'}{\sqrt{1-(\sqrt{x})^2}}$

$$= \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}}.$$

Example 2: If $f(x) = \ell n (\sin^{-1} x^2)$ find $f'(x)$.

Solution: $f'(x) = \frac{1}{(\sin^{-1} x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$$= \frac{2x}{(\sin^{-1} x^2)\sqrt{1-x^4}}.$$

Example 3: If $f(x) = 2x \sec^{-1}x - \text{cosec}^{-1}(x)$ then find $f'(-2)$.

Solution:

$$f'(x) = 2\sec^{-1}(x) - \frac{2x}{|x|\sqrt{x^2-1}} + \frac{1}{|x|\sqrt{x^2-1}}$$

$$f'(-2) = 2.\sec^{-1}(-2) + \frac{2}{\sqrt{3}} + \frac{1}{2\sqrt{3}}$$

$$f'(-2) = \frac{4\pi}{3} + \frac{5}{2\sqrt{3}}.$$

Example 4: Find the derivative of the function

$$y = \frac{e^x}{\tan^{-1} x},$$

Solution: The function has the form $y = f(x)g(x)$, where $f(x) = e^x$ and $g(x) = \tan^{-1}x$. Applying the rule for differentiating a quotient, we get

$$y' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$

$$= \frac{(e^x)' \tan^{-1} x - e^x (\tan^{-1} x)'}{(\tan^{-1})^2 x}$$

$$= \frac{e^x \tan^{-1} x - e^x \frac{1}{1+x^2}}{(\tan^{-1})^2 x}.$$

Example 5: If $y = \sec^{-1}\left(\frac{\sqrt{x+1}}{\sqrt{x-1}}\right)$

$$+ \sin^{-1}\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right), x > 1, \text{ find } \frac{dy}{dx}.$$

Solution: For $x > 1$, $y = \cos^{-1}\frac{\sqrt{x-1}}{\sqrt{x+1}} + \sin\frac{\sqrt{x-1}}{\sqrt{x+1}}$

$\Rightarrow \ y = \frac{\pi}{2}$. Hence, $\frac{dy}{dx} = 0$

Example 6: Differentiate $y = \tan^{-1}\frac{1+x+x^2}{1-x+x^2}$

We write $y = \tan^{-1}u$ where $u = \frac{1+x+x^2}{1-x+x^2}$,

Now $\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$.

We find $1 + u^2 = \frac{2(1+3x^2+x^4)}{(1-x+x^2)^2}$,

$$\frac{du}{dx} = \frac{2(1-x^2)}{(1-x+x^2)^2}.$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1-x^2}{1+3x^2+x^4}.$$

Example 7: If $f(x) = \cot^{-1}\left(\frac{x^2 - x^{-x}}{2}\right)$, then the value of $f'(1)$.

Solution: Putting $u = x^x$ and using logarithmic differentiation, we get $\log u = x \ln x$.

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x.\frac{1}{x} + \ln x \Rightarrow \frac{du}{dx} = x^2(1 + \ln x)$$

Similarly, if $v = x^{-x}$, we get

$$\frac{dv}{dx} = -x^{-x}(1 + \ln x)$$

$$\therefore \quad f'(x) = -\frac{1}{1+\left(\frac{x^2 - x^{-x}}{2}\right)^2}\frac{d}{dx}\left(\frac{x^2 - x^{-x}}{2}\right)$$

$$= -\frac{4}{x^{2x} + x^{-2x} + 2}\left[\frac{1}{2}[(x^x + x^{-x})(1 + \ln x)]\right]$$

$$\Rightarrow \quad f'(1) = -\frac{2}{1+1+2}[2(1 + \ln 1)] = -1.$$

Example 8: If $y = \tan^{-1}\frac{1}{x^2+x+1}$

$$+ \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7}$$

$$+ \tan^{-1}\frac{1}{x^2+7x+13} + \text{.... to n terms,}$$

show that $\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$.

Solution: $t_1 = \tan^{-1}\frac{1}{x^2+x+1} = \tan^{-1}\frac{1}{1+x(x+1)}$

$$= \tan^{-1}\frac{x+1-x}{1+(x+1)x} = \tan^{-1}(x+1) - \tan^{-1}x$$

$$= \tan^{-1}\frac{1}{1+(x+1)(x+2)} = \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}$$

$$= \tan^{-1}(x+2) - \tan^{-1}(x+1)$$

Thus, $t_1 = \tan^{-1}(x+1) - \tan^{-1}x$

$t_2 = \tan^{-1}(x+2) - \tan^{-1}(x+1)$

Similarly $t_3 = \tan^{-1}(x+3) - \tan^{-1}(x+2)$

....

....

$t_n = \tan^{-1}(x+n) - \tan^{-1}(x+n-1)$

Adding, we get $y = t_1 + t_2 + ... + t_n$

$= \tan^{-1}(x+n) - \tan^{-1}x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}.$$

Formula for Differentiation of Inverse Function

If f is any bijective differentiable function, it can be proved that its inverse function f^{-1} is also differentiable, except where its tangents are vertical. This is plausible because the graph of a differentiable function has no corner or kink and so if we reflect it about $y = x$, the graph of its inverse function also has no corner or kink.

Let $y = f(x)$ be a function derivable in its domain $[a, b]$. We suppose that f has an inverse function g so that we have

$$y = f(x) \Leftrightarrow x = g(y).$$

If y be a continuous function of x, then in most of the functions, x will be a continuous function of y.

We have to find a relation between $f'(x)$ and $g'(y)$ for the corresponding values of x and y.

Let Δy be the change in y corresponding to the change Δx in x as determined from $y = f(x)$. The change Δx in x corresponds to the change Δy in y as determined from $x = g(y)$. We suppose that $f'(x) \neq 0$. We have

$$1 = \frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y} \Leftrightarrow \frac{\Delta x}{\Delta y} = 1/\left(\frac{\Delta y}{\Delta x}\right).$$

Let $\Delta x \to 0$.

$$\therefore \quad \frac{dx}{dy} = \frac{1}{dy/dx} \qquad \text{...(1)}$$

Thus $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals of each other.

$$\Rightarrow \quad \frac{dy}{dx}\cdot\frac{dx}{dy} = 1$$

$$\Rightarrow \quad f'(x)\,g'(y) = 1.$$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \text{...(2)}$$

It is important to remember that dy/dx is to be evaluated at x, but dx/dy is to be evaluated at the corresponding value of y namely, $y = f(x)$.

As an example, take the function $y = \sqrt[3]{x}$. Here we have $x = y^3$ and $\frac{dx}{dy} = 3y^2$. Consequently, $\frac{dy}{dx} = \frac{1}{3y^2}$.

Now let us express the derivative $\frac{dy}{dx}$ in terms of the independent variable x. To this end, we substitute into the equality obtained the expression of y as function of x to find $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$.

One can differentiate the function $y = \sqrt[3]{x} = x^{\frac{1}{3}}$ with the aid of the general formula for differentiating a power function to check that the result is the same as above.

When we use the symbol $f^{-1}$for the inverse function and $f(a) = b$ i.e. $f^{-1}(b) = a$ then

$$\left.\frac{df^{-1}(x)}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=a}}$$

i.e. $(f^{-1}(b))' = \dfrac{1}{f'(a)}$.

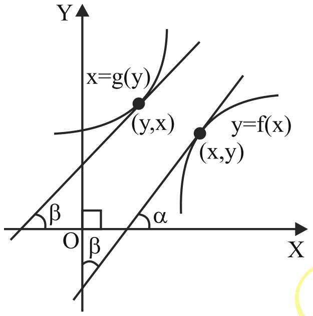
Briefly, the derivatives of two mutually inverse functions are the reciprocals of each other.

This is also expressed geometrically as:

$\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ are respectively the tangent and the cotangent of the angle α which the tangent to the curve $y = f(x)$ makes with the x-axis.

Let α and β denote the angles formed by the tangent and the positive x and y-axes. On the basis of the geometrical meaning of a derivative we have

$$\left.\begin{matrix} f'(x) = \tan\alpha \\ g'(y) = \tan\beta \end{matrix}\right\}$$



From the figure it follows that if $\alpha < \dfrac{\pi}{2}$, then $\beta = \dfrac{\pi}{2} - \alpha$, But if, $\alpha > \dfrac{\pi}{2}$, then, $\beta = \dfrac{3\pi}{2} - \alpha$. Hence, in any case $\tan\beta = \cot\alpha$,

$\Rightarrow \quad \tan\alpha\tan\beta = \tan\alpha\cot\alpha = 1$

$\Rightarrow \quad \tan\alpha = \dfrac{1}{\tan\beta}$.

Substituting the expressions for $\tan\alpha$ and $\tan\beta$ from above, we get

$$f'(x) = \frac{1}{g'(y)}.$$

Example 9: If $y = f(x) = x^3 + x^5$ and g is the inverse of f then find $g'(2)$ (i.e. dx/dy when y=2).

Solution: $\dfrac{dy}{dx} = 3x^2 + 5x^4$

$\Rightarrow \; g'(y) = \dfrac{dx}{dy} = \dfrac{1}{3x^2 + 5x^4}$

when $y = 2$ then $2 = x^3 + x^5 \qquad \Rightarrow x = 1$

$\therefore \quad g'(2) = \left.\dfrac{dx}{dy}\right]_{\substack{x=1\\y=2}} = \dfrac{1}{3+5} = \dfrac{1}{8}$.

Alternative:

$(gof)(x) = x$

$g'[f(x)]\, f'(x) = 1$, when $f(x) = 2$ then $x = 1$

$g'(2) \cdot f'(1) = 1 \qquad$ (but $f'(1) = 8$)

$\therefore \quad g'(2) = 1/8$.

Example 10: Let $f(x) = e^{x^3+x^2+x}$ for any real number x, and let g be the inverse function for f. Find the value of $g'(e^3)$.

Solution: $y = e^{x^3+x^2+x}$

$\dfrac{dy}{dx} = e^{x^3+x^2+x} \cdot (3x^2 + 2x + 1)$

$g'(y) = \dfrac{dx}{dy} = \dfrac{1}{e^{x^3+x^2+x} \cdot (3x^2 + 2x + 1)}$

when $y = e^3$ then $e^3 = e^{x^3+x^2+x}$.

$\Rightarrow \quad x^3 + x^2 + x - 3 = 0$

$\Rightarrow \quad (x - 1)(x^2 + 2x + 3) = 0 \Rightarrow x = 1$

$\therefore \quad g'(e^3) = \dfrac{1}{6e^3}$.

Example 11: If g is an inverse function of f and $f'(x) = \dfrac{1}{1+x^n}$, prove that $g'(x) = 1 + [g(x)]^n$.

Solution: f and g are the inverse of each other.

Hence $(gof)(x) = (fog)(x) = x \qquad$ ...(1)

$f'[g(x)] \,.\, g'(x) = 1$

But $\quad f'(x) = \dfrac{1}{1+x^n}$

$f'[g(x)] = \dfrac{1}{1+[g(x)]^n}$

$\therefore \quad$ From (1) $\dfrac{1}{1+[g(x)]^n}\, g'(x) = 1$

$g'(x) = 1 + [g(x)]^n$.

Example 12: If the function

$$f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

and $g(x) = f^{-1}(x)$, then find the value of $g'\left(\dfrac{-7}{6}\right)$.

Solution: $\dfrac{dy}{dx} = \dfrac{4}{2}e^{\frac{1-x}{2}} + 1 + x + x^2$

$$= 2e^{\left(\frac{1-x}{2}\right)} + x^2 + x + 1$$

$$g'(y) = \frac{dx}{dy} = \frac{1}{2e^{-\left(\frac{x-1}{2}\right)} + x^2 + x + 1}$$

when $y = -\frac{7}{6}$ we have x = 1 by trial.

$$g'\left(-\frac{7}{6}\right) = \frac{1}{2+3} = \frac{1}{5}.$$

Alternative:

We have $(gof)(x) = x$

$$g'(f(x))\ f'(x) = 1$$

When $f(x) = -\frac{7}{6}$, $x = 1$.

$$\Rightarrow \quad g'\left(-\frac{7}{6}\right) f'(1) = 1$$

Hence, $g'\left(-\frac{7}{6}\right) = \frac{1}{f'(1)} = \frac{1}{5}$.

Concept Problems E

1. If $y = \sec^{-1}(2x)$ then find $\left.\frac{dy}{dx}\right]_{x=-1}$.

2. Prove that if $y = \cos^{-1}\frac{9-x^2}{9+x^2}$, then

$$y' = \frac{6x}{(x^2+9)|x|}.$$

3. If $y = \sin^{-1}\sqrt{1-x^2}$ find $\frac{dy}{dx}$.

4. Differentiate the following functions:

(i) $y = \sec^{-1}(x^2)$ (ii) $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

5. If $t = \sin^{-1} 2^s$ find the expression for $\frac{ds}{dt}$ in terms of s, in terms of t.

6. Verify the validity of the relationship $\frac{dy}{dx}\cdot\frac{dx}{dy} = 1$ if x and y are related in the following way:

(i) $y = x^2 + ax + b$ (ii) $y = x^{-n}$

(iii) $y = \ln(x^2 - 1)$

7. If $f(x) = x^3 + 2x^2 + 3x + 4$ and g(x) is the inverse of f(x) then find $g'(4)$.

8. Let f and g are the inverse of each other. If $f(x) = e^x - e^{-x}$ then find the value of $g'(2)$.

9. Find $\frac{dy}{dx}$ if y is equal to:

$$\cos^{-1}\left\{\frac{5x + 12\sqrt{1-x^2}}{13}\right\}, \text{ where } 0 < x < \frac{1}{4}$$

10. If $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, find $\frac{dy}{dx}$.

Practice Problems D

11. Find the derivative of the following functions:

(i) $y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$

(ii) $y = \sin^{-1}\frac{x^2-1}{x^2+1} + \sec^{-1}\frac{x^2+1}{x^2-1}$, $|x| > 1$

(iii) $y = e^{(\tan^{-1}x)^3}$

(iv) $y = \sqrt{(1-x^2)}\ \sin^{-1}x - x$

12. Find the derivative of $\tan^{-1}\frac{\sqrt{x+x^2}}{1-x^{8/9}}$.

13. Let $f(x) = e^x + x$. Then find the value of $\frac{df^{-1}(x)}{dx}$ at $x = \ell n(2e^2)$.

14. If $f(x) = x^5 + 2x^3 + 2x$ and g is the inverse of f then find $g'(-5)$.

15. If $y = e^{\sin^{-1}x}$ and $u = \ln x$, then find $\frac{dy}{du}$.

16. If $u = \sin(m\cos^{-1}x)$ and $v = \cos(m\sin^{-1}x)$ then prove that $\frac{du}{dv} = \sqrt{\frac{1-u^2}{1-v^2}}$.

17. Let $y = \frac{1-x^4}{1+x^4}$. Express $\frac{dx}{dy}$ in terms of x, in terms of y.

Show that the relation $\frac{dy}{dx}\cdot\frac{dx}{dy} = 1$ is valid.

18. Find the derivative of $y = \frac{\sin^{-1}x\cos^{-1}x}{\tan^{-1}x}$.

19. Find $\frac{dy}{dx}$ if $x = \tan\frac{y}{2} - \ln\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$.

4.8 Parametric Differentiation

Derivative of a Parametric Function

Let a function y of x be represented by the parametric equations

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad \text{...(1)}$$

Let us assume that these functions have derivatives and that the function $x = \phi(t)$ has an inverse, $t = \sigma(x)$, which also has a derivative. Then the function $y = f(x)$ defined by the parametric equations may be regarded as a composite function:

$$y = \psi(t),\ t = \sigma(x)$$

t being the intermediate argument (or parameter).

By the rule for differentiating a composite function we get

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{d\psi}{dt}\frac{d\sigma}{dx} \qquad \text{...(2)}$$

From the theorem for the differentiation of an inverse function, it follows that

$$\frac{d\sigma}{dx} = \frac{1}{\frac{d\phi}{dt}}$$

Putting this expression into (2), we have

$$\frac{dy}{dx} = \frac{\frac{d\psi}{dt}}{\frac{d\phi}{dt}}$$

The derived formula permits finding the derivative dy/dx of a function represented parametrically without having to find y as a function of x.

Example 1: The function y of x is given by the parametric equations $\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\}\ (0 \le t \le \pi)$.

Find $\frac{dy}{dx}$ (i) for any value of t, (ii) for $t = \frac{\pi}{4}$.

Solution:

(i) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(a\sin t)'}{(a\cos t)'} = \frac{a\cos t}{-a\sin t} = -\cot t.$$

(ii) $$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1.$$

Example 2: Find the slope of the tangent to the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ at an arbitrary point $(-\pi \le t \le \pi)$.

Solution: The slope of the tangent at each point is equal to $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}$

$$= \cot\frac{t}{2} = \tan\left(\frac{\pi}{2} - \frac{t}{2}\right).$$

Hence, the slope of the tangent to a cycloid at every point is equal to $\tan\left(\frac{\pi}{2} - \frac{t}{2}\right)$, where t is the value of the parameter corresponding to this point. But this means that the angle α of inclination of the tangent to the x-axis is equal to $\frac{\pi}{2} - \frac{t}{2}$ (for values of t lying between $-\pi$ and π).

Derivative of a Function w.r.t. another Function

Let $y = f(x)$ and $z = g(x)$

then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

For example, to differentiate sin x w.r.t. tan x, we write $y = \sin x$ and $z = \tan x$.

Now, $$\frac{dy}{dz} = \frac{d\sin x}{d\tan x} = \frac{\frac{d\sin x}{dx}}{\frac{d\tan x}{dx}} = \frac{\cos x}{\sec^2 x} = \cos^3 x.$$

Example 3: Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1} x$ for $-1 < x < 1$.

Solution:

$$\frac{\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)}{\frac{d}{dx}\tan^{-1}(x)} = \frac{\frac{d}{dx}\left(2\tan^{-1}x\right)}{\frac{d}{dx}\left(\tan^{-1}x\right)}$$

$$= \frac{2\frac{d}{dx}(\tan^{-1}x)}{\frac{d}{dx}\left(\tan^{-1}x\right)} = 2$$

Example 4: Differentiate $\ln\tan x$ with respect to $\sin^{-1}(e^x)$.

Solution: $$\frac{d\left(\ln\tan x\right)}{d\left(\sin^{-1}(e^x)\right)} = \frac{\frac{d}{dx}\left(\ln\tan x\right)}{\frac{d}{dx}\sin^{-1}\left(e^x\right)} = \frac{\cot x\sec^2 x}{e^x.\frac{1}{\sqrt{1-e^{2x}}}}$$

$$= \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x.\cos x}.$$

Example 5: Find the derivative of $f(\sin^{-1}x)$ w.r.t. $g(x^2+x)$ at $x = 0$ if $f'(0) = 2g'(0)$.

Solution: We have

$$\frac{d}{dx}\{f(\sin^{-1}x)\} = f'(\sin^{-1}x).\frac{1}{\sqrt{1-x^2}}$$

and $\frac{d}{dx}\{g(x^2+x)\} = g'(x^2+x).(2x+1)$.

$$\Rightarrow \frac{d\{f(\sin^{-1}x)\}}{d\{g(x^2+x)\}} = \frac{f'(\sin^{-1}x)}{g'(x^2+x)}.\frac{1}{(2x+1)\sqrt{1-x^2}}$$

Hence, the value of the derivative at $x = 0$ is

$$= \frac{f'(0)}{g'(0)}.\frac{1}{2} = 1. \qquad [\because f'(0) = 2g'(0)]$$

Concept Problems F

1. Find $\frac{dy}{dx}$ when
   (i) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$
   (ii) $x = a\left(\frac{1-t^2}{1+t^2}\right),\quad y = b.\left(\frac{2t}{1+t^2}\right)$
   (iii) $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$
2. Find $\frac{dy}{dx}$ if
   (i) $x = 3\cos\theta - \cos^3\theta,\ y = 3\sin\theta - \sin^3\theta$
   (ii) $x = a(\cos t + \ln\tan\frac{t}{2}),\ y = a\sin t$
3. If $x = e^{-t^2}$ and $y = \tan^{-1}(2t+1)$, find $\frac{dy}{dx}$
4. If $x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$, find $\frac{dy}{dx}$ at $t = \frac{1}{2}$.
5. Find derivative of $y = \ell n\, x$ with respect to $z = e^x$.
6. Differentiate $\sin^3 x + \cos^3 x$ with respect to $\sin x + \cos x$

Practice Problems E

7. If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$ then show that $\frac{dy}{dx} = \left(-\frac{y}{x}\right)$.
8. If $x = a\sqrt{\cos 2t}\,\cos t$ and $y = a\sqrt{\cos 2t}\,\sin t$ then, find $\left.\frac{dy}{dx}\right|_{t=\pi/6}$.
9. Prove that the function represented parametrically by the equations $x = 2t + 3t^2$, $y = t^2 + 2t^3$ satisfies the relation $y = y'^2 + 2y'^3$.
10. Prove that the function represented by the equations $x = \frac{1+\ln t}{t^2},\ y = \frac{3+2\ln t}{t}$ satisfies $yy' = 2xy'^2 + 1$.
11. If $x^2 - y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$ then prove that $x^3y\frac{dy}{dx} + 1 = 0$.
12. Find the derivative of $\sqrt{\sin(1+x^2)^2}$ w.r.t. $1+x^2$.
13. Show that the differential coefficient of $e^{\sin^{-1}x}$ w.r.t. $e^{-\cos^{-1}x}$ is independent of x.

4.9 Differentiation of Implicit Functions

Implicit Relation of x and y

So far we have been concerned with the case in which y is expressed explicitly, i.e., directly in terms of x.

There are cases in which y is not expressed directly in terms of x, but its functionality is implied by an algebraic relation connecting x and y.

For example, the equation

$$x^2 + y^2 - a^2 = 0 \tag{1}$$

defines implicitly the following elementary functions

$$y = \sqrt{a^2 - x^2} \tag{2}$$

$$y = -\sqrt{a^2 - x^2} \tag{3}$$

But not every implicitly defined function may be represented explicitly, that is, in the form $y = f(x)$, where $f(x)$ is an elementary function.

For instance, functions defined by the equations

$$y^6 - y - x^2 = 0$$

$$y - x - \frac{1}{4}\sin y = 0$$

are not expressible in terms of elementary functions; that is these equations cannot be solved for y.

Observe that the terms "explicit function" and "implicit function" do not characterize the nature of the function but merely the way it is defined. Every explicit function $y = f(x)$ may also be represented as an implicit function $y - f(x) = 0$.

Assume that the equation $F(x, y) = 0$ specifies y as an implicit function of x. In what follows, we shall consider this function to be differentiable.

Differentiating both sides of the equation $F(x, y) = 0$ with respect to x, we obtain an equation in y'. This equation easily yields y', that is, the derivative of the implicit function.

Suppose we have an implicit relation: $x^3 + y^3 = 3axy$.

To obtain $\frac{dy}{dx}$, we differentiate both sides of the equation w.r.t x.

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right),$$

i.e. $$3(x^2 - ay) + 3(y^2 - ax)\frac{dy}{dx} = 0, \qquad ...(1)$$

giving $$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}.$$

Example 1: Find $\frac{dy}{dx}$ if $x^3 + \ln y - x^2e^y = 0$.

Solution: Differentiating both sides of the equation with respect to x, we obtain

$$3x^2 + \frac{y'}{y} - x^2e^yy' - 2xe^y = 0,$$

i.e. $$y' = \frac{(2xye^y - 3x^2)y}{1 - x^2ye^y}.$$

Example 2: If $x^3 + y^2 = \sin(x + y)$, find $\frac{dy}{dx}$

Solution: Given, $x^3 + y^3 = \sin(x + y)$ differentiating w. r. t. x, we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}\{\sin(x + y)\}$$

or $$\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx}.\frac{dy}{dx} = \frac{d\sin(x+y)}{d(x+y)}.\frac{d(x+y)}{dx}$$

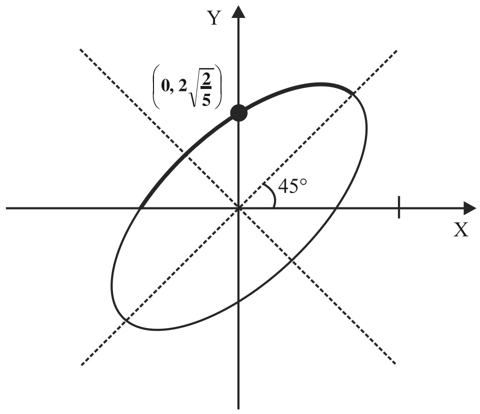
or $$3x^2 + 3y^2\frac{dy}{dx} = \cos(x + y).\left(1 + \frac{dy}{dx}\right)$$

or $$[3y^2 - \cos(x + y)]\frac{dy}{dx} = \cos(x + y) - 3x^2$$

$$\therefore \frac{dy}{dx} = \frac{\cos(x+y) - 3x^2}{3y^2 - \cos(x+y)}.$$

Let us have a deeper understanding of the method.

The set of all points (x,y) that satisfy the equation $5x^2 - 6xy + 5y^2 = 8$ lies on the ellipse shown in the figure. We now find the slope of the line tangent to the ellipse at $\left(0, 2\sqrt{\frac{2}{5}}\right)$.



It is clear from the figure that the set of all pairs (x, y) on the ellipse for which $y > 0$ and $y > x$ (drawn with a heavy curve in the figure) is a differentiable function f(x). This function is implicitly defined by

$$5x^2 - 6xf(x) + 5(f(x))^2 - 8 = 0$$

for every x in the domain of f. Since this is an equality between two functions we obtain by differentiating both sides,

$$10x - 6f(x) - 6xf'(x) + 10f(x)f'(x) = 0$$

Solving for $f'(x)$, we get

$$f'(x) = \frac{3f(x) - 5x}{5f(x) - 3x}.$$

This problem deals with an implicitly defined function whose graph passes through the point $\left(0, 2\sqrt{\frac{2}{5}}\right)$. Hence $f(0) = 2\sqrt{\frac{2}{5}}$ and therefore $f'(0) = \frac{3}{5}$, which is the slope of the desired tangent.

The definition, which we have so far illustrated with equation, is the following :

A function f(x) is defined implicitly by an equation $F(x,y) = 0$, if $F(x, f(x)) = 0$ for every x in the domain of f.

We emphasize that, in general, an equation in x and y defines y as a function of x in many ways. The most we can hope for in the way of uniqueness is that, for a given point (a, b) such that $F(a, b) = 0$, we can choose an open interval containing a which is the domain of precisely one continuous function f(x) defined implicitly by $F(x, y) = 0$ with $f(a) = b$.

Note that in the example the derivative f' of the implicitly defined function was computed without solving the original equation for f. The fact that this is always possible is almost too good to be true – especially for an equation where first solving for y in terms of x is either impractical or even impossible (except by numerical techniques). This method of finding the derivative of an implicitly defined function by differentiating both sides of the equation that defines the function is called implicit differentiation.

Caution

It is important to realize that implicit differentiation is a technique for finding dy/dx that is valid only if y is a differentiable function of x, and careless application of the technique can lead to errors. For example, there is clearly no real-valued function

$y = f(x)$ that satisfies the equation $x^2 + y^2 = -1$, yet formal application of implicit differentiation yields the derivative: $dy/dx = -x/y$.

To be able to evaluate this "derivative," we must find some values for which $x^2 + y^2 = -1$. Because no such values exist, the derivative does not exist.

We now prove that $\frac{d}{dx}x^a = ax^{a-1}$, for any real number a.

Proof In this proof we are trying to remove the restriction that a be rational. Observe first that x^a is certainly a differentiable function, since it is the composition of differentiable functions:

$$x^a = e^{a \ln x}$$

Knowing this, we use implicit differentiation to compute its derivative.

Let $y = x^a$. Then $\ln y = \ln x^a = a \ln x$, and so

$$\frac{d}{dx}\ln y = \frac{d}{dx}a \ln x,$$

$$\frac{1}{y}\frac{dy}{dx} = a\frac{1}{x}, \quad \frac{dy}{dx} = \frac{ay}{x}.$$

Since $y = x^a$, it follows that $\frac{ay}{x} = \frac{ax^a}{x} = ax^{a-1}$.

Thus, we have proved that

$$\frac{d}{dx}x^a = ax^{a-1}, \text{ for any number a.}$$

Example 3: Find $\frac{dy}{dx}$ if $xy - x = 1$.

Solution: We differentiate both sides of $xy - x = 1$.

$$\frac{d}{dx}(xy - x) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(xy) - \frac{d}{dx}(x) = 0. \quad \text{...(1)}$$

By the product rule,

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + \frac{dx}{dx}y = x\frac{dy}{dx} + y.$$

Hence, substituting into (1) we obtain

$$x\frac{dy}{dx} + y - 1 = 0$$

$$\frac{dy}{dx} = \frac{1-y}{x}. \quad \text{...(2)}$$

Remark We observe that the formula for the derivative involves both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x and y coordinates of the point of tangency are known.

If we had explicitly solved for y (which is not always possible), we would have obtained $y = 1 + \frac{1}{x}$

Then, $\frac{dy}{dx} = -\frac{1}{x^2}$.

To see that this is exactly the result obtained by implicit method, note that if $1 + 1/x$ is substituted for y in (2), we get

$$\frac{dy}{dx} = \frac{1-y}{x} = \frac{1-[1+(1/x)]}{x} = -\frac{1}{x^2}.$$

However, it may not always be possible to express the result only in terms of one variable which is either x or y.

It is sometimes possible to simplify the expression of the derivative of an implicit function by using the original equation connecting y and x.

For instance, let us take the implicit function specified by the equation $xy^2 = e^{x+y}$.

The differentiation with respect to x leads to

$y^2 + 2xyy' = e^{x+y}(1+y')$ which implies

$y' = \frac{e^{x+y} - y^2}{2xy - e^{x+y}}$. Now, replacing e^{x+y} by xy^2 we simplify the expression of the derivative as

$$y' = \frac{y(x-1)}{x(2-y)}.$$

In this case, it is difficult to express the derivative solely in terms of the independent variable x. It follows that to find the value of the derivative of an implicit function for a given value of the argument x, one also has to know the value of the function y for the given value of x.

Thus, the derivative of any implicit function specified by an equation involving elementary functions can be determined according to the known differentiation rules irrespective of whether it is possible to represent the function explicitly. In the general case the derivative of such a function is expressed in terms of the independent variable and the function itself.

Steps for Implicit Differentiation

(i) Differentiate both sides of the equation with respect to x.

(ii) Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.

(iii) Factor dy/dx out of the left side of the equation.

(iv) Solve for dy/dx by dividing both sides of the equation by the left hand factor that does not contain dy/dx.

Example 4: Find dy/dx given that

$$y^3 + y^2 - 5y - x^2 = -4.4$$

Solution:

(i) Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(y^3 + y^2 - 5y - x^2) = \frac{d}{dx}(-4)$$

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(y^2) - \frac{d}{dx}(5y) - \frac{d}{dx}(x^2) = \frac{d}{dx}(-4)$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

(ii) Collect the dy/dx terms on the left side of the equation.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

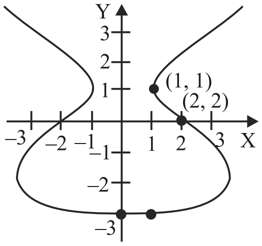
(iii) Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

(iv) Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

The curve represented by the given equation is shown below. We have also tabulated the slope of the graph at several points on the curve.



Point on Graph	Slope of Graph
(2, 0)	$-\frac{4}{5}$
(1, –3)	$\frac{1}{8}$
x = 0	0
(1, 1)	Undefined

Example 5: The point (2, 1) lies on the curve defined by the equation $x^2y + xy^3 = 10$. Assuming that this equation implicitly defines a differentiable function f(x) whose graph passes through (2, 1), compute $f'(2)$.

Solution: Letting y stand for f(x), we obtain by implicit differentiation

$$3x^2y + x^3\frac{dy}{dx} + y^3 + x^3y^2\frac{dy}{dx} = 0$$

Hence $\frac{dy}{dx} = -\frac{3x^2y + y^3}{3xy^2 + x^3}$

At the point x = 2, y = 1, we therefore get

$$\left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{13}{14}$$

Partial Differentiation

It is perceived while differentiating $x^3 + y^3 = 3axy$ that the expressions $3(x^2 - ay)$ and $3(y^2 - ax)$ occurring are algebraically the same as would be given by differentiating the expression $x^3 + y^3 - 3axy$ first with regard to x, keeping y a constant, and second with regard to y, keeping x a constant.

When such processes are applied to a function f(x, y) of two or more variables the results are denoted by the symbols $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Thus in the above example

$$\frac{\partial f}{\partial x} = 3(x^2 - ay) \text{ and } \frac{\partial f}{\partial y} = 3(y^2 - ax)$$

This is termed partial differentiation, and the results are called partial differential coefficients.

It appears that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0,$$

or $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

This proposition is true for all implicit relations between two variables, such as f(x, y) = 0.

A Direct Formula for Implicit Functions

Take all the terms to left side and call left side equal to f(x,y). Let f(x,y) = 0. Then

$$\frac{dy}{dx} = -\frac{\text{differentiate w.r.t. x treating y as constant}}{\text{differentiate w.r.t. x treating x as constant}}$$

For example, if $f(x,y) \equiv x^5 + x^4y + y^3 = 0$,

we have $$\frac{\partial f}{\partial x} = 5x^4 + 4x^3y,$$

$$\frac{\partial f}{\partial y} = x^4 + 3y^2;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{5x^4 + 4x^3y}{x^4 + 3y^2}.$$

Example 6: Find dy/dx if $2x^2 + xy - 3y^2 = x$.

Solution: We have $2x^2 + xy - 3y^2 - x = 0$ ...(1)

Applying the formula $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$, we get

$$\frac{dy}{dx} = -\frac{4x+y-1}{x-6y}$$

$$\frac{dy}{dx} = \frac{1-4x-y}{x-6y}. \qquad ...(2)$$

Caution

When using the result (2), keep in mind that the only values of x and y that can be substituted into the right hand side of result (2) are those values that satisfy the original condition (1).

For instance, we might substitute x = 1, y = 2 to obtain (dy/dx) = (1 – 4 – 2)/(1 – 12) = 5/11.

However, (1, 2) is not a point on the curve $2x^2 + xy - 3y^2 = x$, so the calculation of dy/dx at this point is totally meaningless.

The equation (1) can be written as

$$-3y^2 + xy + (2x^2 - x) = 0$$

which is a quadratic equation in y.

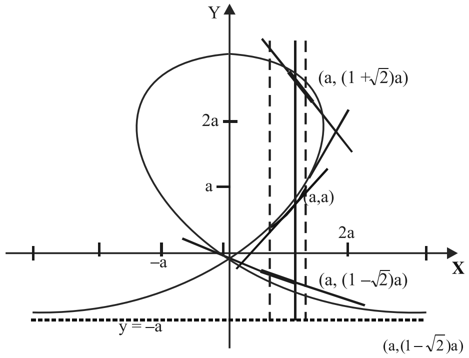
Hence, we could use the quadratic formula to solve this equation for y in terms of x, obtaining

$$y = \frac{-x \pm \sqrt{25x^2 - 12x}}{-6}.$$

Though we are able to find y explicitly in terms of x, the resulting expression is fairly complex, and it still might be best to find dy/dx implicitly as found above.

Example 7: The equation $y^3 + yx^2 + ax^2 - 3ay^2 = 0$ represents the curve, called a trisectrix, as shown in the figure.

Find $\frac{dy}{dx}$ when x = a.



Solution: So stated, the problem is typical. There are three distinct points on the curve with x-coordinate equal to a, which may be found by substituting a = x in equation and then solving for y. The points are (a, a,), (a, (1 + $\sqrt{2}$) a), and (a, (1 – $\sqrt{2}$) a).

As shown in the figure, we may select a small interval about a to serve as the domain of three different implicitly defined function. To find the derivative of each one at x = a, we proceed by implicit differentiation:

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy + 2ax - 6ay\frac{dy}{dx} = 0.$$

Hence, $$\frac{dy}{dx} = \frac{2xy + 2ax}{6ay - x^2 - 3y^2}$$

Thus the derivatives of the three differentiable functions defined implicitly at a are:

$$\left.\frac{dy}{dx}\right|_{(a,a)} = 2,$$

$$\left.\frac{dy}{dx}\right|_{(a,a(1+\sqrt{2}))} = -1 - \frac{\sqrt{2}}{2},$$

$$\left.\frac{dy}{dx}\right|_{(a,a(1-\sqrt{2}))} = -1 + \frac{\sqrt{2}}{2}.$$

Note: In each of the examples of implicit differentiation the existence of an implicitly defined differentiable function has either been assumed outright or justified geometrically from the picture.

Example 8: If $x^3y^2 = \ln(x + y) + \sin(e^x)$, find $\frac{dy}{dx}$.

Solution: $x^3y^2 = \ln(x + y) + \sin(e^x)$ ...(1)

Differentiating w.r.t. x, we get

$$3x^2y^2 + x^2 \cdot 2y\frac{dy}{dx} = \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) + \cos(e^x) \cdot e^x$$

or, $$3x^2y^2 + 2x^2y\frac{dy}{dx} = \frac{1}{x+y} + \frac{1}{x+y}\cdot\frac{dy}{dx} + e^x\cos(e^x)$$

or, $$\left[2x^2y - \frac{1}{x+y}\right]\frac{dy}{dx}$$

$$= \frac{1 + (x+y)e^x\cos(e^x) - 3x^2y^2.(x+y)}{x+y}$$

or, $$\frac{dy}{dx} = \frac{1 + (x+y)e^x\cos(e^x) - 3x^2y^2 - 3x^2y^2}{2x^4y + 2x^3y^2 - 1}.$$

Example 9: If $ax^2 + 2hxy + by^2 = 0$ then prove that

$$\frac{dy}{dx} = -\frac{ax+hy}{hx+by} = \frac{y}{x}.$$

Solution: $2ax + 2by\,y' + 2h(xy' + y) = 0$

$$(ax + hy) + (by + hx)y' = 0$$

$$y' = -\frac{ax+hy}{by+hx} \qquad ...(1)$$

$$ax^2 + hxy + hxy + by^2 = 0$$

$$x(ax + hy) + y(by + hx) = 0$$

$$\therefore \quad -\frac{ax+hy}{hx+by} = \frac{y}{x} \qquad \text{...(2)}$$

Now from (1) and (2), $\frac{dy}{dx} = -\frac{ax+hy}{hx+by} = \frac{y}{x}$.

Example 10: Find $\left(\frac{dy}{dx}\right)$ if $\sin y = \log_{\sin x} \cos x$.

Solution: We have $\sin y = \log_{\sin x} \cos x = \frac{\log_e \cos x}{\log_e \sin x}$.

Differentiating both sides w.r.t. x, we have

$$(\cos y).\frac{dy}{dx} = \frac{(-\tan x).\ln \sin x - (\cot x).\ln \cos x}{(\ln \sin x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(\tan x)\ln \sin x + (\cot x)\ln \cos x}{(\cos y)(\ln \sin x)^2}.$$

Example 11: If $\sin y = x \sin(a + y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Solution: Given, $\sin y = x \sin(a + y)$ ...(1)
Differentiating w.r.t. x, we get

$$\cos y = \frac{dy}{dx} = 1 . \sin(a+y) + x.\cos(x+y)\left(0 + \frac{dy}{dx}\right)$$

$$= \sin(a = y) + x \cos(a+y)\frac{dy}{dx}$$

or $\quad [\cos y - x \cos(a+y)]\frac{dy}{dx} = \sin(a+y)$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)} \qquad \text{...(2)}$$

[Here we have to eliminate x]
Putting the value of x from (1), in (2), we get

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$= \frac{\sin(a+y)}{\frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\sin(a+y)}}$$

$$= \frac{\sin^2(a+y)}{\sin(a+y-y)} = \frac{\sin^2(a+y)}{\sin a}.$$

Example 12: If S_n denotes the sum of n terms of a G.P. whose common ratio is r, prove that

$$(r-1)\frac{dS_n}{dr} = (n-1)S_n - nS_{n-1}.$$

Solution: $S_n = \frac{a(r^n - 1)}{r-1}$ or, $(r-1)S_n = ar^n - a$

Differentiating both sides w.r.t. x, we get,

$$(r-1)\frac{dS_n}{dr} + S_n = nar^{n-1}$$

$$\Rightarrow \quad (r-1)\frac{dS_n}{dr} = nar^{n-1} - S_n$$

$$= n.[\text{nth term of G.P.}] - S_n$$

$$= n(S_n - S_{n-1}) - S_n$$

$$= (n-1)S_n - nS_{n-1}.$$

Example 13: If $x^y = e^{x-y}$, then find $\frac{dy}{dx}$.

Solution: Taking log on both sides

$$y\,\ell n\,x = (x - y)$$

Differentiating w.r.t x, we get

$$\frac{y}{x} + \ln x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \ell n\,x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x-y}{x(1+\ell n\,x)}.$$

Example 14: If $x^y + y^x = 2$ then find $\frac{dy}{dx}$.

Solution: Here $u + v = 2$, where $u = x^y$ and $v = y^x$

$$\Rightarrow \quad \frac{du}{dx} + \frac{dv}{dx} = 0, \qquad \text{...(1)}$$

Now $u = x^y$

$$\Rightarrow \quad \ell n\,u = y\,\ell n\,x$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell n\,x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell n\,x\frac{dy}{dx}\right) \qquad \text{...(2)}$$

For $\quad v = y^x$

$$\ell n\,v = x\,\ell n\,y$$

$$\frac{1}{v}\frac{dv}{dx} = \ell n\,y + \frac{x}{y}\frac{dy}{dx}$$

$$\frac{dv}{dx} = y^x\left(\ell n\,y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \text{...(3)}$$

Putting (2 and (3) in (1), we get

$$\Rightarrow \quad x^y\left(\frac{y}{x} + \ell n\,x\frac{dy}{dx}\right) + y^x\left(\ell n\,y + \frac{x}{y}\frac{dy}{dx}\right) = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\left(y^x \ell n\,y + x^y.\frac{y}{x}\right)}{\left(x^y \ell n\,x + y^x.\frac{x}{y}\right)}.$$

Example 15: If $x\sqrt{(x+y)} + y\sqrt{(1+x)} = 0$, then show that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Solution: We have $x\sqrt{(1+y)} + y\sqrt{(1+x)} = 0$ ...(1)

$\Rightarrow \quad x\sqrt{(1+y)} = -y\sqrt{(1+y)}$

On squaring both sides

$$x^2(1+y) = y^2(1+x)$$

$\Rightarrow \quad x^2 - y^2 + x^2y - xy^2 = 0$

$\Rightarrow \quad (x-y)(x+y+xy) = 0$

$x - y = 0$ gives two points (0, 0) and (–1, –1) only.

We cannot find derivative here.

$\therefore \quad x + y + xy = 0$ ...(1)

$\Rightarrow \quad y = -\dfrac{x}{(1+x)}$ ...(2)

$$\therefore \quad \frac{dy}{dx} = -\left\{\frac{(1+x).1 - x.1}{(1+x)^3}\right\} = -\frac{1}{(1+x)^2}.$$

Alternatively, differentiating both sides of (1) w.r.t.x, we get,

$$y.1 + (1+x)\frac{dy}{dx} = -1 \text{ or, } (1+x)\frac{dy}{dx} = -(y+1)$$

or, $\dfrac{dy}{dx} = -\left(\dfrac{y+1}{1+x}\right) = -\dfrac{\left(\dfrac{-x}{1+x}+1\right)}{1+x}$ [from (2)]

$$= -\frac{1}{(1+x)^2}.$$

Derivative of Functions Represented by Infinite Series

Let $y = \sqrt{\phi(x) + \sqrt{\phi(x) + \sqrt{\phi(x) + \ldots \infty}}}$ then

$y = \sqrt{\phi(x) + y}$, since by taking out one term from an infinite series, it remains unchanged.

$\Rightarrow \quad (y^2 - y) = \phi(x)$

Differentiating both sides w.r.t x, we get

$$(2y-1)\frac{dy}{dx} = \phi'(x)$$

Hence, $\dfrac{dy}{dx} = \dfrac{\phi'(x)}{2y-1}$

Now if $\phi(x) = \sin x$, $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$

then $(2y-1)\dfrac{dy}{dx} = \cos x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y-1}.$$

Example 16: If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\infty}}}$

then prove that $\dfrac{dy}{dx} = \dfrac{y}{2y-x}$.

Solution: We have $x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\infty}}}$

$$y = x + \frac{1}{y} \Rightarrow y^2 = xy + 1$$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = x\frac{dy}{dx} + y \,.\, 1$$

$$\Rightarrow \quad (2y - x)\frac{dy}{dx} = y \qquad \therefore \quad \frac{dy}{dx} = \frac{y}{2y-x}$$

Example 17: Differentiate $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{\ldots\infty}}}$.

Solution: Since $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{\ldots\infty}}}$, $y = \{f(x)\}^y$

$\Rightarrow \quad y = e^{y \ln f(x)}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{y \ln f(x)}\left\{y \cdot \frac{1}{f(x)} \cdot f'(x) + \ln f(x) \cdot \frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \{f(x)\}^y\left\{\frac{y}{f(x)} \cdot f'(x) + \ln f(x) \cdot \frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \{1 - \{f(x)\}^y \ln f(x)\}\frac{dy}{dx} = y\,\{f(x)\}^{y-1} \cdot f'(x)$$

Hence $\dfrac{dy}{dx} = \dfrac{y\{f(x)\}^{y-1} \cdot f'(x)}{1 - \{f(x)\}^y \cdot \ln f(x)}$

$$= \frac{y^2 f'(x)}{f(x)\{1 - y \ln f(x)\}}.$$

Example 18: If $\sqrt{y - \sqrt{y - \sqrt{y - \sqrt{y \ldots\ldots\infty}}}}$

$= \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x \ldots\ldots\infty}}}}$, then prove that

$$\frac{dy}{dx} = \frac{x - y - 1}{x - y + 1}.$$

Solution: Given $\dfrac{-1 + \sqrt{1+4y}}{2} = \dfrac{1 + \sqrt{1+4x}}{2}$,

$\Rightarrow \quad \sqrt{1+4y} = 2 + \sqrt{1+4x}$ ...(1)

We differentiate both sides,

$$\frac{4}{2\sqrt{1+4y}}\frac{dy}{dx} = \frac{1 \cdot 4}{2\sqrt{1+4x}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1+4y}}{\sqrt{1+4x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2+\sqrt{1+4x}}{\sqrt{1+4x}} \qquad ...(2)$$

From (1), $\quad 1 + 4y = 4 + (1 + 4x) + 4\sqrt{1+4x}$

$\therefore \quad y = 1 + x + \sqrt{1+4x}$

$\therefore \quad \sqrt{1+4x} = y - x - 1$

From (2), $\quad \dfrac{dy}{dx} = \dfrac{2+(y-x-1)}{y-x-1} = \dfrac{1+y-x}{y-x-1}$

$$= \frac{x-y-1}{x-y+1}.$$

Example 19: If $y = \dfrac{\sin x}{1+}\dfrac{\cos x}{1+}\dfrac{\sin x}{1+}\dfrac{\cos x}{1+}........\infty$ then find $y'(0)$.

Solution: $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+.........}}}$

$\Rightarrow \quad y = \dfrac{\sin x}{1+\dfrac{\cos x}{1+y}} = \dfrac{\sin x(1+y)}{1+y+\cos x}$, and $y(0) = 0$

$\Rightarrow \quad y(1 + y + \cos x) = \sin x\,(1 + y)$

$\Rightarrow \quad y' + 2yy' + \cos x\, y' - y\sin x = \cos x(1 + y) + \sin x y'$

$\Rightarrow \quad y'(0)\,[1 + 2y + \cos x] - 0 = 1 + 0$

$\Rightarrow \quad 2y'(0) = 1 \Rightarrow \quad y'(0) = 1/2.$

Concept Problems G

1. Find $\dfrac{dy}{dx}$ if

   (i) $y = \cos(x + y)$ (ii) $x^{2/3} + y^{2/3} = a^{2/3}$

   (iii) $x = y\,\ell n\,(x - y)$

2. If $x^y = e^{x-y}$, then prove that $\dfrac{dy}{dx} = \dfrac{\ell n\, x}{(1+\ell n\, x)^2}$.

3. If $\dfrac{x}{x-y} = \ln\dfrac{a}{x-y}$, prove that $\dfrac{dy}{dx} = 2 - \dfrac{x}{y}$.

4. The equation $x\sin xy + 2x^2 = 0$ defines y implicitly as a function of x. Assuming the derivative y' exists, show that it satisfies the equation

$$y'x^2\cos xy + xy\cos xy + \sin xy + 4x = 0.$$

5. For each of the following equations calculate $\dfrac{dy}{dx}$ at the point specified.

   (i) $y^2 = \dfrac{x^2-1}{x^2+1}$, at the point (a,b).

   (ii) $x^{3/2} + y^{3/2} = 2$, at the point (1, 1).

6. If $1 + f(x) + x^3\,[f(x)]^3 = 0$ and $f(1) = 2$, find $f'(1)$.

7. If $g(x) + x\sin g(x) = x^2$ and $g(1) = 0$, find $g'(1)$.

8. Let g be a differentiable function of x that satisfies $g(x) < 0$ and $x^2 + g^2(x) = 10$ for all x.

   (i) Use implicit differentiation to show that

   $$\frac{dg}{dx} = \frac{-x}{g(x)}.$$

   (ii) Show that $g(x) = -\sqrt{10-x^2}$ satisfies the given requirements. Then use the chain rule to verify (i).

9. (i) If $x^2 + y^2 = 6y - 10$ and $\dfrac{dy}{dx}$ exists, show that

   $$\frac{dy}{dx} = \frac{x}{3-y}.$$

   (ii) Show that there are no real numbers x,y that satisfy the equation $x^2 + y^2 = 6y - 10$

   (iii) What can you conclude from the result found in part (i) in light of this observation in part (ii)?

10. If $y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$ find y'.

11. If $y = \dfrac{x}{1+}\dfrac{x}{2+}\dfrac{x}{1+}\dfrac{x}{2+}........\infty$, find y'.

12. Given (i) $xy^3 + x^2y = 6$ (ii) $x^3 + y^2 = \sin^2 y$ find both dy/dx (treating y as a function of x) and dx/dy (treating x as a function of y). How do dy/dx and dx/dy seem to be related? Can you explain the relationship geometrically in terms of the graphs?

13. (i) Factor the left-hand side of the equation $2x^2 - 5xy + 2y^2 = 0$ to show that its graph consists of two straight lines through the origin. Hence the derivative dy/dx has only two possible numerical values (the slopes of these two lines).

    (ii) Calculate dy/dx by implicit differentiation of the equation in part (i). Verify that the expression you obtain yields the proper slope for each of the straight lines of part (i).

Practice Problems F

14. Find $\frac{dy}{dx}$ if

(i) $x^m y^n = (x + y)^{m+n}$

(ii) $x \cos y + y \sin x = \tan(x + y)$

(iii) $x \cos y + y \cos x = \tan^{-1}(x^2)$

15. If $\cos y = x \cos(b + y)$, prove that

$$\frac{dy}{dx} = \frac{\cos^2(b+y)}{\sin b}.$$

16. If $x^y . y^x = 1$, find $\frac{dy}{dx}$.

17. Find $\frac{dy}{dx}$ if $x^{\sin y} + y^{\cos x} = 1$.

18. Differentiate $x^{y^2} + y^2 \ln x - 4 = 0$.

19. Find dy/dx if $2y^3t + t^3y = 1$ and $\frac{dt}{dx} = \frac{1}{\cos t}$.

20. Differentiate $y = (\ln x)^{(\ln x)^{(\ln x)^{\cdots\infty}}}$.

21. (i) The equation $x^3 + y^3 - 6xy = 0$ implicitly defines a differentiable function f(x) whose graph passes through (3, 3). Compute $f'(3)$.

(ii) How many differentiable functions f(x) having a small interval about the number 3 as a common domain are implicitly defined by the equation in (a)?

(iii) Compute $f'(3)$ for each them.

22. Prove that the function

$$y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x + \sqrt{x^2+1}}$$

satisfies the relationship $2y = xy' + \ln y'$.

23. Let $f(x) = \sqrt[3]{\frac{x^4+1}{x^4-2}}$. Find $f'(x)$ by using implicit differentiation to differentiate $[f(x)]^3$.

24. Suppose f is a one-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0. If $f(4) = 5$ and $f'(4) = 2/3$, find $(f^{-1})'(5)$.

4.10 Differentiation by Trigonometric Substitution

In some cases, a preliminary transformation of the function to be differentiated facilitates the process of differentiation. The following substitutions are generally used to simplify expressions involving these terms.

(i) $\sqrt{x^2 + a^2}$ $x = a \tan\theta$ or $a \cot\theta$

(ii) $\sqrt{a^2 - x^2}$ $x = a \sin\theta$ or $a \cos\theta$

(iii) $\sqrt{x^2 - a^2}$ $x = a \sec\theta$ or $a \operatorname{cosec}\theta$

(iv) $\sqrt{\frac{x+a}{a-x}}$ $x = a \cos\theta$

Example 1: Differentiate $\tan^{-1}\frac{\sqrt{(1+x)} - \sqrt{(1-x)}}{\sqrt{(1+x)} + \sqrt{(1-x)}}$.

Solution: Let $x = \cos\theta$, $0 \le \theta \le \pi$

We have $\sqrt{(1+x)} = \sqrt{(1+\cos\theta)}$

$$= \left|\sqrt{2}\cos\frac{\theta}{2}\right| = \sqrt{2}\cos\frac{\theta}{2} \text{ but for } \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right),$$

$$\sqrt{(1-x)} = \sqrt{(1-\cos\theta)} = \sqrt{\left(2\sin^2\frac{\theta}{2}\right)} = \sqrt{2}\sin\frac{\theta}{2}.$$

$$\therefore \quad y = \tan^{-1}\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= \tan{-1}\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right\}$$

since $-\frac{\pi}{4} \le \frac{\pi}{4} - \frac{\theta}{2} \le \frac{\pi}{4}$

$$= \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{(1-x^2)}}.$$

Example 2: Find the derivative of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and mention the points of non-differentiability.

Also find (i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$.

Solution: Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow$ $\theta = \tan^{-1}x$

$\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases}$$

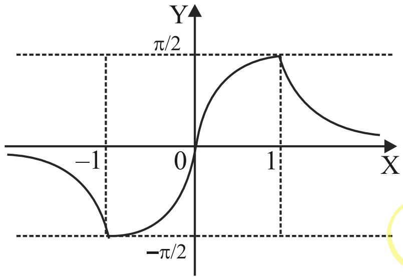
$$\Rightarrow \quad f(x) = \begin{cases} \pi - 2\tan^{-1} x & x > 1 \\ 2\tan^{-1} x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1} x) & x < -1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -\dfrac{2}{1+x^2} & x > 1 \\ \dfrac{2}{1+x^2} & -1 < x < 1 \\ \dfrac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\dfrac{2}{5}$

(ii) $f'\left(\dfrac{1}{2}\right) = \dfrac{8}{5}$

(iii) Since $f'(1^+) = -1$ and $f'(1^-) = +1$, $f'(1)$ does not exist.
f is continuous for all x but non-differentiable at $x = 1, -1$.
The graph of the function is shown below:



Alternative:

This problem can also be solved without any substitution, but in a little tedious way.

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$f'(x) = \frac{1}{\sqrt{1-\dfrac{4x^2}{(1+x^2)^2}}} \cdot \frac{2\{(1+x^2)-2x^2\}}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{(1-x^2)^2}} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f'(x) = \frac{2}{(1+x^2)} \cdot \frac{(1-x^2)}{|1-x^2|}$$

$$\text{Thus } f'(x) = \begin{cases} \dfrac{2}{1+x^2} & |x| < 1 \\ \dfrac{-2}{1+x^2} & |x| > 1 \end{cases}.$$

Example 3: Find the derivative of

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

Solution: Here, the domain is $x \in R - \{1, -1\}$

Let $x = \tan\theta$, $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) - \left\{-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right\}$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$= \begin{cases} \pi + 2\theta, & 2\theta < -\dfrac{\pi}{2} \\ 2\theta \quad , & -\dfrac{\pi}{2} < 2\theta < \dfrac{\pi}{2} \\ -\pi + 2\theta, & 2\theta > \dfrac{\pi}{2} \end{cases}$$

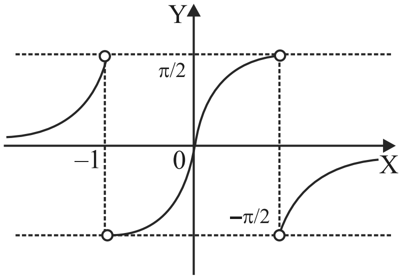
$$= \begin{cases} \pi + 2\tan^{-1} x, & \tan^{-1} x < -\dfrac{\pi}{4} \\ 2\tan^{-1} x \quad , & -\dfrac{\pi}{4} < \tan^{-1} x < \dfrac{\pi}{4} \\ -\pi + 2\tan^{-1} x, & \tan^{-1} x > \dfrac{\pi}{4} \end{cases}$$

{as $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$}

$$\text{So, } y = \begin{cases} \pi + 2\tan^{-1} x, & x < -1 \\ 2\tan^{-1} x, & -1 < x < 1 \\ -\pi + 2\tan^{-1} x, & x > 1 \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2}, & x \in R - \{-1, 1\} \\ \text{does not exist}, & x \in \{1, -1\} \end{cases}$$

The graph of the function is shown below:



Example 4: Find the derivative of $y = \cos^{-1}(4x^3 - 3x)$.

Solution: Here, the domain is $[-1, 1]$.

Let $x = \cos\theta$, $\theta \in [0, \pi]$

$$\Rightarrow \quad y = \cos^{-1}(\cos 3\theta) = \begin{cases} 2\pi - 3\theta, & \pi < 3\theta \le 2\pi \\ 3\theta \quad , & 0 \le 3\theta \le \pi \\ -2\pi + 3\theta, & 2\pi < 3\theta \le 3\pi \end{cases}$$

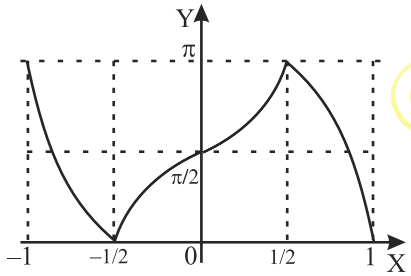
$$= \begin{cases} 2\pi - 3\cos^{-1}x, & \dfrac{\pi}{3} < \cos^{-1}x \le \dfrac{2\pi}{3} \\ 3\cos^{-1}x, & 0 \le \cos^{-1}x \le \dfrac{\pi}{3} \\ -2\pi + 3\cos^{-1}x, & \dfrac{2\pi}{3} < \cos^{-1}x \le \pi \end{cases}$$

So, $y = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 2\pi - 3\cos^{-1}x, & -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ 3\cos^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1}x, & -1 \le x < -\dfrac{1}{2} \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} -\dfrac{3}{\sqrt{1-x^2}}, & \dfrac{1}{2} < |x| < 1 \\ \text{does not exist}, & |x| = \dfrac{1}{2} \\ \dfrac{3}{\sqrt{1-x^2}}, & |x| < \dfrac{1}{2} \end{cases}$$

The graph of the function is shown below:



Example 5: If $y = \sqrt{(a-x)(x-b)}$

$-(a-b)\tan^{-1}\sqrt{\dfrac{a-x}{x-b}}$, then find $\dfrac{dy}{dx}$.

Solution: Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore$ $a - x = a - a\cos^2\theta + b\sin^2\theta = (a-b)\sin^2\theta$, and

$x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$

$\therefore$ $y = (a-b)\sin\theta\cos\theta - (a-b)\tan^{-1}\tan\theta$

$$= \frac{a-b}{2}\sin 2\theta - (a-b)\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(a-b)\cos 2\theta - (a-b)}{(b-a)\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$$

$$= \tan\theta = \sqrt{\frac{a-x}{x-b}}$$

Example 6: Find the derivative of

$\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to

$\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = -\dfrac{\sqrt{3}}{2}$.

Solution: Put $x = \tan\theta$ in the first function.

$$\theta = \tan^{-1}x,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}\right\}$$

$$\Rightarrow u = (\text{first function}) = \tan^{-1}\left(\frac{|\sec\theta| - 1}{\tan\theta}\right)$$

$$[\ |\sec\theta| = \sec\theta\ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)]$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\frac{2\sin^2(\theta/2)}{2\cos(\theta/2)\,.\,\sin(\theta/2)}$$

$$= \tan^{-1}\tan\frac{\theta}{2} \quad [\tan^{-1}(\tan x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)]$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

Put $x = \sin\phi$ in second function.

$$\phi = \sin^{-1}x,\ \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$$

and $v = (\text{second function}) = \tan^{-1}\dfrac{2\sin\phi\cos\phi}{1-2\sin^2\phi}$

$$= \tan^{-1}\tan 2\phi\ [\text{ at } x = -\frac{\sqrt{3}}{2},\ 2\phi = -\frac{2\pi}{3}]$$

$= 2\phi + \pi$, in the neighbourhood of $-\dfrac{2\pi}{3}$

$= 2\sin^{-1}x + \pi$,

$$\therefore \frac{du}{dv} = \frac{du}{dx} \Big/ \frac{dv}{dx} = \left[\frac{1}{2(1+x^2)}\right]\left(\frac{\sqrt{1-x^2}}{2}\right)$$

$$\Rightarrow \left(\frac{du}{dv}\right)\Bigg|_{x=-\sqrt{3}/2} = \frac{1}{14}.$$

Example 7: If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, prove

that $\dfrac{dy}{dx} = \dfrac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$.

Solution: Putting $x = \sin\theta$ and $y = \sin\phi$,

$$-\frac{\pi}{2} \le \theta, \phi \le \frac{\pi}{2}$$

we have,

$$\sqrt{(1-\sin^2\theta)}+\sqrt{(1-\sin^2\phi)} = a\,(\sin\theta-\sin\phi)$$

$$\Rightarrow \quad \cos\theta+\cos\phi = a\,(\sin\theta-\sin\phi)$$

$$\Rightarrow \quad 2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$$

$$= 2a\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$$

$$\therefore \quad \cot\frac{1}{2}(\theta-\phi) = a \quad \text{or} \quad \cos\frac{1}{2}(\theta+\pi) = 0$$

$$\Rightarrow \quad \frac{1}{2}(\theta-\phi) = \cot^{-1}a \quad \Rightarrow \quad \theta-\phi = 2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x-\sin^{-1}y = 2\cot^{-1}a$$

Differentiating, we get

$$\frac{1}{\sqrt{(1-x^2)}}-\frac{1}{\sqrt{(1-y^2)}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

Now, considering $\cos\frac{1}{2}(\theta+\pi) = 0$

$$\Rightarrow \quad \theta+\phi = (2n+1)\pi \quad \Rightarrow \quad \theta = \phi = \pm\pi/2$$

$$\Rightarrow \quad (x, y) = (1, 1), (-1, -1)$$

We cannot find the derivative here.

Alternative:

Using implicit differentiation on

$$\sqrt{(1-x^2)}+\sqrt{(1-y^2)} = a\,(x-y)$$

$$\frac{-x}{\sqrt{1-x^2}}-\frac{y}{\sqrt{1-y^2}}\frac{dy}{dx} = a\left(1-\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(a-\frac{y}{\sqrt{1-y^2}}\right)\frac{dy}{dx} = a+\frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a+\dfrac{x}{\sqrt{1-x^2}}}{a-\dfrac{y}{\sqrt{1-y^2}}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\dfrac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}+\dfrac{x}{\sqrt{1-x^2}}}{\dfrac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}-\dfrac{y}{\sqrt{1-y^2}}}$$

$$\frac{dy}{dx} = \frac{(1-x^2)+\sqrt{(1-x^2)(1-y^2)}+x^2-xy}{\sqrt{(1-x^2)(1-y^2)}+(1-y^2)-xy+y^2}$$

$$\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$= \frac{1+\sqrt{(1-x^2)(1-y^2)}-xy}{1+\sqrt{(1-x^2)(1-y^2)}-xy}\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$

Concept Problems H

1. If $y = \sin^{-1}\left(\dfrac{x^2}{\sqrt{x^4+a^4}}\right)$ then prove that

$$\frac{dy}{dx} = \frac{2xa^2}{x^4+a^4}.$$

2. If $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ then prove that

$$\frac{dy}{dx} = \frac{2}{1+x^2}, \quad x \neq \pm 1$$

3. If $f(x) = \sin^{-1}[2x\sqrt{1-x^2}]$ then find the value of $f'\left(\dfrac{4}{5}\right)$.

4. Find the derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t $\cos^{-1}\dfrac{1-x^2}{1+x^2}$ for $x < -1$.

5. Find the derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\sqrt{1-x^2}$ when $x = 1/2$.

Practice Problems G

6. If $y = \cos^{-1}\left(\frac{2\sqrt{x}}{1+x}\right)$ for $x \in (0, 1)$ then prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{x}\,(1+x)}$.

7. Find the derivative of $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ w.r.t. $\sin^{-1} x$.

8. Find dy/dx if $y = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$,

$x = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, where $-1 < t < 1$.

9. Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. $\tan^{-1} x$.

10. Let $f(x) = \sin^{-1} 2x\sqrt{1-x^2} + \sec^{-1}\frac{1}{\sqrt{1-x^2}}$. Find $f'\left(-\frac{1}{2}\right)$.

11. Define the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\forall\, x \in R$.

12. Differentiate $\tan^{-1}\frac{3x-x^3}{1-3x^2}$ with respect to $\tan^{-1}\frac{2x}{1-x^2}$

13. For what values of x is the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$ is -1.

14. If the differential coefficients of $\sin^{-1}(2x\sqrt{1-x^2})$ at $x = \frac{\sqrt{3}}{2}$ –, 0, $\frac{\sqrt{3}}{2}$ are a, b, c respectively, then find the value of a + b – c.

4.11 Derivatives of Higher Order

Because the derivative of a function is a function, differentiation can be applied over and over, as long as the derivative itself is a differentiable function.

The derivative $y' = dy/dx = f'(x)$ is the first (first order) derivative of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

is called the second (second order) derivative of y with respect to x.

For example, consider $y = x \sin x$

Then, $y' = f'(x) = x \cos x + \sin x$, and

$$y'' = \frac{d^2y}{dx^2} = f''(x) = -x \sin x + \cos x + \cos x$$
$$= -x \sin x + 2 \cos x.$$

If y'' is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) \text{ is the}$$

third (third order) derivative of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}\, y^{(n-1)} = f^{(n)}(x)$$

denoting the nth (nth order) derivative of y with respect to x, for any positive integer n.

Thus the several derivatives of y are written as

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n} \ldots$$

They are often further abbreviated as

$y_1, y_2, y_3, \ldots\ldots y_n \ldots\ldots$

Note:

1. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ does not mean multiplication. It means "the derivative of the derivative."

2. The expression $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ can be also abbreviated as $\left(\frac{d}{dx}\right)^2 y$.

3. Notice that for derivatives higher than the third, the parentheses distinguish a derivative from a power. For example, $f^4 \neq f^{(4)}$.

In general, we can interpret a second derivative as a rate of change of rate of change. The most familiar example of this is acceleration, which we define as follows.

If s = s(t) is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity v(t) of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the acceleration a(t) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

We can interpret the third derivative physically in the case where the function is the position function s = s(t) of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the jerk;

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus, the jerk j is the rate of change of acceleration. it is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

Rules of Higher Order Derivatives

(i) If c is a constant, then $\frac{d^2}{dx^2}(cu(x)) = c\frac{d^2}{dx^2}u(x)$

(ii) $\frac{d^2}{dx^2}(u(x) \pm v(x)) = \frac{d^2}{dx^2}u(x) \pm \frac{d^2}{dx^2}v(x)$

(iii) $\frac{d^2(u\,v)}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

(iv) $\frac{d^3(u\,v)}{dx^2} = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$

(v) If y = f(u) and u = g(x), then

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

Example 1: If $y = x^3 \ell n\, x$ then find y″and y‴.

Solution: $y = x^3 \ell n\, x$

$$y' = 3x^2 \ell n\, x + x^3 . \frac{1}{x}$$
$$y' = 3x^2 \ell n\, x + x^2$$
$$y'' = 6x\, \ell n\, x + 3x^2 . \frac{1}{x} + 2x$$
$$y'' = 6x\, \ell n\, x + 5x$$
$$y''' = 6\, \ell n\, x + 11$$

Example 2: If $y = \left(\frac{1}{x}\right)^x$ then find y″(1)

Solution: $y = \left(\frac{1}{x}\right)^x$ ...(1)

when x = 1, y = 1

Taking log on both sides of (1)

$$\ell n\, y = -x\, \ell n\, x$$

$$\Rightarrow \frac{y'}{y} = -(1 + \ell n\, x)$$

$$\Rightarrow y' = -y(1 + \ell n\, x) \qquad ...(2)$$

Again differentiating w.r.t. to x,

$$y'' = -y'(1 + \ell n\, x) - y . \frac{1}{x}$$

$$\Rightarrow y'' = y(1 + \ln x)^2 - \frac{y}{x} \qquad \text{(using (2))}$$

Now, y″(1) = 0.

Example 3: If y = A sin mx + B cos mx, prove that $y_2 + m^2 y = 0$.

Solution: We have, y = A sin mx + B cos mx ...(1)

Differentiating both sides w.r.t. 'x' we get

y_1 = Am cos mx – Bm sin mx.

Again $y_2 = -Am^2 \sin mx - Bm^2 \cos mx$

$= -m^2 (A \sin mx + B \cos mx)$

$= -m^2y$, from (1).

$\therefore\ y_2 + m^2y = 0.$

Example 4: Evaluate $\frac{d^4}{dx^4}\left[\sqrt{1+x^2}\right]$.

Solution: We have $y = \sqrt{1+x^2}$

Differentiating w.r.t. x, we have

$$y' = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$y' = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

$$y''' = -\frac{3/2}{(1+x^2)^{5/2}}.2x = \frac{-3x}{(1+x^2)^{5/2}}$$

$$y^{(iv)} = \frac{-3(1+x^2)^{5/2} + 3x\frac{5}{2}(1+x^2)^{3/2}(2x)}{(1+x^2)^5} = \frac{12x^2-3}{(1+x^2)^{7/2}}$$

Example 5: A non zero polynomial with real coefficients has the property that f (x) = f′ (x) · f″(x). Find the leading coefficient of f (x)

Solution: Degree of f (x) = n; degree of f′ (x) = n – 1

degree of f″(x) = (n – 2)

Hence n = (n – 1) + (n – 2) = 2n – 3

$\therefore\ n = 3$

Hence $f(x) = ax^3 + bx^2 + cx + d, \qquad (a \neq 0)$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$\therefore \quad ax^3 + bx^2 + cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

$\therefore \quad 18a^2 = a \quad \Rightarrow \quad a = \dfrac{1}{18}.$

Example 6: Let P(x) be a polynomial of degree 4, with $P(2) = -1$, $P'(2) = 0$, $P''(2) = 2$, $P'''(2) = -12$ and $P^{iv}(2) = 24$. Find the value of $P''(1)$.

Solution: Let us assume

$P(x) = a(x-2)^4 + b(x-2)^3 + c(x-2)^2 + d(x-2) + e$

$-1 = P(2) = e$

$0 = P'(2) = d$

$2 = P''(2) = 2c \Rightarrow c = 1.$

$-12 = P'''(2) = 6b \Rightarrow b = -2$

$24 = P^{iv}(2) = 24a \Rightarrow a = 1.$

Thus, $P''(x) = 12(x-2)^2 - 12(x-2) + 2$

$\Rightarrow \quad P''(1) = 12 - 12(-1) + 2 = 26.$

Example 7: Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)\ \forall\ x \in R$. Then prove that $f(x) = 0$ has all roots real and distinct.

Solution: We have $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$f'(x) = 3x^2 + 2x f'(1) + f''(2)$

$\Rightarrow \quad f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow \quad f'(1) + f''(2) + 3 = 0$

$f''(x) = 6x + 2f'(1)$

$\Rightarrow \quad f''(2) = 12 + 2f'(1)$

$f'''(x) = 6 \Rightarrow f'''(3) = 6$

$f'(1) = -5$

$f''(2) = 2.$

Hence, $f(x) = x^3 - 5x^2 + 2x + 6$

Now, $f(-1) = -1 - 5 - 2 + 6 = -2$

$f(0) = 6 \Rightarrow$ one root $\in (-1, 0)$

$f(2) = 8 - 20 + 4 + 6 = -2 \Rightarrow$ One root $\in (0, 2)$

$\Rightarrow$ Two roots are real $\Rightarrow$ All three roots are real.

Also $f(5) = 125 - 125 + 10 + 6 = 16$

$\Rightarrow$ One root $\in (2, 5)$

$\therefore$ One root lies in each of the intervals $(-1, 0), (0, 2), (2, 5)$

Hence all roots are real and distinct.

Example 8: Two functions f & g have first & second derivatives at $x = 0$ & satisfy the relations,

$$f(0) = \frac{2}{g(0)},\quad f'(0) = 2g'(0) = 4g(0),$$

$g''(0) = 5f''(0) = 6f(0) = 3$ then prove that

(i) if $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(0) = \dfrac{15}{4}$

(ii) if $k(x) = f(x) \cdot g(x) \sin x$ then $k'(0) = 2$

(iii) $\displaystyle\lim_{x\to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}.$

Solution:

(i) $h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{gf' - fg'}{g^2}$

$$h'(0) = \frac{g(0)\,f'(0) - f(0)\,g'(0)}{g^2(0)}$$

Since $f(0) = \dfrac{1}{2}$ we get $g(0) = 4$

Hence, $f'(0) = 16$ and $g'(0) = 8$

Thus, $h'(0) = \dfrac{4 \times 16 - 4}{16} = \dfrac{15}{4}$

(ii) $k(x) = f(x) \cdot g(x) \sin x$

$k'(0) = f'(x)\ g(x)\ \sin x$

$\quad + f(x)\ g'(x)\ \sin x + \cos x\ f(x)\ g(x)$

$k'(0) = \cos(0)\ f(0)\ g(0) = 2$

$\displaystyle\lim_{x\to 0} \frac{g'(x)}{f'(x)} = \frac{8}{16} = \frac{1}{2}.$

Second Derivative of Inverse Functions

Suppose $y = f(x)$ and $x = g(y)$ are inverse functions of each other. Let us express $g''(y)$ in terms of derivatives of f(x). We have

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{dx}{dy} = g'(y)$$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)}.$$

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{f'(x)}\right) \cdot \frac{dx}{dy}$$

$$= -\frac{f''(x)}{f'(x)^2} \cdot g'(y)$$

$$\Rightarrow \quad g''(y) = -\frac{f''(x)}{f'(x)^3} \qquad ...(1)$$

which can also be remembered as

$$g''(g(x)) = -\frac{f''(x)}{(f'(x))^3}$$

$$\text{or,} \quad \frac{d^2x}{dy^2} = -\frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3} \qquad ...(2)$$

Similarly, $$\frac{d^3x}{dy^3} = -\frac{\left\{\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{d^2y}\right)^2\right\}}{\left(\frac{dy}{dx}\right)^5} \qquad ...(3)$$

or, $$g'''(y) = -\frac{f'(x).f'''(x) - 3\left(f''(x)\right)^2}{\left(f'(x)\right)^5}. \qquad ..(4)$$

Example 9: If g is the inverse of $f(x) = e^x + x^3 - 1$, then find $g''(e)$.

Solution: $f(x) = e^x + x^3 - 1$

When $f(x) = e$, $x = 1$

$f'(x) = e^x + 3x^2$ and $f''(x) = e^x + 6x$

Now, $$g''(y) = -\frac{f''(x)}{f'(x)^3} = -\frac{e^x + 6x}{(e^x + 3x^2)^3}$$

$$g''(e) = -\left.\frac{e^x + 6x}{(e^x + 3x^2)^3}\right|_{x=1} = -\frac{e+6}{(e+3)^3}.$$

Example 10: Let $f(x) = 1 + x^3$. If $g(x) = f^{-1}(x)$, then find the value of $g'''(2)$.

Solution: $f(x) = 1 + x^3$

When $f(x) = 2$, $x = 1$

$f'(x) = 3x^2$, $f''(x) = 6x$ and $f'''(x) = 6$

$f'(1) = 3$, $f''(1) = 6$ and $f'''(1) = 6$.

Now, $$g'''(9) = -\frac{f'(1).f'''(1) - 3\left(f''(1)\right)^2}{\left(f'(1)\right)^5} = \frac{10}{27}.$$

Example 11: If $y = (\sin^{-1}x)^2$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx} + 2.$$

Solution: $\because$ $y = (\sin^{-1}x)^2$

$$\therefore \quad \frac{dy}{dx} = 2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$$

or, $$\sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x$$

$$\Rightarrow \quad (1 - x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2 = 4y \text{ [from (1)]}$$

Again differentiating both sides w.r.t to x, we get

$$(1 - x^2)\,2.\frac{dy}{dx}.\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$$

Dividing both sides by $2\frac{dy}{dx}$, we have,

$$(1 - x^2)\frac{d^2y}{dx^2} = x.\frac{dy}{dx} + 2.$$

Alternative: We have $y = (\sin^{-1}x)^2$.
Therefore $y_1 = 2(\sin^{-1}x)/\sqrt{(1 - x^2)}$,

or $$y_1^2 = \frac{4(\sin^{-1}x)^2}{1-x^2} = \frac{4y}{1-x^2}, \quad [\because (\sin^{-1}x)^2 = y]$$

or $$y_1^2(1 - x^2) - 4y = 0.$$

Differentiating again,

we get $$2y_1y_2(1 - x^2) - 2xy_1^2 - 4y_1 = 0$$

or $$2y_1[y_2(1 - x^2) - xy_1 - 2] = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get

$$y_2(1 - x^2) - y_1x - 2 = 0.$$

Example 12: $y = \tan^{-1}\left(\frac{x}{1 + \sqrt{1 - x^2}}\right) + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then prove that,

$$4(1 - x^2)^3\left(\frac{d^2y}{dx^2}\right)^2 + 4x = x^2 + 4.$$

Solution: Let $y = u + v$.

Now $$\frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

$$v = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$$

$$= \frac{2\sqrt{\frac{1-x}{1+x}}}{1 + \frac{1-x}{1+x}} = \sqrt{1-x^2}.$$

Hence $$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-2x}{2\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{x-2}{2(1-x)^{3/2}}$$

$$\Rightarrow \quad 4(1 - x^2)^3\frac{d^2y}{dx^2} = x^2 + y - 4x$$

Second Derivative of Parametric Functions

Let a function y of x be represented by the parametric equations $x = x(t)$ and $y = y(t)$.

To find the second derivative $\frac{d^2y}{dx^2}$, we differentiate

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

with respect to x, bearing in mind that t is a function of x:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}$$

$$\text{where } \frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$$

$$\text{Now, } \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$\text{Hence, } \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

This formula may be written in more compact form as follows: if y is a function of x given parametrically by $x=\phi(t)$, $y=\psi(t)$ then

$$\frac{d^2y}{dx^2}=\frac{\phi'(t)\psi''(t)-\psi'(t)\phi''(t)}{(\phi'(t))^3}$$

Example 13: A function y of x is represented parametrically as x = a cos t, y = b sin t

Find $\frac{d^2y}{dx^2}$.

Solution: $\frac{dx}{dt}=-a\sin t,\ \frac{d^2x}{dt^2}=-a\cos t$

$$\frac{dy}{dt}=b\cos t,\ \frac{d^2y}{dt^2}=-b\sin t$$

$$\frac{dy}{dx}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}\left(-\frac{b}{a}\cot t\right)=\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\frac{dt}{dx}$$

$$=\left(\frac{b}{a}\operatorname{cosec}^2 t\right)\frac{1}{\frac{dx}{dt}}=\frac{b}{a}\operatorname{cosec}^2 t.\left(-\frac{1}{a\sin t}\right)$$

$$=-\frac{b}{a^2}\frac{1}{\sin^3 t}.$$

Alternative:

$$\text{Using the formula, } \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(-a\sin t)(-b\sin t)-(b\cos t)(-a\cos t)}{(-a\sin t)^3}$$

$$=-\frac{b}{a^2}\frac{1}{\sin^3 t}.$$

Example 14: Find d²y/dx² (independent of t), of the function defined parametrically as x = sin(ln t), y = cos(ln t).

Solution: We have x = sin(ln t), y = cos(ln t).

Differentiating w.r.t. x, we have

$$\frac{dx}{dt}=\frac{\cos(\ln t)}{t},\ \frac{dy}{dt}=\frac{-\sin(\ln t)}{t}$$

Thus, we have

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=-\tan(\ln t)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\{-\tan(\ln t)\}.\frac{dt}{dx}$$

$$=\frac{-\sec^2(\ln t)}{t}.\frac{t}{\cos(\ln t)}\quad\left[\because \frac{dt}{dx}=\frac{1}{dx/dt}\right]$$

$$=-\sec^3(\ln t)$$

$$\text{Hence, } \frac{d^2y}{dx^2}=\frac{-1}{y^3}.$$

Example 15: If x = a(cos θ + θsinθ),

y = a(sinθ – θcos), then find $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{4}$.

Solution: x = a(cosθ + θsinθ)

$$\therefore\ \frac{dx}{d\theta}=a(-\sin\theta+1.\sin\theta+\theta\cos\theta)=a\theta\cos\theta \quad ...(1)$$

and y = a(sin θ – θ cos θ)

$$\therefore\ \frac{dy}{d\theta}=a[\cos\theta-\{1.\cos\theta+\theta(-\sin\theta)\}]=a\theta\sin\theta \quad ...(2)$$

$$\text{Now } \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\theta.\sin\theta}{a\theta\cos\theta}=\tan\theta$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d(\tan\theta)}{dx}=\frac{d(\tan\theta)}{d\theta}.\frac{d\theta}{dx}$$

$$=\sec^2\theta.\frac{1}{\frac{dx}{d\theta}}=\sec^2\theta.\frac{1}{a\theta.\cos\theta}=\frac{\sec^2\theta}{a\theta}$$

[from (1)]

$$\therefore\ \text{When } \theta=\frac{\pi}{4},$$

$$\frac{d^2y}{dx^2}=\frac{\sec^3\frac{\pi}{4}}{a,\frac{\pi}{4}}=\frac{(\sqrt{2})^3}{a\pi}.4=\frac{8\sqrt{2}}{\pi a}.$$

Example 16: If $x = \cos\theta$, $y = \sin^3\theta$, then find

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + 3\Bigg|_{\theta=\pi/3}.$$

Solution: $\frac{dx}{d\theta} = -\sin\theta$, $\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta$

So, that $\frac{dy}{dx} = -3\sin\theta\cos\theta$

$$\frac{d^2y}{dx^2} = -3\cos 2\theta\ \frac{dx}{d\theta} = \frac{3\cos 2\theta}{\sin\theta}$$

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 9\sin^2\theta\cos^2\theta + \sin^3\theta\ .\ \frac{3\cos 2\theta}{\sin\theta}$$

$$= \quad 9\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos 2\theta$$

So, $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + 3\Bigg|_{\theta=\pi/3} = \frac{57}{16}$

Example 17: If $x = \operatorname{cosec}\theta - \sin\theta$;

$y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that

$$(x^2 + 4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2 + 4) = 0.$$

Solution: $x^2 + 4 = (\operatorname{cosec}\theta - \sin\theta)^2 + 4$

$$= (\operatorname{cosec}\theta + \sin\theta)^2 \qquad ...(1)$$

and $y^2 + 4 = (\operatorname{cosec}^n\theta - \sin^n\theta)^2 + 4$

$$= (\operatorname{cosec}^n\theta + \sin^2\theta)^2 \qquad ...(2)$$

Now, $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$$= \frac{n(\operatorname{cosec}^{n-1}\theta)(-\operatorname{cosec}\theta\cot\theta) - n\sin^{n-1}\theta\cos\theta)}{-\operatorname{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n\,(\operatorname{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta)}{(\operatorname{cosec}\theta\ \cot\theta + \cos\theta)}$$

$$= \frac{n\cot\theta\ (\operatorname{cosec}^n\theta + \sin^n\theta)}{\cot\theta\ (\operatorname{cosec}\theta + \sin\theta)}$$

$$= \frac{n(\operatorname{cosec}^n\theta + \sin^n\theta)}{(\operatorname{cosec}\theta + \sin\theta)} = \frac{n\sqrt{(y^2+4)}}{\sqrt{(x^2+4)}}$$

[from (1) and (2)]

Squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2+4)}{(x^2+4)}$$

or $(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + 4).$

Second Derivative of Implicit Functions

The second derivative of the function y specified by an implicit equation $f(x, y) = 0$ is found by differentiating the equation twice.

For instance, let us find the second derivative of the function y given by the equation $x^2 + y^2 = 1$.

Differentiating the equation twice w.r.t. x yields, in succession, $x + yy' = 0$ and $(x') + (yy')' = 1 + y'^2 + yy'' = 0$

$$\Rightarrow \quad y' = -\frac{x}{y} \text{ and } y'' = -\frac{1+y'^2}{y} = -\frac{x^2+y'^2}{y^3} = -\frac{1}{y^3}.$$

In this example the function y can be expressed explicitly in terms of x, and therefore the differentiation can be performed directly also.

For instance, taking the branch $y = \sqrt{1-x^2}$ we find $y'' = -\frac{1}{\sqrt{(1-x^2)^3}}$ which coincides with the previous result.

Example 18: Find y'' if $x^4 + y^4 = 16$.

Solution: Differentiating the equation implicitly with respect to x, we get $4x^3 + 4y^3y' = 0$

Solving for y' gives $y' = -\frac{x^3}{y^3}$ ...(1)

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = \frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^2)}{(y^3)^2}$$

$$= -\frac{y^3.3x^2 - x^3(3y^2y')}{y^6}$$

If we now substitute equation (1) into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^2} = -\frac{3x^2(y^4+x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}.$$

Example 19: Find the value of y'' at the point $x = 1$ if $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$.

Solution: Differentiating with respect to x,

$$3x^2 - 4xy^2 - 4x^2yy' + 5 + y' = 0.$$

Putting $x = 1$ and $y = 1$, we obtain y' at $x = 1$

$$3 - 4 - 4y' + 5 + y' = 0 \Rightarrow y' = 4/3.$$

Differentiating once again with respect to x,
$6x - 4y^2 - 8xyy' - 8xyy' - 4x^2y'^2 - 4x^2yy'' + y'' = 0$.
Putting x = 1, y = 1 and y′ = 4/3,
we find the value y″ at x = 1.

$$6 - 4 - \frac{64}{3} - \frac{64}{9} - 3y'' = 0, \Rightarrow y'' = -8\frac{22}{27}.$$

Example 20: If $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$ then show that $\frac{dy}{dx} = \tan^2\alpha$.

Solution: We rewrite the given equation as

$$(x+y)\left(\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y}\right) = 1 = (\cos^2\alpha + \sin^2\alpha)^4$$

$$\therefore \quad \frac{y}{x}\cos^4\alpha + \frac{x}{y}\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha = 0$$

$$\text{or} \quad \left(\sqrt{\frac{y}{x}}\cos^2\alpha - \sqrt{\frac{x}{y}}\sin^2\alpha\right)^2 = 0$$

$$\therefore \quad \tan^2\alpha = \frac{y}{x} \text{ or } y = x\tan^2\alpha \qquad \therefore \quad \frac{dy}{dx} = \tan^2\alpha.$$

Example 21: If $\sqrt{x+y} + \sqrt{y-x} = c$ then find the value of $\frac{d^2y}{dx^2}$.

Solution: Differentiating,

$$\frac{1}{2\sqrt{x+y}}\left(1 + \frac{dy}{dx}\right) + \frac{1}{2\sqrt{y-x}}\left[\frac{dy}{dx} - 1\right] = 0.$$

$$\frac{dy}{dx} = \frac{[(\sqrt{x+y}) - (\sqrt{y-x})(\sqrt{x+y} + \sqrt{y-x})]}{c\,(\sqrt{x+y} + \sqrt{y-x})}$$

$$\frac{dy}{dx} = \frac{2x}{c^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{c^2}.$$

Example 22: If $y = e^{ax}\sin bx$, prove that
$y_2 - 2ay_1 + (a^2 + b^2)y = 0$

Solution: Here $y = e^{ax}\sin bx$ ...(1)

$\therefore \quad y_1 = e^{ax}\, b\cos bx + a.e^{ax}\sin bx$.

Replacing $e^{ax}\sin bx$ by y, we get

$y_1 = be^{ax}\cos bx + ay$. ...(2)

Differentiating both sides of (2) w.r.t. x, we get

$y_2 = abe^{ax}\cos bx - b^2e^{ax}\sin bx + ay_1$
$= a(y_1 - ay) - b^2y + ay_1,$

[Since from (2), $be^{ax}\cos bx = y_1 - ay$ and from (1), $e^{ax}\sin bx = y$]

$= 2ay_1 - (a^2 + b^2)y.$

$\therefore \quad y_2 - 2ay_1 + (a^2 + b^2)y = 0$

Example 23: If y = x sin x, prove that

$$x^2.\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + y)\,y = 0$$

Solution: $\because \; y = x\sin x$ ...(1)

$\therefore \quad \frac{dy}{dx} = x\cos x + \sin x$ ...(2)

Again differentiating both sides w. r. t. x we get

$$\frac{d^2y}{dx^2} = x(-\sin x) + \cos x + \cos x = 2\cos x - x\sin x$$

or, $x^2\frac{d^2y}{dx^2} = 2x^2\cos x - x^2.x\sin x$

$= 2x^2\cos x - x^2y$ [from (1)]

$= 2x\left(\frac{dy}{dx} - \sin x\right) - x^2y$ [from (2)]

$= 2x\frac{dy}{dx} - 2x\sin x - x^2y = 2x\frac{dy}{dx} - 2y - x^2y$

$$\therefore \quad x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + x^2)\,y = 0.$$

Example 24: If y is a twice differentiable function of x, transform the equation, $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ by means of the substitution, x = sin t, in terms of the independent variable t.

Solution: $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ ...(1)

$$x = \sin t\,;\quad \frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{\cos t} = \frac{dy}{dt}\cdot\sec t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}\cdot\sec t\right)\cdot\frac{dt}{dx}$$

$$= \left[\sec t\cdot\frac{d^2y}{dt^2} + \frac{dy}{dt}\sec t\cdot\tan t\right]\sec t$$

$$= \sec^2 t\,\frac{d^2y}{dt^2} + \frac{dy}{dt}\sec^2 t\cdot\tan t$$

$\therefore$ (1) becomes

$$\cos^2 t\left[\left(\sec^2 t\,\frac{d^2y}{dt^2}\right) + \frac{dy}{dt}\sec^2 t\cdot\tan t\right]$$

$$- \sin t\cdot\sec t\,\frac{dy}{dt} + y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + \tan t\,\frac{dy}{dt} - \tan t\,\frac{dy}{dt} + y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + y = 0.$$

Example 25: If $y = (A + Bx)\,e^{mx} + (m-1)^{-2}\,e^x$ then show that $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y$ is equal to e^x.

Solution: $y = (A + Bx)\,e^{mx} + (m-1)^{-2}\,e^x$

$$\Rightarrow \frac{dy}{dx} = m\,e^{mx}(A + Bx) + B\,e^{mx} + \frac{e^x}{(m-1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}(A + Bx) + B\,me^{mx} + Bm\,e^{mx} + \frac{e^x}{(m-1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y$$

$$= 2m^2e^{mx}(A + Bx) - 2mB\,e^{mx} - \frac{2m}{(m-1)^2}\,e^x + m^2e^{mx}(A + Bx) + \frac{m^2e^x}{(m-1)^2}$$

$$= \frac{(1 + m^2 - 2m)}{(m-1)^2}\,e^x = e^x.$$

Example 26: Let $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then find the value of $[f(10)]^2 - [\phi(10)]^2$.

Solution: $\frac{d}{dx}\{[f(x)]^2 - [\phi(x)]^2\}$

$= 2\,[f(x)\cdot f'(x) - \phi(x)\cdot\phi'(x)]$

$= 2\,[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)] = 0$

$[\because f'(x) = \phi(x) \text{ and } \phi'(x) = f(x)]$

$\Rightarrow [f(x)]^2 - [\phi(x)]^2 = \text{constant}$

$\therefore [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2$

$= [f(3)]^2 - [f'(3)]^2 = 25 - 16 = 9.$

Example 27: If α is a repeated root of a quadratic equation $f(x) = 0$ and $A(x), B(x), C(x)$ be polynomials of degree > 2, then show that the determinant

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \text{ is divisible by } f(x).$$

Solution: Let $\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

So $\Delta(\alpha) = 0 = \Delta'(\alpha)$, therefore α is a repeated root of $\Delta(x)$ and α is a repeated root of the quadratic equation $f(x) = 0$, so Δ is divisible by $f(x)$.

Example 28: If $y^2 = P(x)$, a polynomial of degree $n > 3$, show that $2\frac{d}{dx}\left(y^2\frac{d^2y}{dx^2}\right) = P(x)\,.\,P'''(x)$

Solution: Given, $y^2 = P(x)$ ...(1)

$\therefore\ 2y\frac{dy}{dx} = P'(x)$ ...(2)

Differentiating again w.r.t. to x, we get

$$2\frac{dy}{dx}\,.\,\frac{dy}{dx} + 2y\,.\,\frac{d^2y}{dx^2} = P''(x)$$

$$\text{or}\quad 2y^2\left(\frac{dy}{dx}\right)^2 + 2y^3\frac{d^2y}{dx^2} = y^2P''(x)$$

$$\text{or}\quad 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$

$$= y^2P''(x) - \frac{1}{2}[P'(x)]^2$$

Differentiating again w.r.t. to x, we get

$$2\frac{d}{dx}\left(y^2\frac{d^2y}{dx^2}\right)$$

$$= 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - \frac{1}{2}2(P'(x).\,P''(x)$$

$$= P'(x)P''(x) + y^2P'''(x) - P'(x)P''(x)$$

$$\left[\because 2y\frac{dy}{dx} = P'(x)\right]$$

$= y^2P'''(x)$

$= P(x)P'''(x) \qquad [\because y^2 = P(x)].$

Example 29: Let f be twice differentiable such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = \{f(x)\}^2 + \{g(x)\}^2$ and $h(5) = 11$, find $h(10)$.

Solution: Given $h(x) = \{f(x)\}^2 + \{g(x)\}^2$

On differentiating both sides w.r.t. x, we get

$h'(x) = 2f(x)\,.\,f'(x) + 2g(x).\,g'(x)$ ...(1)

Now $f'(x) = g(x)$ implies $f''(x) = g'(x)$

$\Rightarrow -f(x) = g'(x)$ ...(2)

$\{\because f''(x) = -f(x)\}$

From (1) and (2) we get,

$h'(x) = 2f(x).g(x) + 2g(x).\ \{-f(x)\}$
[using $f'(x) = g(x)$ and $g'(x) = -f(x)$]

$\therefore \quad h'(x) = 0$

So, h(x) must be a constant function.
But $h(5) = 11$, so, $h(x) = 11$
Hence, $h(10) = 11$.

Concept Problems I

1. Find the derivatives of the second order of the following functions :

(a) $y = x\sqrt{1+x^2}$ (b) $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

(c) $y = e^{-x^2}$.

2. Use implicit differentiation to find the first and second derivative at the indicated point.

(i) $x^2 - y^2 = 1$, at $(\sqrt{2}, 1)$
(ii) $y^2 = 1 - xy$. at (0, 1)
(iii) $\sqrt{x} + \sqrt{y} = 3$ at (1, 4)
(iv) $x^3 + 3xy - y^2 = 5$ at (1, 1)

3. Consider the function $f(x) = \dfrac{x^2}{e^{2x}}$.

Find the solutions set of the equations

(i) $f'(x) = 0$ (ii) $f''(x) = 0$.

4. If $y = \dfrac{1}{x^2}$, then find $\dfrac{d^3y}{dx^3}$.

5. If $x = t^3 + t + 5$ and $y = \sin t$ then find $\dfrac{d^2y}{dx^2}$.

6. Let $y = e^{2x}$. Then find the value of

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right).$$

7. If $y = \sin x + e^x$, then find the value of $\dfrac{d^2x}{dy^2}$.

8. Let $f(x) = x^8 - 2x + 3$ find $\lim\limits_{w\to 2} \dfrac{f'(w) - f'(2)}{w-2}$

9. If $f''(x) = 10$ and $f'(1) = 6$ and $f(1) = 4$ then find $f(-1)$.

10. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

11. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 5y' - 6y = 0$?

12. Find the derivative of the indicated order $(f(\varphi(x)))^{(3)}$.

13. There is a polynomial $P(x) = ax^3 + bx^2 + cx + d$ such that $P(0) = P(1) = -2$, $P'(0) = -1$, and $P''(0) = 10$. Compute a, b, c, d.

14. (a) Show that $y = x \sin x$ is a solution to $y'' + y = 2 \cos x$.
(b) Show that $y = x \sin x$ is a solution of the equation $y^{(4)} + y'' = -2 \cos x$.

Practice Problems H

15. Find the second order derivatives:

(i) $y = \dfrac{1}{3}x^2\sqrt{1-x^2} + \dfrac{2}{3}\sqrt{1-x^2} + x\sin^{-1} x$

(ii) $y = x \ln\left(x + \sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}$

(iii) $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$

(iv) $\begin{cases} x = \cos^{-1}\sqrt{t}, \\ y = \sqrt{t - t^2}. \end{cases}$

16. Find the third order derivatives:

(i) $y = \dfrac{x}{6(x+1)}$ (ii) $y = \dfrac{1}{2}\ln^2 x$

(iii) $y = (2x+3)^3\sqrt{2x+3}$

17. If the line $y = 3x - 1$ is tangent to the graph of the function $f(x) = ax^3 + bx^2 + c$ at the point (1, 2) and $\dfrac{d^2f}{dx^2}(1) = 0$ compute a, b and c.

18. If $z = f(y)$ and $y = g(x)$, show that

$$\frac{d^2z}{dx^2} = \frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{dz}{dy}\frac{d^2y}{dx^2}.$$

Further, if $z = 2y^3 - 3y + 1$ and $y = x^2 - 1$ compute $\dfrac{d^2z}{dx^2}(2)$.

19. If $y = 2\cos t - \cos 2t$ and $x = 2\sin t - \sin 2t$ then prove that $\left(\dfrac{d^2y}{dx^2}\right)_{t=\pi/2} = \dfrac{-3}{2}$.

20. Find a third-degree polynomial Q such that $Q(1) = 1$, $Q'(1) = 3$, $Q''(1) = 6$, and $Q'''(1) = 12$.

21. Find constants A, B and C such that the function $y = Ax^2 + Bx + C$ satisfies the differential equation $y'' + y' - 2y = x^2$.

22. The function g is a twice differentiable function. Find f'' in terms of g, g', and g''.
(i) $f(x) = xg(x^2)$ (ii) $f(x) = g(\sqrt{x})$

23. The function g is a twice differentiable function. Find f'' in terms of g, g', and g''.
(i) $f(x) = xg(x^2)$ (ii) $f(x) = g(\sqrt{x})$

24. Starting with $\frac{dx}{dy} = \frac{1}{dy/dx}$ prove that

$$\frac{d^2x}{dy^2} = -\frac{d^2y/dx^2}{(dy/dx)^3}$$ and deduce that for the parabola

$y^2 = 4ax$, $\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2} = -\frac{2a}{y^3}$.

25. Let g be the inverse of $f(x) = x(\ell nx - 1)$. Prove that

$$g''(x) = -\frac{1}{g(x)\{\ell ng(x)\}^3},\ x > 0$$

26. Find values for the constants a, b and c that will make $f(x) = \cos x$ and $g(x) = a + bx + cx^2$ satisfy the conditions $f(0) = g(0)$, $f'(0) = g'(0)$, and $f''(0) = g''(0)$.

27. Find a polynomial function f(x) such that $f(2x) = f'(x)\ f''(x)$.

28. If $f'(x) = g(x)$ and $g'(x) = -f(x)\ \forall\ x \in R$ and $f(3) = 9 = f'(3)$, then find the value of $f^2(17) + g^2(17)$.

29. Prove that the function

$$y = A\sin(\omega t + \omega_0) + B\cos(\omega t + \omega_0)$$

(A, B, ω, ω_0) constants) satisfies $\frac{d^2y}{dx^2} + \omega^2 y = 0$.

30. Show that the function $y = \sin\ln x + \cos\ln x$ satisfies the equation $x^2y'' + xy' + y = 0$.

31. Prove that the function, represented parametrically by the equations $x = \sin t$, $y = \sin kt$, satisfies the

relation $(1 - x)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2y = 0$.

32. Show that the function $y = e^x + 2e^{2x}$ satisfies the equation $y''' - 6y'' + 11y' - 6y = 0$.

4.12 Successive Differentiation

Since the rules and formulae for computing derivative of any elementary function are known, we can find derivatives of any order of a given elementary function by differentiating it, in succession, the required number of times. In some cases it is possible to derive the general formula for the derivative of the kth order of a given function for an arbitrary k. For brevity, such a derivative is also denoted by $y^{(k)}$.

For example,

(i) If $y = x^n$ then $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$, ...
$y^{(k)} = n(n-1)(n-2)\ldots(n-k+1)x^{n-k}$.
If n is a positive integer, we have
$y^{(n)} = n!$ and $y^{(n+1)} = y^{(n+2)} = \ldots.. = 0$.

(ii) Taking $y = \sin x$ we find $y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$,

$$y'' = \sin(x + \pi),\ \ldots..,\ y^{(k)} = \sin\left(x + \frac{k\pi}{2}\right).$$

Example 1: Find y_n if (i) $y = e^x$ (ii) $y = a^x$

(iii) $y = \frac{1}{x+1}$ (iv) $y = \ln(1 + x)$.

Solution:

(i) For $y = e^x$ we obtain $y^{(n)} = e^x$.

(ii) If $y = a^x$ then $y' = a^x\ln a$, $y'' = a^x(\ln a)^2$, ...
$y^{(n)} = a^x(\ln a)^n$.

(iii) Given $y = \frac{1}{x+1}$. Then $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$,

$$\frac{d^2y}{dx^2} = \frac{2}{(x+1)^2},$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = \frac{(-1)^n n!}{(x+1)^{n+1}}$$

(iv) Differentiating the function $y = \ln(1 + x)$

we get $y' = \frac{1}{1+x}$, $y'' = -\frac{1}{(1+x)^2}$, ...

$$y^{(n)} = \frac{(-1)(-2)\ldots(-n+1)}{(1+x)^n} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}.$$

Note: We have the following results:

(a) $(x^m)^{(n)} = m(m-1)\ldots\ldots.(m-n+1)x^{m-n}$. $(m \geq n)$

(b) $(a^x)^{(n)} = a^x(\ln a)^n$

(c) $(\ln x)^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}$

(d) $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$

(e) $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

Example 2: Prove $(\ln x)^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$.

Solution: $y = \ln x$. Then, we have

$$y' = \frac{1}{x} = x^{-1}$$
$$y'' = (-1)x^{-2}$$
$$y''' = (-1)(-2)x^{-3} = (-1)^2(1.2)x^{-3}$$
$$y^{(n)} = (-1)^{n-1}\, 1.\, 2.\, 3.....(n-1)x^{-n}$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Example 3: Prove $y^{(n)} = \sin\left(x + \dfrac{n\pi}{2}\right)$.

Solution: $y = \sin x$. Then, we have

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$
$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right)$$

....

$$y^{(n)} = \sin\left(x + \frac{n\pi}{2}\right).$$

Example 4: Find y_n if $y = \dfrac{x}{x^3 - 3x + 2}$.

Solution: Now, $y = \dfrac{x}{x^3 - 3x + 2} = \dfrac{x}{(x-1)(x-2)}$.

Resolving y into its partial fractions, we get

$$\frac{x}{(x-1)(x-2)} = \frac{2}{x-2} - \frac{1}{x-1},$$

so that, $y_n = \dfrac{2\,.\,(-1)^n \cdot n!}{(x-2)^{n+1}} + \dfrac{(-1)^{n+1} \cdot n!}{(x-1)^{n+1}}$.

Example 5: If $f(x) = e^{-1/x}$, where $x > 0$. Let for each positive integer n, P_n be the polynomial such that $\dfrac{d^n f(x)}{dx^n} = P_n\left(\dfrac{1}{x}\right)e^{-1/x}$ for all $x > 0$. Show that $P^{n+1}(x) = x^2\left[P_n(x) - \dfrac{d}{dx}P_n(x)\right]$.

Solution: From the given condition of the problem

$$\frac{d^{n+1}f(x)}{dx^{n+1}} = P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{d^n f(x)}{dx^n}\right) = P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow \frac{d}{dx}\left(P_n\left(\frac{1}{x}\right)e^{-1/x}\right) = P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow e^{-1/x}\frac{dP^n\left(\frac{1}{x}\right)}{dx} + P^n\left(\frac{1}{x}\right)\frac{de^{-1/x}}{dx} = P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow e^{-1/x}\frac{dP^n\left(\frac{1}{x}\right)}{dx\left(\frac{1}{x}\right)} \times \frac{d\left(\frac{1}{x}\right)}{dx}$$
$$+ P^n\left(\frac{1}{x}\right)\frac{de^{-1/x}}{d\left(\frac{-x}{x}\right)} \times \frac{d\left(\frac{-x}{x}\right)}{x} = P_{n+1}\, e^{-1/x}$$

$$\Rightarrow -\frac{1}{x^2}\frac{dP^n\left(\frac{1}{x}\right)}{dx\left(\frac{1}{x}\right)} + \frac{1}{x^2}P_n\left(\frac{1}{x}\right) = P_{n+1}\left(\frac{1}{x}\right)$$

Now replace $\left(\dfrac{1}{x}\right)$ by x to get the required result.

$$P^{n+1}(x) = x^2\left[P_n(x) - \frac{d}{dx}P_n(x)\right]$$

Leibniz's Theorem for the nth Derivative of the Product of two Functions of x

Sometimes a function can be written as the product of two functions where it is easy to determine the nth derivative of the two functions separately. In such a case, the nth derivative of the product can be written by using Leibniz's theorem which is as follows:

Theorem If $y = uv$, where u and v are functions of x having derivatives of nth order, then

$$y_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + ...+ {}^nC_r u_{n-r}v_r + ...+ {}^nC_n u v_n$$

where suffixes denote derivatives with respect to x.

Proof We shall prove the theorem by induction on n.

We know that if $y = uv$, then $y_1 = u_1v + v_1u$.

Again, taking derivatives, we get

$y_2 = u_2v + u_1v_1 + v_1u_1 + v_2u = u_2v + {}^2C_1u_1v_1 + {}^2C_2uv_2$.

Thus, the theorem is true for n = 1, and for n = 2.

Suppose the theorem is also true for n = m, i.e.

$$(uv)_m = u_m v + {}^mC_1 u_{m-1}v_1 + {}^mC_2 u_{m-2}v_2 + ... + {}^mC_{r-1}u_{m-r+1}v_{r-1} + {}^mC_r u_{m-r}v_r + ... + {}^mC_m u v_m \quad ...(1)$$

We now show that the theorem is also true for n = m + 1. Taking derivatives of both sides of (1), we get

$$(uv)_{m+1} = u_{m+1}v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1}v_2 + {}^mC_2 u_{m-1}v_2 + {}^mC_2 u_{m-2}v_3 + ... + {}^mC_{r-1}u_{m-r+2}v_{r-1} + {}^mC_{r-1}u_{m-r+1}v_r$$

$$+ {}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r} v_{r+1} + ... + {}^mC_m uv_{m+1}$$
$$= u_{m+1} v + (1 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2)_{u-1} v_2$$
$$+ .... + ({}^mC_{r-1} + {}^mC_r) u_{m-r+1} v_r + ...$$
$$......... + {}^mC_m uv_{m+1}.$$

Since ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$,

$$1 + {}^mC_1 = 1 + m = {}^{m+1}C_1 \text{ and } {}^mC_m = {}^{m+1}C_{m+1},$$

therefore, $(uv)_{m+1} = u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^mC_2 u_{m-1} v_2$
$$+ ... + {}^{m+1}C_r u_{m-r+1} v_r + ... + {}^{m+1}C_{m+1} uv_{m+1}$$

Thus, if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Example 6: Find $y^{(10)}$ if $y = x^2 e^{3x}$.

Solution: This function is a product of two functions x^2 and e^{3x}. Using Leibnitz's formula, we obtain

$$(x^2e^{3x})^{(10)} = x^2\,(e^{3x}) + {}^{10}C_1\,(x^2)'\,(e^{3x})^{(9)}$$
$$+ {}^{10}C_2 (x^2)^{(2)}\,(e^{3x})^{(8)} + ... + (x^2)^{(10)}\,e^{3x}.$$

Since $(x^2)^{(n)} = 0$ for $n \geq 3$, $(e^{3x})^{(n)} = e^{3x}3^n$, it follows that $(x^2e^{3x})^{10}$
$= x^2e^{3x}3^{10} + 10.2xe^{3x}3^9 + 45.2e^{3x}3^8$
$$= 3^9 e^{3x}\,(3x^2 + 20x + 30).$$

The example shows that it is most convenient to use Leibnitz's formula in the case when one of the factors is a polynomial of a not high degree of p. In that case, all term of Leibnitz's formula, beginning with $(p + 2)$, are zero.

Example 7: If $y = x^{n-1}.\ \ln x$ then use induction to prove that, $\dfrac{d^n y}{dx^n} = \dfrac{(n-1)!}{x}\ \forall\ n \in N$.

Solution: Let $P(n)$: $\dfrac{d^n}{dx^n}(x^{n-1}\,.\,\ln x) = \dfrac{(n-1)!}{x}$.

Assume that $P(k)$ is true.

$$P(k+1): \frac{d^{k+1}}{dx^{k+1}}(x^k\,.\,\ln x) = \frac{k!}{x}$$

$$\text{L.H.S. of } P(k+1) = \frac{d^k}{dx^k}(x^{k-1} + \ln x\,.\,k\ x^{k-1})$$

$$= \frac{d^k}{dx^k}(x^{k-1}) + k\,.\,\frac{d^k}{dx^k}(\ln x\,.\,x^{k-1})$$

$$= 0 + k\,.\,\frac{(k-1)!}{x} = \frac{k!}{x}$$

$=$ R.H.S. of $P(k+1)$.

Practice Problems I

1. Evaluate the followings:

   (i) $(\sqrt{x})^{(10)}$ (ii) $\left(\dfrac{x^2}{x-1}\right)^{(6)}$

   (iii) $\left(\dfrac{x-1}{x+1}\right)^{(8)}$ (iv) $(xe^{5x})^{(11)}$

   (v) $(\ln 3x)^{(10)}$

2. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$. Is this true?

3. Let $f(x) = \cos x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

4. Use the Leibnitz formula for computing the following derivatives: $[(x^2 + 1)\sin x]^{(20)}$

5. Prove that $\left(x^{n-1}e^{\frac{1}{x}}\right)^{(n)} = (-1)^n \dfrac{e^{\frac{1}{x}}}{x^{n+1}}$.

6. How many times has the function $y = (x^2 + 1)^{50}$ should be differentiated to result in a polynomial of the 30th degree?

4.13 Derivative of a Determinant

Let $F(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$ where f, g, u and v are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix} \text{ or, } \quad ...(1)$$

$$= \begin{vmatrix} f'(x) & g(x) \\ u'(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) \\ u(x) & v'(x) \end{vmatrix} \quad ...(2)$$

Proof:

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\left\{\begin{vmatrix} f(x+h) & g(x+h) \\ u(x+h) & v(x+h) \end{vmatrix} - \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\begin{vmatrix} f(x+h) & g(x+h) \\ u(x+h) & v(x+h) \end{vmatrix} - \begin{vmatrix} f(x) & g(x) \\ u(x+h) & v(x+h) \end{vmatrix}\right.$$
$$\left. + \begin{vmatrix} f(x) & g(x) \\ u(x+h) & v(x+h) \end{vmatrix} - \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\begin{vmatrix} f(x+h)-f(x) & g(x+h)-g(x) \\ u(x+h) & v(x+h) \end{vmatrix}\right.$$
$$\left. + \begin{vmatrix} f(x) & g(x) \\ u(x+h)-u(x) & v(x+h)-v(x) \end{vmatrix}\right\}$$

$$= \lim_{h\to 0} \left\{\begin{vmatrix} \dfrac{f(x+h)-f(x)}{h} & \dfrac{g(x+h)-g(x)}{h} \\ u(x+h) & v(x+h) \end{vmatrix}\right.$$

$$+\left.\begin{vmatrix} f(x) & g(x) \\ \dfrac{u(x+h)-u(x)}{h} & \dfrac{v(x+h)-v(x)}{h} \end{vmatrix}\right\}$$

$$=\begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$$

In a similar manner we can prove (2).

If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Here, we differentiate one row at a time.
As before, we can also differentiate one column at a time.

$$F'(x)=\begin{vmatrix} f'(x) & g(x) & h(x) \\ l'(x) & m(x) & n(x) \\ u'(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g'(x) & h(x) \\ l(x) & m'(x) & n(x) \\ u(x) & v'(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h'(x) \\ l(x) & m(x) & n'(x) \\ u(x) & v(x) & w'(x) \end{vmatrix}$$

Example 1: Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant, then find $\dfrac{d^3}{dx^3}(f(x))$ at $x=0\cdot$

Solution: On differentiating $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

we get,

$$\frac{d}{dx}(f(x))=\begin{vmatrix} \dfrac{d}{dx}(x^3) & \dfrac{d}{dx}(\sin x) & \dfrac{d}{dx}(\cos x) \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$+\begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix}+\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\frac{d}{dx}(f(x))=\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Again differentiating it, we get

$$\frac{d^2}{dx^2}(f(x))=\begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

+ two determinants which are zero.

Differentiating it again,

$$\frac{d^3}{dx^3}(f(x))=\begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

+ two determinants which are zero.

At $x=0$, $\dfrac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}=0.$

Example 2: Let $f(x)=\begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x)=0$ and that $f(x)=f(0)+kx$ where k denotes the sum of all the cofactors of the elements in f(0).

Solution: $f'(x)=\begin{vmatrix} 1 & 1 & 1 \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$

$$+\begin{vmatrix} a+x & b+x & c+x \\ 1 & 1 & 1 \\ p+x & q+x & r+x \end{vmatrix}+\begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ 1 & 1 & 1 \end{vmatrix}$$

$f''(x)=0$ (obviously – two identical rows)

$f'(x)=k \Rightarrow f(x)=kx+x$, $f(0)=c$

$\Rightarrow f(x)=f(0)+kx$. Note that $f'(x)=k$

$$\Rightarrow f'(0)=k=\begin{vmatrix} 1 & 1 & 1 \\ \ell & m & n \\ p & q & r \end{vmatrix}+\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ p & q & r \end{vmatrix}+\begin{vmatrix} a & b & c \\ \ell & m & n \\ 1 & 1 & 1 \end{vmatrix}$$

$$=(C_{11}+C_{12}+C_{13})+(C_{21}+C_{22}+C_{23})+(C_{31}+C_{32}+C_{33})$$

= sum of cofactors of elements in f(0).

Example 3: If f, g, h are differentiable functions of x and $\Delta(x) = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$, prove that

$$\frac{d\Delta(x)}{dx} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'') & (x^3h'')' \end{vmatrix}.$$

Solution: $\Delta(x) = \begin{vmatrix} f & g & h \\ f + xf' & g + xg' & h + xh' \\ 2f + 4xf' + x^2f'' & 2g + 4xg' + x^2g'' & 2h + 4xh' + x^2h'' \end{vmatrix}$

$(\because\ (x^2f)'' = 2xf + x^2f'\ (x^2f)'' = 2(f + xf') + 2xf' + x^2f''\}$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ 2xf' + x^2f'' & 2xg' + x^2g'' & 2xh' + x^2h'' \end{vmatrix}$$

$R_3 \to R_3 - 2R_2,\ R_2 \to R_2 - R_1$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix} \quad R_3 \to R_3 - 2R_2$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}.$$

$$\therefore \frac{d\Delta(x)}{dx} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'') & (x^3h'')' \end{vmatrix}$$

$$= 0 + x^3 \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + x^3 \times 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}.$$

Example 4: If $g(x) = \dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where f(x) is a polynomial of degree < 3, then prove that

$$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \Big/ \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

Solution: By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$\Rightarrow\ g(x) = \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)}\right]$$

$$\Rightarrow\ g(x) = \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \Big/ \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow\ \frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} \Big/ \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

Example 5: If

$$f(x) = \begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

then find the value of $\dfrac{d^n}{dx^n}(f(x))$ at $x = 0$ for $n = 2m + 1$.

Solution:

$$\frac{d^n}{dx^n}\, f(x)) = \begin{vmatrix} n! & \sin(x+n\pi/2) & -\cos(x+n\pi/2) \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

$$= \frac{d^n}{dx^n}(f(x))\bigg|_{x=0} = \begin{vmatrix} n! & \sin(n\pi/2) & -\cos(n\pi/2) \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

$$= \begin{vmatrix} n! & \sin(n\pi/2) & 0 \\ n! & \sin(n\pi/2) & 0 \\ a & a^3 & a^3 \end{vmatrix} = 0.$$

Practice Problems

1. Let $\Delta(x) = \begin{vmatrix} x^2-1 & x-1 & x-2 \\ 2x^2-1 & 3x & 3x-3 \\ x^2+4 & 2x-1 & 2x-1 \end{vmatrix}$. Prove that $\Delta(x)$ is a first degree polynomial.

2. If $\phi(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ x & x^2 & x^3 \\ 0 & 2 & 6x \end{vmatrix}$ then find $\phi'(1)$.

3. If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ then find $f'(x)$.

4. Prove that $\dfrac{d}{dx}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$ where u,v,w are functions of x and $\dfrac{du}{dx} = u_1, \dfrac{d^2u}{dx^2} = u_2$, etc.

5. If $f_r(x)$, $g_r(x)$, $h_r(x)$; r = 1, 2, 3 are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$; r = 1, 2, 3 and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$ then find $F'(a)$

4.14 Properties of Derivative

Intermediate Value Property of Derivative (Darboux Property)

Theorem Let I be an interval. Then the function $f : I \to R$ is said to have the intermediate value (or Darboux property) if and only if for every $a, b \in I$ and for every k between f(a) and f(b) there exists c between a and b such that f(c) = k.

In other words, if f is differentiable in [a, b] and $f'(a) \neq f'(b)$, then $f'(x)$ takes all values between $f'(a)$ and $f'(b)$ atleast once in (a, b).

Proof Let k be any real number between $f'(a)$ and $f'(b)$. Define the function $\phi(x) = f(x) - kx$,

Then ϕ is differentiable in [a, b] and

$$\phi'(a) = f'(a) - k \text{ and } \phi'(b) = f'(b) - k$$

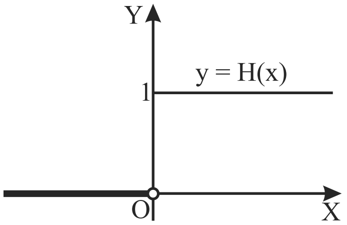
are of opposite signs. Hence by the Intermediate Value Theorem, there exists atleast one point c of (a, b) such that $\phi'(c) = 0$ or $f'(c) - k = 0$ or $f'(c) = k$.

Corollary. If f is differentiable in a closed interval [a, b] and $f'(a)$, $f'(b)$ have opposite signs, then there exists atleast one point c, of the interval (a, b) such that $f'(c) = 0$.

The fact that not every function can be some function's derivative, can be seen from the theorem.

This theorem says that a function cannot be a derivative on an interval unless it has the intermediate value property there.

For example, the Heaviside step function does not have the intermediate value property and cannot be the derivative of a function on the real line.



Test of Constancy of a Function

If the derivative f′(x) of a function exists and is identically equal to 0 for all x in a certain interval, then the function f(x) has a constant value within that interval.

The value of the constant can be obtained by evaluating f(x) at any point in the interval.

Consider the function $f(x) = 2\cos^2 x - \cos 2x$.

$$f'(x) = -4\cos x \sin x + 2\sin 2x = 0 \text{ for all x.}$$

Hence, the function f must be a constant function.

Now $f(0) = 1$

Thus, $f(x) = 1$ for all x.

Example 1: For what values of x is

$$f(x) = 2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2},$$

a constant and find the constant.

Solution: $f(x) = 2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$

$$f'(x) = \frac{2x}{1+x^2}\left(1+\frac{1-x^2}{|1-x^2|}\right)$$

$$= 0 \text{ if } 1 - x^2 < 0$$

i.e. if $x < -1$ or $x > 1$.

Since there are two separate intervals, we use one point in each of these intervals to get the constant(s).

$f\left(-\sqrt{3}\right) = -\pi$, and $f\left(\sqrt{3}\right) = \pi$.

Hence, $f(x) = \begin{cases} -\pi, & \text{if } x < -1 \\ \pi, & \text{if } x > 1 \end{cases}$

Here, we can cover the points x = –1 and 1 also.

$f(-1) = -\pi,\ \ f(1) = \pi$

Thus, $f(x) = \begin{cases} -\pi, & \text{if } x \le -1 \\ \pi, & \text{if } x \ge 1 \end{cases}.$

Differentiation of Identity

When we differentiate both sides of an identity, we get another identity. But nothing sensible is obtained when we differentiate both sides of an equation.

For instance, when we differentiate the identity

$$\cos 2x = \cos^2 x - \sin^2 x$$

we get $-2\sin 2x = -2\cos x \sin x - 2\cos x \sin x$

$\Rightarrow \quad \sin 2x = 2\sin x \cos x$

which we all know that it is an identity.

In the same way, we can differentiate several formulae (identity), to get another formulae.

We can also find several uses of this idea which is explained below with the help of examples:

(i) Summation of Series

Example 2: From the formula for the sum of the geometric progression $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$ $(x \ne 1)$, deduce the formula for the following sums:

(a) $1 + 2x + 3x^2 + \ldots + nx^{n-1}$

(b) $1^2 + 2^2x + 3^2x^2 + \ldots + n^2x^{n-1}$.

Solution: The formula for the sum of a geometric progression represents an identity with respect to x. Equating the derivatives of both sides of the identity, we get,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}.$$

Multiplying both sides of this equality by x and differentiating again, we get $1^2 + 2^2x + \ldots .\ n^2x^{n-1}$

$$= \frac{1 + x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - nx^{n+2}}{(1-x)^3}.$$

Example 3: Prove the identity $\cos x + \cos 3x + \ldots$

$\ldots\ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$ $(x \ne k\pi)$

Hence, prove that

$$\sin x + 3\sin 3x + \ldots\ldots + \cos(2n-1)x \sin(2n-1)x$$

$$= \frac{(2n+1)\sin(2n-1)x - (2n-1)\sin(2n+1)x}{4\sin^2 x}$$

Solution: We have

L.H.S. $= \cos x + \cos 3x + \cos 5x + \ldots\ldots + \cos(2n-1)x$

$$= \frac{1}{\sin x}[\sin x\cos x + \sin x\cos 3x + \sin x\cos 5x + \ldots\ldots + \sin x\cos(2n-1)x]$$

$$= \frac{1}{\sin x}\left[\frac{\sin 2x}{2} + \frac{\sin 4x - \sin 2x}{2} + \frac{\sin 6x - \sin 4x}{2} + \ldots\ldots + \frac{\sin 2nx - \sin(2n-2)x}{2}\right]$$

$$= \frac{\sin 2nx}{2\sin x} = \text{R.H.S.}$$

Differentiating both sides w.r.t. x, we have

$$-\sin x - 3\sin 3x - \ldots . - (2n-1)\sin(2n-1)x$$

$$= \frac{(2\sin x)(2n\cos 2nx) - (2\cos x)(\sin 2nx)}{4\sin^2 x}$$

$\Rightarrow \quad \sin x + 3\sin 3x + \ldots\ldots + (2n-1)\sin(2n-1)x$

$$= \frac{(2\sin 2x\cos x) - 2n(2\sin x\cos 2nx)}{4\sin^2 x}$$

$$= \frac{[\sin(2n+1)x + \sin(2n-1)x] - 2n[\sin(2n+1)x - \sin(2n-1)x]}{4\sin^2 x}$$

which is the desired result.

Example 4: Find the sum of the series:

$\sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x$

Solution: Let, $S = \cos x + \cos 3x + \cos 5x + ...$
$...... + \cos(2k-1)x$.

Here the angles are in AP whose first term is x and common difference is 2x.

$$\therefore\ S = \frac{\sin\left(\frac{k.2x}{2}\right)}{\sin\left(\frac{2x}{2}\right)}.\cos\left\{x + \frac{(k-1)2x}{2}\right\} = \frac{\sin kx}{\sin x}.\cos kx$$

$$\Rightarrow\quad \{\cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x\}$$

$$= \frac{\sin 2kx}{2\sin x} \qquad ...(1)$$

On differentiating (1) w.r.t. x, we get

$$-\{\sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x\}$$

$$= \frac{1}{2}\left\{\frac{2k(\cos 2kx).\sin x - (\sin 2kx).\cos x}{\sin^2 x}\right\}$$

$$\therefore\ [\sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x]$$

$$= -\frac{1}{2\sin^2 x}\,[k\{\sin(2k+1)x - \sin(2k-1)x\}$$

$$-\frac{1}{2}\{\sin(2k+1)x + \sin(2k-1)x\}]$$

$$= \frac{1}{4\sin^2 x}[(2k+1)\sin(2k-1)x - (2k-1)\sin(2k+1)x].$$

Example 5: Find the sum of the series:

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + ... + \frac{2^n.x^{2^n-1}}{1+x^{2^n}}.$$

Solution: We know, $(1-x)(1+x) = (1-x^2)$

$(1-x)(1+x)(1+x^2) = (1-x^4) = (1-x^{2^2})$

$(1-x)(1+x)(1+x^2)(1+x^4) = (1-x^8) = (1-x^{2^3})$

....

....

$(1-x)(1+x)(1+x^2)(1+x^4)........(1+x^{2^n})$
$= (1-x^{2^{n+1}})$

Taking logarithm on both sides, we get

$\ln\{(1-x)(1+x)(1+x^2)(1+x^4).......(1+x^{2^n})\}$
$= \ln(1-x^{2^{n+1}})$

or $\ln(1-x) + \ln(1+x) + \ln(1+x^2) + \ln(1+x^4)$ ....
$+ \ln(1+x^{2^n}) = \ln(1-x^{2^{n+1}})$

On differentiating both sides, we get

$$\frac{1(-1)}{1-x} + \frac{1.1}{1+x} + \frac{1(2x)}{1+x^2} + \frac{1.(4x^3)}{1+x^4} + ...$$

$$.... + \frac{1.2^n.x^{2n-1}}{1+x^{2n}} = \frac{-1.2^{n+1}.x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

or
$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + ... + \frac{2^n.x^{2^n-1}}{1+x^{2^n}}$$

$$= \frac{1}{1-x} - \frac{2^{n+1}.x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

Example 6: If $0 < x < 1$, prove that

$$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + ...\infty$$

$$= \frac{1-2x}{1+x+x^2}.$$

Solution: We have

$(1+x+x^2)(1-x+x^2) = (1+x^2)^2 - x^2 = 1 + x^2 + x^4$

Now, $(1+x+x^2)(1-x+x^2)(1-x^2-x^4)$
$= (1+x^2+x^4)(1-x^2+x^4) = 1 + x^4 + x^8$

Continuing in this way, we have

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)$$

$$\ldots\ldots\ (1-x^{2^{n-1}}+x^{2^n}) = (1+x^{2^n}+x^{2^{n+1}}) \quad ...(1)$$

Now, for $0 < x < 1$, $x^m \to 0$ as $m \to \infty$.

Taking limit $n \to \infty$ in (1), we get

$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\ ... = 1$

Taking logarithm on both sides, we get

$\Rightarrow\ \ln(1+x+x^2) + \ln(1-x+x^2) + \ln(1-x^2+x^4)$
$+ \ln(1-x^4+x^8) + ... = 0$

Differentiating both sides w.r.t. x, we get

$$\frac{1+2x}{1+x+x^2} + \frac{-1+2x}{1-x+x^2} + \frac{-2x+4x^3}{1-x^2+x^4}$$

$$+ \frac{-4x^3+8x^7}{1-x^4+x^8} .... = 0$$

Hence

$$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + ...$$

$$= \frac{1-2x}{1+x+x^2}.$$

(ii) Determination of Coefficients of a Polynomial

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ ...(1)

The constant term is obtained as $a_0 = f(0)$.

We differentiate both sides of (1)

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$ ...(2)

Since this is an identity, we can put x = 0,

hence the coefficient of $x = a_1 = f'(0)$.

We differentiate both sides of (2)

$f''(x) = 2a_2 + 3.2\,a_3x + ... + n(n-1)a_nx^{n-1}$ ...(3)

The coefficient of $x^2 = a_2 = \frac{1}{2}f''(0)$ and so on.

Example 7: Find the coefficient of x in the determinant

$$= \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

Solution: Let

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$= A + Bx + Cx^2 + Dx^3 + \dots$$

Differentiating both sides w.r.t. x and then putting $x = 0$, we get

The coefficient of $x = B$

$$= \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix} = 0 + 0 + 0 = 0$$

Hence, the coefficient of $x = 0$.

(iii) Derivative of Even/Odd Function

If a function f(x) is even then its derivative $f'(x)$ is an odd function in the domain of $f'(x)$.

We differentiate the identity $f(-x) = f(x)$ w.r.t. x :

to get $-f'(-x) = f'(x)$

$\Rightarrow$ $f'(-x) = -f'(x)$. This shows that $f'(x)$ is odd.

For example, the derivative of an even function cos x + 1 is an odd function – sin x.

Similarly, if a function f(x) is odd, then its derivative $f'(x)$ is even in the domain of $f'(x)$.

For example, the derivative of an odd function tan x is an even function sec²x.

Caution

It should be noted that if $f'(x)$ is odd then f(x) is even, but, if $f'(x)$ is even then f(x) need not be odd. Suppose we consider $f'(x) = \cos x$ corresponding to a function $f(x) = 1 - \sin x$, we see that $f'(x)$ is even, although f(x) is neither even nor odd.

(iv) Derivative of Periodic Function

If a function f(x) is a periodic function with period T then $f'(x)$ is also a periodic function with period T.

But, it should be noted that the fundamental period of $f'(x)$ may be smaller than that of f(x).

We differentiate the identity $f(x + T) = f(x)$.

$\Rightarrow$ $f'(x + T) = f'(x)$

Hence, $f'(x)$ is also periodic with period T.

For example, if $f(x) = \tan x$ having period π then its derivative $\sec^2 x$ is also periodic with period π.

If $f(x) = \dfrac{\sin \pi x}{|\sin \pi x|}$, then its fundamental period is 2. But its derivative, $f'(x) = 0$, $x \neq n$, has a fundamental period of 1, which less than the fundamental period of f(x).

Determination of Multiple Roots of a Polynomial

If a polynomial function f(x) contains a factor $(x - \alpha)^r$, then $f'(x)$ contains a factor $(x - \alpha)^{r-1}$. Therefore if f(x) and $f'(x)$ have no common factor, no factor in f(x) will be repeated; hence the equation $f(x) = 0$ has or has not equal roots, according as f(x) and $f'(x)$ have or have not a common factor involving x.

Proof If α be an r-multiple root of $f(x) = 0$, whose degree is n, then $f(x) = (x - \alpha)^r \phi(x)$ where $\phi(x)$ is a polynomial in x of degree n – r and it is not divisible by $x - \alpha$.

Now, $f'(x) = r(x - \alpha)^{r-1}\phi(x) + (x - \alpha)^r\phi'(x)$

$= (x - \alpha)^{r-1} \{r\phi(x) + (x - \alpha)\phi'(x)\}$

$= (x - \alpha)^{r-1} \psi(x)$, where $\psi(x)$

$= r\phi(x) + (x - \alpha)\phi'(x)$.

$\psi(x)$ is not divisible by $(x - \alpha)$. Therefore, α is an $(r - 1)$ multiple root of $f'(x) = 0$. Consequently $(x - \alpha)^{r-1}$ is the highest common divisor of f(x) and $f'(x)$.

If f(x) has no other multiple root, $(x - \alpha)^{r-1}$ is the highest common divisors of f(x) and $f'(x)$. So, to determine the multiple roots of $f(x) = 0$, we find out the H.C.F. of f(x) and $f'(x)$ and if it is of the form $(x - \alpha)^{p-1}(x - \beta)^{q-1}$...., then α, β,.... are the multiple roots of $f(x) = 0$ with multiplicities p, q,...... respectively.

For example, consider

$f(x) = 2x^5 + 5x^4 + 4x^3 + 2x^2 + 2x + 1$,

we have $f'(x) = 10x^4 + 20x^3 + 12x^2 + 4x + 2$,

$f''(x) = 4(10x^4 + 15x^2 + 6x + 1)$

These all vanish for $x = -1$, which is therefore a triple root of $f(x) = 0$.

We find, in fact, that $f(x) = (x + 1)^3(2x^2 - x + 1)$

Note:

1. If f(x) is divisible by $(x - \alpha)^2$, then $f'(x)$ is divisible by $x - \alpha$ and generally, if f(x) is divisible by $(x - \alpha)^r$, then $f'(x)$ is divisible by $(x - \alpha)^{r-1}$.
2. Conversely, if f(x) and $f'(x)$ both divisible by $x - \alpha$, then f(x) is divisible by $(x - \alpha)^2$; and if f(x) is divisible by $x - \alpha$ and $f'(x)$ by $(x - \alpha)^{r-1}$, then f(x) is divisible by $(x - \alpha)^r$.
3. If α is an r-multiple root of $f(x) = 0$, it is then an $(r - 1)$ multiple root of $f'(x) = 0$, an $(r - 2)$ multiple root of $f''(x) = 0$, and so on.
4. The necessary and sufficient conditions for an r-fold root of $f(x) = 0$ are that the functions f(x), $f'(x)$, $f''(x)$, ..., $f^{(r-1)}(x)$ should simultaneously vanish.

Example 8: Factorize $x^4 - 3x^3 + 4x + 12$.

Solution: Let $f(x) = x^4 - 3x^3 + 4x + 12$.

We have $f'(x) = 4x^3 - 18x + 4$.

The usual test leads to the conclusion that $x - 2$ is a common factor of $f(x)$ and $f'(x)$.

From here we infer that $(x-2)^2$ is a factor of $f(x)$. The remaining factors are then easily ascertained; thus we find $f(x) = (x-2)^2 (x+1)(x+3)$.

Example 9: Find the condition that the cubic equation $x^3 + qx + r = 0$ should have a double root.

Solution: Let $\phi(x) = x^3 + qx + r = 0$

we have $\phi'(x) = 3x^2 + q$

$$3x^2 + q = 0 \Rightarrow x = \pm\sqrt{-\frac{1}{3}q}$$

If a double root exists, it must be one of the values

$\pm\sqrt{-\frac{1}{3}q}$ and such a root must satisfy $\phi(x)$.

Substituting in $\phi(x)$, we find

$$r = \pm\frac{2}{3}\sqrt{\left(-\frac{1}{3}q^3\right)}, \text{ or } r^3 = -\frac{4}{27}q^3,$$

which is the required condition

Example 10: Find the condition that the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may have two roots equal.

In this case the equations $f(x) = 0$, and $f'(x) = 0$, that is

$$ax^3 + 3bx^2 + 3cx + d = 0 \qquad ...(1)$$

$$ax^2 + 2bx + c = 0 \qquad ...(2)$$

must have a common root, and the condition required will be obtained by eliminating x between these two equations.

By combining (1) and (2), we have

$$bx^2 + 2cx + d = 0 \qquad ...(3)$$

From (2) and (3), we obtain

$$\frac{x^2}{2(bd - c^2)} = \frac{x}{bc - ad} = \frac{1}{2(ac - b^2)};$$

thus the required conditions is

$$(bc - ad)^2 = 4(ac - b^2)(bd - c^2).$$

Concept Problems

J

1. By differentiating the addition formula $\sin(x + a) = \sin x \cos a + \cos x \sin a$ obtain the addition formula for the cosine function.
2. Find out whether the following statement is true: If $f(x) < g(x)$, then $f'(x) < g'(x)$.
3. Assume $f'(c) = 3$. Find $f'(-c)$ given the following.
   (i) f is an odd function
   (ii) f is an even function.
4. Use differentiation to prove the identity $2\sin^{-1}x = \cos^{-1}(1 - 2x^2)$, $0 \le x \le 1$.
5. Prove the identity $\sin^{-1}\frac{x-1}{x+1} = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.
6. Make sure by differentiation that the derivatives of both members of the equality
   $$\frac{2\sin^2 x - 1}{\cos x} + \frac{\cos x\,(2\sin x + 1)}{1 + \sin x} = \tan x$$
   are identically equal to each other.
7. Consider the cubic equation $x^3 = 3x + 8$. If we differentiate each side with respect to x, we obtain $3x^2 = 3$, which has the two solutions $x = 1$ and $x = -1$. But neither of these is a solution of the origin cubic equation. What went wrong? Why does differentiation of both sides of the cubic equation gives an invalid result?
8. Show that the function
   $$y = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$$
   is a constant (i.e. is independent of x). Find the value of this constant.
9. If f is derivable on [a, b], $f(a) = f(b) = 0$, and $f(x) \ne 0$ for any x in (a, b), then prove that $f'(a)$ and $f'(b)$ must be of opposite signs.
10. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, determine values of the constants a, b, c, d such that $f'(x) = x\cos x$.
11. If $g(x) = (ax^2 + bx + c)\sin x + (dx^2 + ex + f)\cos x$, determine values of the constants a, b, c, d, e, f such that $g'(x) = x^2 \sin x$.
12. Prove the formula $\cos 2x = \cos^2 x - \sin^2 x$ by differentiating the identity $\sin 2x = 2\sin x\cos x$ termwise.
13. Tell what is wrong with this alleged proof that 2 = 1; Observe that $x^2 = x \cdot x = x + x + \dots + x$ (x times). Differentiation with respect to x yields the equation $2x = 1 + 1 + \dots + 1$ (x is). Thus $2x = x$. Setting $x = 1$ shows that $2 = 1$.
14. Find the sum of $\sum_{n=1}^{\infty} n^2 x^{n-1}$, $|x| < 1$,
15. Find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$ if two roots are equal.
16. Prove that the equation $x^5 - 10x^2 + 15x - 6 = 0$ has a triple root.

Practice Problems

K

17. For what values of x is
   $$f(x) = \sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{x^2-1}{x^2+1}$$
   a constant function? Also find the constant (s).
18. Find the sum of series $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots$ to infinite terms, if $|x| < 1$.

19. Let $f(x) = (1 + x)^3 = a_0 + a_1x + a_2x^2 + a_3x^3$, where the constants a_0, a_1, a_2 and a_3 are to be determined.

(i) Compute $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ using the formula $f(x) = (1 + x)^3$.

(ii) Compute the quantities in (a), using the formula $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

(iii) Comparing the result in (a) and (b), show that $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$.

20. Solve the following equations each of which has equal roots:

(i) $4x^3 - 12x^2 - 15x - 4 = 0$,

(ii) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$.

21. If $x^3 + 3x^2 - 9x + c$ is the product of three factors, two of which are identical, show that c is either 5 or – 27 and resolve the given expression into factors in each case.

4.15 L'Hospital's Rule

In the chapter of limits, we have already discussed the L'Hospital's Rule. But its application was restricted because of our limitation in differentiating various functions. Now, having learnt the methods of differentiation we can go ahead to apply this rule in finding limits.

Suppose f and g are differentiable on an open interval containing a, except possibly at a itself. Assume that $g'(x) \neq 0$ for all x in the open interval containing a, except possibly at a. Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, provided the limit on the right side exists or is ∞ or $-\infty$

Example 1: Find $\lim\limits_{x\to 0} \dfrac{x\sin x}{\sqrt{2x^2+3}-\sqrt{3}}$

Solution: We can find the limit by applying L'Hospital's rule:

$$\lim_{x\to 0} \frac{x\sin x}{\sqrt{2x^2+3}-\sqrt{3}} \quad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{x\to 0} \frac{\sin x + x\cos x}{\left(\dfrac{4x}{2\sqrt{2x^2+3}}\right)}$$

$$= \lim_{x\to 0} \frac{(\sqrt{2x^2+3})(\sin x + x\cos x)}{2x}$$

$$\lim_{x\to 0} \frac{\sqrt{2x^2+3}}{2x} \cdot \lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right) = \frac{\sqrt{3}}{2} \cdot 2$$

Example 2: Evaluate $\lim\limits_{x\to 4} \dfrac{\sqrt{2x^2+1}+\sqrt{x-3}-4}{\sqrt{(3x+4)}+\sqrt{5x+5}-9}$

Solution: Using L'Hospital's rule to the $\left(\frac{0}{0}\right)$ form,

$$l = \lim_{x\to 4} \frac{\dfrac{1}{\sqrt{2x+1}} + \dfrac{1}{2\sqrt{x-3}}}{\dfrac{3}{2\sqrt{2x+3}} + \dfrac{5}{2\sqrt{5x+5}}} = \frac{\dfrac{1}{3}+\dfrac{1}{2}}{\dfrac{3}{8}+\dfrac{1}{2}} = \frac{20}{21}$$

Example 3: Evaluate $\lim\limits_{x\to 0^-} \left(1-2^x\right)^{\sin x}$ (0^0)

Solution: Let the limit be l. We have a (0°) form.

$$\therefore \quad \ln l = \lim_{x\to 0} \sin x \ln\left(1-2^x\right) = \lim_{x\to 0} \frac{\ln\left(1-2^x\right)}{\operatorname{cosec} x}$$

using L'Hospital's rule

$$\ln l = \lim_{x\to 0} \frac{+2^x \ln 2}{\left(1-2^x\right)} \cdot \frac{1}{\operatorname{cosec} x \cdot \cot x}$$

$$= -\lim_{x\to 0} \frac{\ln 2}{\left(\dfrac{2^x-1}{x}\right)} \cdot \frac{\sin x \cdot \tan x}{x}$$

Hence, $\ln l = -\lim\limits_{x\to 0} \tan x = 0$

$\Rightarrow \quad l = e^0 = 1.$

Example 4: Evaluate $\lim\limits_{k\to 0} \dfrac{\ln(\ln(1+k^4))}{\ln(\ln(1+k^2))}$.

Solution: Using L'Hospital's rule to the $\left(\frac{\infty}{\infty}\right)$ form,

$$\lim_{k\to 0} \frac{\ln(\ln(1+k^4))}{\ln(\ln(1+k^2))}$$

$$= \lim_{k\to 0} \frac{4k^3}{(\ln(1+4^4))(1+k^4)} \cdot \frac{(\ln(1+4^2))(1+k^2)}{2k}.$$

$$\left(\lim_{k\to 0} \frac{1+k^2}{1+k^4} = 1\right)$$

$$= = \frac{4}{2} \lim_{k\to 0} \frac{k^2 \ln(1+k^2)k^2}{\ln(1+k^4)k^2}$$

$$= = \frac{4}{2} \lim_{k\to 0} \underbrace{\frac{k^4}{\ln(1+k^4)}}_{1} \cdot \underbrace{\frac{\ln(1+k^2)}{k^2}}_{1} = 2$$

Example 5: Find $\lim\limits_{x\to 0} \left(\dfrac{a\tan^{-1}\dfrac{\sqrt{x}}{a} - b\tan^{-1}\dfrac{\sqrt{x}}{a}}{x\sqrt{x}}\right)$

Solution: Using L'Hospital's rule to the $\left(\frac{0}{0}\right)$ form,

$$= \lim_{x\to 0} \frac{\dfrac{a}{\left(1+\dfrac{x}{a^2}\right)}\cdot\dfrac{1}{a}\cdot\dfrac{1}{2\sqrt{x}} - \dfrac{b}{\left(1+\dfrac{x}{b^2}\right)}\cdot\dfrac{1}{b}\cdot\dfrac{1}{2\sqrt{x}}}{\dfrac{3}{2}\cdot\sqrt{x}}$$

$$= \lim_{x\to 0}\left[\frac{a^2}{(a^2+x)\,x} - \frac{b^2}{(b^2+x)}\,\frac{1}{x}\right]\cdot\frac{1}{3}$$

$$= \frac{a^2-b^2}{3a^2b^2}.$$

Practice Problems L

1. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$

(ii) $\displaystyle\lim_{x\to\infty}\frac{\sin\dfrac{2}{3^x}}{\sin\dfrac{3}{2^x}}$

(iii) $\displaystyle\lim_{x\to 0}\frac{\sin x - x + \dfrac{x^3}{6}}{x^5}$

2. Show that if n be a positive integer

(i) $\displaystyle\lim_{x\to 0^+} x^n \ln x = 0$ (ii) $\displaystyle\lim_{x\to 0^+} x(\ln x)^n = 0$

(iii) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^n} = \infty$ (iv) $\displaystyle\lim_{x\to\infty} x^n e^{-x} = 0$

3. Evaluate the limits:

(i) $\displaystyle\lim_{x\to 0}(\text{cosec } x)^{1/\ln x}$ (ii) $\displaystyle\lim_{x\to\frac{\pi}{2}}(\sec x)^{\cot x}$

(iii) $\displaystyle\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$ (iv) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$

4. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1}\frac{x^2 - x.\ell nx + \ell nx - 1}{x-1}$

(ii) $\displaystyle\lim_{x\to 3}\frac{e^x - e^3}{x-3}$

(iii) $\displaystyle\lim_{x\to 0}\frac{e^{\sin x} - \sin x - 1}{x^2}$

(iv) $\displaystyle\lim_{x\to 0}\left(\frac{e^x - 1 - x}{x^2}\right)$

5. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1}\frac{1+\ln x - x}{1-2x+x^2}$

(ii) $\displaystyle\lim_{x\to 0}\frac{x\cos x - \ln(1+x)}{x^2}$

(iii) $\displaystyle\lim_{x\to 0}\frac{1+\sin x - \cos x + \ln(1-x)}{x\tan^2 x}$

(iv) $\displaystyle\lim_{x\to 0}\frac{e^{x\cos x} - 1 - x}{\sin(x^2)}$

6. Find a and b if $\displaystyle\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$

7. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1}\frac{x^x - x}{x - 1 - \ln x}$

(ii) $\displaystyle\lim_{x\to 0}\frac{1-\cos x\cdot\cos 2x\cdot\cos 3x}{x^2}$

(iii) $\displaystyle\lim_{x\to 0}\frac{(\cos ax)^{1/m} - (\cos bx)^{1/n}}{x^2}$

(iv) $\displaystyle\lim_{x\to\pi/2}\frac{\cot x - \cos x}{(\pi - 2x)^3}$

Target Problems for JEE Advanced

Problem 1: If $f(x) = (ln\ x)^x$, find $f'(x)$ using first principles.

Solution: $\displaystyle f'(x) = \lim_{h\to 0}\frac{[\ell n\,(x+h)]^{x+h} - (\ell n\ x)^x}{h}$

$$= \lim_{h\to 0}\frac{e^{(x+h)\,\ell n\,[\ell n\,(x+h)]} - e^{x\,\ell n\,(\ell n\ x)}}{h}$$

$$= e^{x\,ln\,(ln\ x)}\lim_{h\to 0}\left[\frac{e^{(x+h)\,\ell n\,(\ell n\,(x+h)) - x\,\ell n\,(\ell n\ x)} - 1}{h}\right]$$

$$= (\ln x)^x \lim_{\substack{h \to 0 \\ s \to 0}} \left[\frac{e^s - 1}{s} \cdot \frac{s}{h}\right]$$

where $s = (x+h)\ln[\ln(x+h)] - x\ln(\ln x)$

Note that as $h \to 0$, $s \to 0$

$$\text{since } \lim_{s \to 0} \frac{e^s - 1}{s} = 1$$

$$= (\ln x)^x \lim_{h \to 0} \frac{(x+h)\,\ell n\,[\ell n\,(x+h)] - x\,\ell n\,(\ell n\,x)}{h}$$

$$= (\ln x)^x \left[x\left\{\lim_{h\to 0} \frac{\ell n(\ell n(x+h)) - \ell n(\ell n\,x)}{h}\right\}\right.$$

$$\left. + \lim_{h \to 0} \ell n(\ell n\,(x+h))\right] \qquad ...(1)$$

$$\text{Let } l = \lim_{h \to 0} \frac{\ell n\left(\dfrac{\ell n\,(x+h)}{\ell n\,x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ell n\left(\dfrac{\ell n\,x\left(1+\frac{h}{x}\right)}{\ell n\,x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ell n\left(\dfrac{\ell n\,x + \ell n\,\left(1+\frac{h}{x}\right)}{\ell n\,x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ell n\left(1 + \dfrac{\ell n\,\left(1+\frac{h}{x}\right)}{\ell n\,x}\right)}{h}$$

$$= \lim_{h \to 0} l n \left(1 + \frac{\ell n\,\left(1+\frac{h}{x}\right)}{\ell n\,x}\right)^{1/h}$$

$$= \lim_{h \to 0} l n\ e^{\lim\limits_{h \to 0} \frac{1}{h}\left(\frac{\ell n\left(1+\frac{h}{x}\right)}{\ell n\,x}\right)}$$

$$= \lim_{h \to 0} \frac{\ell n\,\left(1+\frac{h}{x}\right)^{1/h}}{\ell n\,x} = \frac{1}{x\,\ell n\,x}$$

Substituting the value of l in (1)

$$= (l n\,x)^x \cdot \left[\left\{x\left(\frac{1}{x\,\ell n\,x}\right)\right\} + \ell n\,(\ell n\,x)\right]$$

$$= (l n\,x)^x \left[\frac{1}{\ell n\,x} + \ell n\,(\ell n\,x)\right].$$

Problem 2: If $y = 1 + \dfrac{c_1}{x - c_1} + \dfrac{c_2 x}{(x-c_1)(x-c_2)} + \dfrac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$

Show that $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{c_1}{c_1 - x} + \dfrac{c_2}{c_2 - x} + \dfrac{c_3}{c_3 - x}\right]$.

Solution: $y = \left(1 - \dfrac{c_1}{x - c_1}\right) + \dfrac{c_2 x}{(x-c_1)(x-c_2)}$

$$+ \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \left[\frac{x}{x - c_1} + \frac{c_2 x}{(x-c_1)(x-c_2)}\right]$$

$$+ \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \frac{x^2}{(x-c_1)(x-c_2)} + \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \frac{x^3}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \frac{x}{x-c_1} \cdot \frac{x}{x-c_2} \cdot \frac{x}{x-c_3}$$

$$\therefore \quad \ln|y| = \ln\left|\frac{x}{x-c_1}\right| + \ln\left|\frac{x}{x-c_2}\right| + \ln\left|\frac{x}{x-c_3}\right|$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{x-c_1}{x} \cdot \frac{x-c_1-x}{(x-c_1)^2} + \frac{x-c_2}{x} \cdot \frac{x-c_2-x}{(x-c_2)^2}$$

$$+ \frac{x-c_3}{x}\ \frac{x-c_3-x}{(x-c_3)^2}$$

$$= -\frac{1}{x}\left[\frac{c_1}{x-c_1} + \frac{c_2}{x-c_2} + \frac{c_3}{x-c_3}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}\left[\frac{c_1}{c_1-x} + \frac{c_2}{c_2-x} + \frac{c_3}{c_3-x}\right].$$

Problem 3: If $y = \tan^{-1}\left(\dfrac{x}{1{\cdot}2 + x^2}\right)$

$+ \tan^{-1}\left(\dfrac{x}{2{\cdot}3 + x^2}\right) + \tan^{-1}\left(\dfrac{x}{3{\cdot}4 + x^2}\right) + ...$ up to n terms.

Find $\dfrac{dy}{dx}$ expressing the answer in two terms.

Solution:

$$T_n = \tan^{-1}\left(\frac{x}{n(n+1) + x^2}\right) = \tan^{-1}\frac{\left(\dfrac{x}{n} - \dfrac{x}{n+1}\right)}{1 + \dfrac{x}{n} \cdot \dfrac{x}{n+1}}$$

$$T_n = \left(\tan^{-1}\frac{x}{n} - \tan^{-1}\frac{x}{n+1}\right)$$

$$y = \sum T_n = \tan^{-1} x - \tan^{-1} \frac{x}{n+1}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{1+\left(\frac{x}{n+1}\right)^2} \cdot \frac{1}{n+1}$$

$$= \frac{1}{1+x^2} - \frac{n+1}{(n+1)^2 + n^2}.$$

Problem 4: Suppose f and g are two functions such that $f, g : R \to R$, $f(x) = \ln\left(1+\sqrt{1+x^2}\right)$ and $g(x) = \ln\left(x+\sqrt{1+x^2}\right)$ then find the value of

$$x\, e^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x) \text{ at } x = 1.$$

Solution: $\ln\left(x+\sqrt{1+x^2}\right) = g(x)$

$$\therefore \quad g'(x) = \frac{1}{\sqrt{1+x^2}} \qquad ...(1)$$

Also $\left(f\left(\frac{1}{x}\right)\right)' = f'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$

Now, $f'(x) = \dfrac{1}{1+\sqrt{1+x^2}} \cdot \dfrac{x}{\sqrt{1+x^2}}$

$$f'\left(\frac{1}{x}\right) = \frac{x}{1+\sqrt{1+x^2}} \cdot \frac{1}{x} \cdot \frac{x}{\sqrt{1+x^2}}$$

$$f'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$$

$$= -\frac{1}{x\sqrt{1+x^2}} \cdot \frac{1}{e^{g(x)}}\, e^{g(x)} \cdot x \cdot \left(f\left(\frac{1}{x}\right)\right)'$$

$$= -\frac{1}{\sqrt{1+x^2}} \qquad ...(2)$$

From (1) and (2)

$$g'(x) + x\, e^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' = 0.$$

Problem 5: Find $\dfrac{dy}{dx}$, if $(\tan^{-1} x)^y + y^{\cot x} = 1$.

Solution: $(\tan^{-1}x)^y + y^{\cot x} = 1$

$$e^{y\,\ell n\,(\tan^{-1} x)} + e^{\cot x\,.\,\ell n\, y} = 1$$

Differentiating both sides,

$$(\tan^{-1} x)^y \left[\frac{y}{\tan^{-1} x\,(1+x^2)} + \ell n\,(\tan^{-1} x)\, y_1\right]$$

$$+\; y^{\cot x}\left[\frac{\cot x}{y}\,.\,y_1 - \ell n\; y\, \operatorname{cosec}^2 x\right] = 0$$

$$(\ell n\,(\tan^{-1} x)\,.\,(\tan^{-1} x)^y + y^{\cot x - 1}\,.\,\cot x)\, y_1$$

$$= y^{\cot x}\,.\,\ell n\, y\; \operatorname{cosec}^2 x - \frac{(\tan^{-1} x)^{y-1}\; y}{1+x^2}$$

$$\frac{dy}{dx} = \frac{y^{\cot x}\,.\,\ell n\, y \operatorname{cosec}^2 x - (\tan^{-1} x)^{y-1}\,.\,y\left(1+x^2\right)^{-1}}{\ell n(\tan^{-1} x).(\tan^{-1} x)^y + y^{\cot x - 1}.\cot x}.$$

Problem 6: If $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \tan^{-1}\left(\dfrac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)$

where $a \in (-\infty, -1) \cup (1, \infty)$, then find $y'\left(\frac{\pi}{2}\right)$.

Solution: Let $\sqrt{a^2-1} = C_1$

and $a + \sqrt{a^2-1} = C_2$

$\Rightarrow \quad C_2 - C_1 = a$

$$\therefore \quad y = \frac{x}{C_1} - \frac{2}{C_1} \tan^{-1}\left(\frac{\sin x}{C_2 + \cos x}\right)$$

$$\frac{dy}{dx} = \frac{1}{C_1} - \frac{2}{C_1} \frac{\left[\left(C_2 + \cos x\right)\cos x + \sin^2 x\right]}{\left[1+\left(\frac{\sin x}{C_2 + \cos x}\right)^2\right]\cdot\left(C_2 + \cos x\right)^2}$$

$$= \frac{1}{C_1} - \frac{2}{C_1} \cdot \frac{C_2 \cos x + 1}{(C_2 + \cos x)^2 + \sin^2 x}$$

$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{2}} = \frac{1}{C_1} - \frac{2}{C_1} \cdot \frac{1}{C_2^2 + 1} = \frac{1}{C_1}\left[\frac{C_2^2 + 1 - 2}{C_2^2 + 1}\right]$$

$$= \frac{1}{C_1} \cdot \frac{C_2^2 - 1}{C_2^2 + 1}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=\frac{\pi}{2}} = \frac{2a^2 + 2a\sqrt{a^2-1} - 2}{\sqrt{a^2-1}\left(2a^2 - \cancel{1} + 2a\sqrt{a^2-1} + \cancel{1}\right)}$$

$$= \frac{2a^2 + 2a\sqrt{a^2-1} - 2}{\sqrt{a^2-1}\left(2a\left(a+\sqrt{a^2-1}\right)\right)}$$

$$= \frac{2(a^2-1) + 2a\sqrt{a^2-1}}{\sqrt{a^2-1}\left(2a\left(a+\sqrt{a^2-1}\right)\right)}$$

$$= \frac{2\sqrt{a^2-1}\left(\sqrt{a^2-1}+a\right)}{\left(\sqrt{a^2-1}\right)(2a)\left(a+\sqrt{a^2-1}\right)} = \frac{1}{a}.$$

Problem 7: If $x = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y + \text{....to } \infty}}}$, prove that

$\frac{dy}{dx} = 2x^2 + y^2 - 3xy.$

Solution: $x = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y + \text{....to } \infty}}}$

$\therefore \quad x = y + \frac{1}{x}$ ...(1)

Differentiating w. r. t. x, we get, $1 = \frac{dy}{dx} - \frac{1}{x^2}$

or $\frac{dy}{dx} = 1 + \frac{1}{x^2}$

or $\frac{dy}{dx} = 1 + (x - y)^2$ $\left[\because \text{ from(1), } \frac{1}{x} = x - y\right]$

$= 1 + x^2 + y^2 - 2xy$ ...(2)

From (1) $x^2 = xy + 1$ $\therefore$ $1 = x^2 - xy$

Putting in (2), we get $\frac{dy}{dx} = x^2 - xy + x^2 + y^2 - 2xy$

Hence $\frac{dy}{dx} = 2x^2 + y^2 - 3xy.$

Problem 8: If $y = \cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}$ then find $y'(x)$.

Solution: $y'(x) = -\frac{1}{\sqrt{1-\left(\frac{x^{2n}-1}{x^{2n}+1}\right)^2}}\frac{d}{dx}\left(\frac{x^{2n}-1}{x^{2n}+1}\right)$

$$= \frac{x^{2n}+1}{\sqrt{(x^{2n}+1)^2-(x^{2n}-1)^2}}$$

$$\times \frac{2nx^{2n-1}(x^{2n}+1) - 2nx^{2n-1}(x^{2n}-1)}{(x^{2n}+1)^2}$$

$$= -\frac{1}{\sqrt{4x^{2n}}}\frac{4nx^{2n-1}}{(x^{2n}-1)} = -\frac{2nx^{2n-1}}{|x^n|(x^{2n}+1)}$$

$$= -\frac{2n(|x|^2)^n}{x|x|^n(x^{2n}+1)} = -\frac{2n|x|^n}{x(x^{2n}+1)}$$

$$= \begin{cases} -\dfrac{2nx^{n-1}}{x^{2n}+1} & \text{if } n \text{ is even} \\ -\dfrac{2nx^n}{|x|(x^{2n}+1)} & \text{if } n \text{ is odd} \end{cases}$$

Problem 9: If $y = \sqrt{\ln\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$, then find $\frac{dy}{dx}$.

Solution: $y = \sqrt{\ln\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$

Putting $\frac{x^2}{3} - 1 = v$, $\sin\left(\frac{x^2}{3}-1\right) = \sin v = u$ and

$\ln\left\{\sin\left(\frac{x^2}{3}-1\right)\right\} = \ln u = z.$

we get $y = \sqrt{z}$, $z = \ln u$, $u = \sin v$ and $v = \frac{x^2}{3} - 1$

$\therefore \quad \frac{dy}{dz} = \frac{1}{2\sqrt{z}}, \quad \frac{dz}{du} = \frac{1}{u}, \quad \frac{du}{dv} = \cos v$ and $\frac{dv}{dx} = \frac{2x}{3}$

Now $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{1}{2\sqrt{z}}\right)\left(\frac{1}{u}\right)(\cos v)\left(\frac{2v}{3}\right) = \frac{x}{3}.\frac{\cos v}{u\sqrt{\ln u}}$

$$= \frac{x\cot\left(\frac{x^2}{3}-1\right)}{\sqrt[3]{\ln\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}.$$

Problem 10: If $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, find whether $f'(0)$ is existent or not.

Solution: Put $2^x = \tan\theta$, $x \in R$

$f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+2\tan\theta}\right) = \sin^{-1}(\sin 2\theta)$

$(x \in R,\ 0 < \theta < \pi/2)$

If $x < 0$, $0 < \theta < \pi/4$,

$f(x) = \sin^{-1}(\sin 2\theta)$

$$= \begin{cases} 2\tan^{-1} 2^x & \text{if } x \le 0 \Rightarrow 0 < \theta \le \pi/4 \\ \pi - 2\tan^{-1} 2x & \text{if } x > 0 \Rightarrow \theta > \pi/4 \end{cases}$$

$\therefore \quad f'(0^-) = \ln 2$ and $f'(0^+) = -\ln 2$.

Hence, $f'(0)$ does not exist.

Problem 11: Find $\dfrac{d^2y}{dx^2}$ for the cycloid whose equation is $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution: We have $x = a(\theta - \sin\theta)$.

$\therefore \quad (dx/d\theta) = a(1 - \cos\theta)$.

Also $y = a(1 - \cos\theta)$;

$\therefore \quad (dx/d\theta) = a\sin\theta$.

$$\text{Now } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\frac{\theta}{2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot\theta/2)$$

$$= \left[\frac{d}{d\theta}(\cot\theta/2)\right]\frac{d\theta}{dx}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{a(1-\cos\theta)}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2a\sin^2\theta/2} = -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}.$$

Problem 12: Find the value of $\dfrac{d^2y}{dx^2}$ if $x^2 - y^4 = 6$.

Solution: $y^4 = x^2 - 6$

$$4y^3\frac{dy}{dx} = 2x \;\Rightarrow\; y^3\frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \quad y^3\frac{d^2y}{dx^2} + 3y^2\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}$$

$$y^3\frac{d^2y}{dx^2} + 3y^2\left(\frac{x}{2y^3}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \quad y^3\frac{d^2y}{dx^2} + 3y^2\frac{x^2}{4y^6} = \frac{1}{2}$$

$$\Rightarrow \quad y^3\frac{d^2y}{dx^2} + \frac{3x^2}{4y^4} = \frac{1}{2}$$

$$\Rightarrow \quad y^3\frac{d^2y}{dx^2} = \frac{1}{2} - \frac{3x^2}{4y^4}$$

$$\Rightarrow \quad y^3\frac{d^2y}{dx^2} = \frac{2y^4 - 3x^2}{4y^4}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2y^4 - 3x^2}{4y^7}.$$

Problem 13: Change the independent variable to θ in the equation $\dfrac{d^2y}{dx^2} + \dfrac{2x}{1+x^2}\dfrac{dy}{dx} + \dfrac{y}{(1+x^2)^2} = 0$, by means of the transformation $x = \tan\theta$.

Solution: We have $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \cos^2\theta\dfrac{dy}{d\theta}$

$$\frac{dx}{d\theta} = \sec^2\theta \ne 0 \qquad \text{for any value of } \theta.$$

$$\text{Again, } \frac{d^2y}{dx^2} = -2\cos\theta\sin\theta\cdot\frac{d\theta}{dx}\cdot\frac{dy}{d\theta} + \cos^2\theta\cdot\frac{d^2y}{d\theta^2}\;\frac{d\theta}{dx}$$

$$= -2\cos\theta\sin\theta\cos^2\theta\cdot\frac{dy}{d\theta} + \cos^2\theta\cdot\frac{d^2y}{d\theta^2}\cdot\cos^2\theta$$

$$= -2\sin\theta\cos^3\theta\cdot\frac{dy}{d\theta} + \cos^4\theta\,\frac{d^2y}{d\theta^2}.$$

Substituting the values of x, dy/dx and d^2y/dx^2 in the given differential equation, we have

$$-2\sin\theta\cos^3\theta\frac{dy}{d\theta} + \cos^4\theta\frac{d^2y}{d\theta^2} + \frac{2\tan\theta}{1+\tan^2\theta}\cdot\cos^2\theta\frac{dy}{d\theta} + \frac{y}{(1+\tan^2\theta)^2} = 0$$

$$\Rightarrow -2\sin\theta\cos^3\theta\frac{dy}{d\theta} + \cos^4\theta\cdot\frac{d^2y}{d\theta^2} + 2\sin\theta\,.\cos^3\theta\frac{dy}{d\theta} + \cos^4\theta\,.\,y = 0$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} + y = 0.$$

Problem 14: If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then express y as an explicit function of x and prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 25y.$$

Solution: $\left(y^{\frac{1}{5}} - y^{-\frac{1}{5}}\right)^2 = \left(y^{\frac{1}{5}} + y^{-\frac{1}{5}}\right)^2 - 4$

$$= 4x^2 - 4 \qquad \left\{\because\ y^{\frac{1}{5}} - y^{-\frac{1}{5}} = 2x\right\}$$

$$\therefore\quad 2y^{\frac{1}{5}} = \left(y^{\frac{1}{5}} + y^{-\frac{1}{5}}\right) + \left(y^{\frac{1}{5}} - y^{-\frac{1}{5}}\right)$$

$$= 2x + 2\sqrt{x^2 - 1}$$

$$\therefore\quad y = \left(x + \sqrt{x^2-1}\right)^5 \qquad \text{...(1)}$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = 5\left(x+\sqrt{x^2-1}\right)^4\left\{1 + \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2-1}}\right\}$$

$$= 5\,\frac{\left(x+\sqrt{x^2-1}\right)^5}{\sqrt{x^2-1}} = \frac{5y}{\sqrt{x^2-1}} \quad \{\text{using (1)}\}$$

$$\therefore\quad (x^2-1)\left(\frac{dy}{dx}\right)^2 = 25y^2$$

Differentiating w.r.t. x,

$$2x\left(\frac{dy}{dx}\right)^2 + (x^2-1).2\,\frac{dy}{dx}.\frac{d^2y}{dx^2} = 50y\,\frac{dy}{dx}$$

$$\text{or,}\quad x\frac{dy}{dx} + \left(x^2-1\right)\frac{d^2y}{dx^2} = 25y$$

$$\left\{\because\ \frac{dy}{dx} \neq 0 \text{ then y is a constant}\right\}$$

Problem 15: Given F(x) = f(x) . ϕ(x) and f ′(x) . f ′(x) . f ′(x) = c then prove that

$$\frac{F'''}{F} = \frac{f'''}{f} + \frac{\varphi'''}{\varphi},\ \text{where c} \in \text{constant, when}$$

f(x), ϕ(x), F(x) are differentiable

Solution: Given F(x) = f(x) . ϕ(x) ...(1)

and f ′(x) = f(x) . ϕ′(x) + f ′(x) . ϕ(x) ...(2)

On differentiating (1) w.r.t. x, we get,

$$F'(x) = f(x) . \phi'(x) + f'(x) . \phi(x)$$

Again differentiating;

$$F''(x) = \{f(x)\,\phi''(x) + f'(x)\,\phi'(x)\} + \{f'(x) . \phi'(x) + f''(x)\,\phi(x)\}$$

$$F''(x) = f(x)\,\phi''(x) + 2f'(x)\,\phi'(x)\} + \{f'(x) . \phi'(x)$$

$$\text{or}\quad F''(x) = f(x)\,\phi''(x) + f''(x)\,\phi(x) + 2c \qquad [\text{using (2)}]$$

Now again differentiating w.r.t. x, we get

$$F'''(x) = f(x) . \phi'''(x) + f'(x) . \phi''(x) + f''(x) . \phi'(x) + f'''(x) . \phi(x) \qquad \text{...(3)}$$

On differentiating (2) w.r.t. x, we get

$$f'(x) . \phi''(x) + \phi'(x) . f''(x) = 0 \qquad \text{...(4)}$$

From (3) and (4), we get,

$$F'''(x) = f(x)\,\phi'''(x) + \phi(x) . f'''(x) + 0$$

$$\text{or}\quad \frac{F'''(x)}{F(x)} = \frac{f(x).\varphi'''(x)}{F(x)} + \frac{\varphi(x).f'''(x)}{F(x)}$$

$$\frac{F'''(x)}{F(x)} = \frac{\varphi'''(x)}{\varphi(x)} + \frac{f'''(x)}{f(x)} \quad [\because F(x) = f(x)\,\phi(x)]$$

$$\text{or}\quad \frac{F'''}{F} = \frac{\varphi'''}{\varphi} + \frac{f'''}{f}$$

Problem 16: Let f(x) = e^{-1/x^2} sin (1/x) when x ≠ 0 and f(0) = 0. Show that at every point f has a differential coefficient and this is continuous at x = 0. The differential coefficient vanishes at x = 0 and at an infinite number of points in the neighbourhood of x = 0. Show that the function f′ like f makes an infinite number of oscillations.

Solution: At x = 0, we have

$$f'(0^+) = \lim_{h\to 0}\frac{e^{-1/h^2}\sin(1/h) - 0}{h} = \lim_{h\to 0}\frac{\sin(1/h)}{he^{1/h^2}}$$

$$= \lim_{h\to 0}\frac{\sin(1/h)}{\left\{1 + \frac{1}{h^2} + \frac{1}{2!}\cdot\frac{1}{h^4} + ...\right\}h}$$

$$= \lim_{h\to 0}\frac{\sin(1/h)}{h + \frac{1}{h} + \frac{1}{2!}\cdot\frac{1}{h^3} + ...} = 0.$$

Similarly f′(0⁻) = 0, Hence f′(0) = 0.

At any point other than zero,

$$f'(x) = (2/x^3)\,e^{-1/x^2}\sin(1/x) - (1/x^2)\,e^{-1/x^4}\cos(1/x)$$

$$= \{(2/x).\sin(1/x) - \cos(1/x)\}\,(1/x^2)\,(1/e^{1/x^2}), \text{ ...(1)}$$

$$\text{Then } f'(0^+) = \lim_{h\to 0}\left(\frac{2}{h}\sin\frac{1}{h} - \cos\frac{1}{h}\right)\cdot\frac{1}{h^2e^{1/h^2}}$$

$$= \lim_{h\to 0}\left(\frac{2\sin(1/h)}{h^3e^{1/h^2}} - \frac{\cos(1/h)}{h^2e^{1/h^2}}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\sin(1/h)}{h^3\left(1+\frac{1}{h^2}+\frac{1}{2!}\cdot\frac{1}{h^4}+...\right)} - \frac{\cos(1/h)}{h^2\left(1+\frac{1}{h^2}+\frac{1}{2!}\cdot\frac{1}{h^4}+...\right)}\right)$$

$$= \frac{\text{some finite quantity}}{\infty} - \frac{\text{some finite quantity}}{\infty} = 0$$

Similarly f′(0⁻) = 0. Hence f′ is continuous at x = 0.

$$\text{Now, if } x = \frac{1}{2n\pi},\ \text{then } f'\left(\frac{1}{2n\pi}\right)$$

$$= (0 - 1)\cdot e^{4n^2\pi^2/4n^2\pi^2}$$

$$= -\,e < 0 \text{ and similarly at}$$

$$x = \frac{1}{(4n+1)\pi}, \quad f'\left(\frac{1}{(2n+1)\pi}\right) > 0$$

Hence between $\frac{1}{2n\pi}$ and $\frac{1}{(2n+1)\pi}$, $f'(x)$ changes from negative to positive values and since f' is continuous in $\left[\frac{1}{2n\pi}, \frac{1}{(2n+1)\,\pi}\right]$, it follows that f' must take all values between

$$f'\left(\frac{1}{2n\pi}\right) \text{ and } f'\left(\frac{1}{(2n+1)\,\pi}\right).$$

Giving successively large values to n, we see that f′ changes from negative to positive, taking all intermediate values in the neighbourhood of zero an infinite number of times. It follows that f′ makes an infinite number of oscillations like f (which can be easily proved) in the neighbourhood of the origin.

Problem 17: A function f(x) is so defined that for all x, $[f(x)]^n = f(nx)$. Prove that $f(x) \,.\, f'(nx) = f'(x) \,.\, f(nx)$, where $f'(x)$ denotes derivative of f(x) w.r.t.x.

Solution: Given $[f(x)]^n = f(nx)$ ...(1)

Differentiating both sides w. r t. x, we get

$$n[f(x)]^{n-1} \,.\, f'(x) = f'(nx) \,.\, (n \,.\, 1)$$

or, $[f(x)^n \,.\, f'(x) = f'(nx)$

Multiplying both sides by f′(x), we get

$$[f(x)^n \,.\, f'(x) = f'(nx) \,.\, f(x)$$

or, $f(nx) \,.\, f'(x) = f'(nx) \,.\, f(x)$ [from (1)]

Problem 18: Using the series

$$\frac{(\sin^{-1}x)^2}{2!} = \frac{x^2}{2!} + \frac{2^2 x^4}{4!} + \frac{2^2.4^2}{6!}x^6 + \frac{2^2.4^2.6^2}{8!}x^8 + \ldots$$

prove that $\frac{2\theta}{\sin 2\theta} = 1 + \frac{2}{3}\sin^2\theta + \frac{2.4}{3.5}\sin^4\theta$

$$+ \frac{2.4.6}{3.5.7}\sin^6\theta + \ldots$$

Solution: On differentiating both sides, we get a new series $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2^2}{3!}x^3$

$$+ \frac{2^2.4^2}{5!}x^5 + \frac{2^2.4^2.6^2}{7!}x^7 + \ldots$$

If we put x = sin θ we may write this as

$$\frac{2\theta}{\sin 2\theta} = 1 + \frac{2^2}{3!}\sin^2\theta$$

$$+ \frac{2^2.4^2}{5!}\sin^4\theta + \frac{2^2.4^2.6^2}{7!}\sin^6\theta + \ldots$$

$$= 1 + \frac{2}{3}\sin^2\theta + \frac{2.4}{3.5}\sin^4\theta + \frac{2.4.6}{3.5.7}\sin^6\theta + \ldots$$

Problem 19: If $x_1, x_1, x_2, x_3, \ldots\ldots\ x_{n-1}$ be n zero's of the polynomial $P(x) = x^n + \alpha x + \beta$, where $x_i \neq x_j\ \forall$ i and $j = 1, 2, 3, \ldots.. (n-1)$. Show that the value of $Q(x) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots\ldots (x_1 - x_{n-1})$, is ${}^nC_2\, x_1^{n-2}$.

Solution: Let $P(x) = x^n + \alpha x + \beta$

As $x_1, x_1, x_2, x_3, \ldots\ldots x_{n-1}$ be zeroes of P(x)

$$P(x) = (x - x_1)^2 (x - x_2)(x - x_3)(x - x_4)$$

$$\ldots\ldots (x - x_{n-1}) = x^n + \alpha x + \beta$$

Let $Q(x) = (x - x_2)(x - x_3)(x - x_4) \ldots.. (x - x_{n-1})$

$$= \frac{P(x)}{(x - x_1)^2}$$

Then $Q(x_1) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots..(x_1 - x_{n-1})$

As Q(x) is a polynomial function hence it is continuous and differentiable at $x = x_1$

$$\therefore\ Q(x_1) = \lim_{x\to x_1} \frac{P(x)}{(x - x_1)^2} = \lim_{x\to x_1} \frac{x^n + \alpha x + \beta}{(x - x_1)^2}$$

Applying L'Hospital's rule twice or otherwise

$$Q(X_1) = \frac{n(n-1)}{2} X_1^{n-2} = {}^nC_2 \cdot X_1^{n-2}.$$

Things to Remember

1. **The Product rule**
   $(uv)' = uv' + vu'$
   $(uvw)' = u'vw + uv'w + uvw'$
2. **The Quotient rule**
   $$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$
3. **The Chain rule**
   (i) If y = f(u) and u = g(x), then y = f(g(x)) then
   $$\frac{dy}{dx} = (f(g(x)))' = f'(g(x)).g'(x))$$
   i.e. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
   (ii) If y = f(x), u = g(v) and v = h(x), then
   $(f(g(h)))' = f'(g(h))g'(h)h'$
   i.e. $\frac{dy}{du} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

4. Logarithmic differentiation

(i) If $y = \frac{u_1 u_2 u_3 ...}{v_1 v_2 v_3 ...}$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \frac{1}{u_3}\frac{du_3}{dx} + ...$$

$$-\frac{1}{v_1}\frac{dv_1}{dx} - \frac{1}{v_2}\frac{dv_2}{dx} - \frac{1}{v_3}\frac{dv_3}{dx} - ....$$

(ii) If $y = u^v$, $y' = vu^{v-1}u' + u^v v'.\ln u$

5. Differentiation of inverse function

(i) $\frac{dx}{dy} = \frac{1}{dy/dx}$, $\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$,

$$\frac{d^3x}{dy^3} = -\frac{\left\{\frac{dy}{dx}.\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{d^2y}\right)^2\right\}}{\left(\frac{dy}{dx}\right)^5}$$

(ii) If g is inverse of f then $g'(y) = \frac{1}{f'(x)}$,

$$g''(y) = -\frac{f''(x)}{f'(x)^3},$$

$$g'''(y) = -\frac{f'(x).f'''(x) - 3\left(f''(x)\right)^2}{\left(f'(x)\right)^5}.$$

6. Differentiation of parametric function

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

7. Rules of higher order derivatives

(i) $\frac{d^2(uv)}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

(ii) $\frac{d^3(uv)}{dx^2} = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$

(iii) If $y = f(u)$ and $u = g(x)$, then

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

8. Leibniz rule

If $y = uv$, then

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + ... + {}^nC_r u_{n-r} v_r + ... + {}^nC_n u v_n$$

9. Differentiation of determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

10. Darboux Theorem

Let I be an interval. Then the function $f : I \to R$ is said to have the intermediate value (or Darboux property) if and only if for every $a, b \in I$ and for every k between fA and fB there exists c between a and b such that fD = k.

11. Derivative of even/odd funtion

If a function f(x) is even then its derivative f′(x) is an odd function in the domain of f′(x).

Similarly, if a function f(x) is odd, then its derivative f′(x) is even in the domain of f′(x).

12. Derivative of periodic funtion

If a function f(x) is a periodic function with period T then f′(x) is also a periodic function with period T. But, it should be noted that the fundamental period of f′(x) may be smaller than that of f(x).

13. Determination of multiple roots of a polynomial

If a polynomial function f(x) contains a factor $(x-a)^r$, then f′(x) contians a factor $(x-a)^{r-1}$.

Objective Exercises

Single Correct Answer Type

1. If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

(A) $\frac{1}{2}(\sqrt{3}+1)$
(B) $2(\sqrt{3}-1)$
(C) $\frac{1}{2}(\sqrt{3}-1)$
(D) none of these

2. Let $f(x) = x[x]$, $x \notin I$, where [.] denotes the greatest integer function, then f′(x) is equal to

(A) 2 x
(B) [x]
(C) 2[x]
(D) none of these

3. Let $f(x) = x^2 \ln(g(x))$ where g(x) is a differentiable positive function on $(0, \infty)$ satisfying $g(2) = 3$, $g'(2) = -4$, then $f'(2)$ equals.

(A) $-\frac{16}{6} + 8\ln(3)$ (B) $-\frac{16}{3} + 4\ln(3)$

(C) $4\ln(3)$ (D) $-\frac{16}{3} + 4\ln(6)$

4. If $x = t\cos t$, $y = t + \sin t$, then $\frac{d^2x}{dy^2}$ at $t = \frac{\pi}{2}$ is equal to

(A) $\left(\frac{\pi+4}{2}\right)$ (B) $-\left(\frac{\pi-4}{2}\right)$

(C) $-\left(\frac{\pi+4}{2}\right)$ (D) $\left(\frac{\pi-4}{4}\right)$

5. If $f(x^2) = x^4 + x^3 + 1$, then $f'(x^4)$

(A) $\frac{4x^2+3x}{2}$ (B) $\frac{4x+3}{2x}$

(C) $x^2\frac{(4x^2+3)}{2}$ (D) None of these

6. If $y = \ln|\sec(e^{x^2})|$, then $\frac{dy}{dx}$ is equal to

(A) $2x\,e^{x^2}(\tan e^{x^2})$ (B) $2x\,e^{x^2}(\sec e^{x^2})$

(C) $x^2 e^{x^2}\tan e^{x^2}$ (D) $e^{x^2}\tan e^{x^2}(\tan e^{x^2})$

7. If $f(x) = \frac{x}{1+e^{1/x}}$, $x \neq 0$, then

(A) $x^3 f'(x) - x^2 f(x) - e^{1/x}(f(x))^2 = 0$

(B) $x^3 f'(x) + x^2 f(x) - e^{1/x}(f(x))^2 = 0$

(C) $x f'(x) - f(x) + e^{1/x}(f(x))^2 = 0$

(D) $x f'(x) - f(x) - e^{1/x}(f(x))^2 = 0$

8. If g is the inverse of f and $f'(x) = \cos 2x$ then $g'(x)$ equals

(A) $2\sec[2\{g(x)\}]$ (B) $\frac{\sec[2\{g(x)\}]}{2}$

(C) $\sec 2(g(x))$ (D) None of these

9. The function $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ satisfies

(A) $y = xy' + \ln y$ (B) $2y = x + \ln y'$

(C) $2y = xy' + \ln y'$ (D) $y + y' + \ln y = 0$

10. If $f(x) = \ln(\ln \sin x)$ then $f'\left(\frac{\pi}{6}\right)$ has the value

(A) $-\frac{1}{\sqrt{3}\,\ell n 2}$ (B) $\frac{\sqrt{3}}{\ell n 2}$

(C) $-\frac{\sqrt{3}}{\ell n 2}$ (D) none

11. Let $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$, then the least value of n for which $\left|\frac{d^n}{dx^n}f(x)\right|_{x=0}$ is non-zero is

(A) 4 (B) 5

(C) 7 (D) 3

12. A function $f: R \to R$ satifies $\sin x \cos y(f(2x+2y) - f(2x-2y)) = \cos x \sin y \,.\, f(2x+2y) + f(2x-2y))$

If $f'(0) = 1/2$, then

(A) $f''(x) - f'(x) = 0$ (B) $4f''(x) + f(x) = 0$

(C) $f''(x) + f(x) = 0$ (D) $4f''(x) - f(x) = 0$

13. If $y = \tan^{-1}\left(\frac{2^x}{1+2^{x+1}}\right)$, then $\frac{dy}{dx}$ at $x = 0$ is

(A) 1 (B) 2

(C) $-\frac{3}{5}\ln 2$ (D) $\frac{\ln 2}{10}$

14. $\lim\limits_{x\to1}\frac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$ where n = 100 is equal to

(A) $\frac{5050}{\pi e}$ (B) $\frac{100}{\pi e}$

(C) $-\frac{5050}{\pi e}$ (D) $-\frac{4950}{\pi e}$

15. If $y = \log_{e^x}(x-2)^2$ for $x \neq 0, 2$ then $y'(3)$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{4}{3}$ (D) None

16. Let $f: [0, \infty) \to [0, \infty)$ be a function defined by $y = f(x) = x^2$ then $\left(\frac{d^2y}{dx^2}\right) \cdot \left(\frac{d^2x}{dy^2}\right)$ is equal to

(A) 1 (B) $\frac{1}{2x^2}$

(C) $\frac{1}{2x^3}$ (D) $-\frac{1}{2x^3}$

17. Let $f: R \to R$ be a differentiable function satisfying $f(y) \,.\, f(x-y) = f(x)$ $\forall$ x, y $\in$ R and $f'(5) = q$ and $\frac{f(0)}{f'(0)} = \frac{1}{p}$ then f(5) is

(A) $\frac{p^2}{q}$ (B) $\frac{p}{q}$

(C) $\frac{q}{p}$ (D) q

18. If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$, then $\frac{dy}{dx}$ is equal to

(A) y
(B) $y + \frac{x^n}{n!}$
(C) $y - \frac{x^n}{n!}$
(D) $y - 1 - \frac{x^n}{n!}$

19. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is

(A) 25
(B) 9
(C) – 15
(D) – 9

20. If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ equals

(A) $\frac{1}{1+[g(x)-x]^2}$
(B) $\frac{1}{2-[g(x)-x]^2}$
(C) $\frac{1}{2+[g(x)-x]^2}$
(D) None of these

21. If $f(x) = x^4 \tan(x^3) - x \ln(1 + x^2)$, then the value of $\frac{d^4(f(x))}{dx^4}$ at x = 0 is

(A) 0
(B) 6
(C) 12
(D) 24

22. If $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\sin x)$ and $\phi(x) = f(f(f(x)))$, then $\phi'(x)$

(A) 1
(B) sin x
(C) 0
(D None

23. If $y = \tan^{-1}\frac{3a^2x - x^3}{a(a^2 - 3x^2)}$ then $\frac{dy}{dx}$ is

(A) $3\left(\frac{a}{a^2 + x^2}\right)$
(B) $2\left(\frac{a}{a^2 - x^2}\right)$
(C) $\left(\frac{a}{a^2 - x^2}\right)$
(D) $\left(\frac{5a}{a^2 - x^2}\right)$

24. If $ax^2 + 2hxy + by^2 = 1$, then $\frac{d^2y}{dx^2}$ equals

(A) $\frac{h^2 + ab}{(hx + by)^3}$
(B) $\frac{h^2 - ab}{(hx + by)^2}$
(C) 1
(D) $\frac{h^2 - ab}{(hx + by)^3}$

25. If a curve is given by $x = a\cos t + \frac{b}{2}\cos 2t$ and $y = a\sin t + \frac{b}{2}\sin 2t$, then the points for which $\frac{d^2y}{dx^2} = 0$ are given by

(A) $\sin t = \frac{2a^2 + b^2}{3ab}$
(B) $\cos t = -\frac{a^2 + 2b^2}{3ab}$
(C) $\tan t = a/b$
(D) None of these

26. If f(x), g(x), h(x) be such that $f'(x) = g(x)$, $g'(x) = h(x)$ & $h'(x) = f(x)$, $f(0) = 1$, $g(0) = h(0) = 0$, then $f^3 + g^3 + h^3 - 3fgh$, equals

(A) 1
(B) 2
(C) 0
(D) 8

27. Let u(x) and v(x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

(A) 1
(B) 0
(C) 7
(D) – 7

28. Suppose f is a differentiable function such that for every real number x, $f(x) + 2f(-x) = \sin x$, then $f'\left(\frac{\pi}{4}\right)$ has the value equal to

(A) $\frac{1}{\sqrt{2}}$
(B) $-\frac{1}{\sqrt{2}}$
(C) $-\frac{1}{2\sqrt{2}}$
(D) $\sqrt{2}$

29. If $f(1) = 3$, $f'(1) = 2$, $f''(1) = 4$, then$(f^{-1})''(3) =$

(A) 1
(B) $-\frac{1}{2}$
(C) –2
(D) none

30. f(x), g(x), h(x) are functions having non-zero derivatives, the derivative of f(x) with respect to g(x) is $\alpha(x)$ and derivative of g(x) with respect to h(x) is $\beta(x)$. Then derivative of h(x) with respect to f(x) equal

(A) $\alpha(x)\,\beta(x)$
(B) $\frac{\alpha(x)}{\beta(x)}$
(C) $\frac{1}{\alpha(x)\beta(x)}$
(D) $\frac{\beta(x)}{\alpha(x)}$

31. If $3f(\cos x) + 2f(\sin x) = 5x$, then $f'(\cos x) =$ (where f′ denotes derivative with respect to cos x)

(A) $-\frac{5}{\cos x}$
(B) $\frac{5}{\cos x}$
(C) $-\frac{5}{\sin x}$
(D) $\frac{5}{\sin x}$

32. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\frac{d^3y}{dx^3}$ equals

(A) $24a^2(at + b)$
(B) $24a(ax + b)^2$
(C) $24a(at + b)^2$
(D) $24a^2(ax + b)$

33. Let $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ then

(A) $f'(1) = 1$ and $g'(1) = 2$
(B) $g'(1) = 2$ and $g'(1) = 1$
(C) $f'(1) = 1$ and $g'(1) = 0$
(D) $f'(1) = 1$ and $g'(1) = 1$

34. Let $f(x)$, $f'(x)$ and $f''(x)$ are all positive $\forall\ x \in [0, 7]$. If $f^{-1}(x)$ exists, then $f^{-1}(5) + 4f^{-1}\left(\frac{29}{5}\right)$ is

(A) always positive (B) always negative
(C) non-negative (D) non-positive

35. If $f(x) = \sin x + \ell n\ x$, then $f'(x^2)$ is equal to

(A) $\frac{1}{2x}\left(\cos x + \frac{1}{x}\right)$ (B) $\cos x^2 + \frac{1}{x^2}$
(C) $2x \sin x^2 + \frac{2}{x}$ (D) None of these

36. If $y = (\tan x)^{(\tan x)^{\tan x}}$, then at $x = \frac{\pi}{4}, \frac{dy}{dx}$ is equal to

(A) 0 (B) 1
(C) 2 (D) None

37. If $f(x) = \frac{a + \sqrt{a^2 - x^2} + x}{\sqrt{a^2 - x^2} + a - x}$ where $a > 0$ and $x < a$, then $f'(0)$ has the value equal to

(A) $\sqrt{a}$ (B) a
(C) $\frac{1}{\sqrt{a}}$ (D) $\frac{1}{a}$

38. If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = K$ then the value of K is equal to

(A) 1 (B) –1
(C) 2 (D) 0

39. If $y = (1 - x)^{-\alpha} e^{-\alpha x}$ $x \neq 1$ then

(A) $(1 - x) y'' + (1 - \alpha x)y' - \alpha y = 0$
(B) $(1 - x) y'' - (1 + \alpha x)y' - \alpha y = 0$
(C) $(1 - x) y'' + (1 + \alpha x)y' - \alpha y = 0$
(D) $(1 - x) y'' + (1 + \alpha x)y' + \alpha y = 0$

40. If y is a function of x then $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$. If x is a function of y then the equation becomes

(A) $\frac{d^2x}{dy^2} + x\frac{dx}{dy} = 0$

(B) $\frac{d^2x}{dy^2} + y\left(\frac{dx}{dy}\right)^3 = 0$

(C) $\frac{d^2x}{dy^2} - y\left(\frac{dx}{dy}\right)^2 = 0$

(D) $\frac{d^2x}{dy^2} - x\left(\frac{dx}{dy}\right)^2 = 0$

41. A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Then the value of $f'(4) =$

(A) 12 (B) 3
(C) 3/2 (D) cannot be determined

42. Let $f(x) = x^n$, $n \in W$. The number of values of n for which $f'(p + q) = f'(p) + f'(q)$ is valid for all positive p and q is

(A) 0 (B) 1
(C) 2 (D) none

43. If $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then $g'(2\pi)$ is equal to

(A) $\frac{7}{3}$ (B) $\frac{3}{7}$
(C) $\frac{30\pi^4 + 4}{3}$ (D) $\frac{3}{30\pi^4 + 4}$

44. Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ has the value equal to

(A) $\sqrt{2} - 1$ (B) $\frac{\sqrt{2} + 1}{\sqrt{2}}$
(C) $2 - \sqrt{2}$ (D) $\sqrt{2} + 1$

45. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 2\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 0 (B) –12
(C) 4 (D) 1

46. let $f(x) = \sin^{-1}(\cos 2x)$ be the equation of a curve then $f(5) + f'(5)$ is equal to

(A) $10 - 7\pi/2$ (B) $12 - 5\pi/2$
(C) $12 - 7\pi/2$ (D) $10 - 5\pi/2$

47. The differentiation of $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$ with respect to $\sqrt{1 - x^2}$ at $x = -1/2$ is

(A) 1 (B) –1
(C) 4 (D) none

48. If $(f(x))^n = f(nx)$, then which of the following is correct

(A) $n(f(x))^{n-1} f'(x) = f'(nx)$
(B) $(f(x))^{n-1} f'(x) = nf'(nx)$
(C) $f(nx) f'(x) = f'(nx) f(x)$
(D) $f(nx) f(x) = f'(nx) f'(x)$

49. If $x^2 + y^2 = a^2$ and $k = 1/a$, then k is equal to

(A) $\frac{y''}{\sqrt{1 + y'}}$ (B) $\frac{|y''|}{\sqrt{(1 + y'^2)^3}}$
(C) $\frac{2y''}{\sqrt{1 + y'^2}}$ (D) $\frac{y''}{2\sqrt{(1 + y'^2)^3}}$

50. Let $f(x) = \frac{x^n(1-x)^n}{n!}$. Then for any integer $k \geq 0$, the k-th derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$

(A) are both 0

(B) are both rational numbers but not necessarily integers

(C) are both integers

(D) do not satisfy any of the foregoing properties

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$ (B) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2x}$

(C) $\frac{1}{2\sqrt{x}}\sqrt{y^2-4}$ (D) $\frac{1}{2\sqrt{x}}\sqrt{y^2+4}$

52. Choose the correct statement:

(A) If u(x) is differentiable then $\frac{d}{dx}|u| = \frac{uu'}{|u|}$, $u \neq 0$

(B) If u(x) = sin bx then $u''(x) + b^2u(x) = 0$

(C) If $g(x) = \sqrt{x(x+n)}$ and $a = \frac{2x+n}{2}$, then $\frac{dg}{dx} = \frac{a}{g}$

(D) none of these

53. If 1 is a double root of the equation $ax^3 + bx^2 + bx + d = 0$, then

(A) a= b = d (B) a + b = 0

(C) b + d = 0 (D) a = d

54. $f(x) = |x^2 - 3|x| + 2|$, then which of the following is/are true

(A) $f'(x) = 2x - 3$ for $x \in (0, 1) \cup (2, \infty)$

(B) $f'(x) = 2x + 3$ for $x \in (-\infty, -2) \cup (-1, 0)$

(C) $f'(x) = -2x - 3$ for $x \in (-2, -1)$

(D) None of these

55. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $z = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, then $\frac{dy}{dz}$ for different x may be equal to

(A) – 2 (B) 1

(C) 2 (D) $\frac{1}{2}$

56. If $y = \frac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1}$ and $\frac{dy}{dx} = ax + b$, then the value of a – b is

(A) $\cot\frac{\pi}{8}$ (B) $\cot\frac{5\pi}{12}$

(C) $\tan\frac{5\pi}{12}$ (D) $\frac{5\pi}{8}$

57. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}x$, then

(A) $f'(10) = 1$

(B) $f'(3/2) = -1$

(C) Domain of f(x) is $x \geq 1$

(D) Range of f(x) is $(-2, -1] \cup (2, \infty)$

58. Write the following statements are true:

(A) If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then $(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + 4)$

(B) If $x = \cos^{-1}\frac{1}{\sqrt{(t^2+1)}}$, $y = \sin^{-1}\frac{t}{\sqrt{(1+t^2)}}$ then $\frac{dy}{dx}$ is independent of t.

(C) If $e^y + xy = e$, then the value of $\frac{d^2y}{dx^2}$ for x = 0 is $1/e^2$

(D) If $\phi(x) = f(x)\,g(x)$, where $f'(x)\,g'(x) = c$, then $\frac{\phi''}{\phi} = \frac{f''}{f} + \frac{g''}{g} + \frac{2c}{fg}$

59. The differential coefficient of $\sin^{-1}\frac{t}{\sqrt{1+t^2}}$ w.r.t. $\cos^{-1}\frac{t}{\sqrt{1+t^2}}$ can be

(A) 1, if t > 0 (B) – 1, if t > 0

(C) $\frac{t}{\sqrt{1+t^2}}$ (D) – 1, if t < 0

60. If $\sqrt{y+x} + \sqrt{y-x} = c$ (where $c \neq 0$), then $\frac{dy}{dx}$ has the value equal to

(A) $\frac{2x}{c^2}$ (B) $\frac{x}{y + \sqrt{y^2 - x^2}}$

(C) $\frac{y - \sqrt{y^2 - x^2}}{x}$ (D) $\frac{c^2}{2y}$

61. Which of the following statements are true?

(A) If $xe^{xy} = y + \sin^2x$, then at $y'(0) = 1$.

(B) If $f(x) = a_0x^{2m+1} + a_1x^{2m} + a_3x^{2m-1} + \ldots\ldots.. + a_{2m+1} = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational coefficients then the equation $f'(x) = 0$ must have a real root. ($m \in N$).

(C) If $(x - r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ...+a_0$ repeated m times where $1 \le m \le n$ then r is a root of the equation $f'(x) = 0$ repeated $(m - 1)$ times

(D) If $y = \sin^{-1}(\cos \sin^{-1} x) + \cos^{-1}(\sin \cos^{-1} x)$ then $\frac{dy}{dx}$ is independent on x.

62. Let α be a repeated root of $p(x) = x^3 + 3ax^2 + 3bx + c = 0$, then

(A) α is a root of $x^2 + 2ax + b = 0$

(B) $\alpha = \frac{c - ab}{2(a^2 - b)}$

(C) $\alpha = \frac{ab - c}{a^2 - b}$

(D) α is a root of $ax^2 + 2bx + c = 0$

63. Let $y = \sqrt{(\sin x + \sin 2x + \sin 3x)^2 + (\cos x + \cos 2x + \cos 3x)^2}$ then which of the following is correct?

(A) $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ is -2

(B) value of y when $x = \frac{\pi}{5}$ is $\frac{3 + \sqrt{5}}{2}$

(C) value of y when $x = \frac{\pi}{12}$ is $\frac{\sqrt{1} + \sqrt{2} + \sqrt{3}}{2}$

(D) y simplifies to $(1 + 2 \cos x)$ in $[0, \pi]$

64. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x - 1}}}{\sqrt{x - 1} - 1}$. x then

(A) $f'(10) = 1$ (B) $f'(3/2) = -1$

(C) domain of f(x) is ≥ 1 (D) none of these.

65. If $f(x) = \sqrt{1 - \sin 2x}$, then $f'(x)$ is equal to

(A) $-(\cos x + \sin x)$, for $x \in (\pi/4, \pi/2)$

(B) $\cos x + \sin x$, for $x \in (0, \pi/4)$

(C) $-(\cos x + \sin x)$, for $x \in (0, \pi/4)$

(D) $\cos x + \sin x$, for $x \in (\pi/4, \pi/2)$

66. If $y = x^{(\ell n x)^{\ell n(\ell n x)}}$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{y}{x}\left(\ell n x^{\ell n x - 1} + 2\ell n x \ \ell n(\ell n x)\right)$

(B) $\frac{y}{x}\left((\ln x)^{\ln(\ln x)}(2\ln(\ln x) + 1)\right)$

(C) $\frac{y}{x \ell n x}\left((\ln x)^2 + 2\ln(\ln x)\right)$

(D) $\frac{y \ell n y}{x \ell n x}\left(2\ln(\ln x) + 1\right)$

67. Let $f(x) = \frac{1 - x^{n+1}}{1 - x}$ and $g(x) = 1 + \frac{2}{x} + \frac{3}{x^2} - ....+ (-1)^n \frac{n+1}{x^n}$. Then the constant term in $f'(x) \times g(x)$ is equal to

(A) $\frac{n(n^2 - 1)}{6}$ when n is even

(B) $\frac{n(n + 1)}{2}$ when n is odd

(C) $-\frac{n}{2}(n + 1)$ when n is even

(D) $\frac{n(n - 1)}{2}$ when n is odd

68. If $f(x) = |\ln|x||$, then $f'(x)$ equals

(A) $\frac{-\text{sgn } x}{|x|}$, for $|x| < 1$, where $x \ne 0$

(B) $\frac{1}{x}$ for $|x| > 1$ and $-\frac{1}{x}$ for $|x| < 1, x \ne 0$

(C) $-\frac{1}{x}$ for $|x| > 1$ and $\frac{1}{x}$ for $|x| < 1$

(D) $\frac{1}{x}$ for $|x| > 0$ and $-\frac{1}{x}$ for $x < 0$

69. If $(x - c)^2 + (y - a)^2 = b^2$, then $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is independent of

(A) b (B) a

(C) c at x = 0 (D) None of these

70. If $y = x^{x^2}$ then $\frac{dy}{dx}$ is equal to

(A) $2 \ln x \, . \, x^{x^2}$

(B) $(2 \ln x + 1) \, . \, x^{x^2}$

(C) $(2 \ln x + 1) \, . \, x^{x^2 + 1}$

(D) $x^{x^2 + 1} \, . \, \ln ex^2$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

71. **Assertion (A) :** Let f(x) = cos (cos (cos (cos (cos (cos (cos (cos x)))))))) and suppose that the number a satisfies the equation a = cos a. Then f ′A cannot be expressed as a polyomial in a.
Reason (R) : $f'A = \sin^8 a = (1 - a^2)^4 = a^8 - 4a^6 + 6a^4 - 4a^2 + 1$.

72. **Assertion (A) :** If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\pi/2 < x < \pi$ then dy/dx equals –2.
Reason (R) : If $\pi/2 < x < \pi$, then $y = 2\pi - 2x$.

73. **Assertion (A) :** Let g(x) = f(x) sin x, where f(x) is a twice differentiable function on $(-\infty, \infty)$ such that $f'(-\pi) = 1$. The value of $g''(-\pi)$ equals – 2.
Reason (R) : We have $g''(x) = f(x)(-\sin x) + f'(x)\cos x + f'(x)\cos x + f''(x)\sin x$. Hence $g''(-\pi) = -2$.

74. **Assertion (A) :** Let f : R → R be a smooth function such that $f'(x) = f(1 - x)$ for all x and f(0) = 1. Then f(x) = cos x + (sec 1 + tan 1) sin x.
Reason (R) : Differentiating the given equation gives $f''(x) = -f(x)$. This has solution of the form f(x) = A cos x + B sin x, when A & B are determined by the boundary conditions.

75. **Assertion (A) :** If two differentiable functions f(x) and g(x) satisfy $\dfrac{f'(x)}{g'(x)} = e^{f(x) - g(x)}$ for all x, then $e^{-f(x)} - e^{-g(x)}$ is constant.
Reason (R): The given equation can be rearranged as $\dfrac{d}{dx}(e^{-f(x)} + e^{-g(x)}) = 0$

76. **Assertion (A):** If the function f(x) – f(2x) has derivative 5 at x = 1 and derivative 7 at x = 2, then the derivative of f(x) – f(4x) at x = 1 is 19.
Reason (R) : Let g(x) = f(x) – f(2x).
Then $\dfrac{d}{dx}(f(x) - f(4x)) = g'(x) + g'(2x)$

77. **Assertion (A) :** If $f(x) = 2\sin^{-1}\sqrt{1-x} + \sin^{-1}\left(2\sqrt{x(1-x)}\right)$, then $f'(x) = 0$, if $x \in \left(0, \dfrac{1}{2}\right)$.
Reason (R) : f(x) is a constant function.

78. **Assertion (A) :** If a differentiable function f(x) satisfies the relation f(x) + f(x – 2) = 0 ∀ x ∈ R, and if $\left(\dfrac{d}{dx}f(x)\right)_{x=a} = b$, then $\left(\dfrac{d}{dx}f(x)\right)_{a+4000} = b$
Reason (R) : f(x) is a periodic function with period 4.

79. If for some differentiable function f, $f(\alpha) = 0$ and $f'(\alpha) = 0$.
Assertion (A) : The sign of f(x) does not change in the neighbourhood of x = α.
Reason (R) : α is repeated root of f(x) = 0

80. Let the function f satisfy the relation, $f(x+y^3) = f(x) + f(y^3)$, ∀ x, y ∈ R and be differentiable for all x.
Assertion (A) : If $f'(2) = a$, then $f'(-2) = a$.
Reason (R) : f(x) is an odd function

Comprehension – 1

A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$, a, b > 0 and a ≠ 1, b ≠ 1 where t ∈ R.

81. Which of the following is not a correct expression for $\dfrac{dy}{dx}$?
(A) $\dfrac{-1}{f(t)^2}$ (B) $-(g(t))^2$
(C) $\dfrac{-g(t)}{f(t)}$ (D) $\dfrac{-f(t)}{g(t)}$

82. The value of $\dfrac{d^2y}{dx^2}$ at the point where f (t) = g(t) is
(A) 0 (B) $\dfrac{1}{2}$
(C) 1 (D) 2

83. The value of $\dfrac{f(t)}{f'(t)} \cdot \dfrac{f''(-t)}{f'(-t)} + \dfrac{f(-t)}{f'(-t)} \cdot \dfrac{f''(t)}{f'(t)}$ ∀ t ∈ R, is equal to
(A) – 2 (B) 2
(C) – 4 (D) 4

Comprehension – 2

Let

$$f(x) = \begin{cases} \max.\ \{t^3 - t^2 + t + 1, 0 \le t \le x\} & 0 \le x \le 1 \\ \min.\ \{3 - t, 1 < t \le x\} & 1 < x \le 2 \end{cases}$$

and g(x)

$$= \begin{cases} \max.\ \{\frac{3}{8}t^4 + \frac{1}{2}t^3 - \frac{3}{2}t^2 + 1, 0 \le t \le x\} & 0 \le x < 1 \\ \min.\ \{\frac{3}{8}t + \frac{1}{32}\sin^2 \pi t + \frac{5}{8}, 1 \le t \le x\} & 1 \le x \le 2 \end{cases}$$

Define f(x) and g(x) explicitly and then answer the following questions.

84. The function f(x), ∀ x ∈ [0, 2] is-
(A) continuous and differentiable
(B) continuous but not differentiable
(C) discontinuous and not differentiable

(D) none of these

85. Which of the following is true?

(A) $\lim\limits_{x\to1^-} fog(x) > \lim\limits_{x\to1^+} gof(x)$

(B) $\lim\limits_{x\to1^-} fog(x) < \lim\limits_{x\to1^+} gof(x)$

(C) $\lim\limits_{x\to1^-} fog(x) = \lim\limits_{x\to1^+} gof(x)$

(D) none of these

86. Let $Z(x) = \frac{d}{dx} f(x)^{g(x)}$ and $Y(x) = \frac{d}{dx} g(x)^{f(x)}$ then Z(x) and Y(x) vanish simultaneously at

(A) $x = -\frac{1}{3}$ (B) $x = 0$

(C) $x = 1$ (D) No real value of x

Comprehension – 3

While finding the sine of a certain angle x, an absent minded professor failed to notice that his calculator was not in the correct angular mode. However he was lucky to get the right answer. The two least positive values of *x* for which the sine of *x* degrees is the same as the sine of *x* radians were found by him as $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$ where m, n, p and q are positive integers. Suppose $\frac{mn}{pq}$ be denoted by the quantity 'L'. Now answer the following questions.

87. The value of (m + n + p + q) is equal to

(A) 720 (B) 900

(C) 1080 (D) 1260

88. If x is measured in radians and $\lim\limits_{x\to\infty}\left(\sqrt{Ax^2+Bx}-Cx\right)$ = L, then the value of $\frac{BC}{A}$ equals (A, B, C∈R)

(A) 4 (B) 2

(C) $\frac{1}{2}$ (D) none

89. Assume that *f* is differentiable for all x. The sign of f' is as follows:

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Let $g(x) = f(10 - 2x)$. The value of $g'(L)$ is

(A) positive

(B) negative

(C) zero

(D) the function g is not differentiable at x = 5

Comprehension – 4

Let f(x) be a differentiable function, we define a mathematical operation

$G * (f(x)) = \lim\limits_{h\to0}\frac{f^m(x+h)-f^m(x)}{h}$, where m is a fixed natural number.

90. G * (f(x) . g(x)) is equal to

(A) $(G * f(x)) . g(x)^m - (G * g(x)) . f(x)^m$

(B) $(G * f(x)) . g(x)^m + (G * g(x)) . f(x)^m$

(C) $(G * f(x)) . g(x)^{m-1} + (G * g(x)) . f(x)^{m-1}$

(D) none of these

91. $G * \left(\frac{f(x)}{g(x)}\right)$ is equal to

(A) $\frac{(G*g(x)).f(x)^m+(G*f(x))g(x)^m}{g(x)^{2m}}$

(B) $\frac{(G*g(x)).f(x)^{m-1}+(G*f(x))g(x)^{m-1}}{g(x)^{2m}}$

(C) $\frac{(G*g(x)).f(x)^m-(G*f(x))g(x)^m}{g(x)^{2m}}$

(D) none of these

92. G * f(g(x)) is equal to

(A) $m f'(g(x)).g'(x)\ f^{m-1}(g(x))$

(B) $m f'(g(x)).g'(x)\ f^{m}(g(x))$

(C) $m f'(g(x)).g(x)\ f^{m-1}(g(x))$

(D) none of these

MATCH THE COLUMNS FOR JEE ADVANCED

93.

Column-I	**Column-II**
(A) If $\sqrt{x^2+y^2} = ae^{\tan^{-1}(y/x)}$, a > 0, assuming y > 0, then, $y''(0) = -\frac{K}{a}e^{-\pi/2}$ where K is	(P) $-\frac{2}{5}$
(B) If y = sin–1 $\frac{2x}{1+x^2}$ then $\frac{dy}{dx}$ at x = –2	(Q) 2
(C) If $\lim\limits_{x\to 0}\frac{4+\sin 2x+A\sin x+B\cos x}{x^2}$ exist then A– B	(R) 5
(D) If f(x) = [2 + 3sinx] 0 < x < π where [ ] denotes GIF then number of points at which function is non differentiable	(S) –4

94.

Column-I	**Column-II**
(A) Suppose that the functions F (x) and G (x) satisfy the following properties F (3) = 2, G(3) = 4, G(0) = 3, F′ (3) = – 1, G′ (3) = 0 ; G′ (0) = 2	(P) – 1
If T(x) = $F(G(x))$ and U(x) = $\ln(F(x))$, then T′(0) + $6U'(3)$ has the value equal to	(Q) $3\sqrt{2}$
(B) $\lim\limits_{x\to-\infty}(x+\sqrt{x^2+2x})$	(R) 2
(C) If y = t^2 at x = t + t^2, then $\frac{1}{2}\frac{d^2y}{dx^2}$ at t = –1 is	(S) 3
(D) If f(x) = x + x^n, a value of n for which f(x) is invertible for all x ∈ R, is	(T) – 5

95.

Column-I	**Column-II**
(A) If y = $3e^{2x} + 2e^{3x}$ and $\frac{d^2y}{dx^2}$ + a. $\frac{dy}{dx}$ + by = 0. where a and b are real numbers, then a + b =	(P) $\frac{\pi}{2}$
(B) $\lim\limits_{x\to 0^+}((x\cos x)^x + (x\sin x)^{1/x}) =$	(Q) –1
(C) If f(x) = $x^{\sin x} + (\sin x)^{\cos x}$, then $f'\left(\frac{\pi}{2}\right)$	(R) 0
(D) Number of positive integer values of x > 4 and satisfying the inequality $\sin^{-1}(\sin 5) < 4x - x^2 + 2$ is	(S) 1

96.

Column–I	**Column–II**
(A) If | g(x) – g(y) | ≤ | x – y |^{99}, (n ∈ N) then g′(x) equals	(P) – 2
(B) Differential coefficient of $\sin^{-1}\frac{2x}{1+x^2}$ with respect to $\cos^{-1}\frac{1-x^2}{1+x^2}$ in the domain of f(x) = $\frac{1}{\sqrt{1-x^2}}$ is	(Q) – 1
(C) If y = $\cot^{-1}(\tan x) + \tan^{-1}(\cot x)$, π/2 < x < π, then dy/dx equals	(R) 0
(D) If y = y (x) and it follows the relation $4xe^{xy} = y + 5\sin^2 x$, then y′ (0) is equal to	(S) 1
	(T) 4

97. Find $\frac{dy}{dx}$

Column–I	**Column–II**
(A) $\sin^{-1}(2x\sqrt{1-x^2}),\ \left(x < -\frac{1}{\sqrt{2}}\right)$	(P) 0
(B) $2\sin^{-1}(\sqrt{1-x} + \sin^{-1}(2\sqrt{x(1-x)}),\ \left(0 < x < \frac{1}{2}\right)$	(Q) $-\frac{2}{1+x^2}$
(C) $\sin^{-1}(3x - 4x^3),\ 0 < x < \frac{1}{2}$	(R) $\frac{3}{\sqrt{1-x^2}}$
(D) $\cos^{-1}\frac{2x}{1+x^2}$ for $\lvert x \rvert > 1$	(S) $-\frac{2}{\sqrt{1-x^2}}$

Review Exercises for JEE Advanced

1. Find the derivative of f(tanx) w.r.t. g(sec x) at $x = \frac{\pi}{4}$ if $f'(1) = 2,\ g'(\sqrt{2}) = 4$

2. Find $\frac{dy}{dx}$, where $y = \left(\sqrt{x}\right)^{x^{x^{\cdots\cdots\text{to } \infty}}}$.

3. If $f(x) = \cos\left\{\frac{\pi}{2}[x] - x^3\right\},\ 1 < x < 2$ and $[x]$ = the greatest integer $\le x$, then find $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$.

4. Let f be defined in $(-\pi/2, \pi/2)$ by $f(x) = x + \tan x$. Show that f has an inverse function g, and that $f'(y) = \frac{1}{2 + (y - g(y))^2}$.

5. Solve the inequality $f'(x) < g'(x)$, if

A $f(x) = x + 3\ln(x - 2),\ g(x) = x + 5\ln(x - 1)$

B $f(x) = e^{2x} - 3x,\ g(x) = 5(e^x - x + 3)$.

6. Find $f'(x)$ if $f(x) = \frac{\sqrt[3]{2x + 2\sqrt{x^2 - 1}}}{\left(\frac{\sqrt{x-1}}{\sqrt{x+1}} + \frac{\sqrt{x+1}}{\sqrt{x-1}} + 2\right)^{1/3}}$.

7. If $x^4 + 7x^2y^2 + 9y^4 = 24xy^3$, show that $\frac{dy}{dx} = \frac{y}{x}$.

8. If $f(x) = \lvert x \rvert^{\lvert \sin x \rvert}$, then find the value of $f'\left(-\frac{\pi}{4}\right)$.

9. If $y = x\ln\left(\frac{x}{a + bx}\right)$ then show that $x^3\frac{d^2y}{dx^2}$ is equal to $\left(x\frac{dy}{dx} - y\right)^2$.

10. If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x \ldots\ldots \infty}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$.

11. If $y = \frac{u(x)}{v(x)}$ then show that v^3y'' equals $\begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$.

12. If $y = \tan^{-1}\frac{x}{1 + \sqrt{1 - x^2}} + \sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$ then find $\frac{dy}{dx}$.

13. Prove $\frac{d^3x}{dy^3} = -\left\{\frac{d^3y}{dx^3}\frac{dy}{dx} - 3\left(\frac{d^2y}{dx^2}\right)^2\right\} \Big/ \left(\frac{dy}{dx}\right)^5$.

If g(x) is inverse function of $f(x) = x^3 + 3x + 4$ then find $g'''(x)$ at $x = 0$.

14. If $f(x) = (\sin x)(\sin 2x) \ldots\ldots\ldots (\sin nx)$, then show that $f'(x)$ is equal to $\sum_{k=1}^{n}(k\cot kx)f(x)$.

15. Let $f(x) = \sin^{-1} 2x\sqrt{1 - x^2} + \sec^{-1}\frac{1}{\sqrt{1 - x^2}}$. Find $f'\left(-\frac{1}{2}\right)$.

16. If $\cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \cos\frac{x}{2^3} \ldots\ldots\ldots\ldots \infty = \frac{\sin x}{x}$ then show that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + \ldots$ is equal to $\frac{(x^2 + 2)\sin x - 2x\cos x}{x^3}$.

17. If g(x) is a polynomial of degree 2 and if g(1) = g(–1) and x_1, x_2, x_3 are is G.P. then show that $g'(x_1), g'(x_2), g'(x_1), g'(x_3)$ are in GP.

18. Let f, g and h are differentiable functions. If f(0) = 1; g(0) = 2; h (0) = 3 and the derivatives of their pair wise products at x = 0 are

$(fg)'(0) = 6;\ (gh)'(0) = 4$ and $(hf)'(0) = 5$

then compute the value of $(fgh)'(0)$.

19. Find the derivative with respect to x of the function:

$(\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \arcsin \dfrac{2x}{1+x^2}$ at $x = \dfrac{\pi}{4}$.

20. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$, prove that $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

21. If $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$ then find $\dfrac{dy}{dx}$.

22. If the substitution $z = \ln\left(\tan\dfrac{x}{2}\right)$ changes the differential equation $D^2y + \cot x \cdot Dy + 4y\,\text{cosec}^2 x = 0$ {where D denotes $\dfrac{d}{dx}$ } to $\dfrac{d^2y}{dz^2} = ky$ then find ‘k’.

23. If $\sqrt{x^2+y^2} = ae^{\tan^{-1} y/x}$, $a > 0$, $y(0) \neq 0$ then find the value of $y''(0)$

24. If $y = (\tan^{-1}x)^2$ then find the value of $(x^2+1)^2 D^2y + 2x(x^2+1)\,Dy$

25. Suppose $a = \dfrac{dy}{dx}$ of $x^2 + y^2 = 4$ at $(\sqrt{2}, \sqrt{2})$, $b = \dfrac{dy}{dx}$ of $\sin y + \sin x = \sin x \cdot \sin y$ at (π, π) and $c = \dfrac{dy}{dx}$ of $2e^{xy} + e^x e^y - e^x - e^y = e^{xy+1}$ at (1, 1), then find the value of (a + b + c).

26. If $y = \cfrac{x}{x + \cfrac{\sqrt[3]{x}}{x + \cfrac{\sqrt[3]{x}}{x + \sqrt[3]{x} + \ldots\ldots\infty}}}$, find $\dfrac{dy}{dx}$.

27. Find f ′(x) if f(x) is

(i) $\dfrac{\sqrt{1+\left(\dfrac{x^2-1}{2x}\right)^2}}{(x^2+1)\dfrac{1}{x^2}}$

(ii) $\sqrt{x+2\sqrt{2x-4}} + \sqrt{x-2\sqrt{2x-4}}$

28. Given functions F, G and H satisfying $F' = G$, $G' = H$, and $H' = F$ Prove that

(i) $(F + G + H)' = F + G + H$

(ii) $(F^3 + G^3 + H^3 - 3FGH)' = 0$

29. Prove that if a and b are positive then

$$\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n \to \sqrt{ab} \text{ as } n \to \infty.$$

30. If $y = e^{\tan^{-1}x}$, show that

$$(x^2+1)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0.$$

31. If $y = \dfrac{ax+b}{x^2+c}$, show that

$(2xy' + y)y'' = 3(xy'' + y')y''$,

where a, b, c are constants.

32. Evaluate the following limits:

(i) $\lim\limits_{x\to 0^+} x^m(\ln x)^n$, $m, n \in N$

(ii) $\lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\ln(1-x)}}$

(iii) $\lim\limits_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$

(iv) $\lim\limits_{x\to 0}\dfrac{1}{x^3}\left(\text{cosec}x - \dfrac{1}{x} - \dfrac{x}{6}\right)$

33. Evaluate the following limits:

(i) $\lim\limits_{x\to a}\dfrac{\sqrt{2a^3x-x^4} - a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$

(ii) $\lim\limits_{x\to 0}\left[\dfrac{1}{x\sin^{-1}x} - \dfrac{1-x^2}{x^2}\right]$

(iii) $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ln(1-x)}{x.\tan^2 x}$

34. If $y = x^{n-1}\ln x$, prove that $x^2y_2 + (3-2n)xy_1 + (n-1)^2y = 0$.

35. Differentiate $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ w.r.t. $\sec^{-1}\dfrac{1}{2x^2-1}$.

Target Exercises for JEE Advanced

1. If $x \in \left(0, \frac{\pi}{2}\right)$, then show that

$$\frac{d}{dx}\cos^{-1}\left\{\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2 x-48\cos^2 x)}\ \sin x\right\}$$

$$= 1 + \frac{7\sin x}{\sqrt{\sin^2 x - 48\cos^2 x}}$$

2. If $\frac{x+b}{2} = a\tan^{-1}(a\,\ell n\, y)$, $a > 0$ prove that

$y\,y'' - y\,y'\,\ell n\, y = (y')^2$.

3. Prove that the expression $S = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ remains unchanged if y is replaced by $\frac{1}{y}$, i.e. if we put $y = \frac{1}{y_1}$,

then $\frac{y_1'''}{y_1'} - \frac{3}{2}\left(\frac{y_1''}{y_1'}\right)^2 = S$.

4. If $D^* f(x) = \lim\limits_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$ where

$f^2(x) = \{f(x)\}^2$ then prove that

(i) $D^*\{f(x)\cdot g(x)\} = f^2(x)\,.\,D^* g(x) + g^2(x)\,.\,D^* f(x)$

(ii) $D^*\left\{\frac{f(x)}{g(x)}\right\}$

$$= \frac{g^2(x)\cdot D^* f(x) - f^2(x)\cdot D^* g(x)}{\{g(x)\}^4}.$$

5. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w. r. t. $\sqrt{1-x^4}$.

6. If $y = \sum\limits_{n=1}^{n}\left(\frac{\cos\left(\frac{n+1}{2}\right)x\,.\,\sin 2nx}{2\sin\frac{x}{2}\cos\frac{nx}{2}} - \cos^2 nx\right)$ show that

$$\frac{dy}{dx} = y\left[n\cot\frac{nx}{2} - (n+1)\tan\frac{(n+1)x}{2} - \cot\frac{x}{2}\right].$$

7. If $u = ln\left(\frac{a+b\tan(x/2)}{a-b\tan(x/2)}\right)$ and

$$v = ln\left(\frac{1}{a^2\cos^2(x/2)-b^2\sin^2(x/2)}\right)$$

then prove that $\left.\frac{du}{dv}\right|_{x=\frac{\pi}{2}} = \frac{ab(a^2-b^2)}{a^2+b^2}$.

8. Show that $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ can be reduced to the form

$$R^{2/3} = \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \frac{1}{\left(\frac{d^2x}{dy^2}\right)^{2/3}}.$$

Also show that, if $x = a\sin 2\theta\,(1+\cos 2\theta)$ & $y = a\cos 2\theta\,(1-\cos 2\theta)$ then the value of R equals to $4a\cos 3\theta$.

9. If $\frac{d}{dx}\{x^n - a_1x^{n-1} + a_2x^{n-1}\ \ldots. + (-1)^n a_n\}\,e^x = x^n e^x$ then find the value of a_r, $0 < r \le n$.

10. If $y = \frac{ax+b}{x^2+c}$, show that $(2xy'+y)\,y''' = 3(xy''+y')y''$, where an overhead dash indicates differentiation w.r.t. x & a, b, c are constants.

11. If $y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$. Show that

$$\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}.$$

12. Find $f'(x)$ if

$$f(x) = \frac{\sqrt{1+\sqrt{1-x^2}}\left(\sqrt{(1+x)^3}-\sqrt{(1-x)^3}\right)}{2+\sqrt{1-x^2}}$$

13. If $f(x) = \frac{1}{\sin x - \sin a} - \frac{1}{(x-a)\cos a}$

then find the value of $\frac{d}{da}\left\{\lim\limits_{x\to a} f(x)\right\} - \lim\limits_{x\to a} f'(x)$.

14. If $x = \tan\frac{y}{2} - 1n\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$, show that

$$\frac{dy}{dx} = \frac{1}{2}\sin y\,(1+\sin y+\cos y).$$

15. If $y = (C_1 + C_2x)\sin x + (C_3 + C_4x)\cos x$. Show that

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

16. Find the sum :

$\cos\theta + 2^2\cos 2\theta + 3^2\cos 3\theta + \ldots + n^2\cos n\theta$.

17. Show that the subtitution $x = \tan\theta$ changes the equation $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\cdot\frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$ to $\frac{d^2y}{d\theta^2} + y = 0$.

18. Prove that if x is positive then $n\ln[\frac{1}{2}(1 + x^{1/n})] \to -\frac{1}{2}\ln x$ as $n \to \infty$.

19. Given $z = \cos^7 x$; $y = \sin x$ then, prove that $\frac{d^3z}{dy^3} = \frac{105}{4}\sin 4x$.

20. Show that the function

$$y = \cos^{-1}\frac{a\cos x + b}{a + b\cos x} - 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right),$$

where $0 < b \le a$, is a constant for $x \ge 0$. Find the value of this constant.

21. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$, $u \in \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$ prove that $2\frac{dy}{dx} + 1 = 0$.

22. If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x), B(x), C(x)$ be the polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

23. Show that

$$\log_e\frac{\sin x}{x} = -\left(\frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \dots\dots\right)$$

and hence, or otherwise, find the value of

$$\lim_{x\to 0}\frac{\log_e\frac{\sin x}{x} + \frac{x^2}{6}}{x(2\sin x - \sin 2x)}.$$

24. If $0 < x < 1$, prove that

$$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \dots\dots\infty = \frac{1+2x}{1+x+x^2}.$$

25. Evaluate the following limits:

(i) $\lim_{x\to\frac{\pi}{2}}\frac{\sin x - (\sin x)^{\sin x}}{1 - \sin x + \ln(\sin x)}$

(ii) $\lim_{x\to 0}\frac{3x\ln\left(\frac{\sin x}{x}\right)^2 + x^3}{(x - \sin x)(1 - \cos x)}$

(iii) $\lim_{x\to 0}\frac{x^{6000} - (\sin x)^{6000}}{x^2.(\sin x)^{6000}}$

26. In a triangle ABC, the side a is expressed in terms of the other two sides b, c and the angle A between them by the formula $a = \sqrt{b^2 + c^2 - 2bc\cos A}$. For b and c constant, side a is a function of the angle A. Show that $\frac{da}{dA} = h_a$ where h_a is the altitude of the triangle corresponding to the base a. Interpret this result geometrically.

27. Let $f(x) = a_1\sin x + a_2\sin 2x + \dots\dots + a_n\sin nx$ where $a_1, a_2, \dots\dots\dots, a_n \in R$ and n is a positive integer. If it is given that $|f(x)| \le |\sin x|\ \forall\ x \in R$; show that $|a_1 + 2a_2 + 3a_3 + \dots\dots + na_n| \le 1$.

28. If $\prod_{r=1}^{n}(x+r) = A_0 + A_1x + \dots\dots\dots + A_nx^n$, prove that

A $A_1 + 2A_2 + \dots + nA_n = (n+1)!\left(\frac{1}{2} + \frac{1}{3} + \dots. + \frac{1}{n+1}\right)$

B $A_1 - 2A_2 + \dots\dots\dots (-1)^{n-1}n.A_n = (n-1)!$

29. If $f(x) = \frac{a}{x+a} + \frac{b}{x+b} - \frac{c}{x+c} + \frac{d}{x+d}$, and $f(x)$ is divisible by x^2 then find the value of $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d}$.

30. Let $f(x) = a_1\tan\frac{x}{1} + a_2\frac{x}{2} + \dots + a_n\tan\frac{x}{n}$ where $a_1, a_2, \dots\dots a_n \in R$ and $n \in N$. If $|f(x)| \le |\tan x|$ for $\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Prove that $\left|\sum_{i=1}^{n}\frac{a_i}{i}\right| \le 1$.

31. Let f be a differentiable function on [a, b], fA = 0 and there exist a real constant c, such that $|f'(x)| \le c|f(x)|$ on [a, b]. Prove that $f(x) = 0$ for all $x \in [a, b]$.

32. If $\phi(x) = \lambda(x).f(x)$ and $\psi(x) = \mu(x).f(x)$ then prove that

$$\begin{vmatrix} f(x) & \varphi(x) & \psi(x) \\ f'(x) & \varphi'(x) & \psi'(x) \\ f''(x) & \varphi''(x) & \psi''(x) \end{vmatrix} = \{f(x)\}^3.\begin{vmatrix} \lambda'(x) & \mu'(x) \\ \lambda''(x) & \mu''(x) \end{vmatrix}.$$

33. If $\lim_{x\to 0}\frac{1 - \cos x\cdot\cos 2x\cdot\cos 3x\dots\dots\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n \in N$).

34. Let R denotes the value of f′ (2) where

$$f(x) = \lim_{n\to\infty} \sum_{n=1}^{n} \tan^{-1}\left(\frac{x}{n(n+1)+x^2}\right)$$

S denotes the sum of all the values of x satisfying the equation

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{6}{17}\right).$$

T denotes the value of the expression 2 sin x + cos x + 4 tan x where $x = 2\tan^{-1}\left(\frac{1}{3}\right)$

Find the value of 5R + 6S + 7T.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks:

1. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ = ... **[IIT - 1982]**

2. If $f_r(x)$, $g_r(x)$, $h_r(x)$ r = 1, 2, 3 are polynomials in x such that $f_rA = g_rA = h_rA$, r = 1, 2, 3

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then F′(x) at

x = a is .... **[IIT - 1985]**

3. If $f(x) = \log_x(\ln x)$, then f′(x) at x = e is.................... **[IIT - 1985]**

4. The derivative of $\sec^{-1}\left(-\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is................. **[IIT - 1986]**

5. If f(x) = |x –2| and g(x) = f[f(x)], then g′(x) = ... for x > 20 **[IIT - 1990]**

6. If $xe^{xy} = y + \sin^2 x$, then at x = 0, $\frac{dy}{dx}$ =................. **[IIT - 1996]**

7. Let F(x) = f(x) g(x) h(x) for all real x, where f(x), g(x) and h(x), are differentiable functions. At some point x_0
$F'(x_0) = 21\,F(x_0)$, $f'(x_0) = 4f(x_0)$,
$g'(x_0) = -7g(x_0)$, $h'(x_0) = kh(x_0)$.
Then k =................. **[IIT - 1997]**

B. True / False:

8. The derivative of an even function is always an odd function. **[IIT - 1993]**

C. Multiple Choice Questions with ONE correct answer:

9. If $y^2 = P(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ equals **[IIT - 1988]**

(A) P‴ (x) + P′(x) (B) P″(x) P‴(x)
(C) P(x) P‴(x) (D) a constant

10. Let f(x) be a quadratic expression which is positive for all the real values of x.
If g(x) = f(x) + f ′(x) + f″(x), then for any real x, **[IIT - 1990]**

(A) g(x) < 0 (B) g(x) > 0
(C) g(x) = 0 (D) g(x) > 0

11. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where P is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at x = 0 is **[IIT - 1997]**

(A) p (B) $p + p^3$
(C) $p + p^3$ (D) independent of p

12. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to **[IIT - 1994]**

(A) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
(B) $\tan x\,(\sin x)^{\tan x - 1}\,.\cos x$
(C) $(\sin x)^{\tan x}\sec^2 x \log \sin x$
(D) tan x (sin x)|tan x = 1

13. If $x^2 + y^2 = 1$ then **[IIT - 2000]**

(A) $yy'' - 2(y')^2 + 1 = 0$
(B) $yy'' + (y')^2 + 1 = 0$
(C) $yy'' - (y')^2 - 1 = 0$
(D) $yy'' + 2(y')^2 + 1 = 0$

14. Let f : (0, ∞) → R and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1+x)$, then f(4) equals **[IIT - 2001]**

(A) 5/4 (B) 7
(C) 4 (D) 2

15. If y is a function of x and log (x + y) – 2xy = 0, then the value of y′(0) is equal to **[IIT - 2004]**

(A) 1 (B) –1
(C) 2 (D) 0

16. $\frac{d^2x}{dy^2}$ equals **[IIT - 2007]**

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

17. Let g(x) = log f(x) where f(x) is a twice differentiable positive function on (0, ∞) such that f(x + 1) = x f(x). Then for N = 1, 2, 3, ...,

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$ **[IIT - 2008]**

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

D. Multiple Choice Question with ONE or MORE THAN ONE correct answer:

18. Let f : (0, 1) → R be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that 0 < b < 1. Then:

(A) f is not invertible on (0, 1) **[IIT - 2011]**

(B) f ≠ f^{-1} on (0, 1) and f′B = $\frac{1}{f'(0)}$

(C) f = f^{-1} on (0, 1) and f′B = $\frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0, 1)

F. Subjective Problems

19. Find the derivative of sin $(x^2 + 1)$ with respect to x from first principle. **[IIT - 1978]**

20. Find the derivative of

$$f(x)=\begin{cases}\frac{x-1}{2x^2-7x+5} & \text{when } x\neq 1\\ -\frac{1}{3} & \text{when } x=1\end{cases} \quad \text{at } x = 1$$

[IIT - 1979]

21. Given $y=\frac{5x}{\sqrt[3]{(1-x)^2}}+\cos^2(2x+1)$; Find $\frac{dy}{dx}$. **[IIT - 1980]**

22. Let $y = e^{x\sin x^3} + (\tan x)^x$ find $\frac{dy}{dx}$ **[IIT - 1981]**

23. Let f be a twice differentiable function such that $f''(x) = -f(x)$, and $f'(x) = g(x)$, $h(x) = [f(x)]^2 + [g(x)]^2$. Find h(10) if h(5) = 11. **[IIT - 1982]**

24. If α be a repeated root of a quadratic equation f(x) = 0 and A(x), B(x) and C(x) be polynomials of degree 3, 4 and 5 respectively, then show that

$$\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix} \text{ is divisible by } f(x),$$

where prime denotes the derivatives. **[IIT - 1984]**

25. Find the derivative with respect to x of the function

$(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ at $x = \frac{\pi}{4}$ **[IIT-1984]**

26. If x = sec θ – cos θ and y = $\sec^n\theta - \cos^n\theta$, then show that

$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$ **[IIT - 1989]**

27. Find dy/dx at x = – 1, when $(\sin y)^{\sin(\pi x/2)} + (\sqrt{3}/2)\ \sec^{-1}(2x) + 2^x \tan\{\ln(x+2)\} = 0$ **[IIT - 1991]**

28. If $y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{(x-c)}+1$ **[IIT - 1998]**

Prove that $\frac{y'}{y}=\frac{1}{x}\left[\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right]$

F. Assertion A and Reason R

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

29. Let f(x) = 2 + cos x for all real x

Assertion (A): For each real t, there exists a point c in [t, t + π] such that f ′D = 0 because.

Reasons (R): f(t) = f(t + 2π) for each real t. **[IIT - 2007]**

30. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x)\sin x$.

Assertion (A): $\lim\limits_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$.

[IIT - 2008]

Reasons (R): $f'(0) = g(0)$.

G. Integer answer type

31. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is.

[IIT - 2011]

Previous Year's Questions (JEE Main Papers)

1. If the surface area of a sphere of radius *r* is increasing uniformly at the rate of 8 cm^2/s then the rate of change of its volume is

(a) constant (b) proportional to $\sqrt{r}$
(c) proportional to r^2 (d) proportional to r

[2013, online]

2. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

(a) lies between 2 and 3
(b) lies between –1 and 0
(c) does not exist
(d) lies between 1 and 2 **[2013]**

3. If f and g are differentiable function in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$ then for some $c \in (0, 1)$

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$
(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$ **[2014]**

4. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc is

(a) $\frac{1}{9}$ (b) $\frac{1}{6}$
(c) $\frac{1}{36}$ (d) $\frac{1}{24}$ **[2014, online]**

5. If Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval $[-1, 1]$ for the point $c = \frac{1}{2}$, then the value of $2a + b$ is

(a) 1 (b) – 1
(c) 2 (d) – 2 **[2014, 2015 online]**

6. The equation of a normal to the curve $\sin y = x\sin\left(\frac{\pi}{3} + y\right)$ at $x = 0$ is

(a) $2x + \sqrt{3}y = 0$ (b) $2y - \sqrt{3}x = 0$
(c) $2y + \sqrt{3}x = 0$ (d) $2x - \sqrt{3}y = 0$

[2015, online]

7. From the top of a 64 m high tower, a stone is thrown upwards vertically with the velocity of 48 m/s. The greatest height (in metres) attained by stone, assuming the value of the gravitational acceleration $g = 32\ m/s^2$, is

(a) 100 (b) 88
(c) 128 (d) 112 **[2015, online]**

8. The distance, from the origin, of the normal to the curve, $x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$, is

(a) 4 (b) $2\sqrt{2}$
(c) 2 (d) $\sqrt{2}$ **[2015, online]**

9. Consider

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right),\ x \in \left(0, \frac{\pi}{2}\right).$$

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point

(a) (0, 0) (b) $\left(0, \frac{2\pi}{3}\right)$
(c) $\left(\frac{\pi}{6}, 0\right)$ (d) $\left(\frac{\pi}{4}, 0\right)$ **[2016]**

10. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$ meets the curve again at a point Q, then the coordinates of Q are

(a) $(16t^2 + 3, -64t^3 - 1)$
(b) $(4t^2 + 3, -8t^3 - 1)$
(c) $(t^2 + 3, t^3 - 1)$
(d) $(t^2 + 3, -t^3 - 1)$ **[2016, online]**

11. Let C be a curve given by $y(x) = 1 + \sqrt{4x - 3}$, $x > \frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$, then a point through which the normal at P passes, is

(a) (1, 7) (b) (3, –4)
(c) (4, –3) (d) (2, 3) **[2016, online]**

12. The tangent at the point (–2, 2) to the curve, $x^2y^2 - 2x = 4(1 - y)$ does not pass through the point:

(a) (4, 1/3) (b) (8, 5)
(c) (–4, –9) (d) (–2, –7) **[2017, online]**

13. A tangent to the curve $y = f(x)$ at $P(x, y)$ meets x-axis at A and y-axis at B. If $AP:BP = 1:3$ and $f(1) = 1$, then the curve also passes through the point

(a) $\left(\frac{1}{3}, 24\right)$ (b) $\left(\frac{1}{2}, 4\right)$

(c) $\left(2, \frac{1}{8}\right)$ (d) $\left(3, \frac{1}{28}\right)$ **[2017, online]**

14. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point

(a) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ **[2017]**

15. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(a) 6 (b) 7/2

(c) 4 (d) 9/2 **[2018, online]**

16. If θ denotes the acute angle between the curves $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection then $|\tan\theta|$ is equal to

(a) 4/9 (b) 7/17

(c) 8/17 (d) 8/15 **[2019]**

17. The tangent to the curve $y = xe^{x^2}$ passing through the point $(1, e)$ also passes through the point

(a) (4/3, 2e) (b) (2, 3e)

(c) (5/3, 2e) (d) (3, 6e) **[2019]**

18. The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point

(a) (1/4, 7/2) (b) (7/2, 1/4)

(c) (–1/8, 7) (d) (1/8, –7) **[2019]**

19. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of x-axis, is

(a) $x = y\cot\theta + 2\tan\theta$

(b) $x = y\cot\theta - 2\tan\theta$

(c) $y = x\tan\theta - 2\cot\theta$

(d) $y = x\tan\theta + 2\cot\theta$ **[2019]**

ANSWERS

CONCEPT PROBLEMS—A

1. $-\frac{6}{49}$

2. (i) $\frac{-a}{(a^2+1)^{3/2}}$ (ii) $1 + \frac{2}{x^3}$

3. (i) $x(2\cos x - x\sin x)$,

(ii) $\frac{-2x}{\sqrt{1-x^4}}$

(iii) $\frac{-\sec^2 x \cdot e^{\sqrt{\tan x}}}{2\sqrt{\tan x}\cdot\sqrt{1-e^{2\sqrt{\tan x}}}}$,

(iv) $\frac{\cos x}{3\cdot\sqrt[3]{\sin^2 x}}$,

(v) $2xe^{x^2}$

CONCEPT PROBLEMS—B

1. $15x^2 + 8x^7 + 3\sec^2 x$.

2. For $f(x) = x^3$ it is not valid.

4. $x = 0, 3$.

5. (i) – 16 (ii) – 20/9

(iii) 20

6. (i) $(x-1)e^x + 1$ (ii) $(x - e + 1)e^x + x + 2$

7. (i) –3 (ii) 1/4

8. $2 + \sqrt{2}$

PRACTICE PROBLEMS—A

11. (i) $(x^2 + 3x - 1)(2x - 3) + (2x + 3)(x^2 - 3x + 1)$

(ii) $\frac{16(t^2-1)t^{15} - 2(t^{16}-1)t}{(t^2-1)^2}$

(iii) $\frac{2x - x^2}{(x^2-2x+2)^2}$

(iv) $3nx^{3n-1}$

12. (i) $\frac{1}{2}\sec^2\frac{x}{2}$

(ii) $e^x((\sin x + \cos x)x + \sin x)$

(iii) $\frac{(1+\tan x)(2x + \sec x\tan x) - \sec^2 x(x^2 + \sec x)}{(1+\tan x)^2}$

13. $a = 2;\ b = -1$ **14.** 0

15. $f'(x) = \begin{cases} 2x+5 & \text{for } x \le 0 \\ 5 & \text{for } 0 < x < 6 \\ 2x & \text{for } x > 6 \end{cases}$

16. 5050 **17.** –2, 3, 6

18. 1 **19.** No.

20. Yes, $27x^4 - 15x^3 + 2x^2 + \frac{7}{x} - \frac{7}{x^4} - 42$

22. A 0 B $-2/3$

CONCEPT PROBLEMS—C

1. $21(x+1)(3x^2+6x+5)^{5/2}$

2. $17, 984$ **3.** $\frac{22}{27}$

4. $\frac{100}{9}$

7. (ii) $\sin x \frac{\cos x}{|\sin x|}$; $x = n\pi,\ n \in I$

(iii) $\frac{x\cos x}{|x|}$; $x = 0.$

8. False **9.** False

10. (i) $\frac{3}{(3x-1)^2+1}$ (ii) $\frac{-1}{x^2+1}$

11. -1

PRACTICE PROBLEMS—B

14. (i) $\frac{x\sec^2(1+x^2)}{\sqrt{\tan(1+x^2)}}$

(ii) $-\frac{\sec^2\sqrt{x+1}.\sin(\tan\sqrt{x+1})}{2\sqrt{x+1}}$

(iii) $\ln\left(x^2\sqrt{x^2+1}\right)$

(iv) $21(x+1)(3x^2+6x+5)^{5/2}$ (v) $\frac{\cos x+\sin x}{\sin x-\cos x}$

16. (i) $\begin{cases} 2x, & |x|>2 \\ -2x, & |x|<2 \end{cases}$ (ii) $\begin{cases} 0, & t<0,\ t>1 \\ 8t-4, & 0<t<1 \end{cases}$

(iii) $\begin{cases} -4y^3+1, & y<1 \\ 4y^3-1, & y>1 \end{cases}$ (iv) $\begin{cases} 0, & x<0 \\ \frac{2}{3x^{2/3}}, & x>0 \end{cases}$

17. (i) 0 (ii) $1+\left.1\middle/\sqrt{6}\right.$

(iii) $\frac{7}{3}$ (iv) $\frac{1}{4}$

19. $1/8\ x^2.$ **20.** 1

21. $f'(x) = (x+1)^{-2}$; $g'(x) = (2x+1)^{-2}$

22. 28 **23.** $10x + 6.$

24. $(x+1)(x^4+2x^3+2x^2+x)$

25. (i) $f'(x) = \frac{1}{x}$

(ii) $f'(x) = \begin{cases} -1 & \text{when} \quad x \le 0, \\ -e^{-x} & \text{when} \quad x > 0. \end{cases}$

26. (i) $3/4$ (ii) does not exist

(iii) -2

28. $f'(x) = \begin{cases} -1, & x \in [1,2), \\ 1, & x \in (2,\infty) \end{cases}$

29. $-2\sqrt{2\pi}$

CONCEPT PROBLEMS—D

1. $y\left(\frac{1}{2x}-\frac{4}{3(1-2x)}+\frac{9}{4(2-3x)}+\frac{16}{5(3-4x)}\right)$

2. $(\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x}+\cot x\ \ell n\, x\right]$

3. (i) $(2\ln x+1)\,.\ x^{x^2+1}$

(ii) $\frac{(\sin x)^{\cos x}.(\cos^2 x-\sin^2 x\log\sin x)}{\sin x}$

4. (i) $\left(1+\frac{1}{x}\right)^x\left[\ell n\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$

(ii) $x^x.\ e^{x^x}(\ell nx+1)$

(iii) $\left(\ell n(\ell nx)+\left(\frac{1}{\ell nx}\right)\right)$

$.(\ell n\, x)^x + x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ \ell nx\right)$

PRACTICE PROBLEMS—C

7. 4 **8.** 0

9. $f'(x) = 0$

10. $\frac{dy}{dx} = (\sin x)^{\tan x}[\sec^2 \times \ln\sin x + 1]$

$+ (\tan x)^{\sin x}[\cos x.\ \ln\tan x + \sec x]$

12. $(\tan x)^{\sin x-1}\cdot\{\sec x\cdot\tan x+\sin x\cdot\ln\tan x\}$

$+ (\ln x)^{ex}\cdot e\left\{\frac{1}{\ln x}+\ln\ln x\right\}$

13. (i) $(\cos x)^{\ln x}\left[\frac{1n(\cos x)}{x} - \tan x\ 1nx\right]$

$+(\ln x)^x\left[\frac{1}{1nx} + 1n(1nx)\right]$

(ii) $e^{x^{e^x}}\ .\ x^{e^x}\left[\frac{e^x}{x}+e^x\,1nx\right] + e^{x^{x^e}}\ x^{e-1}\ x^{x^e}[1+e1nx]$

$+\ x^{e^{e^x}}\ e^{e^x}\left[\frac{1}{x}+e^x\,1nx\right]$

(iii) $\frac{x}{(x^2-2)\sqrt{1-x^2}\ \tan^{-1}\sqrt{1-x^2}}$

14. $\ln(\sin 1)$

CONCEPT PROBLEMS—E

1. $-\dfrac{1}{\sqrt{3}}$ **3.** $-\dfrac{x}{|x|}\times\dfrac{1}{\sqrt{1-x^2}}$

4. (i) $\dfrac{2}{x\sqrt{x^4-1}}$ (ii) $\dfrac{1}{1+x^2}$

5. $\dfrac{1}{2^s\ln 2}\sqrt{1-2^{2s}}$, $\dfrac{1}{\ln 2}\cot t$

7. $\dfrac{1}{3}$ **8.** $\dfrac{1}{2\sqrt{2}}$

9. $\dfrac{-1}{\sqrt{1-x^2}}$ **10.** $-\dfrac{1}{2}$

PRACTICE PROBLEMS—D

11. (i) $\sqrt{\dfrac{6}{\cos 2x+\cos 4x}}$

(ii) 0

(iii) $\dfrac{3e^{(\tan^{-1}x)^3}(\tan^{-1}x)^2}{1+x^2}$

(iv) $\dfrac{-x}{\sqrt{1-x^2}}\sin^{-1}x$

12. $\dfrac{1}{2\sqrt{x(1+x)}}+\dfrac{1}{1+x^2}$ **13.** $\dfrac{1}{3}$

14. $\dfrac{1}{13}$ **15.** $\dfrac{x\,e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

17. $\dfrac{-(1++x^4)^2}{8x^3}$, $-\dfrac{1}{2\sqrt[4]{(1-y)^3(1+y)^5}}$

18. $y'=\dfrac{\left(\dfrac{1}{\sqrt{1-x^2}}\cos^{-1}x-\sin^{-1}x\dfrac{1}{\sqrt{1-x^2}}\right)\tan^{-1}x-\dfrac{\sin^{-1}x\cos^{-1}x}{1+x^2}}{(\tan^{-1})^2x}$

19. $\dfrac{dy}{dx}=\dfrac{1}{2}\sin y\,(1+\sin y+\cos y)$

CONCEPT PROBLEMS—F

1. (i) $\tan t$ (ii) $\dfrac{(t^2-1)b}{2at}$

(iii) $\dfrac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$

2. (i) $-\cot^3\theta$ (ii) $\tan t$

3. $\dfrac{-e^{t^2}}{2t\left(2t^2+2t+1\right)}$ **4.** $\dfrac{5}{4}$

5. $\dfrac{1}{xe^x}$ **6.** $-3\sin x\cos x$

PRACTICE PROBLEMS—E

8. 0 **12.** $\dfrac{(1+x^2)\cos(1+x^2)^2}{\sqrt{\sin(1+x^2)^2}}$

CONCEPT PROBLEMS—G

1. (i) $\dfrac{-\sin(x+y)}{1+\sin(x+y)}$ (ii) $-\left(\dfrac{y}{x}\right)^{1/3}$

(iii) $\dfrac{y(x-y)}{x(x+y)}$

5. (i) $\dfrac{2a}{b(a^2+1)^2}$, (ii) -1

6. $-16/13$ **7.** 1

9. (ii) It is meaningless to find dy/dx.

10. $\dfrac{y\cdot x^{y-1}}{1-x^y\ln x}$ **11.** $\dfrac{1}{y+1}$.

12. (i) $\dfrac{dy}{dx}=-\dfrac{y^3+2xy}{x^2+3xy^2}$, $\dfrac{dx}{dy}=-\dfrac{x^3+3xy^2}{y^3+2xy}$,

$\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$.

PRACTICE PROBLEMS—F

14. (i) $\dfrac{y}{x}$

(ii) $\dfrac{\sec^2(x+y)-\cos y-y\cos x}{\sin x-x\sin y-\sec^2(x+y)}$

(iii) $\dfrac{2x+(y\sin x-\cos y)(1+x^4)}{(1+x^4)(\cos x-x\sin y)}$

16. $\dfrac{dy}{dx}=-\dfrac{(y+x\ln y)}{(x+y\ln x)}\cdot\dfrac{y}{x}$

17. $\dfrac{y^{\cos x}\cdot\sin x\cdot\ln y-x^{\sin y-1}\cdot\sin y}{x^{\sin y}\cdot\cos y\cdot\ln x+y^{\cos x-1}\cdot\cos x}$

18. $y'=-y/(2x\ln x)$

19. $-\dfrac{2y^3+3t^2y}{(6ty^2+t^3)\cos t}$

20. $\dfrac{dy}{dx}=\dfrac{y^2}{x\ln x\,(1-y\ln\ln x)}$

21. (i) -1 (ii) Three

(iii) $-1, -\dfrac{8+2\sqrt{5}}{5+3\sqrt{5}}, -\dfrac{8-2\sqrt{5}}{5-3\sqrt{5}}$

23. $y = \dfrac{-4x^3}{(x^4-2)^{4/3}(x^4+1)^{2/3}}$

24. 3/2

CONCEPT PROBLEMS—H

3. $\dfrac{-10}{3}$ **4.** -1

5. 4

PRACTICE PROBLEMS—G

7. $-\dfrac{1}{2}$ **8.** 1

9. $\dfrac{2x}{|x|}$ **10.** $\dfrac{2}{\sqrt{3}}$

11. $\dfrac{|x|(1-x^2)}{x|1-x^2|} = \begin{cases} 1 & 0<x<1 \text{ or } x<-1 \\ \text{non existent if} & x = 0, 1, -1 \\ -1 & -1<x<0 \text{ or } x>1 \end{cases}$

12. 3/2 if $x \neq \pm\dfrac{1}{\sqrt{3}}, \pm 1$ **13.** $x < -1, x > 1$

14. 2

CONCEPT PROBLEMS—I

1. A $\dfrac{2x^2+3x}{(1+x^2)\sqrt{1+x^2}}$;

B $\dfrac{(1+2x^2)\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} + \dfrac{3x}{(1-x^2)^2}$

D $2e^{-x^2}.(2x^2-1)$

2. (i) $\sqrt{2}, -1$ (ii) $-\dfrac{1}{2}, \dfrac{1}{4}$

(iii) $-2, \dfrac{3}{2}$ (iv) $-\dfrac{6}{5}, -\dfrac{42}{125}$

3. (i) $x = 0,\ x = 1$;

(ii) $x = \dfrac{2+\sqrt{2}}{2},\ x = \dfrac{2-\sqrt{2}}{2}$

4. $-24x^{-5}$

5. $-\dfrac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^3}$ **6.** $-2e^{-2x}$

7. $\dfrac{\sin x - e^x}{(\cos x + e^x)^3}$ **8.** 7.2^9

9. 12 **11.** $r = 1, -6$

12. $\varphi''(x) f'(\varphi(x)) + 3\varphi'(x)\varphi''(x) f''(\varphi(x)) + \varphi'^3(x) f'''(\varphi(x))$

13. $a = -4, b = 5, c = -1, d = -2$

PRACTICE PROBLEMS—H

15. (i) $y'' = 2\sqrt{1-x^2}$

(ii) $y'' = \dfrac{1}{\sqrt{x^2+a^2}}$

(iii) $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4a}\operatorname{cosec}^4\dfrac{t}{2}$

(iv) $\dfrac{d^2y}{dx^2} = -4\sqrt{t-t^2}$

16. (i) $y''' = \dfrac{1}{(x+1)^4}$ (ii) $y''' = (2\ln x - 3)/x^3$

(iii) $y''' = 105\sqrt{2x+3}$

17. $a = -1, b = 3, c = 0.$ **18.** 678

20. $2x^3 - 3x^2 + 3x - 1$

21. $A = -1/2 = B, C = -3/4.$

22. (i) $6xg'(x^2) + 4x^3g''(x^2)$

(ii) $[\sqrt{x}\,g''(\sqrt{x}) - g'(\sqrt{x})]/(4x\sqrt{x})$

23. (i) $f''(x) = 6xg'(x^2) + 4x^3g''(x^2)$

(ii) $f''(x) = [\sqrt{x}\,g''(\sqrt{x}) - g'(\sqrt{x})]/(4x\sqrt{x})$

26. $a = 1, b = 0, c = -1/2$

27. $\dfrac{4x^3}{9}$ **28.** 162

PRACTICE PROBLEMS—I

1. (i) $-\dfrac{1.3.5...17}{2^{10}x^9\sqrt{x}}$ $(x > 0)$

(ii) $\dfrac{720}{(x-1)^7}, (x \neq -1)$

(iii) $-\dfrac{2.8!}{(x+1)^9} (x \neq -1)$

(iv) $5^{10}(5x+11)e^{5x}$

(v) $-\dfrac{9!}{x^{10}} (x > 0)$

2. Yes **3.** 3, 7, 11,......, $(4k-1)$

4. $(x^2-379)\sin x - 40x.\cos x$ **6.** 70.

PRACTICE PROBLEMS—J

2. – 6

3. $2(1 + 2x) . \cos 2(x + x^2)$ **5.** 0

CONCEPT PROBLEMS—J

1. cos (x + a) = cos x cos a – sin x sin a.

2. False

3. (i) = 3 (ii) = –3

7. Not an identify

8. 3/4. Since the function is a constant ($y' = 0$), the value of this constant is equal to that of the given function for any value of x, say x = 0.

10. a = d = 1, b = c = 0

11. a = c = e = 0; b = f = 2 ; d = – 1

13. Not an identify for all $x \varepsilon R$

14. $\dfrac{1+x}{(1-x)^3}$ **15.** $\dfrac{1}{2}, \dfrac{1}{2}, -6$

16. x = 1

PRACTICE PROBLEMS—K

17. $f(x) = \begin{cases} \pi, & 0 \le x \le 1 \\ 0, & x \le -1 \end{cases}$

18. $\dfrac{x}{1-x}$.

19. (i) 1, 3, 6, 6 (ii) $a_0, a_1, 2a_2, 6a_3$

20. (i) $-\dfrac{1}{2}, -\dfrac{1}{2}, 4$ (ii) $3, 3, \pm 2\sqrt{-1}$

21. $(x - 1)^2 (x + 5); (x + 3)^2(x - 2)$

PRACTICE PROBLEMS—L

1. (i) $\dfrac{1}{9}$ (ii) 0

(iii) $\dfrac{1}{2}$

3. (i) $\dfrac{1}{e}$ (ii) 1

(iii) $e^{1/3}$ (iv) 1

4. (i) 2 (ii) e^3

(iii) $\dfrac{1}{120}$ (iv) $\dfrac{1}{2}$

5. (i) $-\dfrac{1}{2}$ (ii) $\dfrac{1}{2}$

(iii) $-\dfrac{1}{2}$ (iv) $\dfrac{1}{2}$

6. $a = -\dfrac{5}{2}, b = -\dfrac{3}{2}$

7. (i) 2 (ii) 7

(iii) $\dfrac{b^2m - a^2n}{2mn}$ (iv) $\dfrac{1}{16}$

OBJECTIVE EXERCISES

1. C
2. B
3. B
4. C
5. C
6. A
7. A
8. C
9. C
10. C
11. D
12. B
13. D
14. C
15. B
16. D
17. C
18. C
19. C
20. C
21. A
22. C
23. A
24. D
25. B
26. A
27. A
28. B
29. B
30. C
31. C
32. D
33. D
34. A
35. B
36. C
37. D
38. D
39. B
40. C
41. B
42. C
43. B
44. C
45. D
46. C
47. C
48. C
49. B
50. C
51. AC
52. ABC
53. BCD
54. ABC
55. C
56. AC
57. ABD
58. ABCD
59. BD
60. ABC
61. ACD
62. ABD
63. AB
64. AB
65. CD
66. BD
67. BC
68. AB
69. BC
70. CD
71. D
72. A
73. A
74. A
75. C
76. C
77. C
78. A
79. D
80. A
81. D
82. D
83. B
84. B
85. A
86. D
87. B
88. A
89. C
90. B
91. C
92. A
93. (A)–(Q); (B)–(P); (C)–(Q); (D)–(R)
94. (A)–(T); (B)–(P); (C)–(P); (D)–(S)
95. (A)–(S); (B)–(S); (C)–(S); (D)–(R)
96. (A)–(R); (B)–(Q); (C)–(P); (D)–(T)
97. (A)–(S); (B)–(P); (C)–(R); (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

1. $\dfrac{1}{\sqrt{2}}$ **2.** $\dfrac{y^3}{2x(1 - y^2 \log x)}$

3. 0

5. A $(3.5\,,\infty)$ B $(-\ln 2, \ln 2)$.

6. $f'(x) = \frac{1}{6}(x^2-1)^{-5/6}\,2x$

8. $\left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}\right)$

12. $\frac{2x-1}{2\sqrt{1-x^2}}$

13. $-2/27$

15. $\frac{2}{\sqrt{3}}$

18. 16

19. $\frac{32}{16+\pi^2}-\frac{8}{\ln 2}$

21. $\begin{cases}\frac{2}{1+x^2}, & x<0\\ \text{does not exist}, & x=0\\ -\frac{2}{1+x^2}, & x>0\end{cases}$

22. -4

23. $\frac{-2e^{-\pi/2}}{a}$

24. 2

25. -3

26. $\frac{\frac{5}{3}x^{\frac{2}{3}}(1-y)}{\left(x^{\frac{5}{3}}+2y\right)}$

27. (i) $f'(x)=\begin{cases}-1/2, & x\in(-\infty,0)\\ 1/2, & x\in(0,\infty)\end{cases}$

(ii) $f'(x)=\begin{cases}0 & , \quad x\in[2,4)\\ 1/\sqrt{x-2}, & x\in(4,\infty)\end{cases}$

32. (i) 0 (ii) e

(iii) $e^{5/2}$ (iv) $\frac{7}{360}$

33. (i) $\frac{16a}{9}$ (ii) $\frac{5}{6}$

(iii) $-\frac{1}{2}$

35. $\begin{cases}-\frac{1}{2}, & -1<x<0, \quad x\neq\frac{1}{\sqrt{2}}\\ \frac{1}{2}. & 0<x<1, \quad x\neq\frac{1}{\sqrt{2}}\end{cases}$

TARGET EXERCISES for JEE ADVANCED

5. $\frac{1+\sqrt{1-x^4}}{x^6}$

9. ${}^{n}P_{r}=\frac{n!}{(n-r!)}$

12. $f'(x) = \sqrt{2}$

13. $\frac{3}{4}\sec^3 a-\frac{5}{12}\sec a$

16. $\frac{n^2}{2}\sin\left(n+\frac{1}{2}\right)\theta\cdot\operatorname{cosec}\frac{\theta}{2}+\frac{\pi}{2}\cos n\theta\cdot\operatorname{cosec}^2\frac{\theta}{2}$

$-\frac{1}{4}\sin n\theta\cdot\operatorname{cosec}^2\frac{\theta}{2}\cdot\cot\frac{\theta}{2}$

18. We have

$n\ln[\frac{1}{2}(1+x^{1/n})] = n\ln[1-\frac{1}{2}(1-x^{1/n})]$

$= \frac{1}{2}n(1-x^{1/n})\ \frac{\ln(1-u)}{u}$ where $u=\frac{1}{2}(1-x^{1/n})$.

20. 0

23. $-\frac{1}{180}$

25. (i) 2 (ii) $-2/5$ (iii) 1000

29. 0

33. $n = 11$

34. 38

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $\frac{2+2x-2x^2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

2. ZERO

3. 1/e

4. -4

5. 1

6. 1

7. 24

8. T

9. C

10. B

11. D

12. A

13. B

14. C

15. A

16. D

17. A

18. CD

20. $\frac{-2}{9}$

21. $\frac{5(3-x)}{3(1-x)^{5/3}}-2\sin(4x+2)$

22. $e^{x\sin x^2}[\sin x^3+3x^3\cos x^3]+(\tan x)^x\left[\frac{2x}{\sin 2x}+\log\tan x\right]$

23. 11

24. $4-8\log_2 e+\frac{32}{16-\pi^2}$

27. 0

29. B

30. B

31. 1

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. D

2. C

3. D

4. C

5. B

6. A

7. A

8. C

9. B

10. D

11. A

12. D

13. C

14. C

15. D

16. D

17. A

18. D

19. A

HINTS & SOLUTIONS

Objective Exercise Solution

SINGLE CORRECT ANSWER TYPE

1. $y = |\cos x| + |\sin x|$

$y = \sin x - \cos x$ is the neighbourhood of $x = \dfrac{2\pi}{3}$

$y' = \cos x + \sin x$

$y' = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}$

Hence, C is correct.

2. $f(x) = x[x]$

$f'(x) = [x],\ x \notin I$

Hence, B is correct.

3. $f(x) = x^2 \ln g(x)$

$$f'(x) = 2x \ln g(x) + \frac{x^2 g'(x)}{g(x)}$$

$$f'(2) = 4 \ln 3 - \frac{16}{3}$$

Hence, B is correct.

4. $x = t\cos t,\ y = t + \sin t$

$$\frac{dx}{dt} = \cos t - t\sin t,\quad \frac{dy}{dt} = 1 + \cos t$$

$$\frac{dy}{dx} = \frac{1+\cos t}{\cos t - t\sin t}$$

$$\frac{dx}{dy} = \frac{\cos t - t\sin t}{1+\cos t}$$

$$\frac{d^2x}{dy^2} = \frac{(1+\cos t)(-\sin t - \sin t - t\cos t) + (\cos t - t\sin t)\sin t}{(1+\cos t)^3}$$

$$= -2 - \pi/2$$

$$= -\frac{(\pi+4)}{2}$$

Hence, C is correct.

5. $f\left(x^2\right) = x^4 + x^3 + 1$

Put $x = x^2$

$$f\left(x^4\right) = x^8 + x^6 + 1$$

$$f'\left(x^4\right) = \frac{8x^7 + 6x^5}{4x^3}$$

$$= 2x^4 + \frac{3}{2}x^2$$

Hence C is correct.

6. $y = \ln|\sec e^{x^2}|$

$$y' = \frac{1}{\sec e^{x^2}} \times \sec e^{x^2} \tan e^{x^2} \times e^{x^2} \times 2x$$

$$= 2xe^{x^2} \tan e^{x^2}$$

Hence A is correct.

7. $f(x) = \dfrac{x}{1+e^{1/x}}$

$$f(x)\left(1+e^{1/x}\right) - x = 0$$

$$f'(x)\left(1+e^{1/x}\right) - \frac{f(x)e^{1/x}}{x^2} - 1 = 0$$

$$x^2 f'(x)\frac{(x)}{f(x)} - f(x)e^{1/x} - x^2 = 0$$

Hence, A is correct.

8. $g'(x) = \dfrac{1}{f'(x)}$

$$= \frac{1}{\cos 2x}$$

$$= \sec 2g(x)$$

Hence, C is correct.

9. $y = \dfrac{x^2}{2} + \dfrac{1}{2}x\sqrt{x^2+1} + \dfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)$

$$2y' = 2x + \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$

$$y' = x + \sqrt{x^2+1}$$

$$\Rightarrow y = \frac{x}{2}(y') + \frac{1}{2}\ln y'$$

Hence C is correct.

10. $f(x) = \ln \sin x$

$$f'(x) = \frac{1}{\ln \sin x} \times \frac{1}{\sin x} \times \cos x$$

$$f'\left(\frac{\pi}{6}\right) = \frac{-1}{\ln 2} \times \sqrt{3}$$

Hence, C is correct.

11. $f(x)=e^x-e^{-x}-2\sin x-\frac{2}{3}x^3$

$f'(x)=e^x+e^{-x}-2\cos x-2x^2$

$f''(x)=e^x-e^{-x}+2\sin x-4x$

$f'''(x)=e^x+e^{-x}+2\cos x-4$

$f^{\prime v}(x)=e^x-e^{-x}-2\sin x$

$f^{v}(x)=e^x+e^{-x}-2\cos x$

$f^{v'}(x)=e^x-e^{-x}+2\sin x$

$f^{v''}(x)=e^x+e^{-x}+2\cos x$

Hence C is correct.

12. $\frac{f(2x+2y)-f(2x-2y)}{f(2x+2y)+f(2x-2y)}=\frac{\cos x\sin y}{\sin x\cos y}$

Applying C & D,

$\frac{f(2x+2y)}{f(2x-2y)}=\frac{\sin(x+y)}{\sin(x-y)}$

$\frac{f(2x+2y)}{\sin(x+y)}$ is constant

$\Rightarrow f(x)=C\sin\frac{x}{2}$

$C=1$ using $f'(0)=\frac{1}{2}$

$\Rightarrow f(x)=\sin\frac{x}{2}$

$f'(x)=\frac{1}{2}\cos\frac{x}{2}$

$f''(x)=\frac{-1}{4}\sin\frac{x}{2}$

$\Rightarrow 4f''(x)+f(x)=0$

Hence, B is correct.

13. $y=\tan^{-1}\left(\frac{2^x}{1+2^x\cdot 2}\right)$

$$y'=\frac{1}{1+\left(\frac{2^x}{1+2^{x+1}}\right)^2}\times\frac{\left(1+2^{x+1}\right)2^x\ln 2-2^x}{\left(1+2^{x+1}\right)^2}2^{x+1}\ln 2$$

$y'_{\text{at } x=0}=\frac{9}{10}\times\frac{\ln 2}{9}=\frac{\ln 2}{10}$

Hence, D is correct.

14. $\lim\limits_{x\to 1}\frac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$

$\lim\limits_{h\to 0}\frac{n(1+h)^{n+1}-(n+1)(1+h)^n+1}{-e(e^h-1)\sin\pi h}$

$\lim\limits_{h\to 0}\frac{n(1+h)^{n+1}-(n+1)(1+h)^n+1}{-\pi eh^2}$

$\lim\limits_{h\to 0}\frac{n(n+1)(1+h)^n-n(n+1)(1+h)^{n-1}}{-2\pi eh}$

$\lim\limits_{h\to 0}\frac{n(n+1)(n)(1+h)^{n-1}-n(n+1)(1+h)^{n-2}(n-1)}{-2\pi e}$

$=\frac{n^2(n+1)-n(n-1)}{-2\pi e}(n+1)=\frac{n(n+1)}{-2\pi e}$

Put $n=100$

$=\frac{-5050}{\pi e}$

Hence C is correct.

15. $y=\log_{e^x}(x-2)^2$

$y=\frac{2\log(x-2)}{\log x}$

$y'=\frac{2(\log x)\times\frac{1}{x-2}-2\log(x-2)\times\frac{1}{x}}{(\log x)^2}$

$y'=\frac{2}{\log 3}$ at $x=3$

Hence D is correct.

16. $y=x^2$

$\frac{dy}{dx}=2x \qquad \frac{dx}{dy}=\frac{1}{2x}$

$\frac{d^2y}{dx^2}=2 \qquad \frac{d^2x}{dy^2}=\frac{-1\times 2}{(2x)^2}\times\frac{dx}{dy}=\frac{-1}{2x^2}\times\frac{1}{2x}$

$\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}=\frac{-1}{2x^3}$

Hence D is correct.

17. Put $x=y=0,\ f(0)=1$

$\Rightarrow f'(0)=p,\ \ f'(5)=q$

$f(y)\ \ f'(x-y)=f'(x)$

Put $y=5=x$

$f(5)\ \ f'(0)=f'(5)$

$f(5)=q/p$

Hence C is correct.

18. $\frac{dy}{dx}=1+\frac{x}{\angle 1}+\frac{x^2}{\angle 2}+\cdots\frac{x^{n-1}}{\angle n-1}$

$=y-\frac{x^n}{\angle n}$

Hence C is correct .

19. $f(x) = e^{ax} + e^{bx}$

$f'(x) = ae^{ax} + be^{bx}$

$f''(x) = a^2e^{ax} + b^2e^{bx}$

Now

$f''(x) - 2f'(x) - 15f(x) = 0$

$\Rightarrow \left(a^2 - 2a - 15\right)e^{ax} + \left(b^2 - 2b - 15\right)e^{bx} = 0$

$\Rightarrow a^2 - 2a - 15 = 0 \quad \& \quad b^2 - 2b - 15 = 0$

a, b are two roots of $x^2 - 2x - 15 = 0$

$ab = -15$

Hence C is correct.

20. $f(x) = x + \tan x$

$$g'(x) = \frac{1}{f'(x)}$$

$$= \frac{1}{1+\sec^2 x}$$

$$= \frac{1}{2+\tan^2 x}$$

$$= \frac{1}{2+(g(x)-x)^2}$$

Hence C is correct.

21. $f(x) = x^4 \tan x^3 - x\ln\left(1+x^2\right)$

Differentiating it four times, using service expansion of $\tan x^3$ & $\ln(1+x^2)$

$f^{IV}(x) = 0$

Hence, A is correct.

22. $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\sin x)$

$$= \frac{\pi}{2}$$

$\phi(x) = f \circ f \circ f(x) = \dfrac{\pi}{2}$

$\phi'(x) = 0$

Hence, C is correct.

23. $y = \tan^{-1}\dfrac{(3a^2x - x^3)}{a^3 - 3x^2a}$

$$y = \tan^{-1}\frac{(3x/a - (x/a)^3}{1-3(x/a)^2} = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$$y' = 3 \times \frac{1}{1+\dfrac{x^2}{a^2}} \times \frac{1}{a} = \frac{3a}{a^2+x^2}$$

Hence, A is correct.

24. $ax^2 + by^2 + 2hxy = 1$

$2ax + 2byy' + 2hy + 2hxy' = 0 \qquad \text{(I)}$

$y' = \dfrac{-(ax+hy)}{hx+by} \qquad \text{(II)}$

Differentiating again eq (I)

$a + b\left(y'\right)^2 + byy'' + hy' + hxy'' + hy' = 0$

$$\Rightarrow y'' = \frac{-(a+b(y')^2+2hy')}{(hx+by)}$$

$$= \frac{-1}{(hx+by)^3}\begin{bmatrix} a(hx+by)^2 + b(ax+hy)^2 \\ -2h(ax+hy)(hx+by) \end{bmatrix}$$

$$= \frac{-1}{(hx+by)^3}\begin{bmatrix} ah^2x^2 + ab^2y^2 + \cancel{2abhxy} + ba^2x^2 + bh^2y^2 \\ +2abhxy - 2ah^2x^2 - \cancel{2abhxy} - 2h^3xy - 2h^2by^2 \end{bmatrix}$$

$$= \frac{-1}{(hx+by)^3}\left[ab^2y^2 + ba^2x^2 - ah^2x^2 - h^2by^2 - 2h^3xy + 2abhxy\right]$$

$$= \frac{-1}{(hx+by)^3}[ab(by^2+ax^2+2hxy) - h^2(ax^2+by^2+2hxy)]$$

$$= \frac{h^2 - ab}{(hx+by)^3}$$

Hence, D is correct.

25. $x = a\cos t + \dfrac{b}{2}\cos 2t, \qquad y = a\sin t + \dfrac{b}{2}\sin 2t$

$\dfrac{dx}{dt} = -a\sin t - b\sin 2t, \qquad \dfrac{dy}{dt} = a\cos t + b\cos 2t$

$$\frac{dy}{dx} = \frac{-(a\cos t + b\cos 2t)}{a\sin t + b\sin 2t}$$

$$\frac{d^2y}{dx^2} = \begin{bmatrix} \dfrac{(a\sin t + b\sin 2t)(+a\sin t + 2b\sin 2t)}{(a\sin t + b\sin 2t)^2} \\ +(a\cos t + b\cos 2t)(a\cos t + 2b\cos 2t) \end{bmatrix} \times \frac{1}{-(a\sin t + b\sin 2t)}$$

For $\dfrac{d^2y}{dx^2} = 0$

$$(a\sin t + b\sin 2t)(a\sin t + 2b\sin 2t) + (a\cos t + b\cos 2t)(a\cos t + 2b\cos 2t) = 0$$

$$\Rightarrow a^2+2b^2+ab\cos t+2ab\cos t=0$$

$$\Rightarrow \cos t=\frac{-\left(a^2+2b^2\right)}{3ab}$$

Hence, B is correct.

26. $f^3+g^3+h^3-3fgh=y$

$$\frac{dy}{dx}=3f^2f_1+3g^2g_1+3h^2h_1-3\,f_1\,g\,h-3\,f\,g_1\,h-3f\,g\,h_1$$

$$\frac{dy}{dx}\Rightarrow 3\left[f^2g+g^2h+h^2f-g^2h-fh^2-f^2g\right]=0$$

So, y is a constant function

So, $y=1$

Hence A is correct.

27. $u(x)=7\,v(x)$

$u'(x)=7v'(x)\Rightarrow p=7.$

$$\text{As } \frac{u(x)}{v(x)}=7 \quad \Rightarrow \left(\frac{u(x)}{v(x)}\right)'=0$$

$\Rightarrow q=0$

$$\Rightarrow \frac{p+q}{p-q}=1$$

Hence, A is correct.

28. $f(x)+2f(-x)=-\sin x$ .... (I)

Put $x=-x$

$f(-x)+2f(x)=-\sin x$ .... (II)

Solve, (I) & (II), we get

$f(x)=-\sin x$

$f'(x)=-\cos x$

$$f'\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

Hence, B is correct.

29. $y=f^{-1}(x)$

$$\text{As } g''(y)=-\frac{f''(x)}{\left(f'(x)\right)^3}=-\frac{4}{8}=-\frac{1}{2}$$

Hence, B is correct.

30. $\dfrac{df(x)}{dg(x)}=\dfrac{b'(x)}{g'(x)}=\alpha(x)$

$$\text{Similarly, } \frac{g'(x)}{h'(x)}=\beta(x)$$

$$\frac{h'(x)}{f'(x)}=\frac{h'(x)}{g'(x)}\times\frac{g'(x)}{f'(x)}=\frac{1}{\beta(x)}\times\frac{1}{\alpha(x)}$$

Hence, C is correct.

31. $3f(\cos x)+2f(\sin x)=5x$ ---(I)

$$\text{Put } x=\frac{\pi}{2}-x$$

$$3f(\sin x)+2f(\cos x)=\frac{5\pi}{2}-5x \qquad \text{---(II)}$$

Solving (I) & (II)

$5f(\cos x)=25x-5\pi$

$f(\cos x)=5x-\pi$

$$f'(\cos x)=\frac{-5}{\sin x}$$

32. $y=at^2+2bt+c, \qquad t=ax^2+2bx+c$

$$y=a\left(ax^2+2bx+c\right)^2+2b\left(ax^2+2bx+c\right)+c$$

$$y'=2a\left(ax^2+2bx+c\right)(2ax+2b)+2b(2ax+2b)$$

$$y''=4a^2\left(ax^2+2bx+c\right)+2a(2ax+2b)^2+4ab$$

$$y'''=4a^2(2ax+2b)+4a(2ax+2b)(2a)$$

$$=(2ax+2b)(12a^2)$$

$$=24a^2(ax+b)$$

Hence, D is correct.

33. $y=f(x)=\left(x^x\right)^x=x^{x^2}$

$\ln y=x^2\ln x$

$$\frac{1}{y}y'=x+2x\ln x$$

$$y'=\left(x^x\right)^x(x+2x\ln x)$$

$f'(1)=1$

Now, $y=g(x)=x^{\left(x^x\right)}$

$\ln y=x^x\ln x$

$$\frac{1}{y}y'=x^{x-1}+(\ln x)\left(x^x\right)(1+\ln x)$$

$$y'=g'(x)=x^{\left(x^x\right)}\left(x^{x-1}+x^x\ln x(1+\ln x)\right)$$

$g'(1)=1$

Hence, D is correct

34. As $f(x), f'(x), f''(x)$ are all +ve $\forall x \in [0,7]$

$\Rightarrow f(x)$ is increasing function and concave up.

and so is $f^{-1}(x)$.

$$\Rightarrow f^{-1}(5)+4f^{-1}\left(\frac{2g}{5}\right) \text{ is always +ve}$$

Hence, A is correct.

35. $f(x)=\sin x+\ln x$

$$f\left(x^2\right)=\sin x^2+\ln x^2$$

$$2xf'\left(x^2\right)=2x\cos x^2+\frac{2}{x}$$

$$f'\left(x^2\right)=\cos x^2+\frac{1}{x^2}$$

Hence, B is correct.

36. $y=(\tan x)^{(\tan x)^{\tan x}}$

$$\ln y=(\tan x)^{\tan x}\ln\tan x$$

$$\frac{1}{y}\frac{dy}{dx}=(\tan x)^{\tan x}\times\frac{\sec^2 x}{\tan x}+\ln\tan x\left[\sec^2 x(1+\ln\tan x)\right]$$

at $x=\frac{\pi}{4}$,

$$\frac{dy}{dx}=2$$

Hence, C is correct.

37. $f(x)=\frac{a+\sqrt{a^2-x^2}+x}{a+\sqrt{a^2-x^2}-x}=1+\frac{2x}{a+\sqrt{a^2-x^2}-x}$

Put $x=a\sin\theta$

$$\Rightarrow\frac{dx}{d\theta}=a\cos\theta$$

$$y=f(x)=1+\frac{2\sin\theta}{1+\cos\theta-\sin\theta}$$

$$\frac{dy}{d\theta}=\frac{(1+\cos\theta-\sin\theta)2\cos\theta}{(1+\cos\theta-\sin\theta)^2}+2\sin\theta(\sin\theta+\cos\theta)$$

$$=\frac{2\cos\theta+2}{(1+\cos\theta-\sin 0)^2}$$

$$\frac{dy}{dx}=\frac{2(1+\cos\theta)}{(1+\cos\theta-\sin\theta)^2}\times\frac{1}{a\cos\theta}$$

At $x=0,\ \theta=0$

$$\frac{dy}{dx}=\frac{1}{a}$$

Hence, D is correct.

38. $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=k$.

As $\frac{dx}{dy}=\frac{1}{dy/dx}$

$$\Rightarrow\frac{d^2x}{dy^2}=\frac{-d^2y/dx^2}{(dy/dx)^3}$$

$$\Rightarrow\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=0$$

Hence, D is correct.

39. $y=(1-x)^{-\alpha}e^{-\alpha x}$

Taking log on both sides,

$\ln y=-\alpha\ln(1-x)-\alpha x$

$$\Rightarrow\frac{1}{y}y'=\frac{+\alpha}{1-x}-\alpha$$

$$\Rightarrow(1-x)y'=\alpha y-\alpha y(1-x)$$

$$\Rightarrow(1-x)y'=\alpha xy.$$

$$\Rightarrow\quad(1-x)y''-y'=\alpha xy'+\alpha y$$

$$\Rightarrow(1-x)y''-(1+\alpha x)y'-\alpha y=0$$

Hence, B is correct.

40. $\frac{d^2x}{dy^2}=-\left(\frac{dx}{dy}\right)^3\frac{d^2y}{dx^2}$

$$=-\left(\frac{dx}{dy}\right)^3(-y)\frac{dy}{dx}$$

$$=y\left(\frac{dx}{dy}\right)^2$$

Hence, C is correct.

41. $f\left(x^2\right)=x^3$

$$2xf'\left(x^2\right)=3x^2$$

$$f'\left(x^2\right)=\frac{3}{2}x$$

Put $x=2$,

$f'(4)=3$

Hence, B is correct.

42. $f'(x)=nx^{n-1}$

$f'(p+q)=f'(p)+f'(q)$

$$\Rightarrow (p+q)^{n-1}=p^{n-1}+q^{n-1} \quad (n\neq 0)$$

So, two values of $n=0$ & $n=2$ are possible.
Hence, C is correct.

43. $f(x)=(2x-3\pi)^5+\frac{4}{3}x+\cos x$

$$g'(y)=\frac{1}{5(2x-3\pi)^4\times 2+\frac{4}{3}-\sin x}$$

Put $x=\frac{3\pi}{2}$ for $y=2\pi$

$$g'(2\pi)=\frac{3}{7}$$

Hence, B is correct.

44. $f(x)=x+\sin x$

$$g'(x)=\frac{1}{1+\cos x}$$

Put $x=\pi/4$

$$g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}}{1+\sqrt{2}}=\sqrt{2}(\sqrt{2}-1)=2-\sqrt{2}$$

Hence, C is correct.

45. $f(x)=\begin{vmatrix}\cos x & \sin x & \cos x\\ \cos 2x & \sin 2x & 2\cos 2x\\ \cos 3x & \sin 3x & 2\cos 3x\end{vmatrix}$

$$f'\left(\frac{\pi}{2}\right)=\begin{vmatrix}-1 & 1 & 0\\ 0 & 0 & -2\\ +3 & -1 & 0\end{vmatrix}$$

$$+\begin{vmatrix}0 & 0 & 0\\ -1 & -2 & -2\\ 0 & 0 & 0\end{vmatrix}+\begin{vmatrix}0 & 1 & -1\\ -1 & 0 & 0\\ 0 & -1 & 6\end{vmatrix}$$

$=1$

46. $f(x)=y=\sin^{-1}\cos 2x$

$$f(5)=\sin^{-1}\cos 10=\frac{\pi}{2}-\cos^{-1}\cos 10$$
$$=\frac{\pi}{2}-(4\pi-10)$$
$$=10-\frac{7}{2}\pi$$

$$f'(x)=\frac{-1}{\sqrt{1-\cos^2 2x}}\times 2\sin 2x$$
$$\Rightarrow f'(5)=2$$

$$f(5)+f'(5)=12-7\frac{\pi}{2}$$

Hence, C is correct.

47. $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Put $x=\cos\theta$

$$y=2\pi-2\theta$$
$$\frac{dy}{d\theta}=-2 \qquad \text{(I)}$$
$$t=\sqrt{1-x^2}=\sin\theta$$
$$\frac{dt}{d\theta}=\cos\theta=-\frac{1}{2} \qquad \text{(II)}$$

Using (I) & (II)

$$\frac{dy}{dt}=-2\times -2=4$$

Hence, C is correct.

48. $(f(x))^n=f(nx)$

Differentiating it

$$(f(x))^{n-1}f'(x)=f'(nx)$$

Multiply by $f(x)$

$$f(nx)f'(x)=f'(nx)f(x)$$

Hence, C is correct.

49. $x^2+y^2=a^2 \qquad \text{(I)}$

$$2x+2yy'=0 \quad\Rightarrow\quad \frac{x}{y}=-y' \qquad \text{(II)}$$

$$1+(y')^2+yy''=0\Rightarrow y=\frac{-(1+(y')^2)}{y''} \qquad \text{(III)}$$

Using, (I), (II) & (III)

$$(y')^2+1=\frac{a^2(y'')^2}{(1+(y')^2)^4}\Rightarrow\frac{1}{a^2}=\frac{(y'')^2}{(1+(y')^2)^2(1+(y')^2)}$$

$$\Rightarrow K=\frac{|y''|}{\sqrt{\left(1+(y')^2\right)^3}}$$

Hence, B is correct.

50. $f(x)=\frac{x^n(1-x)^n}{\angle n}$

$$f'(x)=\frac{\eta x^{n-1}(1-x)^n-nx^n(1-x)^{n-1}}{\angle n}$$

$$f''(x)=\frac{n(n-1)x^{n-2}(1-x)^n-n^2x^{n-1}(1-x)^{n-1}-n^2x^{n-1}(1-x)^{n-1}+n(n-1)x^n(1-x)^{n-2}}{\angle n}$$

By observing, $f^k(0)$ & $f^k(1)$ are always integer.
Hence, C is correct.

MULTIPLE CORRECT ANSWER TYPE

51. $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} - \frac{e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$= \frac{\sqrt{\left(e^{\sqrt{x}} + e^{-\sqrt{x}}\right)^2 - 4}}{2\sqrt{x}} = \frac{\sqrt{y^2 - 4}}{2\sqrt{x}}$$

Hence, A, C are correct.

52. (A) Conceptual

(B) $U(x) = \sin bx$

$U'(x) = b\cos bx$

$U''(x) = -b^2 \sin bx = -b^2 U(x)$

(C) $\dfrac{dg}{dx} = \dfrac{1}{2\sqrt{x(x+n)}} \times (2x+n)$

$\dfrac{dg}{dx} = \dfrac{a}{g}$

Hence, A, B, C are correct.

53. $ax^3 + bx^2 + bx + d = 0$

$3ax^2 + 2bx + b = 0$

$\Rightarrow 3a + 3b = 0$ (as 1 is repeated root)

$\Rightarrow a + b = 0$

Now, $a + b + b + d = 0$

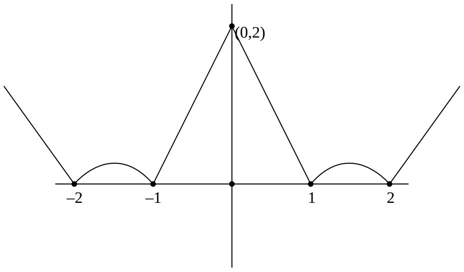
$\Rightarrow b + d = 0$

Hence, B, C, D is correct.

54. $f(x) = \left|x^2 - 3|x| + 2\right|$

$$f(x) = \begin{cases} x^2 - 3x + 2, & 0 < x < 1 \text{ or } x > 2 \\ x^2 + 3x + 2, & -1 < x < 0 \text{ or } x < -2 \\ -\left(x^2 - 3x + 2\right), & 1 < x < 2 \\ -\left(x^2 + 3x + 2\right), & -2 < x < -1 \end{cases}$$

Hence, A, B, C is correct.



55. $y = 2\tan^{-1} x, \qquad z = \dfrac{\pi}{4} + \tan^{-1} x$

$\Rightarrow \quad y = 2z - \dfrac{\pi}{2}$

$\dfrac{dy}{dz} = 2$

Hence, C is correct.

56. $y = \dfrac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1} = x^2 - \sqrt{3}x + 1$

$\dfrac{dy}{dx} = 2x - \sqrt{3} \Rightarrow a = 2, b = -\sqrt{3}$

$a - b = 2 + \sqrt{3}$

Hence, A, C is correct.

57. $f(x) = \dfrac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1} \times x = \dfrac{|\sqrt{x-1} - 1|}{\sqrt{x-1} - 1} \cdot x$

$f(x) = \begin{cases} x, & x \ge 2 \\ -x, & x \in [1, 2) \end{cases}$

Hence, A, B, D is correct.

58.

A) $y^2 + 4 = (\sec^n \theta + \cos^n \theta)^2$ &

$x^2 + 4 = (\sec\theta + \cos\theta)^2$

$\dfrac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$

$\dfrac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2\left(\sec^n\theta\tan\theta + \cos^n\theta\tan\theta\right)^2}{\tan^2\theta\left(\sec\theta + \cos\theta\right)^2}$$

$$= \frac{n^2\left(y^2 + 4\right)}{x^2 + 4}$$

B) Put $t = \tan\theta$

$x = 0, \qquad y = 0$

$\dfrac{dx}{d\theta} = 1 \qquad \dfrac{dy}{d\theta} = 1$

$\Rightarrow \dfrac{dy}{dx} = 1$

C) $e^y + xy = e$

$e^y y' + xy' + y = 0$

$e^y y'' + e^y (y')^2 + y' + xy'' + y' = 0$

For $x = 0,\ y = 1$

$$y'' = \frac{-(e(y')^2 + 2y')}{e}$$

$$= \frac{-1}{e}\left(\frac{1}{e} - \frac{2}{e}\right) = \frac{1}{e^2}$$

D) $\phi(x) = f(x)g(x)$

$\phi'(x) = f'(x)g(x) + f(x)g'(x)$

$\phi''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g'(x)$

$$\frac{\phi''(x)}{\phi(x)} = \frac{f''(x)}{f(x)} + \frac{g''(x)}{g(x)} + \frac{2f'(x)}{f(x)}\frac{g'(x)}{g(x)}$$

Hence, A, B, C, D is correct.

59. Put $t = \tan\theta$

$y = \sin^{-1}\sin\theta, \qquad x = \cos^{-1}\sin\theta$

$y = \theta, \quad x = \frac{\pi}{2} - \theta$

$\frac{dy}{d\theta} = 1, \quad \frac{dx}{d\theta} = -1$

$\frac{dy}{dx} = -1$

Hence, B, D is correct.

60. $\sqrt{y+x} + \sqrt{y-x} = c$

Squaring both sides.

$2y + 2\sqrt{y^2 - x^2} = c^2$

Differentiating

$$2y' + \frac{2(yy' - x)}{\sqrt{y^2 - x^2}} = 0$$

$$y' = \frac{x}{y + \sqrt{y^2 - x^2}} = \frac{2x}{c^2}$$

Rationilising, we get

$$y' = \frac{y - \sqrt{y^2 - x^2}}{x}$$

Hence, A, B, C is correct.

61.

A) $xe^{xy} \mp y + \sin^2 x$

At $x = 0, \quad y = 0$

Differentiating wrt x,

$e^{xy} + xe^{xy}(y + xy') = y' + \sin 2x$

At $x = 0$

$\Rightarrow 1 = y'$

B) Conceptual

C) Conceptual

D) $y = \sin^{-1}\cos\sin^{-1}x + \cos^{-1}\sin\cos^{-1}x$

$= \sin^{-1}\sqrt{1-x^2} + \cos^{-1}\sqrt{1-x^2}$

$= \frac{\pi}{2}$

Hence, A, C, D is correct.

62. $p(x) = x^3 + 3ax^2 + 3bx + c = 0$

$p'(x) = 3x^2 + 6ax + 3b = 3\left(x^2 + 2ax + b\right)$

$\Rightarrow \alpha^3 + 3a\alpha^2 + 3b\alpha + c = 0 \ \ \&\ \ \alpha^2 + 2a\alpha + b = 0$ ......(I)

Using above eqn (I)

$\Rightarrow \alpha^3 + 2a\alpha^2 + b\alpha + a\alpha^2 + 2b\alpha + c = 0$ ....(II)

$\Rightarrow a\alpha^2 + 2b\alpha + c = 0$

from (I) & (II)

$\alpha^2 + 2a\alpha + b = 0 \quad \times\ a$

$a\alpha^2 + 2b\alpha + c = 0 \quad \times 1$

$2\left(a^2 - b\right)\alpha + ab - c = 0$

$$\alpha = \frac{c - ab}{2\left(a^2 - b\right)}$$

Hence, A, B, D, are correct.

63. $y = \sqrt{3 + 2[\sin x \sin 2x + \sin x \sin 3x + \sin 2x \sin 3x + \cos x \cos 2x + \cos x \cos 3x + \cos 2x \cos 3x]}$

$= \sqrt{3 + 2[\cos x + \cos 2x + \cos x]}$

$= \sqrt{3 + 4\cos x + 4\cos^2 x - 2}$

$= \sqrt{4\cos^2 x + 4\cos x + 1}$

$y = |2\cos x + 1|$

At $x = \frac{\pi}{2}$

$y = 2\cos x + 1$

$y' = -2\sin x$

$y' = -2$

At $x = \pi/5$

$$y = 2\cos x + 1$$

$$y' = -2\sin x$$

$$y' = -2\sin \pi/5 = \frac{\sqrt{5}+3}{2}$$

Hence A, B, is correct.

64. Similar to Q. 57

Hence A, B is correct.

65. $f(x) = \sqrt{1-\sin 2x} = |\sin x - \cos x|$

For $x \in (0, \pi/4)$

$$f(x) = \cos x - \sin x$$

$$f'(x) = -(\sin x + \cos x)$$

For $x \in (\pi/4, \pi/2)$

$$f(x) = \sin x - \cos x$$

$$f'(x) = \cos x + \sin x$$

Hence C, D is correct.

66. $y = x^{\ln_x^{\ln\ln_x}}$

$$\ln y = (\ln x)^{\ln\ln_x} \ln x$$

$$\frac{1}{y}y' = \frac{1}{x}(\ln x)^{\ln\ln x} + \cancel{\ln x}(\ln x)^{\ln\ln x}\left(\frac{2\ln\ln x}{x\cancel{\ln x}}\right)$$

$$y' = \frac{y}{x}(\ln x)^{\ln\ln x}(1 + 2\ln\ln x)$$

Hence, B, D is correct.

67. $f(x) = \dfrac{1-x^{n+1}}{1-x}$

$$f(x) = 1 + x + x^2 + \ldots\ldots + x^n$$

$$f'(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$$

$$f'(x)g(x)$$

$$= \left(1 + 2x + 3x^2 + \ldots\ldots + nx^{n-1}\right)\left(1 - \frac{2}{x} + \frac{3}{x^2} + \ldots\right)$$

Constant term =

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots (-1)^{n-1}n^2$$

If n is even

$$-3 - 7 - 11 \text{- - - - - - - - -}_{\frac{n}{2}\text{terms}}$$

$$= -\frac{n}{4}\left(6 + \left(\frac{n}{2} - 1\right)4\right) = -\frac{n(n+1)}{2}$$

If n is odd

$$-3 - 7 - 11 \text{- - - - - - - - - - - -}_{\frac{(n-1)}{2}\text{terms}} + n^2$$

$$= -\frac{(n-1)}{4}\left(6 + \left(\frac{n-1}{2} - 1\right)4\right) + n^2$$

$$= -\frac{n(n-1)}{2} + n^2 = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

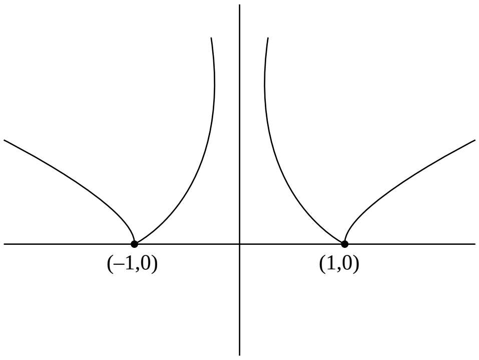
Hence, B, C is correct.

68. $f(x) = |\ln|x||$

$$f(x) = \begin{cases} \ln x, & x \ge 1 \\ -\ln x, & 0 < x < 1 \\ -\ln(-x), & -1 < x < 0 \\ +\ln(-x), & x \le -1 \end{cases}$$

$$f'(x) = \begin{cases} 1/x, & x \ge 1 \\ -1/x, & 0 < x < 1 \\ -1/x, & -1 < x < 0 \\ 1/x, & x \le -1 \end{cases}$$

Hence A, B is correct.



69. $(x-c)^2 + (y-a)^2 = b^2$

$$2(x-c) + 2(y-a)y' = 0 \qquad \text{(I)}$$

$$y' = \frac{c-x}{y-a} \qquad \text{(II)}$$

differentiating eq (1) again

$$1 + (y-a)g'' + (y')^2 = 0$$

As $\dfrac{\left(1+(y')^2\right)^{3/2}}{y''} = \dfrac{\left(1+(y')^2\right)^{3/2}}{1+(y')^2}(a-y)$

$= \sqrt{(c-x)^2+(y-a)^2}$

$= b$

Hence, B, C is correct.

70. $y = x^{x^2}$

$\ln y = x^2 \ln x$

$\frac{1}{y} y' = x + 2x \ln x$

$y' = x^{x^2}(x)(1+2\ln x)$

Hence, C, D is correct.

71. $A : f(x) = \cos(\cos(\cos(\cos(\cdots \cos x))))$

$f'(x) = (\sin \cos \cdots \cos x)(\sin \cos\ \cos \cdots)\text{---------}_{(\sin x)}$

$f'(a) = (\sin a)^8 = \left(\sqrt{1-a^2}\right)^8$

$= \left(1-a^2\right)^4$

Hence, D is correct.

72. $y = \tan^{-1}\cot x + \cot^{-1}\tan x$

$= \pi - \tan^{-1}\tan x - \cot^{-1}\cot x$

$= \pi + (\pi - x) - x$

$= 2\pi - 2x$

$y' = -2$

Hence, A is correct.

73. $g(x) = f(x)\sin x$

$g'(x) = f'(x)\sin x + f(x)\cos x$

$g''(x) = f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$

$g''(-x) = -2$

Hence, A is correct.

74. $f'(x) = f(1-x)$

$f''(x) = -f'(1-x)$

$f''(1-x) = -f'(x) = -f(1-x)$

$f''(x) = -f(x)$

$\Rightarrow f(x) = A\cos x + B\sin x$

As $f(0) = 1 \Rightarrow A = 1$

As $f'(1) = 1 \Rightarrow 1 = -\sin 1 + B\cos 1$

$\Rightarrow B = \frac{1+\sin 1}{\cos 1} = \sec 1 + \tan 1$

Hence, A is correct.

75. $\frac{f'(x)}{g'(x)} = \frac{e^{f(x)}}{e^{g(x)}}$

$\Rightarrow f'(x)e^{-f(x)} = g'(x)e^{-g(x)}$

$\Rightarrow f'(x)e^{-f(x)} - g'(x)e^{-g(x)} = 0$

$\Rightarrow \frac{d}{dx}\left(-e^{-f(x)} + e^{-g(x)}\right) = 0$

Hence, C is correct.

76. $g(x) = f(x) - f(2x)$

$g'(x) = f'(x) - 2f'(2x)$

$g'(2x) = f'(2x) - 2f'(4x)$

Put $x = 1 \,\&\, 2$

$5 = f'(1) - 2f'(2), \quad 7 = f'(2) - 2f'(4)$

Double the second equation and add to the first
We get $f'(1) - 4f(4) = 7 \times 2 + 5 = 19$
Hence, C is correct.

77. $f(x) = 2\sin^{-1}\sqrt{1-x} + \sin^{-1}2\sqrt{x(1-x)}$

$f(x) = \pi - \sin^{-1}2\sqrt{x(1-x)} + \sin^{-1}2\sqrt{x(1-x)}$

$\forall \quad x \in (0, 1/2)$

$= \pi$

$f'(x) = 0$

Hence, C is correct.

78. $f(x) + f(x-2) = 0$

Put $x = x - 2$

$f(x-2) + f(x-4) = 0$

$f(x) = f(x-4)$

$f(x)$ is periodic with period `u'

$\frac{d}{dx}(f(x))_{a+4000} = \frac{d}{dx}(f(x))_a = b$

Hence, A is correct.

79. Conceptual

Hence, D is correct .

80. $f\left(x+y^3\right) = f(x) + f\left(y^3\right)$

put $x = y = 0$

$f(0) = 0$

$f'(x) = \lim_{h\to 0} f\,\frac{(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(h)}{h} = f'(0)$$

$$f(x) = f'(0)x$$

Hence, A is correct.

Comphrehension-1:

$$x = f(t) = a^{\ln b^t}, y = b^{-\ln a^t}$$

$$x = a^{t\ln b}\ ,\ y = b^{-t\ln a}$$

81. $\frac{dx}{dt} = a^{t\ln b} \ln a \ln b, \frac{dy}{dt} = -b^{-t\ln a} \ln b \ln a$

$$\frac{dy}{dx} = \frac{-b^{-t\ln a}}{a^{t\ln b}} = -\frac{g(t)}{f(t)}$$

Hence, D is correct.

82. $f(t) = g(t)$

$$a^{t\ln b} = b^{-t\ln a}$$

$$t \ln b \ln a = -t \ln a \ln b$$

$$\Rightarrow t = 0$$

As $\frac{dy}{dx} = -(g(t))^2$

$$\frac{d^2y}{dx^2} = -2g(t)g'(t) \times \frac{dt}{dx}$$

$$= \frac{+2b^{-t\ln a}b^{-t\ln a} \ln b \ln a}{a^{t\ln b} \ln b \ln a} = 2$$

Hence D is correct.

83. $\frac{f(t)f''(-t)}{f'(t)f'(-t)} + \frac{f(-t)f''(t)}{f'(-t)f'(t)}$

$$= \frac{a^{t\ln b}}{a^{t\ln b} \ln a \ln b} \times \frac{a^{-t\ln b}(\ln a \ln b)^2}{a^{-t\ln b} \ln a \ln b}$$

$$+ \frac{a^{-t\ln b}}{a^{-t\ln b} \ln a \ln b} \times \frac{a^{t\ln b}(\ln a \ln b)^2}{a^{t\ln b} \ln a \ln b}$$

$$= 2$$

Hence, B is correct.

Comphrehension-2:

$$f(x) = \max\left\{t^3 - t^2 + t + 1, \quad 0 \le t \le x\right\}, \quad 0 \le x \le 1$$

$$\min\{3 - t, \quad 1 \le t \le x\}, \quad 1 < x \le 2$$

84. $y = t^3 - t^2 + t + 1$

$$y' = 3t^2 - 2t + 1 > 0$$

$$f(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} \max\left\{\frac{3}{8}t^4 + \frac{t^3}{2} - \frac{3}{2}t^2 + 1, 0 \le t \le x\right\}, & 0 \le x < 1 \\ \min\left\{\frac{3}{8}t + \frac{\sin^2 \pi t}{32} + \frac{5}{8}, 1 \le t \le x\right\}, & 1 \le x \le 2 \end{cases}$$

$$y = \frac{3}{8}t^4 + \frac{t^3}{2} - \frac{3}{2}t^2 + 1$$

$$\frac{dy}{dt} = \frac{3}{2}t^3 + \frac{3t^2}{2} - 3t$$

$$= \frac{3}{2}t\left(t^2 + t - 2\right)$$

$$= \frac{3}{2}t(t+2)(t-1)$$

As $y = \frac{3}{8}t + \frac{\sin^2 \pi t}{32} + \frac{5}{8}$

$$\frac{dy}{dt} = \frac{3}{8} + \pi\frac{\sin 2\pi t}{32} > 0$$

$$g(x) = \begin{cases} 1, & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

Hence, B is correct.

85. Hence, A is correct.

86. Hence, D is correct.

Comprehension-3:

$$\sin x = \sin\frac{\pi x}{180}$$

For least +ve value of x,

$$x + \frac{\pi x}{180} = \pi \quad \text{or} \quad x - 2\pi = \frac{\pi x}{180}$$

$$x = \frac{\pi(180)}{\pi + 180} \quad \text{or} \quad x = \frac{(2\pi)180}{-\pi + 180}$$

$$m = 360,\ n = 180, \quad p = 180,\ q = 180$$

87. $m + n + p + q = 900$

Hence, B is correct.

88. $\lim_{x \to \infty} \sqrt{Ax^2 + Bx} - Cx$

$$= \lim_{x \to \infty} \frac{Ax^2 + Bx - C^2x^2}{\sqrt{Ax^2 + Bx} + Cx} \qquad \left(\because A = C^2\right)$$

$$= \frac{B}{\sqrt{A + C}} = 2$$

$$\Rightarrow \frac{B}{C} = 4$$

$$\Rightarrow \frac{BC}{A} = 4$$

Hence A is correct.

89. $g(x) = f(10 - 2x)$

$g'(x) = -2f'(10 - 2x)$

$g'(2) = -2f'(6) = 0$

Hence, C is correct.

Comprehension-4:

90. $G^{*}f(x) \cdot g(x) = \lim\limits_{h \to 0} \dfrac{f^{m}(x+h)g^{m}(x+h) - f^{m}(x)g^{m}(x)}{h}$

$$\lim_{h \to 0} \frac{f^{m}(x+h)g^{m}(x+h) - g^{m}(x+h)f^{m}(x) + g^{m}(x+h)f^{m}(x) - f^{m}(x)g^{m}(x)}{h}$$

$$\lim_{h \to 0} g^{m}(x+h)\left(\frac{f^{m}(x+h) - f^{m}(x)}{h}\right) + f^{m}(x)\left(\frac{g^{n}(x+h) - g^{m}(x)}{h}\right)$$

$$= (G^{*}f(x))g^{m}(x) \ \ +(G^{*}g(x))f^{m}(x)$$

Hence B is correct.

91. $G^{*}\left(\dfrac{f(x)}{g(x)}\right) = \lim\limits_{h \to 0} \dfrac{\dfrac{f^{m}(x+h)}{g^{m}(x+h)} - \dfrac{f^{m}(x)}{g^{m}(x)}}{h}$

$$= \frac{1}{g^{2m}(x)} \lim_{h \to 0} \frac{f^{m}(x+h)g^{m}(x) - f^{m}(x)g^{m}(x+h)}{h}$$

$$= \frac{1}{g^{2m}(x)} \lim_{h \to 0} f^{m}(x+h)g^{m}(x) - f^{m}(x)g^{m}(x) + f^{m}(x)g^{m}(x) + f^{m}(x)g^{m}(x+h)$$

$$= \frac{\left(G^{*}g(x)\right)f(x)^{m} - \left(G^{*}f(x)\right)g(x)^{m}}{g(x)^{2m}}$$

Hence, C is correct.

92. $G^{*}f(g(x)) = \lim\limits_{h \to 0} \dfrac{f^{m}(g(x+h)) - f^{m}(g(x))}{h}$

$$= \lim_{h \to 0} mf^{m-1}g(x+h) \times f'g(x+h) \times g'(x+h)$$

$$= mf^{m-1}g(x)f'g(x)g'(x)$$

Hence, A is correct.

93. A) $\sqrt{x^{2} + y^{2}} = ae^{\tan^{-1} y/x}$

$$\Rightarrow \frac{x + yy'1}{\sqrt{x^{2} + y^{2}}} = ae^{\tan^{-1} y/x} \times \frac{1}{1 + \left(\frac{y}{x}\right)^{2}} \times \frac{xy' - y}{x^{2}}$$

$x + yy' = xy' - y$

$x + y = xy' - yy'$

$1 + y' = y' + xy'' - (y')^{2} - yy''$

$1 = xy'' - (y')^{2} - yy''$

Now at $x = 0$, $y = ae^{\pi/2}$

$y' = -1$

$$y'' = \frac{1+1}{-ae^{\pi/2}} = \frac{-2}{a}e^{-\pi/2}$$

$k = 2$

B) $y = \sin^{-1}\dfrac{2x}{1 + x^{2}}$

Put $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^{2}\theta$

$y = \sin^{-1}\sin 2\theta$

$= \pi - 2\theta$

$$\frac{dy}{dx} = -2\cos^{2}\theta = \frac{-2}{1+4} = \frac{-2}{5}$$

C) $\lim\limits_{x \to 0} \dfrac{4 + \sin 2x + A\sin x + B\cos x}{x^{2}}$

$\Rightarrow B = -4$

$$\lim_{x \to 0} \frac{2\cos 2x + A\cos x - B\sin x}{2x}$$

$\Rightarrow A = -2$

$\Rightarrow A - B = 2$

D) $f(x) = 2 + [3\sin x]$, $0 < x < \pi$

f(x) is non differentiable at points where 3sinx is an integer

$\Rightarrow \sin x = \dfrac{1}{3}, \dfrac{2}{3}, 1$

So, x have 5 such values

$A \to (Q),\ B \to (P),\ C \to (Q),\ D \to (R)$.

94. A) $T'(x) = F'\left(G(x) \times G'(x)\right)$

$T'(0) = F'(G(0)) \times G'(0) = -2$

$U(x) = \ln F(x)$

$$U'(x) = \frac{F'(x)}{F(x)}$$

$$U'(3) = \frac{F'(3)}{F(3)} = -\frac{1}{2}$$

$T'(0) + 6U'(3) = -5$

B) Put $x = -\frac{1}{y}$, $\lim\limits_{y \to 0}\left(\frac{-1}{y} + \frac{\sqrt{1-2y}}{y}\right)$

$$= \lim_{y \to 0} \frac{-2y}{y(\sqrt{1-2y}+1)} = -1$$

C) $y = t^2$

$$\frac{dy}{dt} = 2t, \qquad \frac{d^2y}{dt^2} = 2$$

$$x = t + t^2 \Rightarrow \frac{dx}{dt} = 1 + 2t, \qquad \frac{d^2x}{dt^2} = 2$$

$$\frac{dy}{dx} = \frac{2t}{1+2t} = 1 - \frac{1}{1+2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{2(1+2t) - 2t(2)}{(1+2t)^3}$$

$$= \frac{2}{(1+2t)^3}$$

$$\frac{d^2y}{dx^2} = -2$$

$$\frac{1}{4}\frac{d^2y}{dx^2} = -1$$

D) $f(x) = x + x^n = x\left(1 + x^{n-1}\right)$

For invertible, $\frac{dy}{dx}$ is > 0 or < 0

$f'(x) = 1 + nx^{n-1}$

$\Rightarrow n - 1 = 2$

$n = 3$

$A \to (T), B \to (P), C \to, D \to (S).$

95.

A) $y = 3e^{2x} + 2e^{3x}$

$y' = 6e^{2x} + 6e^{3x}$

$y'' = 6\left(2e^{2x} + 3e^{3x}\right)$

$y'' + ay' + by = (12 + 6a + 3b)e^{2x} +$

$(18 + 6a + 2b)e^{3x} = 0$

$\Rightarrow 4 + 2a + b = 0 \quad \& \quad 9 + 3a + b = 0$

$a = -5, \quad b = 6$

$a + b = 1$

B) $\lim\limits_{x \to ,0^+} (x\cos x)^x + (x\sin x)^{1/x} = 1$

C) $f'(x) = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

$$+ (\sin x)^{\cos x}\left[-\sin x \ln \sin x + \frac{\cos^2 x}{\sin x}\right]$$

$$f'\left(\frac{\pi}{2}\right) = 1$$

D) $\sin^{-1}\sin 5 < 4x - x^2 + 2$

$5 - 2\pi < 4x - x^2 + 2$

$x^2 - 4x + (3 - 2\pi) < 0$

$$x \in \left(\frac{4 - \sqrt{16 - 4(3-2\pi)}}{2}, \frac{4 + \sqrt{16 - 4(3-2\pi)}}{2}\right)$$

$x \in (2 - \sqrt{1+2\pi},\ 2 + \sqrt{1+2\pi})$

There is + Integral $x >$ 4 that lies in the interval

$A \to (S),\ B \to (S),\ C \to (S),\ D \to (R).$

96.

A) $\left|\frac{g(x) - g(y)}{x - y}\right| \le |x - y|^{98}$

$$\lim_{x \to y}\left|\frac{g(x) - g(y)}{(x - y)}\right| \le \lim_{x \to y}|x - y|^{98}$$

$\Rightarrow |g'(x)| \le 0$

$\Rightarrow g'(x) = 0$

B) Derivative of $\sin^{-1}\frac{2x}{1+x^2}$ with respect to

$\cos^{-1}\frac{(1-x^2)}{1+x^2}$ is -1

C) $y=\cot^{-1}\tan x+\tan^{-1}\cot x$

$y=\pi-\tan^{-1}\tan x-\cot^{-1}\cot x$

$=\pi-(x-\pi)-x$

$=2\pi-2x$

$y'=-2$

D) $4xe^{xy}=y+5\sin^2 x$

$4e^{xy}+4xe^{xy}(y+xy')=y'+5\sin 2x$

Put $x=y=0$

$y'=4$

$A\rightarrow(R),\quad B\rightarrow(Q),\quad C\rightarrow(P),\quad D\rightarrow(T).$

97. A) $y=\sin^{-1}2x\sqrt{1-x^2}=-\pi-2\sin^{-1}x$

$$y'=\frac{-2}{\sqrt{1-x^2}}$$

B) $y=2\sin^{-1}\sqrt{1-x}+\sin^{-1}2\sqrt{x(1-x)}=\pi$

$y'=0$

C) $y=\sin^{-1}\left(3x-4x^3\right)=3\sin^{-1}x$

$$y'=\frac{3}{\sqrt{1-x^2}}$$

D) $y=\cos^{-1}\frac{2x}{1+x^2}=\frac{\pi}{2}-\sin^{-1}\frac{2x}{1+x^2}$

$$y=\frac{\pi}{2}-2\tan^{-1}x$$

$$y'=\frac{-2}{1+x^2}$$

$A\rightarrow(S), B\rightarrow(P), C\rightarrow(R), D\rightarrow(Q).$

Review Exercises Solution

1. $y=f(\tan x)$ wrt $g(\sec x)$

$$\frac{dy}{dx}=f'(\tan x)\sec^2 x\,,\ t=g(\sec x)$$

$$\frac{dt}{dx}=g'(\sec x)\sec x\tan x$$

$$\frac{dy}{dt}=\frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$$

$$=\frac{f'(\tan x)}{g'(\sec x)}\sin x$$

$$=\frac{2\times\sqrt{2}}{4}=\frac{1}{\sqrt{2}}$$

2. $y=(\sqrt{x})^{x^{x\ldots\infty}}$

$y^2=x^{x^{x\ldots\infty}}$

$2\ln y=y^2\ln x$

$$\frac{2}{4}y'=2yy'\ln x+\frac{y^2}{x}$$

$$2y'=2y^2y'\ln x+\frac{y^3}{x}$$

$$y'=\frac{y^3}{\left(2-2y^2\ln x\right)x}$$

3. $f(x)=\cos\left\{\frac{\pi}{2}[x]-x^3\right\},1<x<2$

$f(x)=\sin x^3$

$f'(x)=3x^2\cos x^3$

$$f'\left(\left(\frac{\pi}{2}\right)^{1/3}\right)=0$$

4. $f'(x)=\frac{1}{1+\sec^2 x}$

$$=\frac{1}{2+\tan^2 x}$$

$$=\frac{1}{2+(g(x)-x)^2}$$

5. a) $f(x)=x+3\ln(x-2)$

$$f'(x)=1+\frac{3}{x-2}=\frac{x+1}{x-2}$$

$$g(x) = x + 5\ln(x-1)$$

$$g'(x) = 1 + \frac{5}{x-1} = \frac{x+4}{x-1}$$

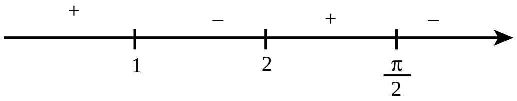
Now $f'(x) < g'(x)$

$$\frac{x+1}{x-2} < \frac{x+y}{x-1}$$

$$\frac{(x+1)(x-1)-(x+4)(x-2)}{(x-1)(x-2)} < 0$$

$$\frac{-2x+7}{(x-1)(x-2)} < 0$$

$$x \in \left(\frac{7}{2}, \infty\right)$$



b) $f(x) = e^{2x} - 3x$

$$f'(x) = 2e^{2x} - 3$$

$$g(x) = 5\left(e^x - x + 3\right)$$

$$g'(x) = 5\left(e^x - 1\right)$$

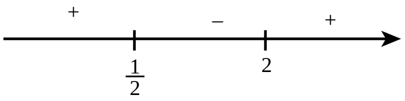
Now, $f'(x) < g'(x)$

$$2e^{2x} - 5e^x + 2 < 0$$

$$\left(e^x - 2\right)\left(2e^x - 1\right) < 0$$

$$\frac{1}{2} < e^x < 2$$

$$-\ln 2 < x < \ln 2$$



6. $$f(x) = \frac{(\sqrt{x+1}+\sqrt{x-1})^{2/3}\left(\sqrt{x^2-1}\right)^{1/3}}{(\sqrt{x+1}+\sqrt{x-1})^{2/3}}$$

$$f(x) = \left(\sqrt{x^2-1}\right)^{1/3}$$

$$f'(x) = \frac{1}{6}\left(x^2-1\right)^{-5/6} \times 2x$$

7. $$x^4 + 7x^2y^2 + 9y^4 - 24xy^3 = 0$$

$$\frac{dy}{dx} = -\frac{\delta F/\delta x}{\delta F/\delta y}$$

$$= -\frac{\left(4x^3 + 14xy^2 - 24y^3\right)}{14x^2y + 36y^3 - 72xy^2} \times \frac{y}{x} \times \frac{x}{y}$$

$$= \frac{y}{x}\frac{\left(24xy^3 - 4x^4 - 14x^2y^2\right)}{\left(14x^2y^2 + 36y^4 - 72xy^3\right)}$$

$$= \frac{y}{x}\frac{\left(24xy^3 - 4\left(24xy^3 - 7x^2y^2 - 9y^4\right) - 14x^2y^2\right)}{\left(14x^2y^2 + 36y^4 - 72xy^3\right)}$$

$$= \frac{y}{x}$$

8. $$f(x) = |x|^{|\sin x|}$$

$f'(x)$ at $x = -\pi/4$

$$y = f(x) = -x^{-\sin x}$$

$$\ln y = -\sin x \ln - x$$

$$\frac{1}{y}y' = -\frac{\sin x}{x} - \cos x \ln(-x)$$

$$y' = (-x)^{-\sin x}\left(\frac{-\sin x}{x} - \cos x \ln(-x)\right)$$

$$y' = (\pi/4)^{1/\sqrt{2}}\left(-\frac{4}{\sqrt{2}\pi} + \frac{1}{\sqrt{2}}\left(\ln\frac{y}{\pi}\right)\right)$$

9. $$y = x\ln\left(\frac{x}{a+bx}\right)$$

$$e^{y/x} = \frac{x}{a+bx}$$

$$e^{y/x} \times \frac{xy'-y}{x^2} = \frac{(a+bx)-x(b)}{(a+bx)^2} = \frac{a}{(a+bx)^2}$$

$$xy' - y = \frac{ax}{a+bx}$$

$$\frac{(a+bx)}{ax} = \frac{1}{xy'-y} \Rightarrow \frac{-1}{x^2} = \frac{-\left(y' + xy'' - y'\right)}{\left(xy'-y\right)^2}$$

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

10. $$y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x + \cdots\infty}}}$$

$$y=\frac{\sin x}{1+\dfrac{\cos x}{1+y}}$$

$$y=\frac{(1+y)\sin x}{1+y+\cos x}$$

$$y+y^2+y\cos x=\sin x+y\sin x$$

$$y'+2yy'+y'\cos x-y\sin x=\cos x+y\cos x+y'\sin x$$

$$y'=\frac{y\sin x+\cos x+y\sin x}{1+2y+\cos x-\sin x}$$

11. $y=\dfrac{u}{v}$

$$y'=\frac{vu'-uv'}{(v)^2}$$

$$v^2y'=vu'-uv'$$

$$v^2y''+2vv'y'=v'u'+vu''-u'v'-uv''$$

$$v^2y''=vu''-uv''-2vu'y'$$

$$v^3y''=v^2u''-uvv''-2v^2u'y'$$

$$v^3y''=v^2u''-uvv''-2v(u')v'+2uvv'$$

$$=2uvv'-uvv''-\left(2vv'u'-v^2u''\right)$$

$$=\begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$$

12. $y=\tan^{-1}\dfrac{x}{1+\sqrt{1-x^2}}+\sin\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$

Put $x=\cos\theta$

$$\frac{dx}{d\theta}=-\sin\theta$$

$$y=\tan^{-1}\frac{\cos^2\theta/2-\sin^2\theta/2}{(\cos\theta/2+\sin\theta/2)^2}+\sin 2\tan^{-1}\tan\frac{\theta}{2}$$

$$y=\frac{\pi}{4}-\frac{\theta}{2}+\sin\theta$$

$$\frac{dy}{d\theta}=-\frac{1}{2}+\cos\theta$$

$$\frac{dy}{dx}=\frac{-1/2+\cos\theta}{-\sin\theta}$$

$$=\frac{-1/2+x}{-\sqrt{1-x^2}}$$

13. $\dfrac{d^2x}{dy^2}=\dfrac{-\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}$

$$\frac{d^3x}{dy^3}=-\left(\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2\right)\bigg/\left(\frac{dy}{dx}\right)^5$$

Now, $g'(y)$ at $y=0$ & $x=-1$

If $f(x)=x^3+3x+4$

$$\frac{dy}{dx}=3x^2+3=6 \text{ at } x=-1$$

$$\frac{d^2y}{dx^2}=6x=-6 \text{ at } x=-1$$

$$\frac{d^3y}{dx^3}=6$$

$$g'(0)=-\frac{\left((6)^2-3(6)^2\right)}{(6)^5}=\frac{2}{6^3}$$

14. $f(x)=\sin x\sin 2x\ldots\sin nx$

$$\ln f(x)=\ln\sin x+\ln\sin 2x+\cdots+\ln\sin nx$$

$$\frac{f'(x)}{f(x)}=\cot x+2\cot 2x+\cdots+n\cot nx$$

$$\Rightarrow f'(x)=\sum_{k=1}^{n}(k\cot kx)f(x)$$

15. $f(x)=\sin^{-1}2x\sqrt{1-x^2}+\sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$

at $x\to-\dfrac{1}{2}$

$$f(x)=2\sin^{-1}x+\pi-\sin^{-1}x$$

$$f'(x)=\frac{1}{\sqrt{1-x^2}}$$

$$f'(-1/2)=2/\sqrt{3}$$

16. $\cos\dfrac{x}{2}\cdot\cos\dfrac{x}{2^2}\cdot\cos\dfrac{x}{2^3}\cdots\infty=\dfrac{\sin x}{x}$

Taking log on both sides & differentiating,

$$-\frac{1}{2}\tan\frac{x}{2}-\frac{1}{2^2}\tan\frac{x}{2^2}\cdots\ \infty=\cot x-\frac{1}{x}$$

$$=\frac{x\cos x-\sin x}{x^2}$$

Again differentiating,

$$\frac{-1}{2^2}\sec^2\frac{x}{2}-\frac{1}{\left(2^2\right)^2}\sec^2\frac{x}{2^2}+\cdots\infty$$

$$=\frac{x(-x\sin x)-(x\cos x-\sin x)}{x^2}$$

$$=\left(1-x^2\right)$$

17. If $g(1)\,\&\,g(-1)$ are equal $\Rightarrow$ coeff of $x = 0$

$g(x)=ax^2+b$

$g'(x)=2ax$

Now, $x_1,\ x_2,\ x_3$ are in GP.

$\Rightarrow 2ax_1,\ 2ax_2,\ 2ax_3$ are in GP

$\Rightarrow g'(x_1), g'(x_2), g'(x_3)$ are in GP

18. $fg'(0)=f(0)g'(0)+f'(0)g(0)$

$=g'(0)+2f'(0)=6$ (I)

$(gh)'(0)=g(0)h'(0)+g'(0)h(0)=4$

$2h'(0)+3g'(0)=4$ (II)

$hf'(0)=h(0)f'(0)+h'(0)f(0)=5$

$=3f'(0)+h'(0)=5$ (III)

Solving (I), (II) & (III)

$f'(0)=2$

$h'(0)=-1, g'(0)=2$

Now $fgh'(0)=f'(0)g(0)h(0)+f(0)g'(0)h(0)+f(0)g(0)h'(0)$

$=12+6-2$

$=16$

19. $y=\log_{\cos x}\sin x\times\dfrac{1}{\log_{\sin x}\cos x}+\sin^{-1}\dfrac{2x}{1+x^2}$

$$y=\left(\frac{\log\sin x}{\log\cos x}\right)^2+\sin^{-1}\frac{2x}{1+x^2}$$

$$y=\left(\frac{\log\sin x}{\log\cos x}\right)^2+2\tan^{-1}x$$

$$y'=\frac{2\log\sin x}{\log\cos x}\times\left(\frac{(\log\cos x)\tan x+(\log\sin x)(\cot x)}{(\log\cos x)^2}\right)+\frac{2}{1+x^2}$$

$$y'\text{ at }x=\frac{x}{4}=\frac{-4}{\left(\log\sqrt{2}\right)}+\frac{2\times16}{16+\pi^2}$$

$$=\frac{32}{16+\pi^2}-\frac{8}{\log 2}$$

20. $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3\left(x^3-y^3\right)$

Let $x^3=\cos p, y^3=\cos q$

$\Rightarrow \sqrt{1-\cos^2 p}+\sqrt{1-\cos^2 q}=a^3(\cos p-\cos q)$

$\Rightarrow \quad \sin p+\sin q=a^3(\cos p-\cos q)$

$\Rightarrow \quad 2\sin\dfrac{p+q}{2}\cos\dfrac{p-q}{2}=2a^3\sin\dfrac{p+q}{2}\sin\left(\dfrac{q-p}{2}\right)$

$\Rightarrow \quad \tan\left(\dfrac{p-q}{2}\right)=-\dfrac{1}{a^3}$

$\Rightarrow \dfrac{p-q}{2}=\tan^{-1}\left(-\dfrac{1}{a^3}\right)$

$\Rightarrow p-q=2\tan^{-1}\left(-\dfrac{1}{a^3}\right)$

$\Rightarrow \cos^{-1}x^3-\cos^{-1}y^3=2\tan^{-1}\left(\dfrac{-1}{a^3}\right)$

Differentiate wrt x

$\Rightarrow \dfrac{-3x^2}{\sqrt{1-x^6}}-\left(\dfrac{-3y^2}{\sqrt{1-y^6}}\right)\dfrac{dy}{dx}=0$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$ Hence proved.

21. $y=x-\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, $x>0, x\neq 0$

Similarly, if $x<0, x\neq 0$

$$y=\cos^{-1}\left(\frac{-1+x^2}{1+x^2}\right)=\pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y=\begin{cases}\pi+2\tan^{-1}x, & x<0\\ \pi-2\tan^{-1}x & x>0\end{cases}$$

$$\frac{dy}{dx}=\begin{cases}\dfrac{2}{1+x^2}, & x<0\\ \dfrac{-2}{1+x^2}, & x>0\end{cases}$$

$\dfrac{dy}{dx}$ does not exist at $x=0$.

22. $z = \ln \tan x/2$

$$\frac{d(z)}{dx} = \frac{1}{2\sin x/2 \cos x/2} = \operatorname{cosec} x$$

$$\frac{d^2z}{dx^2} = -\operatorname{cosec} x \cot x$$

Now, $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx}$

$$\frac{d^2y}{dz^2} = \frac{(dz/dx)d^2y/dx^2 - \frac{dy}{dx}\frac{d^2z}{dx^2}}{(dz/dx)^2} \times \frac{dx}{dz}$$

$$= \frac{\left(\operatorname{cosec} x\left(d^2y\right) + \operatorname{cosec} x \cot x (dy)\right)}{\operatorname{cosec}^2 x} \times \sin x$$

$$= \left(d^2y + \cot x\, dy\right) \times \sin^2 x$$

$$= -4\, y$$

$\Rightarrow k = -4$

23. $\sqrt{x^2 + y^2} = ae^{\tan^{-1} y/x}$

taking log on both sides

$$\frac{1}{2}\ln\left(x^2 + y^2\right) = \tan^{-1}\frac{y}{x} + \ln a$$

Differentiate wrt x .

$$\frac{x + yy'}{x^2 + y^2} = \frac{x^2}{x^2 + y^2} \times \frac{(xy' - y)}{x^2}$$

$$y'' = \frac{-1}{a}e^{-\pi/2}$$

$x + yy' = xy' - y$

Differentiate again wrt x

$\Rightarrow 1 + (y')^2 + yy'' = y' + xy'' - y'$

$\Rightarrow 1 + (y')^2 + yy'' = xy''$

At $x = 0, y = ae^{\pi/2}, y' = -1$

$$y'' = \frac{-2}{ae^{\pi/2}}$$

24. $y = \left(\tan^{-1} x\right)^2$

$$y' = \frac{2\tan^{-1} x}{1 + x^2}$$

$\left(1 + x^2\right)y' = 2\tan^{-1} x$

$$\left(1 + x^2\right)y'' + 2xy' = \frac{2}{1 + x^2}$$

$\left(1 + x^2\right)^2 y'' + 2x\left(1 + x^2\right)y' = 2$

25. $x^2 + y^2 = 4$

$2x + 2yy' = 0$

$$y' = \frac{-x}{y} = -1 = a$$

$\sin y + \sin x = \sin x \sin y$

$y' \cos y + \cos x = y' \sin x \cos y + \sin y \cos x$

$-y' - 1 = 0$

$y' = -1 = b$

$2e^{xy} + e^x \cdot e^y - e^x - e^y = e^{xy+1}$

$2e^{xy}\left(xy' + y\right) + e^{x+y}\left(1 + y'\right) - e^x - e^y y' = (e^{xy+1})\left(xy' + y\right)$

$2e\left(y' + 1\right) + e^2\left(1 + y'\right) - e - ey' = e^2\left(1 + y'\right)$

$\Rightarrow y' = -1 = c$

$a + b + c = -3$

26. $$y = \cfrac{x}{x + \cfrac{\sqrt[3]{x}}{x + \cfrac{\sqrt[3]{x}}{x + \cdots \infty}}}$$

$$\Rightarrow y = \cfrac{x}{x + \cfrac{\sqrt[3]{x}}{\cfrac{x}{y}}}$$

$$\Rightarrow y = \frac{x^{5/3}}{x^{5/3} + y}$$

$\Rightarrow y^2 + x^{5/3}y = x^{5/3}$

Differentiate wrt x

$$\Rightarrow 2yy_1 + \frac{5}{3}x^{2/3}y + x^{5/3}y_1 = \frac{5}{3}x^{2/3}$$

$$\Rightarrow y_1 = \frac{\frac{5}{3}x^{2/3}(-y + 1)}{\left(2y + x^{5/3}\right)}$$

27. i) $$f(x) = \frac{x^2\sqrt{1 + \left(\frac{x^2 - 1}{2x}\right)^2}}{\left(x^2 + 1\right)}$$

$$= \frac{x^2}{\cancel{(x^2 + 1)}} \times \frac{1}{2|x|} \times \cancel{(x^2 + 1)} = \frac{|x|}{2}$$

$$f'(x) = \begin{cases} -\frac{1}{2}, & x < 0 \\ \frac{1}{2}, & x > 0 \end{cases}$$

ii) $y=\sqrt{x+2\sqrt{2(x-2)}}+\sqrt{x-2\sqrt{2(x-2)}}$

$\Rightarrow y=|\sqrt{x-2}+\sqrt{2}|+|\sqrt{x-2}-\sqrt{2}|$

$$y=\begin{cases}2\sqrt{2}, & x\in[2,4)\\ 2\sqrt{x-2}, & x\in(4,\infty)\end{cases}$$

$$y'=\begin{cases}0, & x\in[2,4)\\ \dfrac{1}{\sqrt{x-2}}, & x\in(4,\infty)\end{cases}$$

28. i) $(F+G+H)'$

$=F'+G'+H'$

$=G+H+F$

$=\text{RHS}$

ii) $\left(F^3+G^3+H^3-3FGH\right)'$

$=3F^2F'+3G^2G'+3H^2H'-3F'GH-3FG'H-3FGH'$

$=3F^2G+3G^2H+3H^2F-3G^2H-3FH^2-3F^2G$

$=0$

29. $\lim\limits_{n\to\infty}\left(\dfrac{a^{1/n}+b^{1/n}}{2}\right)^n$

$\Rightarrow e^{\lim\limits_{n\to\infty}\frac{a^{1/n}+b^{1/n}-2}{2\cdot 1/n}}$

$=e^{\frac{1}{2}\lim\limits_{n\to\infty}\left(\frac{a^{1/n}-1}{1/n}+\frac{b^{1/n}-1}{1/n}\right)}$

$=e^{\frac{1}{2}\ln ab}=\sqrt{ab}$

30. $y=e^{\tan^{-1}x}$

Differentiate wrt x.

$y'=\left(\dfrac{1}{1+x^2}\right)e^{\tan^{-1}x}$

$\left(1+x^2\right)y'=y$

Differentiate wrt to *x*.

$\left(1+x^2\right)y''+(2x)y'=y'$

$\Rightarrow\left(1+x^2\right)y''+(2x-1)y'=0$

31. $\left(2xy'+y\right)y'''=3\left(xy''+y'\right)y''$

As $y=\dfrac{ax+b}{x^2+c}$

$\Rightarrow y\left(x^2+c\right)=ax+b$

$\Rightarrow y_1\left(x^2+c\right)+2xy=a$ .... (I)

$\Rightarrow y_{11}\left(x^2+c\right)+y_1 2x+2xy_1+2y=0$

$\Rightarrow\ y''\left(x^2+c\right)+2(2xy_1)+2y=0$ .... (2)

$\Rightarrow\ y'''\left(x^2+c\right)+5\left(xy_{11}+y_1\right)=0$ .... (3)

Now, (eq 2) $\times y_{111}-$ (eq 3)y_{11}

$\Rightarrow 2\left(2xy_1+y\right)y_{111}=6y_{11}\left(xy_{11}+y_1\right)$

$\Rightarrow\left(2xy_1+y\right)y_{111}=3\left(xy_{11}+y_1\right)y_{11}$ Hence proved.

32. i) $\lim\limits_{x\to0^+}x^m(\ln x)^n$

$\lim\limits_{x\to0^+}\dfrac{(\ln x)^n}{x^{-m}}$

Applying L-Hospital repeatedly

$\lim\limits_{x\to0^+}\dfrac{(\ln x)^n}{x^{-m}}\to 0$

ii) $\lim\limits_{x\to1}\left(1-x^2\right)^{\frac{1}{\ln(1-x)}}$

$=e^{\lim\limits_{x\to1}}\dfrac{\ln(1-x^2)}{\ln(1-x)}$

$=e^{\lim\limits_{x\to1}}1+\dfrac{\ln(1+x)}{\ln(1-x)}$

$=e$

iii) $\lim\limits_{x\to7/2}\left(2x^2-9x+8\right)^{\cot(2x-7)}$

$\lim\limits_{e^{x\to7/2}}\dfrac{\left(2x^2-9x+7\right)}{\tan(2x-7)}$

$\lim\limits_{e^{x\to7/2}}\dfrac{(2x-7)(x-1)}{\tan(2x-7)}=e^{5/2}$

iv) $\lim\limits_{x\to0}\dfrac{1}{x^3}\left(\operatorname{cosec}x-\dfrac{1}{x}-\dfrac{x}{6}\right)$

$\lim\limits_{x\to0}\left(\dfrac{6x-6\sin x-x^2\sin x}{6x^4\sin x}\right)$

$=\dfrac{1}{6}\lim\limits_{x\to0}\dfrac{6x-\sin x\left(6+x^2\right)}{x^5}$

$=\dfrac{1}{6}\lim\limits_{x\to0}\dfrac{6x-\left(6+x^2\right)\left(x-\dfrac{x^3}{\angle 3}+\dfrac{x^5}{\angle 5}\cdots\right)}{x^5}=\dfrac{1}{6}\left(\dfrac{-6}{\angle 5}+\dfrac{1}{6}\right)$

33. (i) $\lim\limits_{x\to a}\dfrac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$

$$= \lim_{x\to a}\left(\frac{a^3-2x^3}{\sqrt{2a^3x-x^4}}-\frac{a^{5/3}}{3x^{2/3}}\right)\bigg/\left(-a^{1/4}\frac{3}{4}x^{-1/4}\right)$$

$$= \quad +\frac{8\times 2a}{9}=\frac{16a}{9}$$

(ii) $$\lim_{x\to 0}\left[\frac{1}{x\sin^{-1}x}-\frac{1-x^2}{x^2}\right]$$

$$\lim_{x\to 0}\left(\frac{x-\left(1-x^2\right)\sin^{-1}x}{x^2\sin^{-1}x}\right)$$

Put $x=\sin\theta$

$$\lim_{\theta\to 0}\left[\frac{\sin\theta-\theta\cos^2\theta}{\theta\sin^2\theta}\right]$$

$$\lim_{\theta\to 0}\left[\frac{\sin\theta-\theta\cos^2\theta}{\theta^3}\right]$$

$$\lim_{\theta\to 0}\left[\frac{\cos\theta-\cos^2\theta+\theta\sin 2\theta}{3\theta^2}\right]$$

$$\lim_{\theta\to 0}\frac{1}{3}\left[(\cos\theta)\left(\frac{1-\cos\theta)}{\theta^2}+\frac{\sin 2\theta}{\theta}\right]$$

$$=\frac{5}{6}$$

(iii) $$\lim_{x\to 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x}$$

$$\lim_{x\to 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x^3}$$

$$\lim_{x\to 0}\frac{\left(\cos x+\sin x-\dfrac{1}{1-x}\right)}{3x^2}$$

$$\lim_{x\to 0}\frac{-\sin x+\cos x-\dfrac{1}{(1-x)^2}}{6x}$$

$$\lim_{x\to 0}\frac{1}{6}\left[-\cos x-\sin x-\frac{2}{(1-x)^3}\right]$$

$$=\frac{-1}{2}$$

34. $y=x^{n-1}\ln x$

$y_1=(n-1)x^{n-2}\ln x+x^{n-2}$

Multiply by x^2,

$x^2y_1=(n-1)yx+x^n$

Differentiating wrt x

$x^2y_{11}+2xy_1=(n-1)y+(n-1)xy_1+nx^{n-1}$

$\Rightarrow x^2y_{11}+3xy_1=(n-1)y+nxy_1+nx^{n-1}$

$\Rightarrow x^2y_{11}+(3-2n)xy_1=(n-1)y-nxy_1+nx^{n-1}$

$=(n-1)y-n(n-1)x^{n-1}\ln x$

$=-(n-1)^2y$

$\Rightarrow x^2y_{11}+(3-2n)xy_1+(n-1)^2y=0$

35. $$y=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

Put $x=\cos\theta$

$$y=\tan^{-1}\frac{\cos\theta}{|\sin\theta|}$$

For $0<x<1$

$$y=\frac{\pi}{2}-\theta$$

$$\frac{dy}{d\theta}=-1 \qquad \& \qquad \frac{dx}{d\theta}=-\sin\theta$$

$$\frac{dy}{dx}=\frac{1}{\sin\theta}=\frac{1}{\sqrt{1-x^2}}$$

$$t=\sec^{-1}\frac{1}{2x^2-1}$$

$t=\sec^{-1}\sec 2\theta$

$t=\pi-2\theta$

$$\frac{dt}{d\theta}=-2,\quad \frac{dt}{dx}=\frac{+2}{\sin\theta}$$

$$\frac{dy}{dt}=\frac{dy/dx}{dt/dx}=+\frac{1}{2}$$

When $-1<x<0$

$y=\theta-\pi/2$

$$\frac{dy}{d\theta}=1\quad \frac{dy}{dx}=\frac{-1}{\sin\theta}$$

$\& \, t=+2\theta$

$$\frac{dt}{d\theta}=2,\quad \frac{dt}{dx}=\frac{2}{\sin\theta}$$

$$\frac{dy}{dt}=\frac{-1}{2}$$

Target Exercises Solution

1. $\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2 x-48\cos^2 x)}\sin x\right)$

$=\cos^{-1}\left(7\cos^2 x+\sqrt{(1-7\cos x)(1+7\cos x)}\sin x\right)$

$=\cos^{-1}\left(\cos x\cdot 7\cos x+\sqrt{1-\cos^2 x}\sqrt{1-49\cos^2 x}\right)$

$=\cos^{-1}(\cos x)-\cos^{-1}(7\cos x)$

$=x-\cos^{-1}(7\cos x)$

Now differentiate

2. $\frac{x+b}{2}=a\tan^{-1}(a\ln y)$

Differentiating wrt x.

$\frac{1}{2}=\frac{a^2}{1+a^2\ln^2 y}\times\frac{1}{y}\frac{dy}{dx}$

$2a^2y'=y+a^2y\ln^2 y$ (I)

Differentiating wrt x.

$2a^2y''=y'+a^2y'\ln^2 y+2a^2y'\ln y$

$\Rightarrow a^2=\frac{y'}{2y''-y'\ln^2 y-2y'\ln y}$ (II)

From (I) & (II)

$\frac{y}{2y'-y\ln^2 y}=\frac{y'}{2y''-y'\ln^2 y-2y'\ln y}$

$\Rightarrow 2yy''-yy'\ln^2 y-2yy'\ln y=2(y')^2-yy'\ln^2 y$

$\Rightarrow yy''=(y')^2+yy'\ln y$

3. Put $y=\frac{1}{t}$

$\frac{dy}{dx}=\frac{-1}{t^2}\frac{dt}{dx}$

$\frac{d^2y}{dx^2}=\frac{2}{t^3}\left(\frac{dt}{dx}\right)^2-\frac{1}{t^2}\frac{d^2t}{dx^2}$

$\frac{d^3y}{dx^3}=-\frac{6}{t^4}\left(\frac{dt}{dx}\right)^3+\frac{4}{t^3}\frac{dt}{dx}\frac{d^2t}{dx^2}+\frac{2}{t^3}\frac{d^2t}{dx^2}\frac{dt}{dx}-\frac{1}{t^2}\frac{d^3t}{dx^3}$

Now, $S=\frac{2y'y'''-3(y'')^2}{2(y')^2}$

$Nr=2\left(-\frac{1}{t^2}\frac{dt}{dx}\right)\left(\frac{-6}{t^4}\left(\frac{dt}{dx}\right)^3+\frac{6}{t^3}\frac{dt}{dx}\frac{d^2t}{dx^2}-\frac{1}{t^2}\frac{d^3t}{dx^3}\right)$

$-3\left(\frac{2}{t^3}\left(\frac{dt}{dx}\right)^2-\frac{1}{t^2}\frac{d^2t}{dx^2}\right)^2$

$=\frac{-3}{t^4}\left(\frac{d^2t}{dx^2}\right)^2+\frac{2}{t^4}\cdot\frac{dt}{dx}\frac{d^3t}{dx^3}$

Now, $S=-\frac{3}{2}\left(\frac{d^2t/dx^2}{dt/dx}\right)^2+\frac{d^3t/dx^3}{dt/dx}$

Hence proved.

4. Similar to q.30 (Review exercise) Chapter 3.

5. $y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

Rationalize

$y=\frac{2+2\sqrt{1-x^4}}{2x^2}=\frac{1+\sqrt{1-x^4}}{x^2}$

put $x^2=\sin\theta$

$y=\cot\theta/2$

$\frac{dy}{d\theta}=\frac{-1}{2}\operatorname{cosec}^2\theta/2$

$t=\sqrt{1-x^4}=\cos\theta$

$\frac{dt}{d\theta}=-\sin\theta$

$\Rightarrow\frac{dy}{dt}=\frac{dy}{d\theta}\times\frac{d\theta}{dt}=+\frac{1/2\operatorname{cosec}^2\theta/2}{\sin\theta}$

$\frac{1+\sqrt{1-x^4}}{x^6}==\frac{1}{\left(1-\sqrt{1-x^4}\right)x^2}=\frac{1}{(1-\cos\theta)\sin\theta}$

6. $T_r=\frac{\cos\left(\frac{n+1}{2}\right)x\sin 2nx}{2\sin\frac{x}{2}\cos\frac{nx}{2}}-\cos^2 nx$

$=\frac{2\cos\left(\frac{n+1}{2}x\right)\cos nx\sin nx/2}{\sin\frac{x}{2}}-\cos^2 nx$

$=\frac{(\sin(n+1/2)x+\sin(x/2))\cos nx}{\sin x/2}-\cos^2 nx$

$$= \cos nx[2(\cos x + \cos 2x + \cdots + \cos nx) - \cos nx)$$

$$= \cos nx\left[2\frac{\sin nx/2}{\sin x/2}\frac{\cos(x+1)x}{2} - \cos nx\right]$$

7. $u = \ln\left(\frac{a + b\tan x/2}{a - b\tan x/y}\right)$

$$u = \ln(a\cos x/2 + b\sin x/2) - \ln\left(a\cos\frac{x}{2} - b\sin\frac{x}{2}\right) \quad \ldots\ldots \quad (1)$$

$$v = \ln\left(\frac{1}{a^2\cos^2\frac{x}{2} - b^2\sin^2 x/2}\right)$$

$$= -\left(\ln\left(a\cos\frac{x}{2} + b\sin\frac{x}{2}\right) - \ln\left(a\cos\frac{x}{2} - b\sin\frac{x}{2}\right)\right. \quad \ldots\ldots \quad (11)$$

Differentiating eq (1)

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}\left[\frac{-a\sin x/2 + b\cos x/2}{a\cos x/2 + b\sin x/2}\right] + \frac{1}{2}\frac{(a\sin x/2 + b\cos x/2)}{\left(a\cos\frac{x}{2} - b\sin x/2\right)}$$

at $x = \frac{\pi}{2}$

$$\frac{du}{dx} = \frac{1}{2}\left[\frac{b-a}{a+b} + \frac{a+b}{a-b}\right] = \frac{2ab}{a^2 - b^2}$$

Differentiating eq(2)

$$\frac{dv}{dx} = -\frac{1}{2}\left[\frac{-a\sin x/2 + b\cos x/2}{a\cos x/2 + b\sin x/2}\right] + \frac{1}{2}\left[\frac{a\sin x/2 + b\cos x/2}{a\cos x/2 - b\sin x/2}\right]$$

at $x = \frac{\pi}{2}$

$$\frac{dv}{dx} = \frac{1}{2}\left[\frac{a-b}{a+b} + \frac{a+b}{a-b}\right] = \frac{a^2 + b^2}{a^2 - b^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ab}{a^2 + b^2}$$

8. As $\frac{dx}{dy} = \frac{1}{dy/dx}$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{+1}{dy\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{dx\left(\frac{dy}{dx}\right)} \times \frac{dx}{dy}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \frac{dx}{dy}$$

$$\frac{d^2x}{dy^2} = \frac{-1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 = -\frac{d^2y/dx^2}{d^2x/dy^2}$$

$$\Rightarrow \frac{dy}{dx} = \left(-\frac{d^2y/dx^2}{d^2x/dy^2}\right)^{1/3}$$

Now, $R^{2/3} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^{2/3}}$

$$= \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \frac{\left(d^2y/dx^2\right)^{2/3}}{\left(\frac{d^2x}{dy^2}\right)^{2/3} \times \left(\frac{d^2y}{dx^2}\right)^{2/3}}$$

$$= \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \frac{1}{\left(\frac{d^2x}{dy^2}\right)^{2/3}}$$

Hence proved.

Now, if $x = a\sin 2\theta(1 + \cos 2\theta)$

$$x = a\sin 2\theta + \frac{a}{2}\sin 4\theta$$

$$\frac{dx}{d\theta} = 2a\cos 2\theta + 2a\cos 4\theta = 2a(\cos 2\theta + \cos 4\theta)$$

And, $y = a\ \cos 2\theta - a\ \cos^2 2\theta$

$$= a\cos 2\theta - \frac{a}{2} - \frac{a}{2}\cos 4\theta$$

$$\frac{dy}{d\theta} = -2a\sin 2\theta + 2a\sin 4\theta$$

$$= 2a(\sin 4\theta - \sin 2\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\cos 3\theta\sin\theta}{2\cos 3\theta\cos\theta} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta}{2a(\cos 2\theta + \cos 4\theta)}$$

Now, $R = \frac{\left(1 + \tan^2\theta\right)^{3/2}}{\sec^2\theta}(2a)(\cos 2\theta + \cos 4\theta)$

$\Rightarrow (\sec\theta)(2a)(2\cos 3\theta\cos\theta)$

$\Rightarrow 4a\cos 3\theta$

Hence proved.

9. $\frac{d}{dx}\left\{(x^n - a_1x^{n-1} + a_2x^{n-2} + - - - + (-1)^n a_n)e^x\right\}$

$$\Rightarrow e^x(x^n - a_1x^{n-1} + a_2x^{n-2} + - - - + (-1)^n a_n + nx^{n-1}$$
$$-(n-1)a_1x^{n-2} + - - -)$$

As it is equal to $x^n e^x$

$\Rightarrow a_1 = n$

$a_2 = n-1$

$a_3 = n-2$ & so. on.

$\Rightarrow a_r = n-(r-1) = n+1-r.$

10. $y = \dfrac{ax+b}{x^2+c}$

$\Rightarrow \left(x^2+c\right)y = ax+b$

$\Rightarrow \left(x^2+c\right)y_1 + 2xy = a \quad (1)$

$\Rightarrow \left(x^2+c\right)y_{11} + 4xy_1 + 2x = 0 \quad (11)$

$\Rightarrow \left(x^2+c\right)y_{111} + 6xy_{11} + 4y_1 + 2 = 0 \quad (111)$

Multiply eq(11) by y_{111} & eq(111) by y_{11} & subtract,

We get,

$\left(2xy'+y\right)y''' = 3\left(xy''+y'\right)y''$

Hence proved.

11. $y = \cos^{-1}\sqrt{\dfrac{\cos 3x}{\cos^3 x}}$

$\Rightarrow \cos^2 y = 4-3\sec^2 x$

$\Rightarrow 2\cos y \sin y \dfrac{dy}{dx} = +6\sec^2 x \tan x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{+\,6\sec^2 x \tan x}{\sin 2y}$

$= \dfrac{+6\sin x}{\cos^3 x[2\sqrt{4-3\sec^2 x}\sqrt{3\sec^2 x-3}]}$

$= \dfrac{+6\sin x}{2\sqrt{3}\sin x\sqrt{4\cos^2 x-3}\cos x}$

$= \dfrac{+\sqrt{3}}{\sqrt{\cos 3x \cos x}}$

$\dfrac{dy}{dx} = \dfrac{\sqrt{6}}{\sqrt{\cos 4x + \cos 2x}}$

Hence proved.

12. $f(x) = \dfrac{\sqrt{1+\sqrt{1-x^2}}\,(\sqrt{(1+x)^3}-\sqrt{(1-x)^3})}{2+\sqrt{1-x^2}}$

$= \dfrac{(\sqrt{1+\sqrt{1-x^2}})(\sqrt{1+x}-\sqrt{1-x})(2+\sqrt{1-x^2})}{(2+\sqrt{1-x^2})}$

$= \dfrac{1}{\sqrt{2}}\sqrt{(\sqrt{1+x}+\sqrt{1-x})^2}\,(\sqrt{1+x}-\sqrt{1-x})$

$f(x) \Rightarrow \sqrt{2}x$

$\Rightarrow \quad f'(x) = \sqrt{2}.$

13. $f(x) = \dfrac{1}{\sin x - \sin a} - \dfrac{1}{(x-a)\cos a}$

$f(x) = \dfrac{(x-a)\cos x - \sin x + \sin a}{(x-a)\cos a(\sin x - \sin a)}$

$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \dfrac{-(x-a)\sin x}{(x-a)\cos a(\cos x) + \cos a(\sin x - \sin a)}$

$= \lim\limits_{x\to a} \dfrac{-\sin x}{\cos a\cos x + \cos a\left(\dfrac{\sin x - \sin a}{x-a}\right)}$

$= \dfrac{-\sin a}{\cos^2 a + \cos^2 a} == -2\dfrac{1}{2}\sec a \tan a$

& $f'(x) = \dfrac{1}{(x-a)^2\cos a} - \dfrac{\cos x}{(\sin x - \sin a)^2}$

$= \dfrac{(\sin x - \sin a)^2 - \cos a\cos x(x-a)^2}{(x-a)^2\cos a(\sin x - \sin a)^2}$

$2(\sin x - \sin a)\cos x - 2(x-a)\cos a\cos x$

$\lim\limits_{x\to a} f'(x) = \dfrac{+\cos a \sin x(x-a)^2}{2(x-a)\cos a(\sin x - \sin a)^2}$

$+2\cos a(x-a)^2\dfrac{(\sin x - \sin y)}{\cos x}$

14. $x = \tan\dfrac{y}{2} - 2\ln\left(1+\tan\dfrac{y}{2}\right) + \ln\tan\dfrac{y}{2}$

$\dfrac{dx}{dy} = \dfrac{1}{2}\sec^2\dfrac{y}{2} - \dfrac{\sec^2 y/2}{1+\tan y/2} + \dfrac{1}{2}\dfrac{\sec^2 y/2}{\tan y/2}$

$= \dfrac{\sec^2 y/2}{2}\left[\dfrac{1+\tan y/2}{\tan y/2}\right] - \dfrac{\sec^2 y/2}{1+\tan y/2}$

$\Rightarrow \sec^2\dfrac{y}{2}\left[\dfrac{(1+\tan y/2)}{2\tan y/2} - \dfrac{1}{1+\tan y/2}\right]$

$\Rightarrow \quad \sec^2\dfrac{y}{2}\left[\dfrac{(1+\tan y/2)^2 - 2\tan y/2}{2\tan y/2(1+\tan y/2)}\right]$

$\dfrac{dx}{dy} \Rightarrow \dfrac{\sec^2 y/2}{2\sin y/2(\sin y/2 + \cos y/2)}$

$\dfrac{dy}{dx} = 2\sin\dfrac{y}{2}\cos^2\dfrac{y}{2}\left[\sin\dfrac{y}{2} + \cos\dfrac{y}{2}\right]$

$$\Rightarrow \frac{\sin y}{2}[\sin y+1+\cos y]$$

Hence proved.

15. $y = c_1\sin x + c_2 x\sin x + c_3\cos x + c_4 x\cos x$

$$y_1 = c_1\cos x + c_2 x\cos x + c_2\sin x - c_3\sin x + c_4\cos x$$
$$- c_4 x\sin x$$
$$y_{11} = -c_1\sin x + c_2\cos x - c_2 x\sin x + c_2\cos x - c_3\cos x$$
$$- c_4\sin x - c_4 x\cos x$$

$$\Rightarrow y_{11} = -y + 2c_2\cos x - 2c_4\sin x \qquad (1)$$

$$\Rightarrow y_{111} = -y_1 - 2c_2\sin x - 2c_4\cos x$$

$$\Rightarrow y_{1V} = -y_{11} - 2c_2\cos x + 2c_4\sin x \qquad (11)$$

Using eq (1) in (11)

$$y_{1V} = -y_{11} - y - y_{11}$$

$$y_{1V} + 2y_{11} + y = 0$$

Hence proved.

16. Let $y = \cos\theta + \cos 2\theta + \cdots + \cos n\theta$

$$\text{Now} \quad -\frac{d^2y}{d\theta^2} = \cos\theta + 2^2\cos 2\theta + \cdots + n^2\cos n\theta$$

$$\text{So, } y = \frac{\sin n\theta/2}{\sin\theta/2}\cos\left((n+1)\frac{\theta}{2}\right)$$

$$y = \frac{1}{2}\left[\frac{\sin\left(n\theta+\frac{\theta}{2}\right)+\sin\left(-\frac{\theta}{2}\right)}{\sin\theta/2}\right]$$

$$y = \frac{1}{2}\frac{\sin(n\theta+\theta/2)}{\sin\theta/2} - \frac{1}{2}$$

$$\frac{dy}{d\theta} = \frac{1}{2\sin^2\theta/2}\left[\begin{array}{r}\sin\theta/2\ \cos(n\theta+\theta/2)\left(n+\frac{1}{2}\right)\\ -\frac{1}{2}\sin(n\theta+\theta/2)\cos\theta/2\end{array}\right]$$

$$\frac{dy}{d\theta} = \left(\frac{n+1/2}{2}\right)\left[\frac{\cos(n\theta+\theta/2)}{\sin\theta/2}\right] - \frac{1}{4}\frac{\sin(n\theta+\theta/2)}{\sin\frac{\theta}{2}}\cot\theta/2$$

$$\frac{d^2y}{d\theta^2} = \frac{n^2}{2}\sin\left(n+\frac{1}{2}\right)\theta\ \text{cosec}\frac{\theta}{2} + \frac{n}{2}\cos n\theta\cdot\text{cosec}^2\frac{\theta}{2}$$
$$-\frac{1}{4}\sin n\theta\,\text{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}$$

17. As $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{dy}{d\theta}\cos^2\theta$$

$$\frac{d^2y}{dx^2} = 2\cos\theta(-\sin\theta)\frac{d\theta}{dx}\cdot\frac{dy}{d\theta} + \cos^2\theta\frac{d^2y}{d\theta^2}\cdot\cos^2\theta$$

$$\frac{d^2y}{dx^2} = -2\sin\theta\cos^3\theta\frac{dy}{d\theta} + \cos^4\theta\frac{d^2y}{d\theta^2}$$

$$\text{Now As } \frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{y}{\left(1+x^2\right)^2} = 0$$

$$\Rightarrow -2\sin\theta\cos^3\theta\frac{dy}{d\theta} + \cos^4\theta\frac{d^2y}{d\theta^2} + 2\sin\theta\cos^3\theta\frac{dy}{d\theta}$$
$$+\frac{y}{\left(1+\tan^2\theta\right)^2} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Hence proved.

18. $f(x) = n\ln\frac{\left(1+x^{1/n}\right)}{2}$

$$\Rightarrow \ln\left(\frac{1+x^{1/n}}{2}\right)^n \quad -(1)$$

$$\text{Now } \lim_{n\to\infty} f(x) = \lim_{n\to\infty}\ln\left(\frac{1+x^{1/n}}{2}\right)^n$$

$$= \ln\lim_{n\to\infty}\left(\frac{1+x^{1/n}}{2}\right)^n$$

$$= \ln e^{\lim\limits_{n\to\infty}\left(\frac{x^{1/n}-1}{2}\right)^n}$$

$$= \ln e^{\lim\limits_{n\to\infty}\frac{x^{1/n}-1}{2(1/n)}}$$

$$= -\frac{1}{2}\ln x$$

Hence proved.

19. $z = \cos^7 x,\ y = \sin x$

$$\frac{dz}{dx} = -7\cos^6 x\ \sin x, \quad \frac{dy}{dx} = +\cos x$$

$$\frac{dz}{dy} = -7y(z)^{5/7}$$

$$\frac{d^2z}{dy^2} = -7(z)^{5/7} - 5y(z)^{-2/7}\frac{dz}{dy}$$

$$\frac{d^2z}{dy^2} = -7(z)^{5/7} + 35y^2(3)^{3/7}$$

$$\frac{d^3z}{dy^3} = -5(z)^{-2/7}\frac{dz}{dy} + 70y(z)^{3/7} + 15y^2 \cdot z^{-4/7}\frac{dz}{dy}$$

$$= 35y(z)^{3/7} + 70y(z)^{3/7} - 105y^3z^{1/7}$$

$$= 35\sin x\ \cos^3 x + 70\sin x\cos^3 x - 105\sin^3 x\cos x$$

$$= 105\sin x\cos x[\cos 2x]$$

$$= \frac{105}{4}\sin 4x$$

Hence proved.

20. $y = \cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right) - 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\cdot \quad (1)$

Now as $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) = \cos^{-1}\left(\frac{1-\frac{a-b}{a+b}\tan^2 x/2}{1+\frac{a-b}{a+b}\tan^2 x/2}\right)$

$$= \cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2 x/2}{(a+b)+(a-b)\tan^2 x/2}\right)$$

$$= \cos^{-1}\left(\frac{a\left(1-\tan^2 x/2\right)+b\left(1+\tan^2 x/2\right)}{a\left(1+\tan^2 x/2\right)+b\left(1-\tan^2 x/2\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right) - (11)$$

Using eq (11) in (1)

so, $y = 0 =$ constant function.

21. $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\left(\frac{1}{2u^2-1}\right)$

put $u = \sin\theta$

$y = \theta$ & $x = \sec^{-1}(-\sec 2\theta)$

$y = \theta$ & $x = \pi - 2\theta$

$\frac{dy}{d\theta} = 1$ & $\frac{dx}{d\theta} = -2$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{2}$

$\Rightarrow 2\frac{dy}{dx} + 1 = 0$

Hence proved.

22. Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Now, $g(\alpha) = 0$ & $g'(\alpha) = 0$

$\Rightarrow g(x) = (x-\alpha)^2\,\theta(x)$

$\Rightarrow g(x)$ is $\div$ by $f(x)$

Hence proved.

23. As $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\cdots$

Now,

$$\log\frac{\sin x}{x} = \log\left(1+\left(\frac{\sin x}{x}-1\right)\right)$$

$$= \left(\frac{\sin x}{x}-1\right) - \frac{1}{2}\left(\frac{\sin x}{x}-1\right)^2 + \frac{1}{3}\left(\frac{\sin x}{x}-1\right)^3\cdots$$

$$= \left(1-\frac{x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}\cdots\ -1\right) - \frac{1}{2}\left(1-\frac{x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}\cdots\ -1\right)^2 + \cdots$$

$$= \left(\frac{-x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}+\cdots\right) - \frac{1}{2}\left[\frac{-x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}+\cdots\right)^2 + \cdots$$

$$= -\left[\frac{x^2}{6}+\frac{x^4}{180}+\cdots\right]$$

$$\lim_{x\to 0}\frac{\log\frac{\sin x}{x}+\frac{x^2}{6}}{x(2\sin x)(1-\cos x)}$$

$$\lim_{x\to 0}\frac{-\left[\frac{x^4}{180}+\frac{x^6}{2835}+\cdots\right]}{x(2\sin x)(1-\cos x]}$$

$$\Rightarrow \frac{-1}{180}\times\frac{1}{2}\times 2 = -\frac{1}{180}$$

24. Let $y = \ln\left(1-x+x^2\right)+\ln\left(1-x^2+x^2\right)+\cdots+\cdots$ n terms

Now, $\lim\limits_{n\to\infty}\frac{-dy}{dx} = \frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{+4x^3-8x^7}{1-x^4+x^8}+\cdots$

so, $y = \ln\left(1-x+x^2\right)\left(1-x^2+x^4\right)\cdots$

$$= \frac{\ln\left(\left(1+x+x^2\right)\left(1-x+x^2\right)\left(1-x^2+x^4\right)\cdots \text{n terms}\right)}{(1+x+x^2)}$$

$$= \frac{\ln\left(\left(1+x^2+x^4\right)\left(1-x^2+x^4\right)\cdots\right)\text{n terms}}{1+x+x^2}$$

$$= \ln\frac{\left(1-\left(x^2\right)^n+\left(x^4\right)^n\right)}{1+x+x^2}$$

$$y = \ln\left(1-\left(x^2\right)^n+\left(x^4\right)^n\right)-\ln\left(1+x+x^2\right)$$

As $x \in (0,1)$ & $n \to \infty$

$$y = -\ln\left(1+x+x^2\right)$$

$$-\frac{dy}{dx} = \frac{2x+1}{1+x+x^2}$$

Hence proved.

25.

i) $$\lim_{x\to\frac{\pi}{2}} \frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln\sin x}$$

Applying L-Hospital Rule,

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x-(\sin x)^{\sin x}[\cos x\ln\sin x+\cos x]}{\left(-\cos x+\frac{\cos x}{\sin x}\right)}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\left(1-(\sin x)^{\sin x}(\ln\sin x+1)\right)}{(1-\sin x)}\sin x$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\begin{array}{c}\cos x\left(1-(\sin x)^{\sin x}(\ln\sin x+1)\right)\\ -\sin x\left[(\sin x)^{\sin x}\left(\frac{\cos x}{\sin x}\right)+(\ln\sin x+1)^2\cos x(\sin x)^2\right]\end{array}}{-\cos x}$$

$\Rightarrow 2$

ii) As $\ln\left(\frac{\sin x}{x}\right)=-\left(\frac{x^2}{6}+\frac{x^4}{180}+\cdots\right)$

$$\lim_{x\to 0}\frac{6x\ln\left(\frac{\sin x}{x}\right)+x^3}{(x-\sin x)(1-\cos x)}$$

$$=-\lim_{x\to 0}\frac{x^5}{30(x-\sin x)(1-\cos x)}$$

$$=\frac{-1}{30}\lim_{x\to 0}\left(\frac{x^3}{x-\sin x}\right)\left(\frac{x^2}{(1-\cos x)}\right)=\frac{-1}{30}\times 6\times 2=\frac{-2}{5}$$

iii) $$\lim_{x\to 0}\frac{x^{6000}-(\sin x)^{6000}}{x^2(\sin x)^{6000}}$$

$$\lim_{x\to 0}\frac{1-\left(\frac{\sin x}{x}\right)^{6000}}{x^2\left(\frac{\sin x}{x}\right)^{6000}}$$

$$\lim_{x\to 0}\frac{\left(1-\left(\frac{\sin x}{x}\right)^{6000}\right)}{x^2}$$

$$\lim_{x\to 0}\frac{\left(1-\left(1-x^2/\underline{|3}+x^4/\underline{|5}\ +\cdots\right)^{6000}\right)}{x^2}$$

$$\lim_{x\to 0}\frac{1-\left(1-6000\frac{x^2}{\underline{|3}}+\cdots\right)}{x^2}$$

$\Rightarrow 1000$

26. As $a=\sqrt{b^2+c^2-2bc\cos A}$

$\Rightarrow a^2=b^2+c^2-2bc\cos A$

$\Rightarrow 2a\frac{da}{dA}=+2bc\sin A$

$\Rightarrow \frac{da}{dA}=\frac{bc\sin A}{a}=\frac{\text{Area of}\Delta}{a/2}$

$=$ height of the $\triangle$ corredponding to base 'a'.

Hence proved.

27. $f(x)=a_1\sin x+a_2\sin 2x+\cdots+a_n\sin nx$

As $f'(0)=a_1+2a_2+3a_3+\cdots+na_n$

So, $f'(x)=a_1\cos x+2a_2\cos 2x+\cdots+na_n\cos nx$

Now, As $|f(x)|\le|\sin x|$

$\Rightarrow|f(0)|\le 0$

$\Rightarrow|f(0)|=0$

Now $f'(0)=\lim\limits_{n\to 0}\frac{f(h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{f(h)}{h}$

$\Rightarrow|f'(0)|=\lim\limits_{h\to 0}\left|\frac{f(h)}{h}\right|\le\lim\limits_{h\to 0}\left|\frac{\sin h}{h}\right|\le 1$

$\Rightarrow|f'(0)|\le 1$

$\Rightarrow\left|a_1+2a_2+3a_3+\cdots+na_n\right|\le 1.$

28. a) As $(x+1)(x+2)\cdots(x+n)=A_0+A_1x+\cdots+A_nx^n$

Differentiating,

$$(x+2)(x+3)\cdots(x+n)+(x+1)(x+3)\cdots(x+n)+\cdots$$
$$=A_1+2A_2x+\cdots+nA_nx^{n\text{-}c}\qquad(1)$$

Put $x=1$

$$\underline{|n+1}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)=A_1+2A_2+\cdots+nA_n$$

Hence proved.

b) In eq 1, Put $x=-1$

$1\cdot 2\ \cdots\ (n-1)=A_1-2A_2+(-1)^n nA_n$

$\Rightarrow A_1-2A_2+\cdots+(-1)^n nA_n=\underline{|n-1}$

Hence proved.

29. $f(x)=\frac{a}{x+a}+\frac{b}{x+b}-\frac{c}{x+c}+\frac{d}{x+d}$

$$=\frac{ax+bx+2ab}{(x+a)(x+b)}+\frac{dx\text{-}cx}{(x+c)(x+d)}$$

$$=\frac{\begin{array}{c}((a+b)x+2ab)\left(x^2+(c+d)x+cd\right)\\ +(d\text{-}c)x\left(x^2+(a+b)x+ab\right)\end{array}}{(x+a)(x+b)(x+c)(x+d)}$$

if $f(x)$ is divisible by x^2, then

$(a+b)cd+(2ab)(c+d)+ab(d-c)) = 0$ &

$2abcd = 0$

$\Rightarrow acd+bcd+2abc+2abd+abd-abc = 0$

$\Rightarrow abc+acd+bcd+abd = -3abd$

$$\Rightarrow \frac{1}{d}+\frac{1}{b}-\frac{1}{a}+\frac{1}{c} = -\frac{3}{c}$$

30. As $|f(x)| \le |\tan x| \Rightarrow |f(0)| \le 0 \Rightarrow |f(0)| = 0$

$$\text{Now } \left|f'(0)\right| = \lim_{h\to 0}\left|\frac{f'(h)|}{h}\right| \le \lim_{h\to 0}\left|\frac{\tan h}{h}\right| = 1$$

$$\text{so, } f(x) = a_1\tan x + a_2\tan\frac{x}{2}+\cdots+a_n\tan\frac{x}{n}$$

$$f'(x) = a_1\sec^2 x+\frac{a_2}{2}\sec^2\frac{x}{2}+\cdots+\frac{a_n}{n}\sec^2\frac{x}{n}$$

taking Modulus and

Put $x = 0$

$$\left|a_1+\frac{a_2}{2}+\cdots+\frac{a_n}{n}\right| = \left|f'(0)\right| \le 1$$

Hence proved.

31. As $|f'(x)| \le c|f(x)|$

Put $x = 0$

$\Rightarrow |f'(a)| \le 0 \Rightarrow f'(a) = 0$

32. As $\phi(x) = d(x)f(x)$

$\phi'(x) = d(x)f'(x)+d'(x)f(x)$

$\phi''(x) = 2d'(x)f'(x)+d(x)f''(x)+d''(x)f(x)$

similarly

$\psi(x) = \mu(x)f(x)$

$\psi'(x) = \mu(x)f'(x)+\mu'(x)f(x)$

$\psi(x) = 2\mu'(x)f'(x)+\mu(x)f''(x)+\mu''(x)f(x)$

$$\text{Now } \begin{vmatrix} f(x) & \phi(x) & \psi(x) \\ f'(x) & \phi'(x) & \psi'(x) \\ f''(x) & \phi''(x) & \psi''(x) \end{vmatrix}$$

$c_2 \to c_2 - d(x)c_1$ & $c_3 \to c_3 - \mu(x)c_1$

$$\begin{vmatrix} f(x) & 0 & 0 \\ f'(x) & d'(x)f(x) & \mu'(x)f(x) \\ f''(x) & 2d'(x)f'(x)+d''(x)f(x) & 2\mu'(x)f'(x)+\mu''(x)f(x) \end{vmatrix}$$

$$= f(x)\begin{vmatrix} d'(x)f(x) & \mu'(x)f(x) \\ 2d'(x)f'(x)+d''(x)f(x) & 2\mu'(x)f'(x)+\mu''(x)f(x) \end{vmatrix}$$

$$= \frac{f^2(x)}{2f'(x)}\begin{vmatrix} 2d'(x)f'(x) & 2\mu'(x)f'(x) \\ 2d'(x)f'(x)+d''(x)f(x) & 2\mu'(x)f'(x)+\mu''(x)f(x) \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$$= \frac{f^2(x)}{2f'(x)}\begin{vmatrix} 2d'(x)f'(x) & 2\mu'(x)f'(x) \\ d''(x)f(x) & u''(x)f(x) \end{vmatrix}$$

$$= (f(x))^3\begin{vmatrix} d'(x) & \mu'(x) \\ d''(x) & \mu''(x) \end{vmatrix}$$

Hence proved.

33. $$\lim_{x\to 0}\frac{1-\cos x\cos 2x....\cos nx}{x^2}$$

Apply L-Hospital rule

$$\lim_{x\to 0}\left(\frac{\sin x\cos 2x\cdots\cos nx}{2x}+\frac{2\cos x\sin 2x\cos 3x\cdots\cos nx+\cdots}{2x}\right)$$

$$= \frac{1}{2}+\frac{(2)^2}{2}+\frac{(3)^2}{2}+\frac{(4)^2}{2}+\cdots\frac{(n)^2}{2}$$

$$\Rightarrow \frac{n(n+1)(2n+1)}{12} = 253$$

$\Rightarrow n = 11$

34. $$\text{As } \tan^{-1}\left(\frac{x}{n(n+1)+x^2}\right) = \tan^{-1}\left(\frac{1/x}{\frac{n}{x}\left(\frac{n+1}{x}\right)+1}\right)$$

$$= \tan^{-1}\left(\frac{\frac{n+1}{x}-\frac{n}{x}}{1+\frac{n}{x}\left(\frac{n+1}{x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{n+1}{x}\right)-\tan^{-1}\left(\frac{n}{x}\right)$$

$$\text{Now } f(x) = \lim_{n\to\infty}\sum_{n=1}^{n}\tan^{-1}\left(\frac{x}{n(n+1)+x^2}\right)$$

$$= \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{n+1}{x}\right)-\tan^{-1}\left(\frac{1}{x}\right)\right)$$

$$= \frac{x}{2}-\tan^{-1}\left(\frac{1}{x}\right)$$

$$f'(x) = +\frac{1}{x^2}\left(\frac{1}{1+1/x^2}\right)$$

$$R = f'(x) = \frac{1}{5} \qquad (1)$$

$$\text{As } \tan^{-1}(x+1)+\tan^{-1}(x-1) = \tan^{-1}\left(\frac{6}{17}\right)$$

Taking tan on both sides

$$\Rightarrow \frac{x+1+x-1}{1-\left(x^2-1\right)}=\frac{6}{17}$$

$\Rightarrow 2.17x=-6x^2+12$

$\Rightarrow 6x^2+2.17x-12=0$

$\Rightarrow S=-\dfrac{17.2}{6}$ (11)

As $2\sin x+\cos x+4\tan x$

$$\Rightarrow \frac{2(2\tan x/2)}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}+\frac{8\tan x/2}{1-\tan^2 x/2}$$

$$\Rightarrow \frac{4/3}{1+1/9}+\frac{8/9}{1+1/9}+\frac{8/3}{1-1/9}$$

$$=\frac{4\times 9}{30}+\frac{8}{10}+3$$

$\Rightarrow \dfrac{36+24+90}{30}=5=T$ (111)

$\rightarrow 5R+6S+7T=1+(-17)2+35=2$

Previous Year's Question Solution

1. $y=f\left(\dfrac{2x-1}{x^2+1}\right)$

$f^{-1}(y)=\dfrac{2x-1}{x^2+1}$

$$\frac{dy}{dx}=\frac{2\left(x^2+1\right)-(2x-1)(2x)}{\left(x^2+1\right)^2}\times f'(y)$$

$$=\frac{2+2x-2x^2}{\left(x^2+1\right)^2}\times f'(y)$$

$$=\frac{2+2x-2x^2}{\left(x^2+1\right)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$$

2. Differentiating $F(x)$ & put $x=a$.

$F'(x)$ at $x=a=0$

3. $f(x)=\dfrac{\ln\ln x}{\ln x}$

$$f'(x)=\frac{1/x-(\ln\ln x)/x}{(\ln x)^2}$$

$=\dfrac{1}{e}$ (at $x=e$)

4. Put $x=\cos\theta$

$y=\sec^{-1}\left(\dfrac{-1}{2x^2-1}\right)=\pi-2\theta$

$\dfrac{dy}{d\theta}=-2$

$t=\sqrt{1-x^2}=\sin\theta$

$\dfrac{dt}{d\theta}=\cos\theta$

$\dfrac{dy}{dt}=\dfrac{-2}{\cos\theta}=\dfrac{-2}{x}$

at $x=\dfrac{1}{2},\dfrac{dy}{dt}=-4$

5. $g(x)=f\circ f(x)$

$g'(x)=f'f(x)\times f'(x)$

For $x>20$

$g'(x)=1$

6. $xe^{xy}=y+\sin^2 x$

Differentiating wrt x

$e^{xy}+xe^{xy}\left(xy'+y\right)=y'+\sin 2x$

Put $x=0, y=0$

$1=y'$

7. $F(x)=f(x)g(x)h(x)$

$F'(x)=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$

$F'(x_0)=(k-3)f\left(x_0\right)g\left(x_0\right)h\left(x_0\right)$

$\Rightarrow k-3=21$

$k=24$

8. Conceptual problem

9. $y^2=P(x)$

$2yy'=p'(x)$

$$2(y')^2+2yy''=p''(x)$$

$$(P'(x))^2+4y^3y''=2y^2p''(x)$$

$$\Rightarrow 2y^3y''=p(x)p''(x)+\frac{(p'(x))^2}{2}$$

Differentiate wrt x

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)=p(x)p'''(x)$$

Hence, C is correct.

10. $f(x)=ax^2+bx+c$

$f'(x)=2ax+b$

$f''(x)=2a$

$g(x)=ax^2+(b+2a)x+2a+b+c$

$D=(b+2a)^2-4a(2a+b+c)$

$=b^2+4a^2+4ab-8a^2-4ab-4ac$

$=b^2-4ac-4a^2$

<0

Hence, B is correct.

11. $f(x)=-p^3x^3-6p^3\sin x+\left(6p^2+p\right)\cos x$

$f'(x)=-3p^3x^2-6p^3\cos x-\left(6p^2+p\right)\sin x$

$f''(x)=-6p^3x+6p^3\sin x-\left(6p^2+p\right)\cos x$

$f''(x)=-6p^3+6p^3\cos x+\left(6p^2+p\right)\sin x$

at $x=0$

$f'''(x)=0$

Hence, D is correct.

12. $y=(\sin x)^{\tan x}$

$\ln y=\tan x\ln\sin x$

$\frac{dy}{dx}=(\sin x)^{\tan x}\left(1+\sec^2x\ln\sin x\right)$

Hence, A is correct.

13. $x^2+y^2=1$

$2x+2yy'=0$

$1+(y')^2+yy''=0$

Hence, B is correct.

14. $F'(x)=f(x)$

$F\left(x^2\right)=x^2(1+x)$

$F'\left(x^2\right)\times 2x=2x(1+x)+x^2$

$F'\left(x^2\right)=1+x+\frac{x}{2}$

$f(4)=F'(4)=4$

Hence, C is correct.

15. For $x=0,\ y=1$

$\log(x+y)-2xy=0$

$\frac{1+y'}{x+y}-2y-2xy'=0$

Put $x=0, y=1\Rightarrow y'=1$

Hence, A is correct.

16. $\frac{d^2x}{dy^2}=-\left(\frac{d^2y}{dx^2}\right)\Big/\left(\frac{dy}{dx}\right)^3$

Hence, D is correct.

17. $g(x+1)=\log f(x+1)=\log x+g(x)$

$\Rightarrow g''(x+1)-g''(x)=\frac{-1}{x^2}$

$g''\left(1+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4$

$g''\left(2+\frac{1}{2}\right)-g''(1+1/2)=-4/9$

$g''(N+1/2)-g''(N-1/2)=-\frac{4}{(2N-1)^2}$

Adding all terms,

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left(1+\frac{1}{9}+\cdots+\frac{1}{(2N-1)^2}\right)$

Hence, A is correct.

18. $f(x)=\frac{b-x}{1-bx}$

$f'(x)=\frac{-(1-bx)+(b-x)b}{(1-bx)^2}=\frac{b^2-1}{(1-bx)^2}<0$

So, f(x) is one to one function

As $y=\frac{b-x}{1-bx}$

$y-bxy=b-x$

$x=\frac{b-y}{1-by}$

$$\text{So },f^{-1}(x)=\frac{x-b}{bx-1}$$

$$\Rightarrow f=f^{-1} \text{ on } (0,1)$$

$$f'(b)=\frac{1}{b^2-1}(f'(0)=b^2-1)$$

$$> f'(b)=\frac{1}{f'(0)}$$

Hence, C, D is correct.

19. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\sin((x+h)^2+1)-\sin\left(x^2+1\right)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{(x+h)^2+x^2+2}{2}\right)\sin\left(\frac{(x+h)^2-x^2}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{(x+h)^2+x^2+2}{2}\right)\sin\left(x+\frac{h}{2}\right)(h)}{(x+h/2)h}\times(x+h/2)$$

$$=2x\cos\left(x^2+1\right)$$

20. $f(x)=\begin{cases}\dfrac{1}{(2x-5)}, & x\neq 1\\ -\dfrac{1}{3}, & x=1\end{cases}$

$$\frac{dy}{dx}=\lim_{x\to 1}\frac{-2}{(2x-5)^2}=-\frac{2}{9}$$

21. $y=\dfrac{5x}{(1-x)^{2/3}}+\cos^2(2x+1)$

$$y'=5(1-x)^{-2/3}+\frac{10}{3(1-x)^{5/3}}-4\cos(2x+1)\sin(2x+1)$$

$$=\frac{5(5-3x)}{3(1-x)^{5/3}}-2\sin(4x+2)$$

22. $y=e^{x\sin x^3}+(\tan x)^x$

Let $v=e^{x\sin x^3}$, $\quad t=(\tan x)^x$

$$\frac{dy}{dx}=\frac{dv}{dx}+\frac{dt}{dx}$$

$$\frac{dy}{dx}=e^{x\sin x^3}\left(\sin x^3+3x^3\cos x^3\right)$$

$$+(\tan x)^x\left(\ln\tan x+\frac{x}{\sin x\cos x}\right)$$

23. $h(x)=f^2(x)+g^2(x)$

$h'(x)=2f(x)f'(x)+2g(x)g'(x)$

$=2\,f(x)\,g(x)+2\,g(x)\,(-f(x))=0$

So, h(x) is a constant function

Hence, h(10) = 11.

24. Let $g(x)=\begin{vmatrix}A(x) & B(x) & c(x)\\ A(\alpha) & B(\alpha) & c(\alpha)\\ A'(\alpha) & B'(\alpha) & c'(\alpha)\end{vmatrix}$

Now $,g(\alpha)=0,\quad g'(\alpha)=0$

$\Rightarrow$ x is a repeated root of g(x)

$\Rightarrow g(x)=(x-\alpha)^2Q(x)$

Hence, $g(x)$ is divisible by $f(x)$

25. Similar to Q.19 (Review exercise)

26. $x=\sec\theta-\cos\theta$

$$\frac{dx}{d\theta}=\sec\theta\tan\theta+\sin\theta$$

$$=\tan\theta(\sec\theta+\cos\theta)$$

$$y=\sec^n\theta-\cos^n\theta$$

$$\frac{dy}{d\theta}=n\sec^n\theta\tan\theta+n\cos^{n-1}\theta\sin\theta$$

$$=n\tan\theta\left[\sec^n\theta+\cos^n\theta\right]$$

$$\frac{1}{\sin y}+\frac{\cancel{\sqrt{3}}}{\cancel{2}}\times\frac{\cancel{2}\pi}{\sqrt{3}}$$

$$\sin y=-\sqrt{3}/\pi$$

$$\frac{dy}{dx}=\frac{n\left[\sec^n\theta+\cos^n\theta\right]}{\sec\theta+\cos\theta} \qquad \text{(I)}$$

$$\text{Now,}\quad y^2+4=\left(\sec^n\theta+\cos^n\theta\right)^2 \qquad \text{(II)}$$

üt^{2}üü $= \quad \theta+ \quad \theta)^2$ (III)

Squaring (I) & using (II) & (III)

$$\left(\frac{dy}{dx}\right)^2=\frac{n^2\left(y^2+4\right)}{x^2+4}$$

Hence Proved.

27. $(\sin y)^{\sin\frac{\pi x}{2}}+\frac{\sqrt{3}}{2}\sec^{-1}2x+2^{x}\tan(\ln(x+2))=0$

Differentiate wrt x

$$(\sin y)^{\sin\frac{\pi x}{2}}\left[\frac{\pi}{2}\cos\frac{\pi x}{2}\ln\sin y+\sin\frac{\pi x}{2}\cot y\frac{dy}{dx}\right]$$

$$+\sqrt{3}\times\frac{1}{2x\sqrt{4x^{2}-1}}+2^{x}\ln 2\tan\ln(x+2)$$

$$+\frac{2^{x}\sec^{2}\ln(x+2)}{x+2}=0$$

Put $x=-1,\quad \sin y=\frac{-\sqrt{3}}{\pi}$

$\Rightarrow y'=0$

28. $y=\frac{ax^{2}}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{x-c}+1$

upon solving

$$y=\frac{x}{x-a}*\frac{x}{x-b}\times\frac{x}{x-c}$$

Taking log on both sides.

$$\ln y=\ln\left(\frac{x}{x-a}\right)+\ln\left(\frac{x}{x-b}\right)+\ln\left(\frac{x}{x-c}\right)$$

Differentiate wrt x .

$$\frac{dy}{dx}=\frac{y}{x}\left[\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right]$$

Hence proved.

29. $f(x)=2+\cos x$

$f'(x)=-\sin x$

So, there exist a t in $[t,t+\pi]$ where $f'(c)=0$ as every interval of π, $f'(x)$ will be zero atleast one.

Hence, B is correct.

30. $\lim_{x\to0}g(x)\cot x-g(0)\operatorname{cosec}x$

$$\lim_{x\to0}\frac{g(x)\cos x-g(0)}{\sin x}$$

$$\lim_{x\to0}\frac{g'(x)\cos x-g(x)\sin x}{\cos x}$$

$=g'(0)=0$

Now $f(x)=g(x)\sin x$

$f'(x)=g'(x)\sin x+g(x)\cos x$

$$f''(x)=g''(x)\sin x+g'(x)\cos x+g'(x)\cos x-g(x)\sin x$$

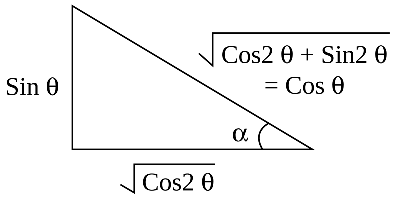
$f''(0)=0$

Hence, B is correct.

31. $f(\theta)=\sin\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)=\tan\theta$

Now

$$\Rightarrow\frac{d\,f(\theta)}{d(\tan\theta)}=1$$


$\sqrt{\text{Cos}2\ \theta+\text{Sin}2\ \theta}$
$=\text{Cos}\ \theta$
Sin θ
α
$\sqrt{\text{Cos}2\ \theta}$

Previous Year's Questions (AIEEE/JEE MAINS)

1. Area of sphere 'A' $=4\pi r^{2}$

$$\frac{dA}{dt}=8\pi r\frac{dr}{dt}$$

$$\Rightarrow r\frac{dr}{dt}=\frac{1}{\pi}$$

Now, Volume $V=\frac{4}{3}\pi r^{3}$

$$\frac{dv}{dt}=4\pi r^{2}\frac{dr}{dt}=4r$$

Hence, D is correct.

2. $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0$

$\Rightarrow f(x)$ can't have distinct real roots.

Hence, C is correct.

3. Let $h(x) = f(x) - 2g(x)$

Now $h(0) = f(0) - 2g(0) = 2$

$h(1) = f(1) - 2g(1) = 2$

Using Rolle's Theorem, there exist some $C \in (0,1)$

Where, $h'(c) = 0$

$\Rightarrow f'(c) - 2g'(c) = 0$

Hence, D is correct.

4. Volume $(v) = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

when $v = 288\pi, r = \left(\frac{288 \times 3}{4}\right)^{1/3}$

Using (1), $\frac{dr}{dt} = \frac{4\pi}{4\pi} \times \frac{4^{2/3}}{(288 \times 3)^{2/3}} = \frac{1}{6}$

Hence, B is correct.

5. $f(x) = 2x^3 + ax^2 + bx$ in$[-1,1]$

$f(-1) = a - b - 2,\ f(1) = a + b + 2$

Now, $f(1) = f(-1)$

$\Rightarrow b = -2$

$f'(x) = 6x^2 + 2ax + b$

$f'(c) = \frac{3}{2} + a + b = 0$

$\Rightarrow \quad a = \frac{1}{2}$

$\Rightarrow 2a + b = -1$

Hence, B is correct.

6. As $\sin y = x \sin\left(\frac{\pi}{3} + y\right)$

For $x = 0 \Rightarrow y = 0$

diffrentiate wrt x ,

$$\cos y \frac{dy}{dx} = \sin\left(\frac{\pi}{3} + y\right) + x\cos\left(\frac{\pi}{3} + y\right)\frac{dy}{dx}$$

Put $x = 0, y = 0$

$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{3}}{2}$

slope of normal $= -\frac{2}{\sqrt{3}}$

eqn $\Rightarrow y = -\frac{2}{\sqrt{3}}x$

Hence, A is correct.

7. At max height v = 0

Now, $v^2 = u^2 - 2gh$

$0 = (48)^2 - 2 \times 32 \times h$

$\Rightarrow h = 36$

So Max height = 36 + 64 = 100 m.

Hence, A is correct.

8. As $x = 2\cos t + 2t\sin t$

$\frac{dx}{dt} = -2\sin t + 2\sin t + 2t\cos t = 2t\cos t$

As $y = 2\sin t - 2t\cos t$

$\frac{dy}{dt} = 2\cos t - 2\cos t + 2t\sin t = 2t\sin t$

$\Rightarrow \frac{dy}{dx} = \tan t$

eqn of normal $\Rightarrow y - 2\sin t + 2t\cos t =$
$-\cot t(x - 2\cos t - 2t\sin t)$

$y + (\cot t)x = 2\sin t + 2\frac{\cos^2 t}{\sin t} = 2\operatorname{cosec} t$

Distance from $(0,0) = 2$

Hence, C is correct.

9. $y = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}} = \tan^{-1}\left(\frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{1+\tan x/2}{1-\tan x/2}\right) = \frac{\pi}{4} + \frac{x}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$

eqn of normal $\Rightarrow y - \left(\frac{\pi}{4} + \frac{\pi}{12}\right) = -2\left(x - \frac{\pi}{6}\right)$

$\Rightarrow y + 2x = \frac{2\pi}{6} + \frac{\pi}{4} + \frac{\pi}{12} = \frac{2\pi}{3}$

Hence, B is correct.

10. As $x = 4t^2 + 3$

$\frac{dx}{dt} = 8t$

$\&\ y = 8t^3 - 1$

$\frac{dy}{dt} = 24t^2$

$\Rightarrow \frac{dy}{dx} = 3t$

eqn of tangent $\Rightarrow y - 8t^3 + 1 = 3t\left(x - 4t^2 - 3\right)$

$\Rightarrow y = 3tx - 4t^3 - 9t - 1$

As it again meets curve at $Q(t_1)$

$\Rightarrow 8t_1^3 - 1 = 3t\left(4t_1^2 + 3\right) - 4t^3 - 9t - 1$

$\Rightarrow 8t_1^3 - (12t)t_1^2 + 4t^3 = 0$

$\Rightarrow t_1 = t$ or $\frac{t}{2}$

$\Rightarrow Q = \left(t_1^2 + 3, -t_1^3 - 1\right)$

Hence, D is correct.

11. $y = 1 + \sqrt{4x - 3}$

$\frac{dy}{dx} = \frac{2}{\sqrt{4x-3}} = \frac{2}{3}$

$\Rightarrow 9 = 4x - 3$

$\Rightarrow x = 3$

$\Rightarrow y = 4$

eqn of normal $\Rightarrow y - 4 = \frac{-3}{2}(x - 3)$

$\Rightarrow 2y - 8 = -3x + 9$

$\Rightarrow 2y + 3x = 17$

Hence, A is correct.

12. As $x^2y^2 - 2x = 4 - 4y$

$\Rightarrow 2xy^2 + 2x^2y\frac{dy}{dx} - 2 = -4\frac{dy}{dx}$

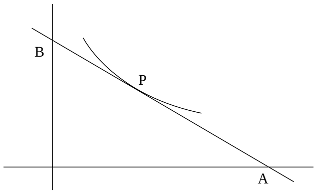
For pt$(-2, 2)$

$\Rightarrow \frac{dy}{dx} = \frac{18}{20} = \frac{9}{10}$

eqn of tangent $\Rightarrow 10(y - 2) = 9(x + 2)$

$10y = 9x + 38$

13.



Eqn of tangent

$y - y_1 = m\left(x - x_1\right)$

Pt A $\equiv \left(x_1 - \frac{y_1}{m}, 0\right)$

Pt B $\equiv \left(0, y_1 - mx_1\right)$

As AP:PB = 1:3

$\Rightarrow y_1 - mx_1 = 3y_1$

$\Rightarrow \quad -\frac{dy}{dx}x = 3y$

$\Rightarrow -\frac{1}{3}\ln y = \ln cx$

$\Rightarrow \quad c^3x^3y = 1$

$y = \frac{1}{x^3}$ as $f(1) = 1$

Hence,C is correct.

14. $y(x-2)(x-3) = x + 6$

Put $x = 0 \Rightarrow y = 1$

$\Rightarrow y\left(x^2 - 5x + 6\right) = x + 6$

$\Rightarrow y(2x - 5) + \frac{dy}{dx}\left(x^2 - 5x + 6\right) = 1$

$\frac{dy}{dx}$ at $x = 0 = \frac{6}{6} = 1$

eqn of normal $\Rightarrow y - 1 = -x$

$\Rightarrow x + y = 1$

Hence, C is correct.

15. $y^2 = 6x$ & $9x^2 + by^2 = 16$

$2y\frac{dy}{dx} = 6$ & $18x + 2by\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{6}{2y}$ & $\frac{dy}{dx} = -\frac{9x}{by}$

as $m_1m_2 = -1$

$\Rightarrow \frac{6}{2y} \times \left(\frac{-9x}{by}\right) = -1$

$\Rightarrow -27x = -by^2$

$\Rightarrow -27x = -b(6x)$

$\Rightarrow b = \frac{9}{2}$

Hence, D is correct.

16. $y = 10 - x^2$

$\frac{dy}{dx} = -2x$ .... (1)

$\& y = 2 + x^2$

$\frac{dy}{dx} = 2x$ .... (11)

Point of intersection is $10 - x^2 = 2 + x^2$

$x = \pm 2$

$y = 6$

Angle between curves $= \left|\frac{4 - (-4)}{1 - 16}\right| = \left|\frac{8}{15}\right| = \frac{8}{15}$

Hence, D is correct.

17. $y = xe^{x^2}$

$\frac{dy}{dx} = e^{x^2}\left(1 + 2x^2\right)$

$$\frac{dy}{dx} \text{ at } x = 1 = 3e$$

eqn of tangent $\Rightarrow y - e = 3e(x - 1)$

$\Rightarrow \quad y = 3ex - 2e$

Hence, A is correct.

18. $y = x^2 - 5x + 5$

$$\frac{dy}{dx} = 2x - 5 = 2$$

$$\Rightarrow x = \frac{7}{2}$$

$$y = \frac{49}{4} - \frac{35}{2} + 5 = \frac{49 - 70 + 20}{4} = \frac{-1}{4}$$

eqn of tangent

$$\Rightarrow y + \frac{1}{4} = 2\left(x - \frac{7}{2}\right)$$

$\Rightarrow 4y + 1 = 8x - 28$

$\Rightarrow 4y = 8x - 29$

Hence, D is correct.

19. $x^2 = 8y$

$$\Rightarrow 2x = 8\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x_1}{4} = \tan\theta$$

$\Rightarrow y - y_1 = \tan\theta(x - x_1)$

$$y - \frac{1}{8}\left(16\tan^2\theta\right) = x\tan\theta - 4\tan^2\theta$$

$y = x\tan\theta - 2\tan^2\theta$

$\cot\theta y = x - 2\tan\theta$

Hence, A is correct.

5

TANGENT AND NORMAL

5.1 Introduction

In the previous chapter, we have seen how to find derivatives of various functions. We now attempt to apply the concept of derivative to problems involving rate of change and slope of tangent.

Instantaneous Rate of Change

If $y = f(x)$, then the average rate of change of y (per unit change in x) on the interval $[x, x + \Delta x]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of y with respect to x is the limit of the average rate of change, as $\Delta x \to 0$. Thus the instantaneous rate of change of y with respect to x is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x).$$

Suppose f(x) is differentiable at $x = x_0$. Then the instantaneous rate of change of $y = f(x)$ with respect to x at x_0 is the value of the derivative of f at x_0.

That is,

$$\text{Instantaneous rate of change} = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}.$$

For example, if $f(x) = x^2 + 2$, then $f'(2) = 4$. We interpret the number 4 as the rate of change of f(x) with respect to x there. It is apparent that at (2, 6) a small change in x produces a corresponding change four times as great in f(x).

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to time at $t = a$. In other words, $f'(a)$ is the velocity of the particle at time $t = a$. The speed of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

5.2 Rate Measurement

If the quantity y varies with respect to another quantity x satisfying some relation $y = f(x)$, then $f'(x)$ or $\frac{dy}{dx}$ represents rate of change of y with respect to x.

For example, $\frac{dV}{dt}$ and $\frac{dS}{dt}$ represent the rate of change of volume V and surface area S w.r.t. time.

Example 1: The displacement 's' of a particle at time 't' is expressed as $s = \frac{1}{2}t^3 - 6t$. Find the acceleration at the time when the velocity vanishes.

Solution: $s = \frac{1}{2}t^3 - 6t$

Thus velocity, $v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$

and acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t$.

The velocity vanishes when $\frac{3t^2}{2} - 6 = 0$

$\Rightarrow \quad t^2 = 4 \Rightarrow \quad t = 2.$

Thus, the acceleration when velocity vanishes is

$a = 3t = 6.$

Example 2: On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Solution: Given $x^3 = 12y$, differentiating with respect to y, we get

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dx}{dy} = \frac{12}{3x^2}$$

To find the interval in which the abscissa changes at a faster rate than the ordinate, we must have

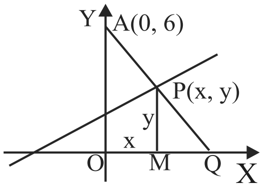
$$\left|\frac{dx}{dy}\right| > 1 \Rightarrow \left|\frac{12}{3x^2}\right| > 1$$

$$\Rightarrow \frac{4}{x^2} > 1 \Rightarrow \frac{4-x^2}{x^2} > 0$$

$$\Rightarrow \quad x \in (-2, 2) - \{0\}.$$

Thus $x \in (-2, 2) - \{0\}$ is the required interval in which abscissa changes at a faster rate than the ordinate.

Example 3: A point P(x, y) moves along the line whose equation is $x - 2y + 4 = 0$ in such a way that y increases at the rate of 3 units/sec. The point A(0, 6) is joined to P and the segment AP is prolonged to meet the x-axis in a point Q. Find how fast the distance from the origin to Q is changing when P reaches the point (4, 4).



Solution: The rate of change of y is given and it is desired to find the rate of change of OQ, which we denote by z. If MP is perpendicular to the x-axis, MP = y and OM = x. The triangles OAQ and MPQ are similar, hence

$$\frac{z}{6} = \frac{z-x}{y} \Rightarrow yz = 6z - 6x \Rightarrow z = \frac{6x}{6-y}$$

Substituting the value of x from the equation of the given line, we have

$$z = \frac{12(y-2)}{6-y}$$

$$\frac{dz}{dt} = \frac{48}{(6-y)^2}\frac{dy}{dt}$$

Setting y = 4 and $\frac{dy}{dt} = 3$, we obtain $\frac{dz}{dt} = 36$

that is, z is increasing at the rate of 36 units/sec.

Example 4: Find out at which points x, the rate of change of the function $f(x) = 3x^5 - 15x^3 + 5x - 7$ is minimum.

Solution: The rate of change of a function at a certain point is equal to the derivative of the function at this point

$$f'(x) = 15x^4 - 45x^2 + 6 = 15[(x^2 - 1/2)^2 + 1/12].$$

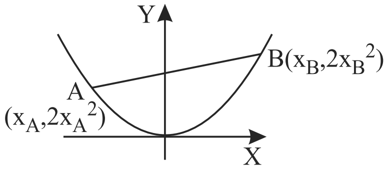
The minimum value of $f'(x)$ is attained at $x = 1/\sqrt{2}$. Hence the minimum rate of change of the function f(x) is at the point $x = 1/\sqrt{2}$ and equals 5/4.

Example 5: The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y = 2x^2$. Let x_A and x_B be the x-coordinate of the ends. At the moment when A is at (0, 0) and B is at (1, 2), find the value of the derivative $\frac{dx_B}{dx_A}$.

Solution: We have $y = 2x^2$

$$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$$

or $\quad (x_B - x_A)^2 + 4(x_B^2 - x_A^2)^2 = 5$



Differentiating w.r.t. x_A and denoting $\frac{dx_B}{dx_A} = D$

$$2(x_B - x_A)(D - 1) + 8(x_B^2 - x_A^2)(2x_B D - 2x_A) = 0$$

Put $x_A = 0,\ x_B = 1$

$$2(1 - 0)(D - 1) + 8(1 - 0)(2D - 0) = 0$$

$$2D - 2 + 16D = 0 \Rightarrow D = 1/9.$$

Example 6: x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

Solution: Given x and y are sides of two squares thus the area of two squares are x^2 and y^2

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}\cdot\frac{dy}{dx}$ ...(1)

where the given curves is, $y = x - x^2$

$\Rightarrow \frac{dy}{dx} = 1 - 2x$ ...(2)

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1 - 2x)$ [from (1) and (2)]

or $\quad \frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x}$

$$\Rightarrow \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

Example 7: A spherical balloon is being inflated. The radius r of the balloon is increasing at the rate of 0.2 cm/s when r = 5cm. At what rate is the volume V of the balloon increasing at that instant ?

Solution: Given dr/dt = 0.2 cm/s when r = 5 cm, we want to find dV/dt at that instant. Because the volume of the balloon is $V = \frac{4}{3}\pi r^3$, we see that $dV/dr = 4\pi r^2$.

So, the chain rule gives

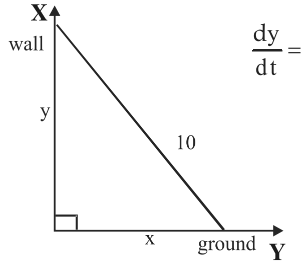
$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi (5)^2(0.2)^2 \approx 62.83 \text{ (cm}^3\text{/s)}$$

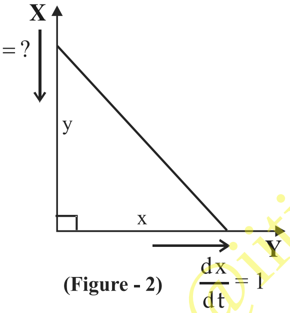
at the instant when r is 5cm.

Example 8: A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

Solution: We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground.



(Figure - 1)



(Figure - 2)

Note that x and y are both functions of t (time). We are given that dx/dt = 1 ft/s and we are asked to find dy/dt when x = 6 ft (see Figure 2).

In this problem, the relationship between x and y is given by the Pythagoras Theorem :

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When x = 6, the Pythagoras theorem gives y = 8 and so, substituting these values and dx/dt = 1, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.

Note: In solving this problem, we obtained the equation $x^2 + y^2 = 100$ and differentiated implicitly with respect to t. We then inserted values for x and y into the resulting equation to solve for dy/dt. This is correct.

It is a mistake to insert the value for x or y before differentiation. For example, if we first insert the value for x , we obtain $6^2 + y^2 = 100^2$.

When we differentiate now we get $0 + 2y\frac{dy}{dt} = 0$ because the derivative of a constant is zero. This yields dy/dt = 0 which is, of course, wrong.

Another common error is to assign a letter to the length of the ladder. If we assign the letter ℓ, say, we obtain $x^2 + y^2 = \ell^2$. Then, differentiating with respect to t yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt}$$

This is technically correct as long as you recognize that $d\ell/dt = 0$ because $\ell = 100$ which is a constant. However, if you fail to set $d\ell/dt = 0$, it is not possible to solve for dy/dt.

In short, we should

(i) remember to differentiate before inserting specific values into the equation, and

(ii) and not turn constants into variables.

Example 9: A man walks along a straight path at the speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight ?

Solution: We draw the figure and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the distance from the man to the point on the path closest to the searchlight and the perpendicular to the path.

We are given that dx/dt = 4 ft/s and are asked to find dθ/dt when x = 15. The equation that relates x and θ can be written from the figure.

$$\frac{x}{20} = \tan\theta \Rightarrow x = 20\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\frac{d\theta}{dt}$$

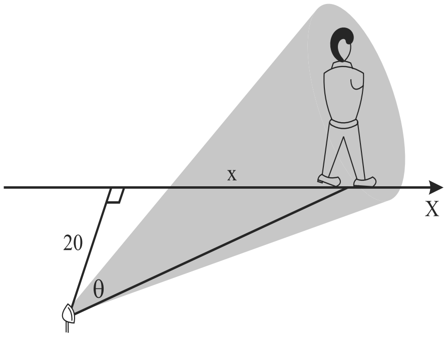
so $$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\frac{dx}{dt} = \frac{1}{20}\cos^2\theta(4) = \frac{1}{5}\cos^2\theta$$

when x = 15, the length of the beam is 25,

so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

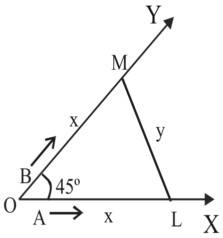
The searchlight is rotating at a rate of 0.128 rad/s.



Example 10: Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.

Solution: Let L and M be the positions of men A and B at any time t.

Since velocities are same OL = OM = x(say)
Let LM = y.



Then OM = x. It is given that $\frac{dx}{dt}$ = v and we need to find $\frac{dy}{dt}$ from Δ LOM.

$$\cos 45^\circ = \frac{OL^2 + OM^2 - LM^2}{2 \, . \, OL \, . \, OM}$$

or, $$\frac{1}{\sqrt{2}} = \frac{x^2 + x^2 - y^2}{2 \, . \, x \, . \, x} = \frac{2x^2 - y^2}{2x^2}$$

or, $\sqrt{2x^2} = 2x^2 - y^2$ or, $(2 - \sqrt{2})\, x^2 = y^2$

∴ $y = \frac{dy}{dt} = \sqrt{2 - \sqrt{2}}\; x$

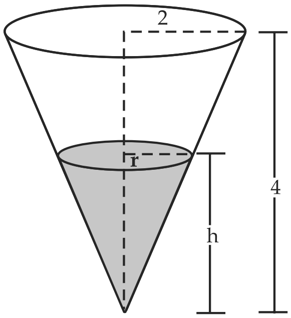
Differentiating w. r. t. we get

$$\frac{dy}{dr} = \sqrt{2\sqrt{2}}\,\frac{dx}{dt} = \sqrt{2 - \sqrt{2}}\; v \quad \left[\because \frac{dx}{dt} = v\right]$$

∴ The two men are being separated from each other at the rate of $\sqrt{2-\sqrt{2}}$ v .

Example 11: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep.

Solution: We first sketch the cone and label it as in the figure. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.



We are given that dV/dt = 2m³/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

But it is the very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in the figure to write

$$\frac{r}{h} = \frac{2}{4} \qquad \Rightarrow r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we differentiate each side with respect to t :

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so $$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

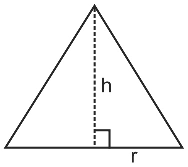
Substituting h = 3 m and dV/dt = 2m³/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}$$

The water level is rising at a rate of 8/(9π) ≈ 0.28 m/min.

Example 12: Sand is pouring from pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution: $v = \frac{1}{3}\pi r^2 h$



but $h = \frac{r}{6}$

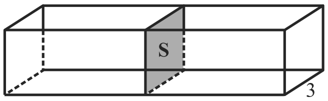
$$\Rightarrow \quad v = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$$

$$\frac{dv}{dt} = 36\pi h^2 . \frac{dh}{dt}$$

when, $\frac{dv}{dt} = 12$ cm³/s and $h = 4$ cm

$$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi} \text{ cm/sec.}$$

Example 13: A rectangular tank has a sliding panel S that divides it into two adjustable tanks of width 3 ft. See figure. Water is poured into the left compartment at the rate of 5 ft³/min. At the same time S is moved to the right at the rate of 3 ft/min. When the left compartment is 10 ft long it contains 70 ft³ of water. Is the water level rising or falling and how fast?



Solution: Let x be the length of the left compartment; let y and V be the depth and the volume of the water in the left compartment. Then x, y and V are all functions of time.

Given : $\frac{dx}{dt} = 3$ ft/min, $\frac{dV}{dt} = 5$ ft³/min.

We need to compute dy/dt when $x = 10$ and $V = 70$.

At any instant $V = 3xy$.

$$\frac{dV}{dt} = 3x\frac{dy}{dt} + 3\frac{dx}{dt}y.$$

Substitute the data at the instant in question :

$$5 = 3 \cdot 10 \cdot \frac{dy}{dt} + 3 \cdot 3 \cdot y,$$

$$\frac{dy}{dt} = \frac{5-9y}{30}$$

At the given instant, $V = 70$ and $x = 10$, hence $y = 7/3$. Therefore,

$$\frac{dy}{dt} = \frac{5-9(7/3)}{30} = -\frac{16}{30}.$$

The water level is falling at the rate of $\frac{8}{15}$ ft./min.

Alternative: Instead of differentiating the equation $V = 3xy$, we solve for y first, and then differentiate :

$$y = \frac{1}{3}\frac{V}{x},$$

$$\frac{dy}{dt} = \frac{1}{3}\frac{x\frac{dV}{dt} - V\frac{dx}{dt}}{x^2}$$

At the given instant, $\frac{dy}{dt} = \frac{1}{3}\frac{10 \cdot 5 - 70 \cdot 3}{10^2} = -\frac{16}{30}.$

Example 14: The radius of the base of a right circular cone is decreasing at the rate of 4 inch/min. and the height is increasing at the rate of 6 inch/min. At what rate is the volume changing when the height is 12 inch and the radius is 6 inch ?

Solution: Let V, r and h denote the volume, radius and height, respectively, of the cone.

We are given $dr/dt = -4$ and $dh/dt = 6$; we have to find dV/dt when $h = 12$ and $r = 6$.

The formula for the volume of a cone is $V = \frac{\pi r^2 h}{3}$.

Differentiating this with respect to time t, we have

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi rh\frac{dr}{dt}$$

Substituting the given values of the variables, we obtain

$$\frac{dV}{dt} = \frac{\pi}{3}(36)(6) + \frac{2}{3}\pi(6)(12)(-4)$$

$$= 72\pi - 192\pi = -120\pi = -377$$

indicating that the volume is decreasing at the rate of about 377 cu. inch/min.

Concept Problems A

1. If the side of an equilateral triangle increases uniformly at the rate of 3ft/s, at what rate is the area increasing, when the side is 10ft ?
2. Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm.
3. y is related with x as follows : $y^2 = 12x$. The argument x increases uniformly at a rate of two units per second. What is the rate of increase in y for $x = 3$?
4. At what point of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease at the same rate at which the abscissa increases?
5. A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall.
6. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.
7. The position function of a particle is given by $s = t^3 - 4.5t^2 - 7t$, $t \geq 0$. When does the particle reach a velocity of 5 m/s?
8. Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm³/s. Find the rate at which slant

height of water is decreasing when the height of water is $\sqrt{2}$ cm.

9. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment.

10. A point moves in a straight line, $s = \sqrt{t}$. Prove that the motion is decelerated and that the acceleration a is proportional to the cube of the velocity v.

11. If the velocity v of a point moving on the x-axis is given by $v(x) = F(x)$, show that the acceleration a is given by $a(x) = F(x)F'(x)$.

12. Prove that if a body moves according to the law $s(t) = ae^{t} + be^{-t}$, then its acceleration is equal to the distance covered.

13. A particle is moving along the parabola $y^2 = 12x$ at the uniform rate of 10 cm/s. Find the components of velocity parallel to each of the axes when the particle is at the point (3, 6).

14. A triangle has two of its vertices at (a, 0), (0, b) and the third (x, y) is movable along the line $y - x = 0$. If A be the area of the triangle, prove that

$$\frac{dA}{dx} = \frac{a+b}{2}.$$

15. A boat sails parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse?

16. The quantity of charge q in coulombs that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s.

Practice Problems A

17. A circle is drawn with its centre on a given parabola and touching its axis. Prove that if the point of contact recedes with a constant velocity from the vertex, the rate of increase of the area of the circle is also constant.

18. An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R km, how fast is the area of the earth, visible from the plane increasing at 3 min. after it started ascending. Take visible area $A = \dfrac{2\pi R^2 h}{R+h}$ where h is the height of the plane in kms. above the earth.

19. Water is dripping out from a conical funnel of semi-vertical angle $\pi/4$, at the uniform rate of 2cm³/s. through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

20. The thin lens equation in physics is $\dfrac{1}{s} + \dfrac{1}{S} = \dfrac{1}{f}$ where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens ? Is the image moving away from the lens or toward the lens ?

21. A car is traveling on a straight road that is 120 km long. For the first 100 km the car travels at an average velocity of 50 km/h. Show that no matter how fast the car travels for the final 20 km it cannot bring the average velocity up to 60 km/h for the entire trip.

22. At time $t = 0$, a car moves into the passing lane to pass a slow-moving truck. The average velocity of the car from $t = 1$ to $t = 1 + h$ is

$$v_{avg} = \frac{3(h+1)^{2.5} + 580h - 3}{10h}.$$

Estimate the instantaneous velocity of the car at t=1, where time is in seconds and distance is in feet.

23. It is projected that x months from now, the population of a certain town will be

$P(x) = 2x + 4x^{3/2} + 5{,}000$

(a) At what rate will the population be changing with respect to time, 9 months from now ?

(b) At what percentage rate will the population be changing with respect to time, 9 months from now.

24. Suppose a person carrying a 20-candlepower light walks towards a wall in such a way that at time t(seconds) the distance to the wall is $s(t) = 28 - t^2$ meters.

(a) How fast is the illuminance on the wall increasing when the person is 19 m from the wall?

(b) How far is the person from the wall when the illuminance is changing at the rate of 1lux/s ?

25. The diameter of a hemispherical bowl is 18 inch. If the depth of the water in it is increasing at the rate of $\dfrac{1}{8}$ inch/sec. When it is 8 inch deep, how fast is the water flowing in ?

26. Water flows into a hemispherical tank of radius 10 feet (flat side up). At any instant, let h denote the depth of the water, measured from the bottom, r the radius of the surface of the water, and V the volume of the water is the tank. Compute dV/dh at the instant when h = 5 feet. If the water flows in at a constant rate of $5\sqrt{3}$ cubic feet per second, compute dr/dt, the rate at which r is changing, at the instant t when h = 5 feet.

27. The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as the radius. When the radius is 1 of the altitude is 6 feet. When the radius is 6 feet, the volume is increasing at a rate of 1 cubic feet per second. When the radius is 36 feet, the volume is increasing at a rate of n cubic feet per second, where n is an integer. Compute n.

28. A particle moves according to a law of motion $s = f(t) = t^3 - 12t^2 + 36t$, $t \ge 0$, where t is measured in seconds and s in meters.

(a) Find the acceleration at time t and after 3s.

(b) Graph the position, velocity and acceleration functions for $0 \le t \le 8$

(c) When is the particle speeding up ? When is it slowing down ?

29. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm² ?

30. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m³/min, how fast is the water level rising when the water is 30 cm deep?

31. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

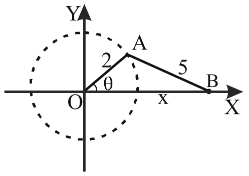
32. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later ?

33. A particle moves along the elliptical path given by $4x^2 + y^2 = 4$ in such a way that when it is at the point $(\sqrt{3}/2, 1)$, its x-coordinate is increasing at the rate of 5 units per second. How fast is the y-coordinate changing at that instant ?

34. A pebble dropped into a pond causes a circular ripple. Find the rate at which the radius of the ripple is changing at a time when the radius is one foot and the area enclosed by the ripple is increasing at the rate of 4 ft²/s.

35. It is estimated that the annual advertising revenue received by a certain news paper will be $R(x) = 0.4x^2 + 3x + 160$ thousand dollars when its circulation is x thousand. The circulation of the paper is currently 10,000 and the increasing at a rate of 2,000 per year. At what rate will the annual advertising revenue be increasing with respect to time, 2 years from now ?

36. A rod OA 2m long is rotating counter-clockwise in a plane about O at the rate of 3 rev/s, as shown in the figure.



The rod AB is attached to OA at A and the end B slides along the x-axis. Suppose AB is 5 meters long. What are the velocity and the acceleration of the motion of the point B along the x-axis ?

5.3 Approximation

Differential of a Function

If the value of the function y(x) changes to $y(x + \Delta x)$ as x changes to $x + \Delta x$, the increment of the function is defined as

$$\Delta y = y(x + \Delta x) - y(x)$$

and its differential is defined as

$$dy = \lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} y(x+\Delta x) - y(x)$$

$$= \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}.\Delta x = y'(x)dx$$

For example if $y = \frac{4}{3}\pi x^3$ then

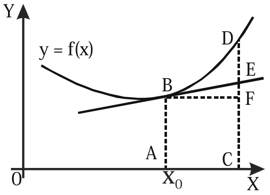
$$dy = \left(\frac{dy}{dx}\right)dx = \frac{4}{3}\pi(3x^2)dx = 4\pi x^2 dx.$$

This gives meaning to dx and dy as separate quantities. Let x be fixed, we define dx to be an independent variable equal to Δx, the change in x. Thus, dx, called the differential of x, is an independent variable equal to the change in x. That is, we define dx to be Δx. Then, if f is differentiable at x, we define dy, called the differential of y, by the formula

$$dy = f'(x)dx \text{ or, equivalently, } df = f'(x)dx$$

Geometrical Meaning of Differential

The figure shows the graph of a differentiable function f(x) in the neighbourhood of a point x_0.



Geometrically, the expressions Δx, $f(x_0)$, $f(x_0 + \Delta x)$, and $\Delta f = f(x_0 + \Delta x) - f(x_0)$ are the lengths of the line segments AC, AB, DC and DF respectively.

The triangle BEF is bounded by the horizontal line BF, the vertical line EF, and the tangent to the curve BE.

From the geometrical meaning of the derivative, $f'(x_0)$ is the tangent of the angle EBF, while $f'(x_0)\,\Delta x = df$ is the length of the line segment EF.

Thus geometrically, the differential is equal to the increment in the ordinate of the tangent from the point x_0 to the point $x_0 + \Delta x$.

We see that $dx = \Delta x$, whereas dy is the rise of a tangent relative to the same change in x (Δy and dy are not the same thing).

Note that the separation of the increment of the function Δf into two parts $\Delta f = f'(x_0)\,\Delta x + o(\Delta x)$ corresponds to the partition of the line segment DF, DF = FE + ED. The length of the line segment EF is equal to the differential, and the length of the line segment ED is an infinitesimal of higher order than Δx. Indeed, as can be seen from the figure, the segment ED in DE tends to zero as $\Delta x \to 0$.

Example 1: Compare the increment and the differential of the function $y = 2x^3 + 5x^2$.

Solution: We find

$$\Delta y = 2(x + \Delta x)^3 + 5(x + \Delta x)^2 - 2x^3 - 5x^2$$
$$= (6x^2 + 10x)\,\Delta x + (6x + 5)\Delta x^2 + 2\Delta x^3,$$
$$dy = (6x^2 + 10x)dx.$$

The difference between the increment Δy and the differential dy is an infinitesimal of the higher order of smallness relative to Δx, and it is equal to $(6x + 5)\Delta x^2 + 2\Delta x^3$.

Linearizing a Function

In a small neighbourhood of the point x_0, the graph of the function $y = f(x)$ and its tangent at x_0 are close together. The tangent is the simpler of the two curves under consideration.

We can often replace the function f(x) by the function whose graph is the tangent to the given function at the point x_0. The function f(x) is then said to be linearized in the neighbourhood of x_0. The term "linearization" arises because the function whose graph is a tangent line is linear in x, that is, has the form $y = ax + b$.

The equation of the tangent line is readily found as that of the straight line passing through the point $(x_0, f(x_0))$ in a given direction. The slope of the tangent is $f'(x_0)$ and hence $y - f(x_0) = f'(x_0)(x - x_0)$ is the equation of the tangent, and

$y = f'(x_0)(x - x_0) + f(x_0)$ is the function whose graph is the tangent line. Clearly,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

(here the sign ≈ means an approximate equality in the small neighbourhood of the point x_0).

Thus, the approximate value of a function f at x in the small neighbourhood of the point x_0 is given by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Example 2: Find the linearization (i.e. tangent line approximation) of the function $f(x) = \sqrt{x+3}$ at a = 1 and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates ?

Solution: The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \dfrac{1}{4}$.

We see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{(when x is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

and $$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

We see that our approximations are overestimates because the tangent line lies above the curve.

Example 3: Compute the approximate value of $\sqrt[3]{1.06}$.

Solution: Consider the function $y = f(x) = \sqrt[3]{x}$.

We have $x_0 = 1$, $\Delta x = 0.06$, $x = x_0 + \Delta x = 1.06$,

$$\sqrt[3]{1.06} = f(x_0 + \Delta x) = f(x).$$

We have $f(x) \approx f(x_0) + f'(x_0)\,\Delta x$,

$$\sqrt[3]{1.06} \approx \sqrt[3]{1} + f'(x_0)\,\Delta x.$$

We find that $f'(x_0) = (1/3)\ x_0^{-2/3}$,

that is, $f'(x_0) = (1/3) \cdot 1^{-2/3} = 1/3$.

Thus, $\sqrt[3]{1.06} \approx 1 + (1/3) \cdot 0.06 = 1.02$.

Example 4: Find the approximate value of $(1.999)^6$.

Solution: Let $f(x) = x^6$

Now, $f(x + \Delta x) - f(x) = f'(x) \,.\, \Delta x = 6x^5\,\Delta x$

We may write, $1.999 = 2 - 0.001$

Taking $x = 2$ and $\Delta x = -0.001$, we have

$f(1.999) - f(2) = 6(2)^5 \times -0.001$

$\Rightarrow$ $f(1.999) = f(2) - 6 \times 32 \times 0.001$

$= 64 - 64 \times 0.003 = 64 \times 0.997$

$= 63.808$ (approx).

Example 5: Calculate the approximate value of the area of a circle whose radius is equal to 3.02 m.

Solution: We make use of the formula $S = \pi R^2$.

Putting $R = 3$, $\Delta R = 0.02$, we obtain

$$\Delta S = dS = 2\pi R \cdot \Delta R = 2\pi \cdot 3 \cdot 0.02 = 1.12\,\pi.$$

Consequently, the approximate value of the area of the circle is equal to $9\pi + 0.12\pi = 9.12\pi \approx 28.66$ (sq.m).

In a different form, the approximate value of a function f at x + h is given by

$$f(x + h) - f(x) \approx f'(x) \,.\, h$$

i.e. $f(x + h) \approx f(x) + f'(x) \,.\, h$

Example 6: Find the approximate value of $(0.007)^{1/3}$.

Solution: Let $f(x) = (x)^{1/3}$

Now, $f(x + h) - f(x) \approx f'(x) \,.\, h = \dfrac{h}{3x^{2/3}}$

we may write, $0.007 = 0.008 - 0.001$

Taking $x = 0.008$ and $h = -0.001$, we have

$$f(0.007) - (0.008)^{1/3} \approx -\frac{0.001}{3(0.008)^{2/3}}$$

or $$f(0.007) \approx (0.008)^{1/3} - \frac{0.001}{3(0.2)^2}$$

or $$f(0.007) \approx 0.2 - \frac{0.001}{3(0.04)} = 0.2 - \frac{1}{120} = \frac{23}{120}$$

Hence, $(0.007)^{1/3} \approx \dfrac{23}{120}$.

5.4 Error

The derivative is also used to estimate the maximum error in a calculation that is based on figures obtained by imperfect measurement. Here, the approximation formula discussed earlier, is used to study propagation of error, which is the term used to describe an error that accumulates from other errors in an approximation. Let us first consider propagation of error in a volume measurement.

Suppose we measure the side of a cube and find it to be 10 cm long. From this we conclude that the volume of the cube is 10^3 = 1,000 cm^3. If the original measurement of the side is accurate to within 2%, approximately how accurate is the calculation of volume ?

The volume of the cube is $V(x) = x^3$, where x is the length of a side. If we take the length of a side to be 10 when it is really $10 + \Delta x$, the error is Δx; and the corresponding error when computing the volume will be ΔV, given by

$$\Delta V = V(10 + \Delta x) - V(10) \approx V'(10)\Delta x$$

Now, $V'(x) = 3x^2$, so $V'(10) = 300$. Also, the measurement of the side can be off by as much as 2% – that is, by as much as $0.02(10) = 0.2$ cm is either direction. Substituting $\Delta x = \pm 0.2$ in the incremental approximation formula for ΔV, we get

$$\Delta V = 3(10)^2(\pm 0.2) \approx \pm 60$$

Thus, the propagated error in computing the volume is approximately ±60 cm^3. Hence the maximum error in the measurement of the side is $|\Delta x| = 0.2$ and the corresponding maximum error in your calculation for the volume is

$$|\Delta V| \approx V'(10)|\Delta x| = 300(0.2) = 60$$

This says that, at worst, the calculation of the volume as 1,000 cm^3 is off by 60 cm^3, or 6% of the calculated volume.

Absolute, Relative and Percentage Errors

The absolute (propagated) error in a measured or approximated value is defined to be the remainder when the approximate value is subtracted from the true value. Hence

actual value = approximate value + error.

If x_0 represents the measured value of a variable and $x_0 + \Delta x$ represents the exact value, then Δx is the error in measurement.

The difference between $f(x + \Delta x)$ and $f(x)$ is called the absolute error and is defined by

$$\Delta f = f(x + \Delta x) - f(x)$$

The relative error is the ratio of the (absolute) error to the true value.

$$\text{relative error} = \frac{\text{error}}{\text{value}}$$

and may be given as either a numerical fraction or as a percentage of the value.

The relative error is $\dfrac{\Delta f}{f} \approx \dfrac{df}{f}$

The percentage error is $100\left(\dfrac{\Delta f}{f}\right)\%$

In the above example, the approximate propagated error in measuring volume is ±60, and the approximate relative error is $\Delta V/V = \pm 60/10^3 = \pm 0.06$.

Example 1: The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of atmost 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere ?

Solution: If the radius of the sphere is r, then its volume is $V = \dfrac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2\, 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3.

Note: Although the possible error in the above example may appear to be rather large, a better picture of the error is given by the relative error, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V}=\frac{dV}{V}=\frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3}=3\frac{dr}{r}$$

Thus, the relative error in the volume is about three times the relative error in the radius. In the example the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as percentage error of 0.24% in the radius and 0.7% in the volume.

Example 2: A hemispherical bowl of radius 10 in. is filled with water to a depth of x inches. The volume V of water in the bowl is given by the formula

$$V=\frac{\pi}{3}(30x^2-x^3).$$

Suppose that you measure the depth of water in the bowl to be 5 in. with a maximum possible measured error of $\frac{1}{16}$ in. Estimate the maximum error in the calculated volume of water in the bowl.

Solution: The error in the calculated volume V(5) is the difference $\Delta V = V(x) - V(5)$ between the actual volume V(x) and the calculated volume.

We do not know the depth x of water in the bowl. We are given only that the difference $\Delta x = x - 5$

between the actual and the measured depth is numerically at most $\frac{1}{16}$ in. : $|\Delta x| \le \frac{1}{16}$.

Now, $V'(x)=\frac{\pi}{3}(60x-3x^2)=\pi(20x-x^2)$,

The linear approximation

$$\Delta V \approx dv = V'(5)\,\Delta x \text{ at } x = 5 \text{ gives}$$

$$\Delta V \approx \pi(20\,.\,5-5^2)\,\Delta x = 75\pi\,\Delta x.$$

With the common practice in science of writing $\Delta x = \pm\frac{1}{16}$ to signify that $-\frac{1}{16} \le \Delta x \le \frac{1}{16}$, this gives

$$\Delta V \approx (75\pi)\left(\pm\frac{1}{16}\right) \approx \pm 14.73 \text{ (in.}^3\text{)}.$$

The formula gives the calculated volume $V(5) \approx 654.50$ in.^{3}, but we now see that this may be in error by almost 15 in.^{3} in either direction.

In the above example a relative error in the measured depth x of

$$\frac{\Delta x}{x}=\frac{\frac{1}{16}}{5}=0.0125=1.25\%$$

leads to a relative error in the estimated volume of

$$\frac{dV}{V}\approx\frac{14.73}{654.50}\approx 0.0225 = 2.25\%.$$

The relationship between these two relative errors is of some interest. The formulas for dV and V in the above example give

$$\frac{dV}{V}=\frac{\pi(20x-x^2)\Delta x}{\frac{1}{3}\pi(30x^2-x^3)}=\frac{3(20-x)}{30-x}\cdot\frac{\Delta x}{x}.$$

When x = 5, this gives

$$\frac{dV}{V}=(1.80)\frac{\Delta x}{x}.$$

Hence, to approximate the volume of water in the bowl with a relative error of at most 0.5%, for instance, we would need to measure the depth with a relative error of at most (0.5%)/1.8, thus with a relative error of less than 0.3%.

Example 3: Two sides a, b of a triangle are given and the included angle C is measured. Find the error in the computed length of the third side c due to a small error in the angle C.

Solution: We have

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

and therefore, supposing C and c alone to vary,

$$c\,dc = ab\sin C\,dC,$$

whence $dc=\frac{ab}{c}\sin C.dC = a\sin B\,dC.$

Concept Problems B

1. Compute the approximate value of $(1.06)^{3.02}$.
2. Find the approximate value of $\sqrt[3]{127}$.
3. A person measures the radius of a sphere to be 6 in and use the formula $V=\frac{4}{3}\pi r^3$ to calculate the volume. If his measurement of the radius is accurate to within 1%, approximately how accurate (to the nearest percent) is his calculation of the volume?
4. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

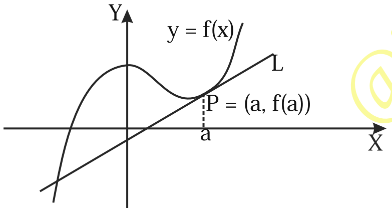
5. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.
   (a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error ?
   (b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error ?

5.5 Tangent and Normal

Tangent

Earlier we have understood that the derivative of f at the point (x, y) on the curve $y = f(x)$ i.e. $f'(x)$ represents the slope (gradient) of the tangent to that curve. We defined the slope of the tangent to a curve at a point as the limit of secant slopes passing through that point. This limit is called the derivative of the curve. The slope of the tangent to a curve at a point is called the slope of the curve at that point. The basic property of a tangent line is that it indicates the direction of a curve at a point.

If $y = f(x)$ is differentiable for $x = a$ and $P(a, f(a))$ lies on the graph of $y = f(x)$, the line through P which has its slope equal to $f'(a)$ is tangent to the graph at P. If the function has an infinite derivative at the point $x = a$ i.e. $\lim\limits_{x\to a} |f'(x)| = \infty$ then, the line tangent to the graph of the function $y = f(x)$ at the point $P(a, f(a))$ is perpendicular to the x-axis and we call such a line as a vertical tangent.



Definition The tangent line to the graph of f at the point $P(a, f(a))$ is
(i) the line on P with slope $f'(a)$ if $f'(a)$ exists ,
(ii) the line $x = a$, if $\lim\limits_{x\to a}\left|\dfrac{f(x)-f(a)}{x-a}\right| = \infty$.

If neither (i) nor (ii) holds, then the graph of f does not have a tangent line at the point $P(a, f(a))$.

In case $f'(a)$ exists, then

$$y - f(a) = f'(a)(x - a)$$

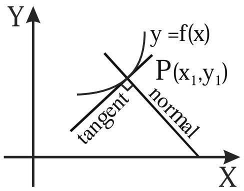
is an equation of the tangent line to the graph of f at the point $P(a, f(a))$.

Note:

1. The above definition based on single-valued functions causes no trouble in practice. If we desire to study the tangents to the graph of $y^2 = x + 2$, we merely consider the curve to be made up of two branches, one given by $y = \sqrt{x+2}$ and the other by $y = -\sqrt{x+2}$. To each of these branches the above definition is applicable.
2. The tangent line to the curve $y = x^3$ at the origin is the x- axis, which crosses the curve itself.
   It may seem strange that a tangent line can cross the curve, as this tangent line does. In high school geometry, where only tangent lines to circles were considered, the tangent line never crossed the curve.
3. A tangent line to a graph may intersect the graph at points other than the point of tangency. For example, the line $y = 1$ is a tangent to the graph of $y = \sin x$, even through it intersects this graph infinitely often.

Normal

The normal to the graph of f at the point P is the line that is perpendicular to the tangent to the graph at P.



The normal line N to the graph of a function f at the point $P(a, f(a))$ is defined to be the line through P perpendicular to the tangent line.

It follows that if $f'(a) \neq 0$ the slope of N is $-1/f'(a)$ and

$y - f(a) = -\dfrac{1}{f'(a)}(x - a)$ is an equation of N.

If $f'(a) = 0$, then N is the vertical line $x = a$; and if the tangent line is vertical, then N is the horizontal line $y = f(a)$.

Note:

1. The point $P(x_1,y_1)$ satisfies the equation of the curve and the equations of tangent and normal. For the tangent, P is called the point of contact and for the normal it is called foot of normal.
2. Tangent to a curve at the point $P(x_1, y_1)$ can be drawn, even though the derivative at P does not exist. e.g., $x = 0$ is a vertical tangent to $y = x^{2/3}$ at (0, 0), where $|dy/dx| = \infty$.

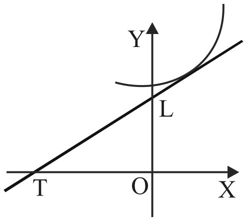
Intercepts Made by the Tangent on the Coordinate Axes

The equation of tangent at P(x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \text{...(1)}$$

Putting Y = 0 in (1), we get

$$X = x - y\frac{dx}{dy}$$



Hence the intercept OT that the tangent cuts off from the x-axis is $x - y\dfrac{dx}{dy}$.

By putting X = 0 in (1), we get

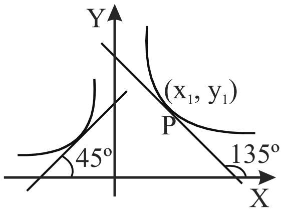
$$Y = y - x\frac{dy}{dx}$$

Hence the intercept OL that the tangent cuts off from the y–axis is $y - x\dfrac{dy}{dx}$.

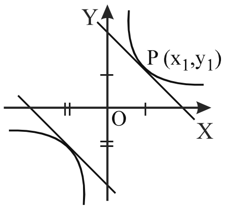
Note that OL is called **initial ordinate** of the tangent to the curve.

Note:

1. If the tangent at any point on the curve is equally inclined to both the axes, then dy/dx = ±1.



2. If the tangent at any point on the curve cuts both the axes at equal distances from origin, then dy/dx = ±1 or it passes through the origin.
3. If the tangent at any point makes equal intercepts on the coordinate axes, then dy/dx = –1 or it passes through the origin.



Example 1: Find the equation of tangent to $y = e^x$ at x = 0.

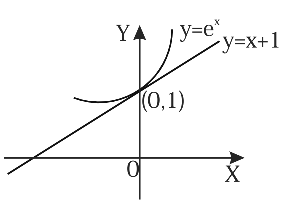
Solution: At x = 0 , $y = e^0 = 1$

Hence, the point of tangent is (0, 1)

$$\frac{dy}{dx} = e^x \quad\Rightarrow\quad \left.\frac{dy}{dx}\right|_{x=0} = 1$$

Hence equation of tangency is (y – 1) = 1 (x – 0)

$\Rightarrow$ y = x + 1



Example 2: Find the equation of all straight lines which are tangent to curve $y = \dfrac{1}{x-1}$ and which are parallel to the line x + y = 0.

Solution: Suppose the tangent is at (x_1, y_1) and it has slope – 1.

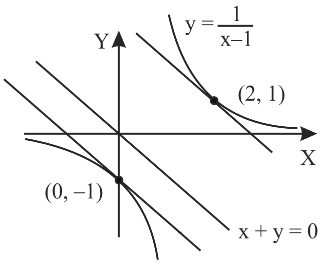
$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,\,y_1)} = -1 \quad\Rightarrow\quad -\frac{1}{(x_1-1)^2} = -1.$$

$\Rightarrow$ $x_1 = 0$ or 2 $\quad\Rightarrow\quad$ $y_1 = -1$ or 1.

Hence tangent at (0, – 1) and (2, 1) are the required lines with equations

(y + 1) = – 1(x – 0) and (y – 1) = – 1 (x – 2)

$\Rightarrow$ x + y + 1 = 0 and y + x = 3.



Example 3: Find the equation of the normal to $y = x^3 - 3x$, which is parallel to 2x + 18y = 9.

Solution: The curve is $y = x^3 - 3x$. ...(1)

$\Rightarrow$ $dy/dx = 3x^2 - 3$

The normal is parallel to the line 2x + 18y = 9, then the slope of the normal = slope of the given line

$$= -\frac{1}{9} = -\frac{1}{(dy/dx)}.$$

$\Rightarrow$ $dy/dx = 9 \Rightarrow 3x^2 - 3 = 9 \Rightarrow x = \pm 2$.

From (1), when x = 2, y = 2 and when x = –2, y = –2.

Hence, the required normals are

y–2 = –(1/9)(x – 2) and y + 2 = –(1/9)(x+2)

$\Rightarrow$ x + 9y = 20 and x + 9y + 20 = 0.

Example 4: Find equation of normal to the curve $y = |x^2 - |x||$ at x = – 2.

Solution: In the neighbourhood of x = – 2, $y = x^2 + x$.

Hence the point of contact is (– 2, 2).

$$\frac{dy}{dx} = 2x+1 \Rightarrow \left.\frac{dy}{dx}\right|_{x=-2} = -3.$$

So the slope of normal at (– 2, 2) is $\frac{1}{3}$.

Hence the equation of normal is

$$y-2 = \frac{1}{3}(x+2) \qquad \Rightarrow 3y = x+8.$$

Example 5: Find the equation of normal to the curve $x + y = x^y$, where it cuts the x–axis.

Solution: The given curve is $x + y = x^y$.

At the x–axis, $y = 0$, $\Rightarrow x + 0 = x^0 \quad \Rightarrow x = 1.$

∴ The point is A(1, 0)

Now we differentiate $x + y = x^y$ taking log on both sides

$\Rightarrow \ln(x+y) = y \ln x$

$$\therefore \quad \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y\,.\,\frac{1}{x} + (\ln x)\,\frac{dy}{dx}$$

Putting x = 1, y = 0 $\qquad \left\{1+\frac{dy}{dx}\right\} = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1$$

∴ Slope of normal = 1

Equation of normal is, $\quad y - 0 = 1(x - 1)$

$\Rightarrow y = x - 1.$

Example 6: Find the initial ordinate of the tangent to the curve $y = \sqrt{x+3}$ at the point (1, 2).

Solution: $\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{1}{2\sqrt{x+3}} = \frac{1}{4}.$

The equation of the tangent at (1, 2) is

$$y-2 = \frac{1}{4}(x-1)$$

For y intercept, put $x = 0$ in this equation to get

$$y = 2 - \frac{1}{4} = \frac{7}{4}.$$

∴ The initial ordinate of the tangent $= \frac{7}{4}$.

Example 7: For the curve $xy = c^2$, prove that the portion of the tangent intercepted between the coordinate axes is bisected at the point of contact.

Solution: Let the point at which tangent is drawn be (α, β) on the curve $xy = c^2$.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = -\frac{\beta}{\alpha}$$

Thus, the equation of tangent is,

$$y-\beta = -\frac{\beta}{\alpha}(x-\alpha)$$

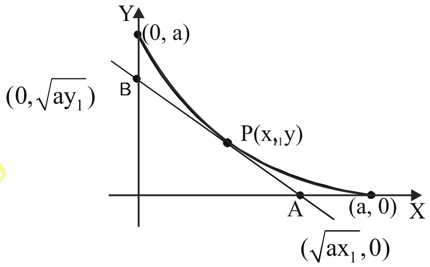
$$\Rightarrow y\alpha - \alpha\beta = -x\beta + \alpha\beta \Rightarrow \quad x\beta + y\alpha = 2\alpha\beta$$

$$\Rightarrow \frac{x}{2\alpha} + \frac{y}{2\beta} = 1.$$

It is clear that the tangent line cuts x and y-axis at A(2α, 0) and B(0, 2β) and the point (α, β) bisects AB.

Example 8: Prove that sum of intercepts of the tangent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate axes is constant.

Solution: Let $P(x_1, y_1)$ be a variable point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$



The equation of tangent at point P is

$$(y-y_1) = -\frac{\sqrt{y_1}}{\sqrt{x_1}}\,(x-x_1)$$

$$\Rightarrow \frac{y}{\sqrt{y_1}} - \sqrt{y_1} = -\frac{x}{\sqrt{x_1}} + \sqrt{x_1}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a} \qquad (\text{since } \sqrt{x_1} + \sqrt{y_1} = \sqrt{a}\,)$$

Hence point A is $(\sqrt{ax_1}, 0)$ and coordinates of point B is $(0, \sqrt{ay_1})$. Thus, the sum of intercepts

$$= \sqrt{a}\,(\sqrt{x_1} + \sqrt{y_1}\,) = \sqrt{a}\,.\sqrt{a} = a, \text{ is a constant.}$$

Example 9: At what points on the curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, the tangents make equal angles with the coordinate axes ?

Solution: The given curve is $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$ ...(1)

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = 2x^2 + x$

∵ The tangent makes equal angles with coordinate axes we have

$$\frac{dy}{dx} = \pm 1 \qquad \text{or} \qquad 2x^2 + x = \pm 1$$

or $2x^2+x+1 \neq 0$ and $2x^2+x-1=0$

or $2x^2+2x-x-1=0$

(If $2x^2+x+1=0$ then x is imaginary)

or $(2x-1)(x+1)$

∴ $x=\frac{1}{2},-1.$

From (1), for $x=\frac{1}{2}$, $y=\frac{2}{3}.\frac{1}{8}+\frac{1}{2}.\frac{1}{4}=\frac{5}{24}$

and for $x=-1$, $y=-\frac{2}{3}+\frac{1}{2}=-\frac{1}{6}$

Hence, the points are $\left(\frac{1}{2},\frac{5}{24}\right)$ and $\left(-1,\frac{1}{6}\right)$.

Example 10: Find the point on the curve $ay^2=x^3$ in the fourth quadrant where normal to the curve makes equal non-zero intercepts with the axes.

Solution: Let the point at which normal is drawn be (x_1, y_1). Then it must satisfy $ay^2=x^3$,

i.e., $ay_1^2=x_1^3$ or $y_1=\pm\sqrt{\frac{x_1^3}{a}}$ ...(1)

Now, differentiating both sides of the given curve with respect to x we get,

$$2ay\frac{dy}{dx}=3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{3x_1^2}{2ay_1}=\frac{3x_1^2}{\pm 2a\sqrt{\frac{x_1^3}{a}}}=\pm\frac{3}{2}\sqrt{\frac{x_1}{a}}$$

Thus, the slope of normal $=\mp\frac{2}{3}\sqrt{\frac{a}{x_1}}$.

We know that the slope of the line making equal intercepts with the axes = – 1.

$\Rightarrow \mp\frac{2}{3}\sqrt{\frac{a}{x_1}}=1$ [using + sign which means that y_1 is negative from (1)]

$\Rightarrow x_1=\frac{4a}{9}$ and $y_1=\frac{-8a}{27}$

Hence, the required point is $\left(\frac{4a}{9},\frac{-8a}{27}\right)$.

Example 11: The tangent to $y=ax^2+bx+\frac{7}{2}$ at (1, 2) is parallel to the normal at the point (–2, 2) on the curve $y=x^2+6x+10$. Find the value of a and b.

Solution: $\left.\frac{dy}{dx}\right|_{(1,2)}=2ax+b=2a+b$

$\frac{dy}{dx}=2x+6$ for $y=x^2+6x+10$

Slope of normal at $(-2,-2)=-\left.\frac{1}{2x+6}\right|_{(-2,2)}=-\frac{1}{2}$.

$\Rightarrow 2a+b=-1/2$ ...(1)

Point (1 , 2) satisfies $y=ax^2+bx+7/2$

$\Rightarrow 2=a+b+7/2$

$\Rightarrow a+b=-3/2$ ...(2)

On solving (1) and (2), we get a = 1 and b = –5/2.

Tangents to Second Degree Curves

There is a direct method for finding equation of tangent to the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ at the point (x_1, y_1).

We need to replace the following in the above equation:

x^2 by xx_1.
y^2 by yy_1.
$2x$ by $x+x_1$.
$2y$ by $y+y_1$.
$2xy$ by x_1y+xy_1.

The equation of tangent is,

$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0.$

Also, the equation of normal at (x_1, y_1) is

$$\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}$$

Example 12: Find the equation of tangent at point $P(x_1, y_1)$ to the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Solution: Here the equation of curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a second degree equation in x and y.

The equation of tangent is obtained by replacing x^2 to xx_1 and y^2 to yy_1

i.e., $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$

Example 13: Find the equation of tangents to the ellipse $3x^2+y^2+x+2y=0$ which are perpendicular to the line $4x-2y=1$.

Solution: Since, the tangent is the perpendicular to the line $4x-2y=1$,

∴ (slope of tangent) × (slope of given line) = –1

∴ $\frac{dy}{dx}.2=-1$ ∴ $\frac{dy}{dx}=-\frac{1}{2}$ ...(1)

The given equation $3x^2+y^2+x+2y=0$ ...(2)

$\Rightarrow 6x+2y\frac{dy}{dx}+1+2\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(6x+1)}{2(y+1)} \qquad ...(3)$$

Let (x_1, y_1) be the point of contact of the tangent and the curve.

Note that if we use the above method, the equation of tangent can be easily obtained :

$$3xx_1 + yy_1 + (x + x_1)/2 + (y + y_1) = 0.$$

Its slope is $-\frac{(6x_1+1)}{2(y_1+1)}$, same as found above.

From (1) and (3), we get

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{(6x_1+1)}{2(y_1+1)} = -\frac{1}{2} \text{ i.e., } y_1 = 6x_1 \qquad ...(4)$$

Substituting this in (2) [since the points lie on the curve] we get,

$$3x_1^2 + 36x_1^2 + x_1 + 12x_1 = 0$$

i.e., $13x_1(3x_1 + 1) = 0 \quad \Rightarrow \quad x_1 = 0, -1/3$

Using (4), $x_1 = 0$ and $y_1 = 0$

and $x_1 = -1/3$ and $y_1 = -2$.

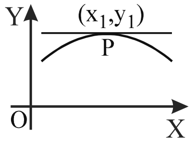
Hence, the points where the tangents have slope $-1/2$ are $P(0, 0)$ and $Q(-1/3, -2)$.

Equation of tangents at P, Q are $y = -\frac{1}{2}x$ i.e., $x + 2y = 0$ and

$y + 2 = -\frac{1}{2}\left(x + \frac{1}{3}\right)$ i.e., $3x + 6y + 13 = 0$ respectively.

Horizontal Tangent

If the tangent at any point P on the curve is parallel to the axis of x then $dy/dx = 0$ at the point P. Such a tangent is called horizontal tangent.



Example 14: Let $y = (x - 2)(x^2 + 4x - 7)$. Find all points on this curve where the tangent is horizontal.

Solution: The tangent will be horizontal when $dy/dx = 0$, because a horizontal line has slope 0.

$$\frac{dy}{dx} = (x - 2)(2x + 4) + (1)(x^2 + 4x - 7)$$

$$= 2x^2 - 8 + x^2 + 4x - 7$$

$$= 3x^2 + 4x - 15 = (3x - 5)(x + 3)$$

Thus, $\frac{dy}{dx} = 0$ when $x = \frac{5}{3}$ or $x = -3$.

The points on the curve at which the tangent is horizontal are $\left(\frac{5}{3}, \frac{-22}{27}\right)$ and $(-3, 50)$.

Example 15:

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what points on the curve is the tangent line horizontal?

Solution:

(a) Differentiating both sides $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, we get

$$3x^2 + 3y^2y' = 6y + 6xy'$$

or $\quad x^2 + y^2y' = 2y + 2xy'$

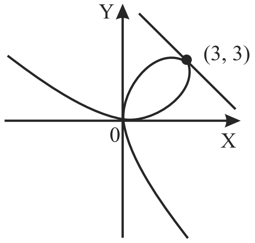
We now solve for y' :

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2,$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$, $y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$.



So the equation of tangent to the folium at (3, 3) is

$$y - 3 = -1(x - 3) \quad \text{or} \quad x + y = 6.$$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$. Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. so either $x = 0$ or $x^3 = 16$.

If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$.

Thus, the tangent is horizontal at (0, 0) and at $(2^{4/3}, 2^{5/3})$.

Example 16: Find the points where the tangents to the curve whose equation is $y^2 = x^2(2 - x^2)$ are parallel to the x-axis, and sketch the curve.

Solution: Solving for y, we obtain $y = \pm x\sqrt{2 - x^2}$. Thus y is a two-valued function of x which is defined for $|x| \le \sqrt{2}$.

Let us take first the case when the sign is positive. Writing the equation in the form $y = x(2 - x^2)^{1/2}$ and differentiating, we have

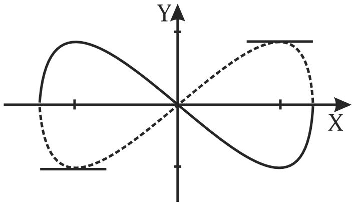
$$\frac{dy}{dx} = \frac{x}{2}(2 - x^2)^{-\frac{1}{2}}(-2x) + (2 - x^2)^{\frac{1}{2}}$$

$$= \frac{2(1 - x^2)}{(2 - x^2)^{1/2}}$$

The tangents are parallel to the x-axis if the derivative is equal to 0. We obtain $x = -1$ and 1. The corresponding values of y are -1 and 1, and so the required points are $(-1, -1)$ and $(1, 1)$.

It is unnecessary to repeat the work for $y = -x\sqrt{2-x^2}$, since only the sign of dy/dx is changed and so we have the same values of x as before. But now the corresponding values of y are 1 and -1, and we have two more points, $(-1,1)$ and $(1, -1)$, where the tangents are parallel to the x-axis.

From the given equation, we find that the x-intercepts are 0 and $\pm\sqrt{2}$. For $x = \pm\sqrt{2}$, dy/dx becomes infinite. Hence at the corresponding points the slope becomes infinite and the tangents are perpendicular to the x-axis.



In the figure, the heavy line is the graph of $y = x\sqrt{2-x^2}$ and the dotted line is the graph of $y = -x\sqrt{2-x^2}$.

Vertical Tangent

If the tangent at any point P (x_1, y_1) on the curve $y = f(x)$ is perpendicular to the x-axis (parallel to y-axis), then slope of the tangent is infinite

i.e. $\left|\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}\right| = \infty$ or, $\left(\dfrac{dx}{dy}\right)_{(x_1,y_1)} = 0$.

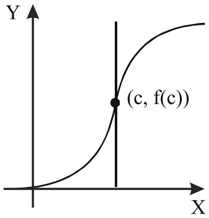
A vertical tangent line occurs at a place on a continuous curve where the slopes of secant line approach ∞ or approach – ∞. Since an infinite limit is a special way of saying that a limit does not exist, a function f is not differentiable at a point of vertical tangency.

Definition If f is continuous at $x = c$ and

$$\lim_{\Delta x\to 0}\left|\frac{f(c+\Delta x)-f(c)}{\Delta x}\right| = \infty$$

then the vertical line, $x = c$, passing through $(c, f(c))$ is a vertical tangent line to the graph of f.

For example, the function shown in the figure below has a vertical tangent line at $(c, f(c))$.

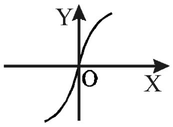


The graph of f has a vertical tangent line at (c, f(c)).

For example, the y-axis is the vertical tangent to the curve $f(x) = \sqrt[3]{x}$ at the point (0, 0) because the function has an infinite derivative at $x = 0$.

$$f'(0^+) = \lim_{h\to 0}\frac{h^{1/3}}{h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = \infty$$

$$f'(0^-) = \lim_{h\to 0}\frac{(-h)^{1/3}}{-h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = \infty$$



In other words, a curve has a vertical tangent at $x = c$, when $\lim_{x\to c}|f'(x)| = \infty$.

If the domain of f is a closed interval [a, b], then we can extend the definition of vertical tangent to include the endpoints, by considering continuity and one-sided derivatives from the right (for $x = a$) and from the left (for $x = b$).

$$\lim_{\Delta x\to 0^+}\left|\frac{f(a+\Delta x)-f(a)}{\Delta x}\right| = \infty$$

$$\lim_{\Delta x\to 0^-}\left|\frac{f(b+\Delta x)-f(b)}{\Delta x}\right| = \infty$$

Graphs with Vertical Tangents

Note the following four important cases :

(i) $f'(x) = \lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \infty$ (Fig. 1)

(ii) $f'(x) = \lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = -\infty$ (Fig. 2)

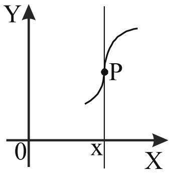


Fig. 1

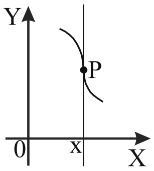


Fig. 2

(iii) $f'(x^-) = \lim_{\substack{\Delta x\to 0\\ \Delta x<0}}\dfrac{\Delta y}{\Delta x} = -\infty$,

$f'(x^+) = \lim_{\substack{\Delta x\to 0\\ \Delta x>0}}\dfrac{\Delta y}{\Delta x} = \infty$

The left tangent is perpendicular to the x-axis and is directed downward, while the right tangent is perpendicular to the x–axis and is directed upward (Fig.3)

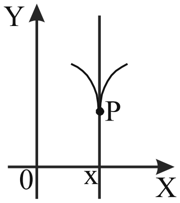


Fig. 3

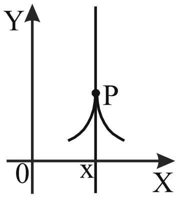


Fig. 4

(iv) $f'(x^-) = \lim\limits_{\substack{\Delta x\to 0\\ \Delta x<0}} \dfrac{\Delta y}{\Delta x} = \infty$,

$f'(x^+) = \lim\limits_{\substack{\Delta x\to 0\\ \Delta x>0}} \dfrac{\Delta y}{\Delta x} = -\infty$

Here, the left tangent is perpendicular to the x-axis and is directed upward, while the right tangent is perpendicular to the x–axis and is directed downward (Fig. 4).

Consider the following functions :

(i) $f(x) = \text{sgn}\, x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{1-0}{h} = \infty$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{-1}{-h} = \infty$

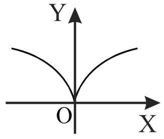
The function f (x) = sgn x has no vertical tangent at x = 0, even if it has infinite derivatives, because of discontinuity at x = 0,

(ii) $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$ has no vertical tangent at x= 0.

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{1-1}{h} = 0$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{0-1}{h} = -\infty$.

(iii) $f(x) = x^{2/3}$ has a vertical tangent at x = 0.
$f(x) = x^{2/3}$



$f'(0^+) = \lim\limits_{h\to 0} \dfrac{h^{2/3}}{h} = \infty$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{(-h)^{2/3}}{-h} = -\lim\limits_{h\to 0} \dfrac{1}{h^{2/3}} = -\infty$

Example 17: Show that the curve $y - e^{xy} + x = 0$ has a vertical tangent at (1, 0).

Solution: $\dfrac{dy}{dx} - e^{xy}\left(y + x\dfrac{dy}{dx}\right) + 1 = 0$

$\Rightarrow \dfrac{dy}{dx}(1 - x\,e^{xy}) = y\,e^{xy} - 1$

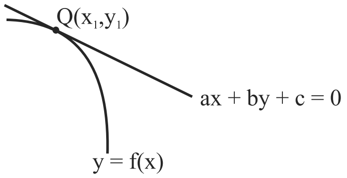
For a vertical tangent ,

$\dfrac{dy}{dx} = \dfrac{y\,e^{xy} - 1}{1 - x\,e^{xy}} = \infty \Rightarrow \dfrac{dx}{dy} = 0$

$\Rightarrow 1 - x\,e^{xy} = 0.$

This equation is satisfied by the point (1, 0). Hence, the curve has a vertical tangent at this point.

Condition for a Given Line to be Tangent to a Curve



Let us find the condition for the line ax + by + c = 0 be a tangent to the curve y = f(x).

Let $P(x_1, y_1)$ be the point on the curve where the line touches the curve.

Then P lies on the curve $\Rightarrow y_1 = f(x_1)$ ...(1)

Also P lies on the line $\Rightarrow ax_1 + by_1 + c = 0$ ...(2)

Further, slope of the line

= slope of tangent to the curve at P

$\Rightarrow -\dfrac{a}{b} = \left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}$ ...(3)

On eliminating x_1 and y_1 from the above three equations, we get the required condition.

Example 18: Show that the straight line x cosα + y sinα = p touches the curve $xy = a^2$, if $p^2 = 4a^2 \cos\alpha \sin\alpha$.

Solution: Let the line touch the curve at the point $P(x_1, y_1)$ on the curve.

$\Rightarrow x_1\cos\alpha + y_1\sin\alpha = p$ ...(1)

and $x_1y_1 = a^2$ ...(2)

Differentiating $xy = a^2$ w.r.t. x, we get $\dfrac{dy}{dx} = -\dfrac{y}{x}$

Now, slope of the line = slope of the tangent to the curve at $P(x_1, y_1)$

$\Rightarrow -\dfrac{y_1}{x_1} = -\dfrac{\cos\alpha}{\sin\alpha}$ ...(3)

From (1) and (3), $x_1\cos\alpha + x_1\cos\alpha = p$

$\Rightarrow 2\cos\alpha\, x_1 = p$ and $2\sin\alpha\, y_1 = p$

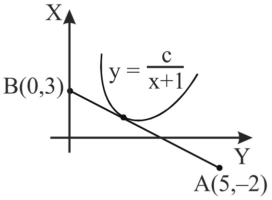
From (2)

$\Rightarrow (2\cos\alpha)(a)(2\sin\alpha)(a) = p^2$

$\Rightarrow p^2 = 4a^2\cos\alpha\sin\alpha.$

Example 19: Find value of c such that line joining the points (0, 3) and (5, – 2) becomes tangent to curve $y = \dfrac{c}{x+1}$.

Solution:



Equation of the line joining A and B is x + y = 3.

Solving this line with the curve we get

$$3 - x = \frac{c}{x+1}$$

$\Rightarrow \quad x^2 - 2x + (c-3) = 0$ ...(1)

For tangency, roots of this quadratic equation must be coincident. Hence D = 0.

$\Rightarrow \quad 4 = 4(c-3) \quad \Rightarrow \quad c = 4.$

Study Tip

If a line touches an algebraic curve then on solving the equation of line and tangent we get atleast two repeated roots corresponding to the point of contact.

Putting c = 4, equation (1) becomes

$$x^2 - 2x + 1 = 0 \qquad \Rightarrow \qquad x = 1$$

Hence the point of contact becomes (1, 2).

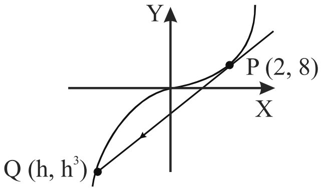
Example 20: Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Solution: The equation of tangent at (2, 8) is

$$y = 12x - 16$$

Solving this with $y = x^3$, we get

$$x^3 - 12x + 16 = 0$$



This cubic must give all points of intersection of line and curve $y = x^3$ i.e., points P and Q.

But, since the line is tangent at P, so x = 2 will be a repeated root of the equation $x^3 - 12x + 16 = 0$ and another root will be x = h(say).

Using relation between roots and coefficients,

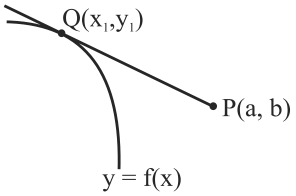
sum of roots = 2 + 2 + h = 0

$\Rightarrow \quad h = -4$

Hence, the coordinates of Q are (– 4, – 64).

Tangents from an External Point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of the possible tangents to the curve y = f(x), passing through (a, b) can be found by first finding the point of contact Q of the tangent with the curve.



Let point Q be (x_1, y_1).

Since Q lies on the curve we have $y_1 = f(x_1)$ ...(1)

Also, the slope of PQ $= \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$

$$\Rightarrow \frac{y_1 - b}{x_1 - a} = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} \qquad ...(2)$$

Solving (1) and (2), we get the point of contact (x_1, y_1).

Note: For a conic, a cubic, a quartic, the maximum number of tangents which can be drawn from a given point is 2, 6, 12 respectively.

Example 21: At what point of the curve $y = x^2 - 5x + 6$ should a tangent to that curve be drawn for the tangent to pass through the point M(1, 1)?

Solution: Let the point of tangency be (x_0, y_0)

The equation of tangent at (x_0, y_0) is satisfied by (1, 1).

Hence, $\quad 1 - y_0 = (2x_0 - 5)(1 - x_0)$,

Also, $\quad y_0 = x_0^2 - 5x_0 + 6$.

Substituting y_0 from the second equation into the first, we get a quadratic equation $x_0^2 - 2x_0 = 0$.

Hence the required points have the coordinates (2, 0) and (0, 6).

Example 22: Find the coordinates of the points on the curve $y = x^2 + 3x + 3$ the tangent at which passes through the origin.

Solution: From the equation of the curve, we have dy/dx = 2x + 3.

∴ The equation of the tangent to the given curve at the point (x_1, y_1) is $y - y_1 = (2x_1 + 3)(x - x_1)$. If this tangent passes through the origin (0,0), we have $0 - y_1 = (2x_1 + 3)(0 - x_1)$

i.e., $y_1 = 2x_1^2 + 3x_1$. ...(1)

Since the point (x_1, y_1) lies on the given curve, therefore

$$y_1 = x_1^2 + 3x_1 + 4. \qquad ...(2)$$

From (1) and (2), we have $2x_1^2 + 3x_1 = x_1^2 + 3x_1 + 4$ or $x_1^2 = 4$ or $x_1 = \pm 2$. Putting $x_1 = 2$ and – 2 in (1), we get $y_1 = 14$ and 2 respectively. Hence the required points are (2, 14) and (–2, 2).

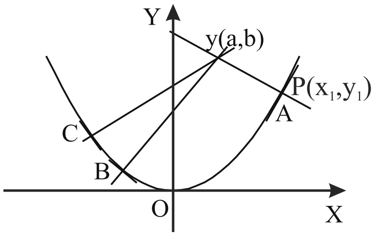
Example 23: Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point (a, b). Find the sum of their x-coordinates.

Solution: $y = x^2, \left.\frac{dy}{dx}\right]_{(x_1,y_1)} = 2x_1$

$$\therefore \quad m \text{ (slope of normal)} = \frac{-1}{2x_1}$$

Equation of normal at (x, x_1^2) is

$$y - x_1^2 = \frac{-1}{2x_1}(x - x_1) \qquad ...(1)$$



As equation (1) passes through (a, b), so

$$b - x_1^2 = \frac{-1}{2x_1}(a - x_1)$$

$\Rightarrow \quad 2x_1(b - x_1^2) = x_1 - a$

$\Rightarrow \quad 2x_1^3 + x_1(1 - 2b) - a = 0$ ...(2)

$\therefore$ Sum of all the x-coordinates = 0

(as the coefficient of x_1^2 in equation (2) is 0)

Concept Problems C

1. At what point is the tangent to the curve $y = \ln x$ inclined to the x-axis at an angle of $\frac{\pi}{4}$?
2. In the curve $y = ce^{\frac{x}{c}}$, if ψ be the angle which the tangent at any point makes with the axis of x, prove that $y = c \tan \psi$.
3. Find the equations of the tangent and the normal to the curve $y = \sqrt[3]{x-1}$ at the point (1, 0).
4. Find the x-coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point (2, 0).
5. Find whether $f(x) = \sqrt{|x|}$ has a vertical tangent at x = 0.
6. Show that the slope of the curve whose equation is $y = \frac{4}{x-1}$ at the point where x = 3 is –1. At what other point is the slope also – 1 ?
7. Find all values of x for which the line that is tangent to $y = 3x - \tan x$ is parallel to the line $y - x = 2$.
8. The tangent to the curve $y = f(x)$ at (x_0, y_0) exists only when $\lim\limits_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists. Is this true?
9. The curve $y = ax^2 + bx + c$ passes through (1, 2) and is tangent to the line $y = x$ at origin. Find the value of $a + b + c$ is.
10. Find the smallest slope of the normal to the curve $y = 2\tan\left(\frac{\pi x}{4}\right)$ on the interval $-2 < x < 2$.
11. Show that the tangents to the parabola $y^2 = 2px$ at the points where $x = p/2$ are perpendicular to each other.
12. Find the equation of the tangent to the curve $y = x^4 + 2e^x$ at (0, 2).
13. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.
14. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.
15. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points (–2, 6) and (2, 0).

16 Find the equations of the tangent and normal to $y = \frac{8a^3}{4a^2 + x^2}$ at the point where x = 2a.

17. Show that the normal to the curve $3y = 6x - 5x^3$ drawn to the point $\left(1, \frac{1}{3}\right)$ passes through the origin.
18. Find the equations of those tangents to the hyperbola $4x^2 - 9y^2 = 36$ which are perpendicular to the straight line $2y + 5x = 10$.
19. Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is – 1? If so, find an equation for the line and the point of tangency. If not, why not?
20. Show that the equations of the tangent and normal at the point (x_0, y_0) of the circle $x^2 + y^2 = a^2$, can be reduced to the forms $xx_0 + yy_0 = a^2$ and $xy_0 - yx_0 = 0$.

Practice Problems B

21. Let $f(x) = (x^3 - 2x^2)(x + 2)$
    (a) Find an equation for the tangent to the graph of f at the point where x = 1.
    (b) Find an equation for the normal line to the graph of f at the point where x = 0.
22. Find the abscissa of the point where the graph of the given function has a horizontal tangent.
    (i) $f(x) = x\sqrt{1-3x}$ (ii) $f(x) = x^2e^{1-3x}$
23. Find all points on the curve $(x^2 + y^2)^{3/2} = \sqrt{x^2 + y^2} + x$ where the tangent is vertical.
24. The straight line $y = -\frac{3}{4}x - \frac{3}{32}$ is known to be a tangent to the graph of the function $f(x) = \frac{1}{2}x^4 - x$. Find the coordinates of the point of tangency.

25. Write the equations of the tangent and the normal to the curve $x^3 + y^2 + 2x - 6 = 0$ at the point with ordinate $y = 3$.

26. Show that any tangent to the curve $y = \frac{1}{2}\sqrt{x - 4x^2}$ intersects with the y-axis at a point equidistant from the point of tangency and the origin.

27. Find the slope of the curve whose equation is $y = \frac{x^2}{x^2 + 4}$ at the points where $x = 0$ and $x = \pm 2$. What is the limit of the slope as $x \to \infty$.

28. Find the normal to $\sqrt{xy} = a + x$, which makes intercepts of equal length upon the coordinate axes.

29. Find the points at which the graph of the equation $25x^2 + 16y^2 + 200x - 160y + 400 = 0$ has a vertical or horizontal tangent.

30. If f and g are differentiable functions and $y = f(g(x)$ has a horizontal tangent at $x = 1$ then the tangent to the graph of g at $x = 1$ or the tangent to the graph of f(x) at $x = g(1)$ is horizontal. Is this true?

31. A chord of the parabola $y = x^2 - 2x + 5$ joins the points with the abscissas $x_1 = 1, x_2 = 3$. Derive the equation of the tangent to the parabola parallel to the chord.

32. Show that the normals to the curve $y = x^2 - x + 1$, drawn at the points with abscissas $x_1 = 0$, $x_2 = -1$ and $x_3 = \frac{5}{2}$, intersect at one point.

33. At the points of intersection of the straight line $x - y + 1 = 0$ and the parabola $y = x^2 - 4x + 5$ normals are drawn to the parabola. Find the area of the triangle formed by the normals and the chord joining the mentioned points of intersection.

34. Find the length of the perpendicular from the origin on the tangent at the point x, y of the curve. $x^4 + y^4 = c^4$.

35. Write down the equations of the tangents and normals to the curve $y (x^2 + a^2) = ax^2$ at the points where $y = \frac{a}{4}$.

36. Show that any tangent to the curve $y = x^5 + 8x + 1$ makes an acute angle with the x-axis.

37. Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

5.6 Tangent to parametric curves

If the equation of the curve be given in the parametric form say $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{g'(t)}{f'(t)}$$

∴ The equation of tangent at any point 't' on the curve is given by : $y - g(t) = \frac{g'(t)}{f'(t)} (x - f(t))$

and the equation of normal at point 't' is given by

$$y - g(t) = -\frac{f'(t)}{g'(t)} (x - f(t))$$

Note that in several problems, the introduction of parametric coordinates from our side helps in solving easily.

Some common parametric coordinates :

(a) $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow x = a\cos^3\theta\ ,\ y = a\sin^3\theta$.

(b) $\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow x = a\cos^4\theta\ ,\ y = a\sin^4\theta$.

(c) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1 \Rightarrow x = a(\sin\theta)^{2/n},\ y = b(\sin\theta)^{2/n}$.

(d) $c^2 (x^2 + y^2) = x^2 y^2 \Rightarrow x = c\sec\theta$ and $y = c\,\text{cosec}\theta$.

(e) $a y^2 = x^3, \Rightarrow x = a t^2$ and $y = a t^3$.

Example 1: Find the equation of the tangent to the curve $\begin{cases} x = t^2 + 3t - 8 \\ y = 2t^2 - 2t - 5 \end{cases}$ at the point P(2, –1).

Solution: First we determine the value of t corresponding to the given values of x and y. This value must simultaneously satisfy the two equations

$$\begin{cases} t^2 + 3t - 8 = 2 \\ 2t^2 - 2t - 5 = -1 \end{cases}.$$

The roots of the first equation are $t_1 = 2$; $t_2 = -5$, the roots of the second equation $t_1 = 2$; $t_2 = -1$.

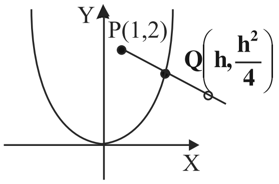
Hence, to the given point there corresponds the common value $t = 2$. Now we determine the value of the derivative at the point P :

$$y'|_{x=2} = \left(\frac{y'(t)}{x'(t)}\right)_{t=2} = \left(\frac{4t - 2}{2t + 3}\right)_{t=2} = \frac{6}{7}.$$

The equation of the tangent to the curve at the point P(2, –1) is $y + 1 = \frac{6}{7}(x - 2)$.

Example 2: Find the equation of all possible normals to the parabola $x^2 = 4y$ drawn from the point P(1, 2).

Solution: Let point $Q\left(h, \frac{h^2}{4}\right)$ be the feet of the normal to the curve $x^2 = 4y$.



Differentiating w.r.t. x $\Rightarrow 2x = 4\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{2}$

$\Rightarrow$ Slope of the normal at Q = $-\left.\dfrac{dx}{dy}\right|_{x=h} = -\dfrac{2}{h}$.

Now, m_{PQ} = slope of the normal at Q.

$$\Rightarrow \frac{\dfrac{h^2}{4}-2}{h-1} = -\frac{2}{h}$$

$$\Rightarrow \frac{h^3}{4} - 2h = -2h + 2 \Rightarrow h^3 = 8 \Rightarrow h = 2.$$

Hence, the coordinates of point Q is (2, 1) and so the equation of the required normal is x + y = 3.

Note: The equation gives only one real value of h. Hence there is only one foot of normal implying that only one real normal is possible from the point (1, 2).

Example 3: Prove that all the normals to the curve x = a cost + at sint and y = a sint – at cost are at a distance a from the origin.

Solution: x = a cos t + at sin t

$$\Rightarrow \frac{dx}{dt} = -a\sin t + at\cos t + a\sin t = at\cos t$$

y = a sin t – at cos t

$$\Rightarrow \frac{dy}{dt} = a\cos t + at\sin t - a\cos t = at\sin t$$

$$\Rightarrow \frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t.$$

Hence the equation of the normal at any point ‘t’ on the curve is

$$(y - (a\sin t - at\cos t)) = -\frac{1}{\tan t}(x - (a\cos t + at\sin t))$$

$$\Rightarrow y\sin t - a\sin^2 t + at\sin t\cos t = -x\cos t + a\cos^2 t + at\sin t\cos t$$

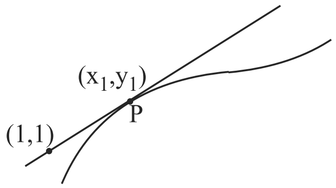
$$\Rightarrow x\cos t + y\sin t = a.$$

The distance of the normal from (0, 0)

$$= \frac{a}{\sqrt{\sin^2 t + \cos^2 t}} = a.$$

Example 4: Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as x = 2t – t^2 and y = t + t^2.

Solution:



$$\frac{dx}{dt} = 2 - 2t; \quad \frac{dy}{dt} = 2t + 1$$

$$\left.\frac{dy}{dx}\right]_t = \frac{2t+1}{2(1-t)} = \text{slope of line PQ}$$

$$\Rightarrow \frac{y_1 - 1}{x_1 - 1} = \frac{2t+1}{2(t-1)}$$

$$\Rightarrow \frac{t^2+t-1}{(t-1)^2} = \frac{2t+1}{2(t-1)} \Rightarrow t = \frac{1}{3}.$$

$$\left.\frac{dy}{dx}\right]_{t=1/3} = \frac{2t+1}{2(1-t)} = \frac{(2/3)+1}{2(2/3)} = \frac{5}{3}\cdot\frac{3}{4} = \frac{5}{4}.$$

$\therefore$ The equation of tangent is $y - 1 = \dfrac{5}{4}(x - 1)$.

$\Rightarrow$ 4y – 4 = 5x – 5 $\Rightarrow$ 5x – 4y = 1 .

Further, at t = 1, dy/dx $\to \infty$, in which case tangent is perpendicular to the x-axis and its equation is x = 1.

Example 5: Prove that normal of the curve $y^2 = 4ax$ touches the curve $27ay^2 = 4(x - 2a)^3$.

Solution: The equation of the normal at any point $(am^2, -2am)$ of the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$.

The parametric equations of the curve $27ay^2 = 4(x - 2a)^3$ can be written as

$y = 2at^3$, $x = 2a + 3at^2$.

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6at^2}{6at} = t.$$

$\therefore$ The equation of tangent at the point 't' is

$y = 2at^3 = t(x - 2a - 3at^2)$

or $y = tx - 2at - at^3$,

which is identical with (1) if t = m.

Hence the normal at $(am^2, -2am)$ to the parabola $y^2 = 4ax$ is a tangent at $(2a + 3am^2, 2am^3)$ to the curve $27ay^2 = 4(x - 2a)^3$.

Example 6: Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real point for x > 1. If the tangent at the point t is inclined at an angle ϕ to OX. Prove that $3t = \tan\phi + \sec\phi$. If the tangent at P(–2, 2) meets the curve again at Q, prove that the tangents at P and Q are at right angles.

Solution: Given curve is $x = 1 - 3t^2$ ...(1)

$y = t - 3t^3$ ...(2)

From (1) and (2), y = tx

or $x = 1 - 3\left(\dfrac{y}{x}\right)^2$

$\Rightarrow x^3 = x^2 - 3y^2$

Since all powers of y are even, so curve is symmetrical about x-axis.

For $x > 1 \Rightarrow 1 - 3t^2 > 1$

$\Rightarrow \quad -3t^2 > 0$ Impossible

From (1) and (2),

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1-9t^2}{-6t} = \tan\phi \text{ (given)} \quad ...(3)$$

$\therefore \quad \sec^2\phi = 1 + \tan^2\phi$

$$= 1 + \left(\frac{1-9t^2}{6t}\right)^2 = \left(\frac{1+9t^2}{6t}\right)^2$$

$$\therefore \quad \sec\phi = \frac{1-9t^2}{6t}$$

Adding (3) and (4) we get $\tan\phi + \sec\phi = 3t$

$\because \quad P(-2, 2)$, we have

$1 - 3t^2 = -2$ and $2 = t - 3t^3 \Rightarrow t = -1$.

$$\therefore \quad \left.\frac{dy}{dx}\right|_{t=-1} = \frac{1-9(-1)^2}{-6(-1)} = -\frac{4}{3}$$

Equation of tangent at (–2, 2) is

$$y - 2 = -\frac{4}{3}(x + 2)$$

Since the tangent meets the curve again,

$$t - 3t^3 - 2 = -\frac{4}{3}(1 - 3t^2 + 2)$$

$\Rightarrow \quad 3t - 9t^3 - 6 = -12 + 12t^2$

$\Rightarrow \quad 9t^3 + 12t^2 - 3t - 6 = 0$

$\Rightarrow \quad (t + 1)^2 (3t - 2) = 0$

Therefore the tangent at $t = -1$ meets the curve again at $t = \frac{2}{3}$, i.e. $Q\left(-\frac{1}{3}, -\frac{2}{9}\right)$.

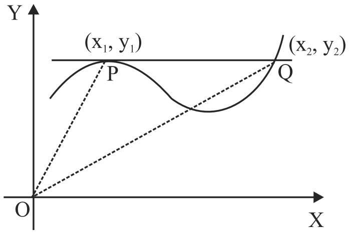
$$\therefore \quad \left.\frac{dy}{dx}\right|_{t=2/3} = \frac{1-9\left(\frac{4}{9}\right)}{-6\left(\frac{2}{3}\right)} = \frac{3}{4}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{t=-1} \times \left.\frac{dy}{dx}\right|_{t=-2/3} = -1$$

Hence the tangents at P and Q are at right angles.

Tangents Intersecting the Curve Itself

The tangent at P meeting the curve again at Q.



$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{P} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 7: In the tangent at (1,1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then find P.

Solution: $2y\frac{dy}{dx} = (2 + x)^2 - 2x(2 - x)$,

$$\text{so} \quad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{2}[1-2] = -\frac{1}{2}.$$

Therefore, the equation of tangent at (1, 1) is

$$y - 1 = -\frac{1}{2}x(-1)$$

$$\Rightarrow \quad 2x - 2 = -x + 1 \Rightarrow y = \frac{-x+3}{2}$$

The intersection of the tangent and the curve is given by

$(1/4)(-x + 3)^2 = x(4 + x^2 - 4x)$

$\Rightarrow \quad x^2 - 6x + 9 = 16x + 4x^3 - 16x^2$

$\Rightarrow \quad 4x^3 - 17x^2 + 22x - 9 = 0$

$\Rightarrow \quad (x - 1)(4x^2 - 13x + 9) = 0$

$\Rightarrow \quad (x - 1)^2 (4x - 9) = 0$

Since x = 1 is already the point of tangency, x = 9/4 and

$y^2 = \frac{9}{4}\left(2 - \frac{9}{4}\right)^2 = \frac{9}{64}$. Hence, p is $\left(\frac{9}{4}, \frac{3}{8}\right)$.

Example 8: The tangent at a point P_1 other than (0, 0) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissa of P_1, P_2, P_3,...,P_n, form a GP.

Also find the ratio of $\frac{\text{area}(\Delta P_1P_2P_3)}{\text{area}(\Delta P_2P_3P_4)}$.

Solution: Let $P_1(x_1, y_1)$ be a point on the curve

$y = x^3$ ...(1)

$y_1 = x_1^3$ ...(2)

Now, $\frac{dy}{dx} = 3x^2$

$\therefore$ Slope of the tangent at $P_1 = m_1 = 3x_1^2$

$\therefore$ Equation of the tangent at $P_1(x_1, y_1)$ is

$y - x_1^3 = 3x_1^2(x - x_1)$

$\Rightarrow \quad y = 3x_1^2 x - 2x_1^3$ ...(3)

Solving (1) and (3), we get

$x^3 - 3x_1^2 x - 2x_1^3 = 0$

$\Rightarrow \quad (x - x_1)(x^2 + xx_1 - 2x_1^2) = 0$

$\Rightarrow \quad (x - x_1)(x - x_1)(x + 2x_1) = 0$

$\Rightarrow \quad x = x_1$ (neglecting) or $x = -2x_1$

$x_2 = -2x_1$, $y_2 = x_2^3 = -8x_1^3$

$\therefore$ $P_2(x_2, y_2)$ is $(-2x_1, -8x_1^3)$.

Now, we find P_3, the point where the curve meets the tangent at P_2.

Slope of the tangent at $P_2 = \left(\dfrac{dy}{dx}\right)_{(x_2,y_2)}$

$$= 3x_2^2 = 3.4x_1^2 = 12x_1^2$$

$\therefore$ Equation of tangent at P_2 is,

$$y - x_2^3 = 3x_2^2(x - x_2) \qquad ...(4)$$

To get $P_3 = (x_3, y_3)$, solve (1) and (4)

$\therefore$ $P_3 = (x_3, y_3) = (-2x_2, -8x_2^3) = (4x_1, 64x_1^3)$ and so on.

$\therefore$ The abscissa of $P_1, P_2, P_3, ...$ are given by $x_1, -2x_1, 4x_1, -8x_1, ..,$ which is a GP with a common ratio of -2.

Now, area $(\Delta P_1P_2P_3) = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\text{area } (\Delta P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix}$$

$$\text{area } (\Delta P_1P_2P_3) = \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

Similarly,

$$\text{area } (\Delta P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$\text{area } (\Delta P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & 512x_1^3 & 1 \end{vmatrix}$$

$$\text{area } (\Delta P_2P_3P_4) = 8x_1^4\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$$\therefore \quad \frac{\text{area } (\Delta P_1P_2P_3)}{\text{area } (\Delta P_2P_3P_4)} = \frac{1}{16}.$$

Example 9: If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos\phi - x \sin\phi = a \cos 2\phi$.

Solution: The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(1)

Differentiating both sides w.r.t.x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}.$$

Slope of normal $= -\dfrac{dx}{dy} = \dfrac{x^{1/3}}{y^{1/3}} = \tan\phi$ (given)

$\therefore$ $x = y\tan^3\phi$ ...(2)

From (1) and (2), $y^{2/3}(1 + \tan^2\phi) = a^{2/3}$

$\Rightarrow$ $y^{2/3} = a^{2/3}\cos^2\phi$

$\therefore$ $y = a\cos^3\phi$ and $x = a\sin^3\phi$

Therefore equation of normal is

$y - a\cos^3\phi = \tan\phi\,(x - a\sin^3\phi)$

$\Rightarrow$ $y\cos\phi - a\cos^4\phi = x\sin\phi - a\sin^4\phi$

$\Rightarrow$ $y\cos\phi - x\sin\phi = a(\cos^4\phi - \sin^4\phi)$

$= a(\cos^2\phi + \sin^2\phi)(\cos^2\phi - \sin^2\phi)$

$= a \,.\, 1 \,.\, \cos 2\phi$

Hence $y\cos\phi - x\sin\phi = a\cos 2\phi$.

Example 10: Show that the normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at $P(0, -3)$ meets the curve again at two points. Find the equations of the tangents to the curve at these points.

Solution: Here, $5x^5 - 10x^3 + x + 2y + 6 = 0$ ...(1)

Differentiating w.r.t x, $25x^4 - 30x^2 + 1 + 2\dfrac{dy}{dx} = 0$

$\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{1}{2}(25x^4 - 30x^2 + 1)$.

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{(0,-3)} = -\dfrac{1}{2}$

Hence, the equation of the normal at $P(0, -3)$ is

$$y + 3 = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(0,-3)}} \cdot (x - 0)$$

$\Rightarrow$ $y + 3 = 2x$ ...(2)

Solving (1) and (2),

$5x^5 - 10x^3 + x + 2(2x - 3) + 6 = 0$

or $5x^5 - 10x^3 + 5x = 0 \Rightarrow x(x^4 - 2x^2 + 1) = 0$

or $x(x^2 - 1)^2 = 0$

$\therefore$ $x = 0, 1, 1, -1, -1$.

The repetition of roots 1 and – 1 signifies that the normal at P(0, – 3) touches the curve again at the two points (1, – 1) and (–1, – 5).

The equation of the tangents at these points is same as the normal at P(0, – 3) i.e. $y + 3 = 2x$.

Locus Problems

Example 11: Prove that all points of the curve $y^2 = 4a\{x + a\sin(x/a)\}$ at which the tangent is parallel to the axis of x lie on a parabola.

Solution: The given curve is

$y^2 = 4a\{x + a\sin(x/a)\}$. ...(1)

Differentiating (1), we get

$$2y\,(dy/dx) = 4a\,\{1+\cos(x/a)\}$$

or $dy/dx = (2a/y)\,\{1+\cos(x/a)\}$.

Suppose that the tangent to (1) at the point (x_1, y_1) is parallel to x-axis. Then dy/dx at (x_1,y_1) is 0

$\Rightarrow$ $(2a/y_1)\,\{1 + \cos(x_1/a)\} = 0$

$\Rightarrow$ $\cos(x_1/a) = -1$. ...(2)

Also (x_1,y_1) lies on (1). Therefore we have

$$y_1^2 = 4a\,\{x_1 + a\sin(x_1/a)\} = 4a\,(x_1 + 0),$$

$[\because$ from (2), $\cos(x_1/a) = -1 \Rightarrow \sin(x_1/a) = 0]$

$\Rightarrow$ $y_1^2 = 4ax_1$

Hence the point of contact (x_1, y_1) lies on $y^2 = 4ax$ which is a parabola.

Example 12: Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on $x^2y^2 = x^2 - y^2$.

Solution: Differentiating the equation of the given curve $y = \sin x$, we get

$$dy/dx = \cos x.$$

Let (x_1, y_1) be the point of contact of a tangent drawn from the origin to the curve $y = \sin x$. We have dy/dx at $(x_1, y_1) = \cos x_1$. Therefore the equation of the tangent to the curve $y = \sin x$ at the point (x_1, y_1) on it is

$$y - y_1 = \cos x_1 \cdot (x - x_1)$$

Since this tangent passes through the origin i.e., the point (0, 0), therefore

$0 - y_1 = \cos x_1 \cdot (0 - x_1)$ or $y_1 = x_1 \cos x_1$

or $(y_1/x_1) = \cos x_1$ ...(1)

Again the point (x_1,y_1) lies on the curve $y = \sin x$. Therefore $y_1 = \sin x_1$. ...(2)

Squaring and adding (1) and (2), we get

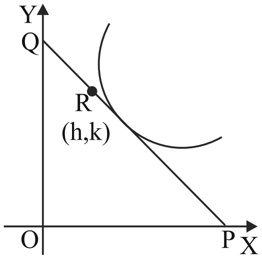
$$(y_1/x_1)^2 + y_1^2 = 1 \text{ or } x_1^2y_1^2 = x_1^2 - y_1^2.$$

Hence, generalizing we get the locus of (x_1, y_1) as

$$x^2y^2 = x^2 - y^2.$$

Example 13: The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the midpoint of PQ is a circle.

Solution: The given curve is $x = a\cos^3\theta$, $y = a\sin^3\theta$.



$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$$

$\therefore$ Equation of tangent at 'θ'

$$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$$

$\Rightarrow$ $\dfrac{y}{\sin\theta} - a\sin^2\theta = -\dfrac{x}{\cos\theta} + a\cos^2\theta$

$\Rightarrow$ $\dfrac{x}{\cos\theta} + \dfrac{y}{\sin\theta} = a$

or $\dfrac{x}{(a\cos\theta)} + \dfrac{y}{(a\sin\theta)} = 1$

$\therefore$ $P \equiv (a\cos\theta, 0)$ and $Q \equiv (0, a\sin\theta)$

If mid point of PQ is R(h, k), then

$$2h = a\cos\theta \text{ and } 2k = a\sin\theta$$

$\therefore$ $(2h)^2 + (2k)^2 = a^2$ or, $h^2 + k^2 = a^2/4$

Hence locus of midpoint is $x^2 + y^2 = a^2/4$ which is a circle.

Tangent at Origin

If a curve passing through the origin is given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.

For example, if the equation of a curve is $x^2 - 4y^2 + x^4 + 3x^3y + 3x^2y^2 + y^4 = 0$, the tangents at the origin are given by $x^2 - 4y^2 = 0$, or $x + 2y$ and $x - 2y = 0$.

In the curve $x^2 + y^2 + ax + by = 0$, $ax + by = 0$, is the equation of the tangent at the origin;

and in the curve $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, $x^2 - y^2 = 0$ is the equation of a pair of tangents at the origin.

If the equation of a curve is $x^2 + y^2 - x^2y^2 = 0$, by equating to zero the terms of the lowest degree in the equation we get $x^2 + y^2 = 0$ which does not represent any real line. There are no tangents at the origin. Note that (0, 0) is an isolated point of the curve.

Let the equation of the curve when arranged according to ascending powers of x and y be

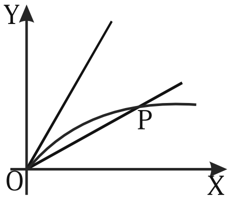
$$(a_1x + a_2y) + (b_1x^2 + b_2xy + b_3y^2) + (c_1x^3 + c_2x^2y + \ldots) = 0, \quad \ldots(1)$$

where the constant term is absent since the curve passes through the origin.

Let P (x,y) be any point on the curve. The slope of the chord OP is y/x. Therefore the equation to OP is.

$$Y = (y/x)\,X,$$

where (X,Y) are current coordinates.



As $P \to O$, i.e., as $x \to 0$ and $y \to 0$, the chord OP tends to the tangent at O.

Excluding for the present, the case when the tangent is the y-axis i.e., when $\lim_{x\to 0}\left(\frac{y}{x}\right) = \pm\infty$, we have the equation of the tangent at O as

$$Y = mX \text{ where } \lim_{x\to 0}\left(\frac{y}{x}\right) = m. \qquad \text{...(2)}$$

Case I : Let $a_2 \neq 0$. Dividing (1) by x and taking limit as $x \to 0$, we get

$$a_1 + a_2 m = 0 \qquad \text{...(3)}$$

Eliminating m between (2) and (3), we get

$$a_1X + a_2Y = 0.$$

as the equation of tangent at the origin to the curve (1). Replacing the current coordinates X, Y by x,y this equation becomes

$$a_1x + a_2y = 0, \qquad \text{...(4)}$$

which is obviously the equation obtained by equating to zero the lowest degree terms in (1).

If $a_2 = 0$, then a_1 is also zero from (3), and we get the next case.

Case II : Let $a_1 = 0$, $a_2 = 0$, but b_2 and b_3 are not both zero. Dividing (1) by x^2 and taking limit as $x \to 0$, we get

$$b_1 + b_2 \lim_{x\to 0}\left(\frac{y}{x}\right) + b_3 \lim_{x\to 0}\left(\frac{y}{x}\right)^2 = 0,$$

or $\quad b_1 + b_2m + b_3m^2 = 0 \qquad$ ...(5)

where $\lim_{x\to 0}\left(\frac{y}{x}\right) = m.$

Equation (5) is a quadratic in m, showing that there are two tangents at the origin in this case. Eliminating m between (2) and (5), we get

$$b_1x^2 + b_2xy + b_3y^2 = 0, \qquad \text{...(6)}$$

as the equation of the tangents at the origin to (1) in this case. In equation (6), we have taken x, y as current coordinates. Obviously the equation (6) is obtained by equation to zero the lowest degree terms in this equation of the curve (1), where $a_1 = a_2 = 0$.

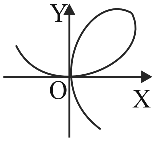
If $b_2 = b_3 = 0$, then by (5), $b_1 = 0$.

Case III : If $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, we can show by the same process that the rule still holds; and so on If tangent at the origin is the y-axis, we can easily show by supposing the axes of x and y to be interchanged for a moment, that the rule is still true.

Hence the equation of the tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

Note: The same line could be the tangent as well as normal to a given curve at a given point.

For example, in the curve $x^3 + y^3 - 3xy = 0$ (Folium of Descartes) the line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.



Concept Problems D

1. Write the equations of the tangent to the curve $x = t \cos t$, $y = t \sin t$ at the origin and at the point $t = \frac{\pi}{4}$.
2. Find the slope of the normal to the curve $x = 1 - a \sin\theta$, $y = b \cos^2\theta$ at $\theta = \frac{\pi}{2}$.
3. Derive the equations of the tangent and the normal to the parabola $x^2 = 4ay$ at the point (x_0, y_0). Show that the tangent at the point with the abscissa $x_0 = 2$ am has the equation $x = \frac{y}{m} + m$
4. Find the slope of the tangent to the curve $r^2 = a^2 \cos 2\theta$, where $x = r\cos\theta$ and $y = r\sin\theta$, at the point $\theta = \frac{\pi}{6}$.
5. Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).
6. Find all values of x_0 for each of which the tangents to the graphs of the functions $f(x) = 3\cos 5x$, $g(x) = 5\cos 3x + 2$ at the points with abscissa x_0 are parallel.
7. Find the coordinates of the points of intersection of the x-axis and the tangents to the graph of the function $y = \frac{x+1}{x-3}$ which form an angle $3\pi/4$ with the x-axis.
8. Calculate the area of the triangle bounded by the coordinate axes and the tangent to the graph of the function $y = x/(2x - 1)$ at the point with abscissa $x = 1$.
9. Find an equation of the tangent to the hyperbola $y = \frac{x+9}{x+5}$, which passes through the origin.
10. A triangle has two of its angular points at (a, 0), (0, b), and the third (x,y) is moveable along the line $y = x$. Show that if A be its area $2dA/dx = a + b$, and interpret this result geometrically.

Practice Problems C

11. Derive the equations of the tangent and the normal to the curve $x = 2 \ln \cot t + 1$, $y = \tan t + \cot t$ at $t = \pi/4$.

12. For the curve represented parametrically indicate the relation between the parameter t and the angle α between the tangent to the given curve and the x-axis.

(i) $\begin{cases} x = \cos t + t \sin t - \dfrac{t^2}{2}\cos t, \\ y = \sin t - t \cos t - \dfrac{t^2}{2}\sin t; \end{cases}$

(ii) $x = a \cos t\sqrt{2\cos 2t}, \; = a \sin t\sqrt{2\cos 2t},$.

13. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

14. Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real points for $x > 1$. If the tangent at the point 't' is inclined at an angle θ to the x-axis, prove that $3t = \tan\theta + \sec\theta$. If the tangent at P $(-2, 2)$ meets the curve again at Q, prove that tangent at P and Q are mutually perpendicular.

15. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts of intercepts α, β on the coordinate axes, where $\alpha^2 + \beta^2 = 61$, find a.

16. For the curve $xy^n = a^{n+1}$, find the value of n so that the area of the triangle included between the axes and any tangent is constant.

17. There are four points on the curve $y = x^4 - 8x^2$ where the associated tangent line passes through $(-\frac{11}{3}, 49)$. Find the x coordinates of these points.

18. Show that the coordinates of the point of inter-section of the tangents to the curve $y = 1 - x^2/a^2$ drawn through the points with the ordinates $y = 0$ do not depend on the parameter a. Find the coordinates of the intersection point.

19. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

20. For the function $y = x^3 - 3x^2 - 7x + 6$, find all points at each of which the tangents to the graph of the function, cut off on the positive x-axis a line segment, half that on the negative y-axis. Find the lengths of the cut-off segments.

21. Two tangents to the graph of the function $y = \sqrt{17(x^2+1)}$ intersect at right angles at a certain point of the y-axis. Write the equations of the tangents.

22. Show that the tangent to the parabola $y = ax^2$ (for $a \neq 0$) at the point where $x = c$, will intersect the x-axis at the point $(c/2, 0)$. Where does it intersect the y-axis ?

23. Find the equation of tangent to the curve

$$y = \frac{|x-2|}{2x+1} - \frac{1}{|x-3|} \text{ at } P\left(1, -\frac{1}{6}\right).$$

24. Write down the equation of the tangents at the origin in the following curves.

(a) $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

(b) $x^5 + y^5 = 5ax^2y^2$

(c) $(y - a)^2 \dfrac{x^2+y^2}{y^2} = b^2$.

5.7 Angle of Intersection

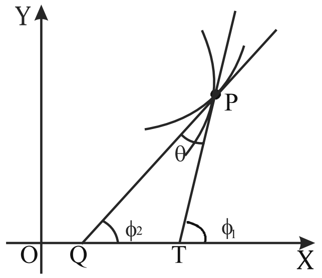
The angle of intersection of two curves is defined as the angle between their tangents at their point of intersection.

Let the equation of the two curves be

$f(x, y) = 0$ ...(1), and

$g(x, y) = 0$. ...(2)

Suppose $P(x_1, y_1)$ is a point of intersection of (1) and (2). Let $m_1 = \tan\phi_1$ and $m_2 = \tan\phi_2$ be the slopes of the tangents at the point $P(x_1, y_1)$ to the curves (1) and (2) respectively.



Then $m_1 = (dy/dx)_1$ at the point $P(x_1, y_1)$

and $m_2 = (dy/dx)_2$ at the point $P(x_1, y_1)$.

Suppose θ is the angle of intersection of (1) and (2) at the point $P(x_1, y_1)$, then $\theta = \phi_1 - \phi_2$

$\tan\theta = \tan(\phi_1 - \phi_2)$

The acute angle θ between the tangents is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

Note:

1. If $m_1 = m_2$, the angle of intersection θ is 0°. In this case the two curves have the same tangent at the point (x_1, y_1) and thus the two curves touch each other at the point (x_1, y_1).
2. If $m_1 m_2 = -1$, the angle of intersection is 90°.
   If $m_1 = \infty$, $m_2 = 0$ or if $m_1 = 0$, $m_2 = \infty$, again the angle of intersection θ is 90°.
   If the angle of intersection of two curves is 90°, we say that the curves intersect **orthogonally**.

The curves (1) and (2) intersect orthogonally at the point (x_1, y_1) if $(dy/dx)_1 \cdot (dy/dx)_2 = -1$, at the point (x_1, y_1).

3. If (1) and (2) intersect at more than one points, we should find their angle of intersection at each point.
4. Let the two curves be $y = f(x)$ and $y = g(x)$ and let the point of intersection be $P(x_1, y_1)$.

and let $m_1 = \frac{dy}{dx} = f'(x_1)$ and $m_2 = \frac{dy}{dx} = g'(x_1)$

If angle between the two curves is θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{f'(x_1) - g'(x_1)}{1 + f'(x_1) g'(x_1)}\right|$$

Example 1: Find the angle of intersection of the curves $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

Solution: The given curves are

$$x^2 - y^2 = a^2, \qquad ...(1)$$

and $$x^2 + y^2 = a^2\sqrt{2}. \qquad ...(2)$$

Solving (1) and (2) to get their points of intersection, we have

$$x^2 = \frac{a^2(\sqrt{2}+1)}{2} \text{ and } y^2 = \frac{a^2(\sqrt{2}-1)}{2}$$

Hence, $x^2y^2 = \frac{a^2(2-1)}{4} = \frac{a^4}{4}$.

Differentiating the equation (1),we get $dy/dx = (2x)/(2y) = x/y$ and differentiating the equation (2), we get $dy/dx = -x/y$.

If θ is the angle of intersection of (1) and (2), at their point of intersection (x, y), then

$$\theta = \tan^{-1}\left|\frac{x/y-(-x/y)}{1+(x/y)(-x/y)}\right| = \tan^{-1}\left|\frac{2xy}{y^2-x^2}\right|$$

$$= \tan^{-1}\left|\frac{\sqrt{(4x^2y^2)}}{y^2-x^2}\right| = \tan^{-1}\left|\frac{\sqrt{a^4}}{-a^2}\right|,$$

[∵ at the point of intersection (x,y), we have $x^2 - y^2 = a^2$ and $4x^2y^2 = a^4$]

$$= \tan^{-1}|-1| = \tan^{-1}1 = \pi/4.$$

Example 2: Show that the curves $y = 2\sin^2x$ and $y = \cos 2x$ intersect at $x = \pi/6$. What is their angle of intersection ?

Solution: Given curves are $y = 2\sin^2 x \qquad ...(1)$

and $y = \cos 2x \qquad ...(2)$

Solving (1) and (2), we get $2\sin^2 x = \cos 2x$

$\Rightarrow \quad 1 - \cos 2x = \cos 2x$

$\Rightarrow \quad \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow \quad 2x = \pm\frac{\pi}{3}$

$\therefore \quad x = \pm\frac{\pi}{6}$ are the points of intersection

From (1), $\frac{dy}{dx} = m{-}2\sin 2x = m_2$ (say)

If angle of intersection is θ, then

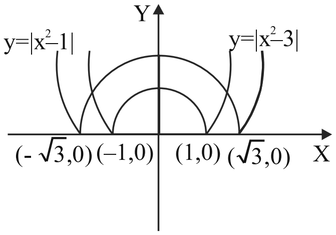
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{4\sin 2x}{1 - 4\sin^2 2x}\right|$$

$$\therefore \quad (\tan\theta)_{x=\pm\pi/6} = \left|\frac{4\times\left(\pm\frac{\sqrt{3}}{2}\right)}{1-4\times\frac{3}{4}}\right| = \left|\frac{\pm2\sqrt{3}}{-2}\right| = \sqrt{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}.$$

Example 3: Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Solution:



The points of intersection are $(\pm\sqrt{2}, 1)$

Since the curves are symmetrical about y-axis,

the angle of intersection at $(-\sqrt{2}, 1)$

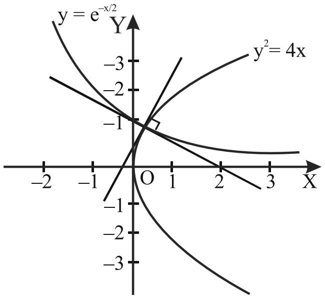
= the angle of intersection at $(\sqrt{2}, 1)$.

At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$.

$$\therefore \quad \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \Rightarrow \theta = \tan^{-1}\frac{4\sqrt{2}}{7}.$$

Example 4: Find the angle between the curves $y^2 = 4x$ and $y = e^{-x/2}$.

Solution:



Let the curves intersect at point (x_1, y_1).

For $y^2 = 4x$, $\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2}{y_1} = m_1$

and for $y = e^{-x/2}$, $\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{1}{2}e^{-x_1/2} = -\frac{y_1}{2} = m_2$

We see that $m_1m_2 = -1$. Hence, angle of intersection between the curves = 90°.

Note that we have not actually found the intersection point but we can see from the graph that the curves intersect at a point.

Example 5: Find the values of a if the curves $x^2/a^2 + y^2/4 = 1$ and $y^3 = 16x$ cut each other orthogonally.

Solution: The two curves are

$x^2/a^2 + y^2/4 = 1$ ...(1)

$dy/dx = -4x/(a^2y) = m_1$

$y^3 = 16x$ ...(2)

$dy/dx = 16/(3y^2) = m_2$

The two curves cut orthogonally, if $m_1m_2 = -1$

$\Rightarrow \quad [-4x/(a^2y)][16/(3y^2)] = -1$

$\Rightarrow \quad 64x = 3a^2y^3 \quad \Rightarrow 64x = 3a^216x$, using (2)

$\Rightarrow a^2 = 4/3$.

Hence, $a = \pm 2/\sqrt{3}$.

Orthogonal Curves

Two curves are said to be orthogonal if the angle of intersection of the two curves is 90°, wherever they intersect i.e. their tangents are perpendicular at each point of intersection. In the next example we use implicit differentiation to show that two families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family.

Example 6: The equation $xy = c, c \neq 0$ ...(1) represents a family of hyperbolas. Different values of the constant c give different hyperbolas.

The equation $x^2 - y^2 = k, \ k \neq 0$ ...(2) represents another family of hyperbolas. Show that every curve in the family (1) is orthogonal to every curve in the family (2), that is, the families are orthogonal trajectories of each other.

Solution: Implicit differentiation of equations (1) and (2) give

$$x\frac{dy}{dx} + y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \qquad \text{...(3)}$$

$$2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y} \qquad \text{...(4)}$$

From (3) and (4) we see that at any point of intersection of curves from each family, the slopes of the tangents are negative reciprocals of each other. Therefore, the curves intersect at right angles, that is, they are orthogonal.

Example 7: Show that the angle between the tangents at any point P and the line joining P to the origin O is the same at all points of the curve

$\ell n\,(x^2 + y^2) = c \tan^{-1}(y/x)$ where c is constant.

Solution: Let the point P on the curve be (x, y).

$\ln(x^2 + y^2) = c\tan^{-1}(y/x)$

Differentiating both sides w.r.t. x, we get

$$\frac{2x + 2yy'}{(x^2 + y^2)} = \frac{c(xy' - y)}{(x^2 + y^2)}$$

$$\Rightarrow \quad y' = \frac{2x + cy}{cx - 2y} = m_1 \text{ (say)}$$

Slope of OP $= \dfrac{y}{x} = m_2$ (say)

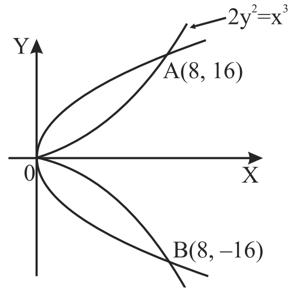
Let the angle between the tangents at P and OP be θ

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\dfrac{2x + cy}{cx - 2y} - \dfrac{y}{x}}{1 + \dfrac{2xy + cy^2}{cx^2 - 2xy}}\right| = \frac{2}{c}.$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{2}{c}\right)$$ which is independent of x and y.

Example 8: Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$.

Solution: Solving $2y^2 = x^3$ and $y^2 = 32x$ we get (0, 0), (8, 16) and (8, – 16)



For $\sqrt{2}\,y = x\sqrt{x}$ or $\sqrt{2}\,y = -x\sqrt{x}$ the slope of tangent at O is

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 0$$

For $y^2 = 32x$, the slope of tangent at O is

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \infty$$

Hence, the angle of intersection at O is 90°.

Alternatively, see that tangent at the origin on $2y^2 = x^3$ is $y^2 = 0$ i.e. $y = 0$ i.e. x-axis and tangent to $y^2 = 32\,x$ at the origin is $x = 0 \Rightarrow$ angle between the two curves at the origin is 90°.

Now, consider the points A and B.

The angles of intersection at these points are equal because of symmetry in the graphs about x-axis.

$$\text{For } 2y^2 = x^3, \quad \left.\frac{dy}{dx}\right|_A = \frac{3x^2}{4y} = \frac{3}{4}\cdot\frac{64}{16} = 3$$

$$\text{For } y^2 = 32x, \quad \left.\frac{dy}{dx}\right|_A = \frac{32}{2y} = \frac{16}{16} = 1$$

$$\therefore \quad \tan\theta = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}.$$

The angle of intersection at A and B is $\tan^{-1}\dfrac{1}{2}$.

Note: If the curves intersect at more than one point, then the angle of intersection between the curves is mentioned with reference to the point of intersection.

Example 9: Find the condition that curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ may cut each other orthogonally.

Solution: $ax_1^2 + by_1^2 = 1$ ...(1)

$a'x_1^2 + b'y_1^2 = 1$ ...(2)

Now subtracting (2) from (1), we get

$(a - a')x_1^2 + (b - b')y_1^2 = 0.$ ...(3)

Differentiating both sides w.r.t. x of the given equations,

$$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \quad \frac{dy}{dx} = -\frac{ax_1}{by_1} = m_1 \text{ (say)}$$

Similarly for the second curve

$$\frac{dy}{dx} = -\frac{a'x_1}{b'y_1} = m_2 \text{ (say)}$$

For orthogonality, $m_1m_2 = -1$.

$$\Rightarrow \quad \frac{aa'x_1^2}{bb'y_1^2} = -1$$

$\Rightarrow \quad a'x_1^2 = -bb'y_1^2$ ...(4)

Dividing (3) by (4), we have

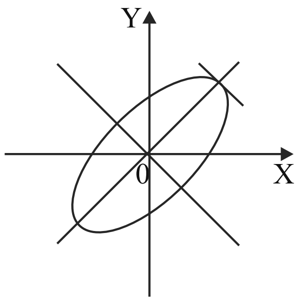
$$\frac{a-a'}{aa'} = \frac{b-b'}{bb'} \quad \Rightarrow \quad \frac{1}{a'} - \frac{1}{a} = \frac{1}{b'} - \frac{1}{b}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

We should remember the above condition, so that we can easily find whether the ellipse and / or hyperbolas represented by the above equations intersect orthogonally or not.

Example 10: Find the condition for the line y = mx to cut the conic $ax^2 + 2hxy + by^2 = 1$ at right angles. Hence find the direction of the axes of the conic.

Solution:



$$ax^2 + 2hxy + by^2 = 1$$

$$\cancel{2}ax + \cancel{2}h\left[x\frac{dy}{dx} + y\right] + \cancel{2}by\frac{dy}{dx} = 0$$

$$(xh + by)\frac{dy}{dx} = -(ax + hy)$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

For the line y = mx to cut the curve at right angles,

$$-\left(\frac{ax + hy}{hx + by}\right).\, m = -1$$

$$m\left[\frac{ax + h\cdot mx}{hx + b\cdot mx}\right] = 1$$

$m(a + hm) = h + bm$

$m^2h + (a - b)m - h = 0$ ...(1)

Hence, (1) is the required condition.

The axes of the conic are $y = m_1x$ and $y = m_2x$ where m_1 and m_2 are roots of the equation (1).

Concept Problems E

1. Find the angle of intersection of the curves:
   (i) $y = x^2$ and $6y = 7 - x^3$ at (1, 1)
   (ii) $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$.
2. Find the acute angle between the curves y = sin x and y = cos x.
3. At what angles do the curves y = sin x and y = sin 2x intersect the x-axis at the origin?
4. Find the angle at which the parabolas $y = (x - 2)^2$ and $y = -4 + 6x - x^2$ intersect.
5. Sketch the intersecting graphs of the equations and show that they are orthogonal
   (i) $\begin{matrix} y^2 = x^3 \\ 2x^2 + 3y^2 = 5 \end{matrix}$ (ii) $\begin{matrix} x^3 = 3(y-1) \\ x(3y-29) = 3 \end{matrix}$
6. If the curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then show that $a^2 = \frac{4}{3}$.
7. Show that the curve $y = e^{|x|}$ cannot have a tangent line at the point x = 0. What is the angle between the one-sided tangents to this curve at the indicated point?

Practice Problems D

8. Find the angles at which the given lines intersect.

(i) $y = \frac{x+1}{x+2}$ and $y = \frac{x^2+4x+8}{16}$

(ii) $y = (x-2)^2$ and $y = 4x - x^2 + 4$

9. Find the angle at which the given curves intersect:

(i) $y = x^2$ and $\begin{cases} x = \frac{5}{3}\cos t, \\ y = \frac{5}{4}\sin t; \end{cases}$

(ii) $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$ and $\begin{cases} x = \frac{at^2}{1+t^2} \\ y = \frac{at\sqrt{3}}{1+t^2} \end{cases}$

10. Show that the given curves are orthogonal:

(i) $2x^2 + y^2 = 3,\ x = y^2$

(ii) $x^2 - y^2 = 5,\ 4x^2 + 9y^2 = 72$

11. Show that the given families of curves are orthogonal trajectories of each other :

(i) $x^2 + y^2 = r^2,\ ax + by = 0$

(ii) $y = cx^2,\ x^2 + 2y^2 = k$

12. If the curves $ay + x^2 = 7$ and $y = x^3$ cut each other orthogonally at a point, then find the value a.

13. Find the angle of intersection of curves,
$y = [\ |\sin x| + |\cos x|\ ]$ and $x^2 + y^2 = 5$
where [.]denotes the greatest integer function.

14. Let f(x) and g(x) be two functions which cut each other orthogonally. At their common point of intersection x_1, both f(x) and g(x) are equal to n, $n \in N - \{1\}$; also $|f'(x_1)| = |g'(x_1)|$ at the common point of intersection, then find $\lim\limits_{x \to x_1} [f(x)g(x)]$, where [.] denotes the greatest integer function.

5.8 Common Tangents

A line which touches two given curves is called a common tangent to the curves. There are two cases :

(i) The point of contact of the common tangent is same for both the curves (see figure 1). Here, the curves touch each other. In such a case the slope of the tangent is equal to the slope of both the curves at the point of contact.

i.e. $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}^{C_1} = \left.\frac{dy}{dx}\right|_{(x_1,y_1)}^{C_2}$

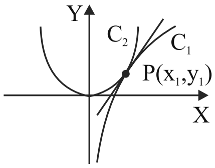


Figure 1

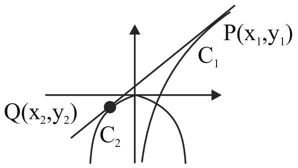


Figure 2

(ii) The points of contact of the common tangent is different for the two curves (see figure 2). In such a case the slope of the tangent is equal to the slope of the curves at their respective points of contact.

i.e. $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}^{C_1} = \left.\frac{dy}{dx}\right|_{(x_2,y_2)}^{C_2} = \frac{y_1 - y_2}{x_1 - x_2}$

Example 1: Prove that the curves xy = 4 and $x^2 + y^2 = 8$ touch each other at two points and find the equation of their common tangents at these points.

Solution: Equation of the given curves are

$$xy = 4 \quad \text{...(1)}$$

and $$x^2 + y^2 = 8 \quad \text{...(2)}$$

Putting the value of y from (1) in (2), we get

$$x^2 + \frac{16}{x^2} = 8 \Rightarrow x^4 + 16 = 8x^2$$

$$\Rightarrow x^4 - 8x^2 + 16 = 0 \Rightarrow (x^2 - 4)^2 = 0$$

$$\therefore x = \pm 2$$

From (1) when x = 2, y = 2

and when x = –2, y = –2

Hence points of intersection of the two curves are (2, 2) and (–2, –2).

From (1), $1.y + x\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} \quad \text{...(3)}$$

From (2), $2x + 2y\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \quad \text{...(4)}$$

Slope of the tangent to the curve (1) at point (2, 2)

$$m_1 = -\frac{2}{2} = -1 \quad \text{[from (3)]}$$

Slope of tangent to the curve (2) at point (2, 2)

$$m_2 = -\left(\frac{2}{2}\right) = -1 \quad \text{[from (4)]}$$

∵ $m_1 = m_2$, therefore, the two curves have a common tangent at (2, 2) i.e. they touch each other at (2, 2). The equation of common tangent is

$$y - 2 = -1(x-2) \Rightarrow x + y = 4.$$

At point (–2, –2) :

Slope of tangent to curve (1), $m_3 = -\left(\dfrac{-2}{-2}\right) = -1$

Slope of tangent to curve (2), $m_4 = -\left(\dfrac{-2}{-2}\right) = -1$

Since $m_3 = m_4$, hence the two curves touch each other again at (–2, –2).

The equation of common tangents are

$$y - 2 = -1(x-2) \Rightarrow x + y = 4 \text{ and}$$
$$y + 2 = -1(x+2) \Rightarrow x + y = -4.$$

Example 2: Prove that the curves $y = e^{-ax}\sin bx$, $y = e^{-ax}$ touch at the points for which $bx = 2m\pi + \pi/2$, where m is an integer.

Solution: The given curves are

$$y = e^{-ax}\sin bx \qquad ...(1)$$

and $y = e^{-ax}$ ...(2)

Solving the equations (1) and (2) for x, we get

$$e^{-ax}\sin bx = e^{-ax}$$

or $\sin bx = 1$. [∵ $e^{-ax} \neq 0$ for any real number x]

Now the general solution of the equation sin bx = 1 is given by $bx = 2m\pi + \pi/2$, where m is any integer.

∴ At the points where the curves (1) and (2) meet, we have $bx = 2m\pi + \pi/2$.

Now $\left(\dfrac{dy}{dx}\right)_1 = -ae^{-ax}\sin bx + be^{-ax}\cos bx$

and $\left(\dfrac{dy}{dx}\right)_2 = -ae^{-ax}$.

At the points where the curves (1) and (2) meet we have sin bx = 1 and so cos bx = 0.

∴ At the points where the two curves meet, we have $\left(\dfrac{dy}{dx}\right)_1 = -ae^{-ax} = \left(\dfrac{dy}{dx}\right)_2$

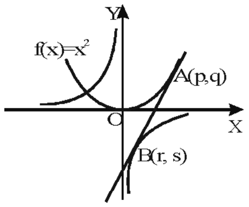
i.e., the two curves have the same tangent at these points.

Hence the two curves touch at the points for which $bx = 2m\pi + \pi/2$.

Example 3: There is a point (p, q) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $g(x) = \dfrac{-8}{x}$ where $p > 0$ and $r > 0$. If the line through (p, q) and (r, s) is tangent to both the curves at these points respectively, then find the value of (p + r).

Solution: $y = x^2$ and $y = -\dfrac{8}{x}$

$$\Rightarrow \quad q = p^2 \text{ and } s = -\frac{8}{r} \qquad ...(1)$$



Equating $\dfrac{dy}{dx}$ at A and B, we get

$$2p = \frac{8}{r^2} \quad \Rightarrow \quad pr^2 = 4 \qquad ...(2)$$

Now $m_{AB} = \dfrac{q-s}{p-r} \Rightarrow 2p = \dfrac{p^2 + \frac{8}{r}}{p-r}$

$$\Rightarrow \quad p^2 = 2pr + \frac{8}{r} \qquad \Rightarrow p^2 = \frac{16}{r}$$

$$\Rightarrow \frac{16}{r^4} = \frac{16}{r} \qquad \text{[from (2)]}$$

$\Rightarrow r = 1\ (r \neq 0) \Rightarrow p = 4$

∴ r = 1, p = 1.

Hence, p + r = 5.

Example 4: A tangent drawn to the curve $C_1 \equiv y = x^2 + 4x + 8$ at its point P touches the curve $C_2 \equiv y = x^2 + 8x + 4$ at its point Q. Find the coordinates of the point P and Q, on the curves C_1 and C_2.

Solution: Let y = mx + c be tangent to

$$y = x^2 + 4x + 8 \qquad ...(1)$$

and $y = x^2 + 8x + 4$ ...(2)

Solving with (1)

$$mx + c = x^2 + 8x + 4$$
$$x^2 + (8-m)x + 4 - c = 0.$$

For tangency, the roots must be coincident

$$D = 0 \Rightarrow (8-m)^2 - 4(4-c) = 0$$
$$\Rightarrow \quad 64 + m^2 - 16m = 16 - 4c$$
$$m^2 - 16m + 48 + 4c = 0 \qquad ...(3)$$

Similarly, $mx + c = x^2 + 4x + 8$

$$x^2 + (4-m)x + 8 - c = 0$$
$$D = 0 \text{ gives } (4-m)^2 = 4(8-c)$$
$$m^2 - 8m + 16 = 32 - 4c$$
$$m^2 - 8m - 16 + 4c = 0$$
$$m^2 + 4c = 8m + 16 \qquad ...(4)$$

From (3) and (4)

$$8m + 16 - 16m + 48 = 0$$
$$8m = 64 \Rightarrow m = 8, \quad c = 4$$

Hence the equation of PQ is y = 8x + 4.

Now $8x + 4 = x^2 + 4x + 8$

$\Rightarrow \quad x^2 - 4x + 4 = 0$

$\Rightarrow \quad x = 2, y = 20 \Rightarrow P(2, 20)$

Also, $8x + 4 = x^2 + 8x + 4$

$\Rightarrow \quad x^2 = 0 \Rightarrow \quad x = 0, \; y = 4 \Rightarrow Q(0, 4)$.

Example 5: Find the common tangent to the two curves $y = x^2 - 5x + 6$ and $y = x^2 + x + 1$.

Solution: Let the common tangent be $y = ax + b$.
On solving with both curves :

$$ax + b = x^2 - 5x + 6 \text{ and}$$
$$ax + b = x^2 + x + 1$$

and putting $D = 0$ gives

$$a^2 + 10a + 4b + 1 = 0$$

and $a^2 - 2a + 4b - 3 = 0$

$\Rightarrow a = -1/3$ and $b = 5/9$

$\therefore$ The common tangent is $3x + 9y = 5$.

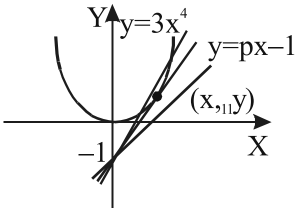
Note: In the above problems, we obtain a quadratic equation while solving the equation of the common tangent with the curve. This enables us to use D = 0 in finding the condition for tangency. In other cases we equate the value of the derivatives with the slope of the tangent.

Number of Solutions of Equations

We know that the equation f(x) = g(x) can be solved by finding the points of intersection of the curves y = f(x) and y = g(x). The concept of tangency between these curves is an important tool in finding the number of roots. The following examples illustrate this method.

Example 6: Find possible values of p such that the equation $3x^4 = px - 1$ has exactly one solution.

Solution: Consider the curves $y = 3x^4$ and $y = px - 1$.For one solution, the two curves must intersect at only one point. Hence, they should touch each other.



The line $y = px - 1$ should be tangent to $y = 3x^4$ at the point (x_1, y_1)

$\Rightarrow \quad 3x_1^4 = px_1 - 1$ and $12x_1^3 = p$

$\Rightarrow \quad 3x_1^4 = 12x_1^3.x_1 - 1$

$\Rightarrow \quad x_1 = \pm\dfrac{1}{\sqrt{3}}$

Hence, $p = 12x_1^3 = \pm\dfrac{4}{\sqrt{3}}$.

These are the values of p such that the equation $3x^4 = px - 1$ has exactly one solution.

Note that if $|p| < \dfrac{4}{\sqrt{3}}$ then there is no solution and if $|p| > \dfrac{4}{\sqrt{3}}$ then there are two solutions.

Example 7: For what values of c does the equation $ln\, x = cx^2$ have exactly one solution ?

Solution: Let's start by graphing $y = ln\, x$ and $y = cx^2$ for various values of c. We know that for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$.

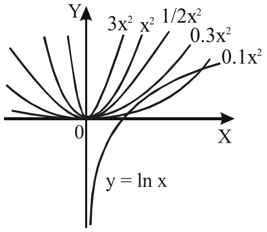


Figure 1

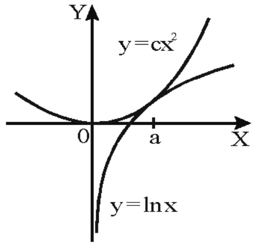


Figure 2

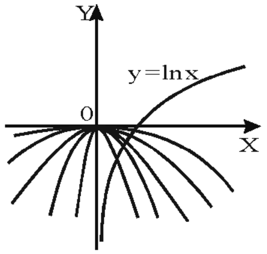


Figure 3

Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them do not intersect $y = \ell n x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words, $\ell n\, a = ca^2$, so a is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when x = a. That means the curves $y = ln\, x$ and $y = cx^2$ have the same slope when x = a. Therefore, $\dfrac{1}{a} = 2ca$.

Solving the equation $ln\ a = ca^2$ and $1/a = 2ca$, we get $ln\ a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$.

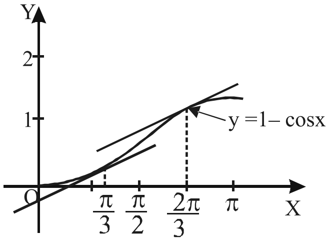
Thus, $a = e^{1/2}$ and $c = \frac{ln\ a}{a^2} = \frac{ln\ e^{1/2}}{e} = \frac{1}{2e}$.

For negative values of c we have the situation illustrated in Figure 3. All parabolas $y = cx^2$ with negative values of c intersect $y = ln\ x$ exactly once. And let us not forget about $c = 0$. The curve $y = 0\ x^2 = 0$ is just the x-axis, which intersects $y = ln\ x$ exactly once.

To summarize, the required values of c are $c = 1/(2e)$ and $c \le 0$.

Example 8: Find the values of a if equation $1–\cos x = \frac{\sqrt{3}}{2}|x|+a$, $x\in (0, \pi)$ has exactly one solution.

Solution:



The equation $1–\cos x = \frac{\sqrt{3}}{2}|x|+a$ has root where the curves $C_1 : y = 1 – \cos x$ and $C_2 : y = \frac{\sqrt{3}}{2}|x|+a$ intersect.

For one solution, we find the condition that the two curves touch each other.

Slope of C_1 is sin x and for $x > 0$ slope of C_2 is $\frac{\sqrt{3}}{2}$. Thus, at the point of contact, $\sin x = \frac{\sqrt{3}}{2}$

$$\Rightarrow \quad x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}.$$

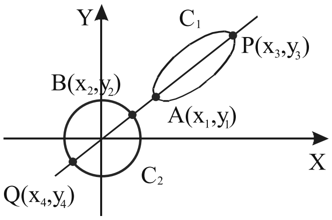
Hence, the point of contact is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ or $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$.

When $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, we get $a = \frac{1}{2} - \frac{\pi}{2\sqrt{3}}$

When $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$, we get $a = \frac{3}{2} - \frac{\pi}{\sqrt{3}}$.

Shortest Distance

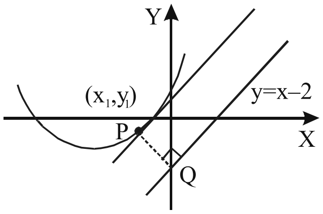
The shortest distance between two non-intersecting curves is found along the common normal to the two curves. In fact, if the two curves also have the largest distance between them, then it is also found along the common normal to the two curves. This can be established with the help of the concept of maxima-minima.



In the figure shown above, we notice that the shortest distance between the curves is AB and the largest distance between them is PQ, both of which are found along a common normal. Note that the common normal may be different in the two cases.

Example 9: Find the shortest distance between the line $y = x – 2$ and the parabola $y = x^2 + 3x + 2$.

Solution: Let $P(x_1, y_1)$ be the point on the parabola which is closest to the line $y = x – 2$



The shortest distance PQ is found along the common normal. Here, it is a line perpendicular to the given line. Thus, the slope of tangent to the parabola at P must be parallel to the given line.

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \text{slope of line}$$

$$\Rightarrow \quad 2x_1 + 3 = 1$$

$$\Rightarrow \quad x_1 = – 1 \quad \Rightarrow y_1 = 0$$

Hence the point (– 1, 0) is closest to the given line and its perpendicular distance from the line $y = x – 2$ gives the shortest distance $= \frac{3}{\sqrt{2}}$.

Example 10: Find the point on the, curve, $3x^2 – 4x^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.

Solution: Slope of the given line $3x + 2y + 1 = 0$ is (–3/2). Let us locate the point on the curve at which the tangent is parallel to given line.

Differentiating the curve both sides with respect to x we get,

$$6x – 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{4y_1} = \frac{-3}{2} \qquad ...(1)$$

[since parallel to $3x + 2y = 1$]

Also the point (x_1,y_1) lies on $3x^2 - 4y^2 = 72$

$$\Rightarrow \quad 3x_1^2 - 4y_1^2 = 72$$

$$\Rightarrow \quad 3\frac{x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2}$$

$$\Rightarrow \quad 3(4) - 4 = \frac{72}{y_1^2} \qquad \text{[from (1)]}$$

$$\Rightarrow \quad y_1^2 = 9 \qquad \Rightarrow y_1 = \pm 3$$

The required points are (–6,3) and (6,–3).
The distance of (–6, 3) from the given line,

$$= \frac{|-18+6+1|}{\sqrt{13}} = \frac{11}{\sqrt{3}}$$

and the distance of (6, 3) from the given line,

$$= \frac{|18-6+1|}{\sqrt{13}} = \frac{13}{\sqrt{3}} = \frac{11}{\sqrt{3}}.$$

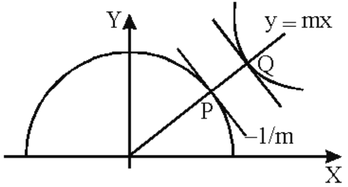
Thus, (–6, 3) is the required point.

Example 11: Let P be a point on the curve C_1: $y = \sqrt{2-x^2}$ and Q be a point on the curve C_2: $xy = 9$, both P and Q lie in the first quadrant. If 'd' denotes the minimum value between P and Q, find the value of d^2.

Solution: Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.
PQ will be the minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$
Let the normal at P be $y = mx$ $(m > 0)$ ...(1)
Solving it with $xy = 9$

$$mx^2 = 9 \quad \Rightarrow \quad x = \frac{3}{\sqrt{m}},\ y = \frac{9\sqrt{m}}{3}$$

$$\therefore \quad Q \equiv \left(\frac{3}{\sqrt{m}}, 3\sqrt{m}\right)$$



Differentiating $xy = 9$, we have

$$x\frac{dy}{dx} + y = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_Q = -\frac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m$$

The tangent at P and Q must be parallel

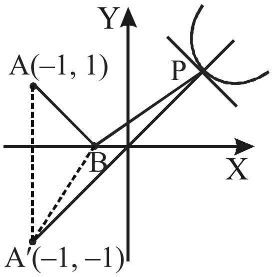
$$\therefore \quad -m = -\frac{1}{m} \;\Rightarrow\; m^2 = 1 \;\Rightarrow\; m = 1$$

$\therefore$ The normal at P and Q is $y = x$.

Solving the normal with the curves, we get
P(1, 1) and Q(3, 3)
$\therefore$ $(PQ)^2 = d^2 = 4 + 4 = 8$.

Example 12: Find the length of the shortest path that begins at the point (– 1, 1), meets the x-axis and then ends at a point on the parabola $(x - y)^2 = 2(x + y - 4)$,

Solution: The axis of parabola is $x - y = 0 \Rightarrow y = x$



For AB + BP to be minimum, A'B + BP should be minimum where A' is the reflection of A about x-axis.

$\therefore$ A'BP are collinear and for minimum distance A'P should be normal to the parabola.

Since A' is on the axis of the parabola and axis itself is a normal, A'P will be minimum if P is the vertex.

Vertex of the parabola is the point of intersection of the axis $x - y = 0$ and the tangent at the vertex $x + y - 4 = 0$, which is P(2, 2).

$$\therefore \quad A'P = \sqrt{(-1-2)^2 + (-1-2)^2} = 3\sqrt{2}.$$

Hence, the length of the shortest path is $3\sqrt{2}$.

Example 13: Find the minimum value of

$$(x_1 - x_2)^2 + \left(\frac{x_1^2}{20} - \sqrt{(17-x_2)(x_2-13)}\right)^2,$$

where $x_1 \in (0, \infty)$, $x_2 \in (13, 17)$.

Solution: The given expression resembles with
$(x_1 - x_2)^2 + (y_1 - y_2)^2$,

where $y_1 = \dfrac{x_1^2}{20}$ and $y_2 = \sqrt{(17-x_2)(x_2-13)}$

Thus, we can think about two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$ respectively. Let D be the distance between P_1 and P_2 then the given expression simply represents D^2.

Now, as per the requirement, we have to locate the point on these curves (in the first quadrant) such that the distance between them is minimum.

Since the shortest distance between two curves always occurs along the common normal, it implies that we have to locate a point $P(x_1, y_1)$ on the parabola $x^2 = 20y$ such that normal drawn to parabola at this point passes through (15, 0), the centre of the circle.

Now, equation of the normal to the parabola at

(x_1, y_1) is $\left(y - \frac{x_1^2}{20}\right) = \frac{-10}{x_1}(x - x_1)$.

It should pass through (15, 0).

$\Rightarrow \quad x_1^3 + 200x_1 - 3000 = 0$

$\Rightarrow \quad x_1 = 10 \quad \Rightarrow \quad y_1 = 5$

$\Rightarrow \quad D = \sqrt{(10-15)^2 + 5^2} - 2 = (5\sqrt{2} - 2)$

The minimum value of the given expression is

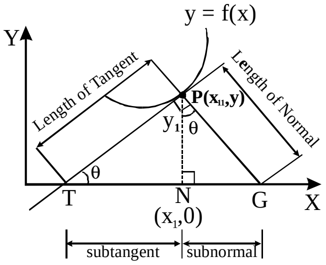
$D^2 = (5\sqrt{2} - 2)^2 = 54 - 20\sqrt{2}$.

Practice Problems E

1. Show that the graphs of equations $y = 3x^2$, $y = 2x^3 + 1$ are tangent at the point (1, 3), i.e., that they have a common tangent line at this point.
2. Determine equations of the common tangent and common normal to $x^2 + y^2 = 2ax$ and $y^2 = 2ax$.
3. Sketch the graph of the two equations $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both the graphs. Find the equations of these lines.
4. If $y = 1 + \frac{x^2}{a^3}$ and $y = 4\sqrt{x}$ have only one point in common. Find a.
5. If $y = e^x$ and $y = kx^2$ touches each other, find k.
6. Find the least distance from the point M(0, –2) to points (x, y) such that $y = \frac{16}{\sqrt{3}x^3} - 2$, $x > 0$.
7. Prove that the straight line $y = 2x - 1$ touches the curve $y = x^4 + 2x^2 - 3x^2 - 2x + 3$ at two distinct points.
8. Prove that the curve $y = x^4 - 6x^3 + 13x^2 - 11x + 4$ touches the straight line $y = x$ twice; and find the abscissa of the points of contact of the remaining tangents from the origin.
9. Find the points on the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ where the tangent is parallel to $y = 2x$ and prove that two of these points have the same tangent.
10. Prove that the curves $y_1 = f(x)$ $(f(x) > 0)$ and $y_2 = f(x) \cos x$, where f(x) is a differentiable function, are tangent to each other at the common points.
11. It is easy to see that the equation $\sin x = ax$, where a is positive, has no real roots except $x = 0$ if $a \geq 1$, and a finite number of roots, which increases as a diminishes, if $a < 1$. Prove that the values of a for which the number of roots changes are the values of $\cos\xi$, where ξ is a positive root of the equation $\tan\xi = \xi$.
12. If $x^4 + y^4 = a^4$, prove that $x^2 + y^2$ is a maximum when $x = \pm y$. Prove that its greatest radial deviation from the circle $x^2 + y^2 = a^2$ is 0.189 a.

5.9 Length of Tangent

Consider a point $P(x_1, y_1)$ on the curve $y = f(x)$. Let PT be the tangent to the curve at P and PG be the normal at P. PN is the ordinate of point P.



Let slope of tangent $m = \tan\theta = \left.\frac{dy}{dx}\right|_{P(x_1, y_1)}$

and $\cot\theta = \left.\frac{dx}{dy}\right|_{P(x_1, y_1)}$

Hence, equation of tangent is $(y - y_1) = m(x - x_1)$

Putting y = 0 we get the x-intercept of tangent as $x_1 - \frac{y_1}{m}$.

Similarly the x-intercept of normal is $x_1 + my_1$.

(a) Length of Tangent : PT is defined as length of tangent. In ΔPNT,

$$PT = |y_1 \operatorname{cosec}\theta| = \left|y_1\sqrt{(1+\cot^2\theta)}\right|$$

$$= \left|y_1\sqrt{1+\left(\frac{dx}{dy}\right)_P^2}\right| = \left|y_1\frac{\sqrt{1+\left(\frac{dy}{dx}\right)_P^2}}{\left(\frac{dy}{dx}\right)_P}\right|$$

$$\therefore \text{ Length of Tangent} = \left|y_1\sqrt{1+\left(\frac{dx}{dy}\right)_P^2}\right|$$

(b) Length of Normal : PG is defined as length of normal. In ΔPNG,

$PG = |y_1 \sec\theta|$

$$= \left|y_1\sqrt{(1+\tan^2\theta)}\right| = \left|y_1\sqrt{1+\left(\frac{dy}{dx}\right)_P^2}\right|$$

$$\therefore \quad \text{Length of Normal} = \left|y_1\sqrt{1+\left(\frac{dy}{dx}\right)_P^2}\right|$$

The projection of the line segment PT on the x-axis, i.e. TN, is called the subtangent and similarly the projection of the line segment PG on x-axis, i.e. NG is called the subnormal.

(c) Subtangent : TN is defined as the subtangent.

In ΔPTN,

$$TN = |y_1\cot\theta| = \left|\frac{y_1}{\tan\theta}\right| = \left|\frac{y_1}{\left(\frac{dy}{dx}\right)_P}\right| = \left|y_1\left(\frac{dx}{dy}\right)_P\right|$$

$$\therefore \quad \text{Subtangent} = \left|y_1\left(\frac{dx}{dy}\right)_P\right|$$

(d) Subnormal : NG is defined as subnormal.

$$\text{In } \Delta PNG,\ NG = |y_1\tan\theta| = \left|y_1\left(\frac{dy}{dx}\right)_P\right|$$

$$\therefore \text{Subnormal} = \left|y_1\left(\frac{dy}{dx}\right)_P\right|$$

These lengths can also be described using $f'(x_1)$ in place of $\left.\frac{dy}{dx}\right|_{P(x_1,\,y_1)}$.

(a) Length of tangent (PT) = $\left|\dfrac{y_1\sqrt{1+(f'(x_1))^2}}{f'(x_1)}\right|$

(b) Length of normal (PG) = $\left|y_1\sqrt{1+(f'(x_1))^2}\right|$

(c) Subtangent (TN) = $\left|\dfrac{y_1}{f'(x_1)}\right|$

(d) Subnormal (NG) = $|y_1 f'(x_1)|$

Example 1: Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.

Solution: Here $y(0) = -1$, $\frac{dy}{dx} = 3x^2 + 6x + 4$

$$m = \left.\frac{dy}{dx}\right|_{x=0} = 4$$

$$\text{Length of tangent} = \left|y_1\sqrt{1+\frac{1}{m^2}}\right|$$

$$= \left|-1\sqrt{1+\frac{1}{16}}\right| = \frac{\sqrt{17}}{4}.$$

Example 2: Find the length of normal to the curve,

$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$, at $\theta = \frac{\pi}{2}$.

Solution: Here, $\frac{dx}{d\theta} = a(1 + \cos\theta)$ and $\frac{dy}{d\theta} = a(\sin\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\pi/2} = \frac{\sin\pi/2}{1+\cos\pi/2} = 1.$$

The length of normal

$$= \left\{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}_{\theta=\pi/2} = a\left(1-\cos\frac{\pi}{2}\right)\sqrt{1+1^2} = \sqrt{2}a$$

Example 3: Prove that for the curve $y = be^{x/a}$, the subtangent at any point is always constant.

Solution: $y = be^{x/a}$

Let the point be (x_1, y_1)

$$\Rightarrow \quad m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{b.e^{x_1/a}}{a} = \frac{y_1}{a}.$$

$$\text{Now, subtangent} = \left|\frac{y_1}{m}\right| = \left|\frac{y_1}{y_1/a}\right| = a = \text{constant}.$$

Example 4: Find the equations of the tangent and normal, the lengths of tangent and normal, the subtangent and subnormal for the ellipse

$x = a\cos t,\ y = b\sin t$

at the point $P(x_1, y_1)$ for which $t = \frac{\pi}{4}$.

Solution: From the given equation we find

$$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = b\cos t,$$

$$\frac{dy}{dx} = -\frac{b}{a}\cos t \Rightarrow \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = -\frac{b}{a}$$

We find the coordinates of the point of tangency P :

$$x_1 = \frac{a}{\sqrt{2}},\ y_1 = \frac{b}{\sqrt{2}}$$

The equation of the tangent is

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

or $\quad bx + ay - ab\sqrt{2} = 0.$

The equation of the normal is

$$y-\frac{b}{\sqrt{2}}=\frac{a}{b}\left(x-\frac{a}{\sqrt{2}}\right)$$

or $(ax-by)\sqrt{2}-a^2+b^2=0$.

The lengths of the tangent and the normal are

$$PT=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\sqrt{\left(-\frac{b}{a}\right)+1}\right|=\frac{1}{\sqrt{2}}\sqrt{a^2+b^2}$$

$$PN=\left|\frac{b}{\sqrt{2}}\sqrt{1+\left(-\frac{b}{a}\right)^2}\right|=\frac{b}{a\sqrt{2}}\sqrt{a^2+b^2}$$

The subtangent and subnormal are

$$TN=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\right|=\frac{a}{\sqrt{2}},\quad NG=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\sqrt{\left(-\frac{b}{a}\right)}\right|=\frac{b^2}{a\sqrt{2}}.$$

Example 5: For the curve $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$ find the subtangent, subnormal and length of normal at (x, y).

Solution: We have $y'=\frac{1}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right)$.

Hence, subtangent $=\left|\frac{y}{y'}\right|=\left|c\,\frac{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}{e^{\frac{x}{c}}-e^{-\frac{x}{c}}}\right|$

Subnormal $=|yy'|=\left|\frac{c}{4}\left(e^{\frac{2x}{c}}-e^{-\frac{2x}{c}}\right)\right|$

Length of normal $=\left|y\sqrt{1+(y')^2}\right|=\left|\frac{y^2}{c}\right|$.

Example 6: What should be the value of n in the equation of curve $y=a^{1-n}.x^n$, so that the subnormal may be of constant length ?

Solution: The given curve is $y=a^{1-n}.x^n$. Taking logarithm of both sides, we get

$$\ell n\,y=(1-n)\,\ell n\,a+n\,\ell n x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=0+\frac{n}{x}\Rightarrow\frac{dy}{dx}=\frac{ny}{x}\qquad\text{...(1)}$$

$$\therefore \text{Subnormal}=\left|y\frac{dy}{dx}\right|$$

$$=\left|y.\frac{ny}{x}\right|\qquad\text{[from (1)]}$$

$$=\frac{ny^2}{x}=n.\frac{(a^{1-n}x^n)^2}{x}\qquad(\because y=a^{1-n}.x^n)$$

$$=n\,.\,a^{2-2n}\,.\,x^{2n-1}$$

Since subnormal is to be constant, x should not appear in its expression i.e., $2n-1=0$.

$\therefore n=\frac{1}{2}$ and the curve is the parabola $y^2=ax$.

Concept Problems F

1. Given a parabola $y^2=4x$. At the point (1, 2) calculate the subtangent, subnormal, and the lengths of tangent and normal.
2. For any curve $y=f(x)$, prove that
$$\frac{(\text{length of normal})^2}{(\text{length of tangent})^2}=\frac{\text{subnormal}}{\text{subtangent}}$$
3. Show that the subnormal in the hyperbola $x^2-x^2=a^2$ at any point is equal to the abscissa of the point.
4. Find the subnormal to the curve $y^2=x^3$ at the point (4, 8).
5. In the curve $y=b\sin\frac{x}{a}$, the subtangent is $a\tan\frac{x}{a}$, the subnormal is $\frac{b^2}{2a}.\sin\frac{2x}{a}$, and the normal is $b\sin\frac{x}{a}.\sqrt{\left(1+\frac{b^2}{a^2}\cos\frac{x}{a}\right)}$.
6. Find the length of the tangent to the curve $y=x^2-6x$ at (2, –8) from the point of tangency to the point where it cuts the x-axis.
7. Show that in the curve $by^2=(x+a)^3$ the square of the subtangent varies as the subnormal.
8. Show that in the curve $y=be^{-a/x}$ the subtangent varies as the square of the abscissa.

Practice Problems F

9. Find the lengths of tangent and normal, the subtangent and subnormal for the cycloid x=a ($\theta - \sin\theta$), y=a($1-\cos\theta$) at the point where $\theta = \frac{\pi}{2}$.

10. Find the lengths of the tangent, normal, subtangent and subnormal to the astroid $x = a\sin^3 t$, $y = a\cos^3 t$ at an arbitrary point.

11. Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

12. Show that in the rectangular hyperbola $x^2 - y^2 = a^2$ the length of the normal at any point is equal to the radius vector of this point.

13. Show that the subtangents of the ellipse and the $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ at points with the same abscissas are equal. What procedure of construction of the tangent to the ellipse follows from this?

14. For the curve $x^{m+n} = a^{m-n} y^{2n}$, where a is a positive constant and m, n are positive integers, prove that the mth power of subtangent varies as nth power of subnormal.

15. A curve is given by the equations $x = at^2$, $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P and Q. Show that the locus of the point of intersection of the tangents at P and Q is $4y^2 = 3ax - a^2$.

16. Consider the curve $y = c\,e^{\frac{x}{a}}$. Prove that the subtangent is constant and the tangent at (x_1, y_1) on the curve intersects the x-axis at the distance of $(x_1 - a)$ from the origin. Also find the equation of normal to the curve where the curve cut y-axes.

17. In the curve $x = a\left(\cos t + \log\tan\frac{1}{2}t\right)$, y = a sin t, show that the portion of the tangent between the point of contact and the x-axis is of constant length.

18. Prove that the perpendicular drawn from the foot of the ordinate to the tangent of a curve is

$$y + \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

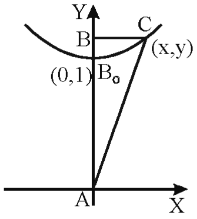
Hence show that in the curve $y = \frac{e^x + e^{-x}}{2}$, this perpendicular is constant.

19. Show that the subnormal at any point of the curve y = x ln (cx) (c is an arbitary constant) is the fourth proportional for the abscissa, ordinate and the sum of the abscissa and the ordinate of this point.

Target Problems for JEE Advanced

Problem 1: A variable triangle ABC in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin & the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Solution: $A = \frac{xy}{2} = \frac{x}{2}\left(1 + \frac{7x^2}{36}\right)$



$$\frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}x^2\right)\frac{dx}{dt}$$

At $t = \frac{7}{2}$, $y = 2 \cdot \frac{7}{2} = 7 \Rightarrow AB = 8$

When y = 8 , x = 6

$$\Rightarrow \frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}3.6\right)\frac{dx}{dt} = 11 \cdot \frac{dx}{dt}$$

$$\text{Also } \frac{dy}{dt} = 2 = \frac{14x}{36}\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{36}{7x} = \frac{6}{7}$$

$$\Rightarrow \frac{dA}{dt} = 11 \cdot \frac{6}{7} = \frac{66}{7}\ \text{cm}^2/\text{sec}.$$

Problem 2: Find the approximate value of $(33)^{1/5}$.

Solution: Let $f(x) = x^{1/5}$ ...(1)

$$\therefore \quad f'(x) = \frac{1}{5}x^{3/5-1} = \frac{1}{5}x^{-4/5} \qquad ...(2)$$

Now f(a + h) = f(a) + hf ′(a) [approx.]

$\therefore \quad (a + h)^{1/5} = a^{1/5} + h.\ \frac{1}{5}(a)^{-4/5}$ ...(3)

Putting a = 32 and h = 1 in (3), we get

$$(33)^{1/5} = (32)^{1/5} + 1\ .\ \frac{1}{5}(32)^{-4/5} = (2^5)^{1/5} + \frac{1}{5}(2^5)^{-4/5}$$

$$= 2 + \frac{1}{5}(2^{-4}) = 2 + \frac{1}{5} \times \frac{1}{16}\ .$$

$$= 2 + \frac{1}{50} = 2.0125 \text{ (approx.)}.$$

Problem 3: In an acute triangle ABC if sides a, b be constants and the base angles A and B vary, show that

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

Solution: $\frac{a}{\sin A} = \frac{b}{\sin B}$

or $\quad$ b sin A = a sin B

$\quad$ b cos A dA = a cos B dB

$$\frac{dA}{a\cos B} = \frac{dB}{b\cos A}$$

$$\Rightarrow \quad \frac{dA}{a\sqrt{1 - \sin^2 B}} = \frac{dB}{b\sqrt{1 - \sin^2 A}}$$

$$\frac{dA}{a\sqrt{1 - \frac{b^2 \sin^2 A}{a^2}}} = \frac{dB}{b\sqrt{1 - \frac{a^2 \sin^2 B}{b^2}}}\ ;$$

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

Problem 4: A certain container is modeled by a right circular cylinder whose height is twice the radius of the base. The radius is measured to be 17.3 cm, with a maximum measurement error of 0.02 cm. Estimate the corresponding propagated error, the relative error, and the percentage error when calculating the surface area S.

Solution: We have

$$S = 2\pi r 2r + \pi r^2 + \pi r^2 = 6\ \pi r^2$$

where r is the radius of the cylinder's base. Then the approximate propagated error is

$\Delta S \approx S'(r)\Delta r = 12\pi r \Delta r = 12\pi\ (17.3)\ (\pm 0.02) \approx \pm 13.04$

Thus, the maximum error in the measurement of the surface area is about 13.04 cm². Is this a large or a small error ? The relative error is found by computing the ratio

$$\frac{\Delta S}{S} = \frac{12\pi r \Delta r}{6\pi r^2} = 2r^{-1}\Delta r$$

$$= 2(17.3)^{-1}(\pm 0.02) \approx \pm 0.0023$$

This tells us that the maximum error of approximately 13.04 is fairly small relative to the surface area S. The corresponding percentage error is found by

$$100\left(\frac{\Delta S}{S}\right)\% = 100(\pm\ 0.0023)\% = \pm\ 0.23\%$$

This means that the percentage error is about ± 0.23%.

Problem 5: A curve in the plane is defined by the parametric equations $x = e^{2t} + 2e^{-t}$ and $y = e^{2t} + e^t$. Find an equation for the line tangent to the curve at the point t = *l*n 2

Solution: when $t = \ell n\ 2\ \ x = e^{\ell n\ 4} + 2\ e^{-\ell n\ 2} = 4 + 1 = 5$

and $\quad y = e^{\ln 4} + e^{\ln 2} = 6$

$\therefore \quad$ The point is (5, 6)

Now, $y = e^{2t} + e^t \quad \Rightarrow \quad \frac{dy}{dt} = 2e^{2t} + e^t$

$\quad x = e^{2t} + 2e^{-t} \Rightarrow \quad \frac{dx}{dt} = 2(e^{2t} - e^{-t})$

$$\frac{dy}{dx} = \frac{2e^{2t} + e^t}{2(e^{2t} - e^{-t})} = \frac{2(4) + 2}{2\left(4 - (1/2)\right)} = \frac{10}{7}\ .$$

$\therefore \quad$ The equation of tangent is $y - 6 = \frac{10}{7}(x - 5) \quad \Rightarrow$

$7y - 42 = 10x - 50 \Rightarrow 10x - 7y = 8.$

Problem 6: Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0.

Solution: $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

When x = 0, y = 1

Now $y = e^{y\ \ln(1+x)} + \sin^{-1}(\sin^2 x)$

$$\frac{dy}{dx} = (1 + x)^y\left[\frac{y}{1+x} + \ln(1+x)\frac{dy}{dx}\right] + \frac{\sin 2x}{\sqrt{1 - \sin^4 x}}$$

$$\frac{dy}{dx} = 1 \text{ at } (0,\ 1).$$

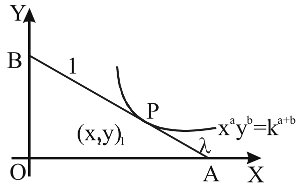
$\therefore \quad$ slope of normal = – 1

Hence, the equation of the normal is y – 1 = – 1(x – 0)

$\quad$ x + y – 1 = 0.

Problem 7: In the curve $x^a y^b = k^{a+b}$, (a b > 0) prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in constant ratio.

Solution: Let $P(x_1,\ y_1)$ be the point of contact of the tangent.



Here, $x^a y^b = k^{a+b}$

∴ a log x + b log y = (a + b) log k.

Differentiating, $\frac{a}{x}+\frac{b}{y}\frac{dy}{dx}=0$

or $\frac{dy}{dx}=-\frac{ay}{bx}$ ∴ $\left(\frac{dy}{dx}\right)_{x_1,y_1}=\frac{-ay_1}{bx_1}$

∴ The equation of the tangent at $P(x_1, y_1)$ is

$$y-y_1=\left(\frac{dy}{dx}\right)_{x_1,y_1}.(x-x_1)$$

or $y-y_1=\frac{-ay_1}{bx_1}(x-x_1)$ ...(1)

Solving with y = 0, $-y_1=\frac{-ay_1}{bx_1}(x-x_1)$

or $bx_1=a(x-x_1)$, ∴ $x=\frac{(a+b)x_1}{a}$

Solving (1) with x = 0, $y-y_1=\frac{-ay_1}{bx_1}(-x_1)$

or $y=y_1+\frac{ay_1}{b}=\frac{(a+b)y_1}{b}$

∴ $A=\left(\frac{a+b}{a}x_1, 0\right)$ and $B=\left(0, \frac{a+b}{b}y_1\right)$

Let P divide AB in the ratio λ : 1

Then $P=\left(\frac{\lambda.0+1.\frac{a+b}{b}x_1}{\lambda+1}, \frac{\lambda.\frac{a+b}{b}y_1+1.0}{\lambda+1}\right)$

$$=\left(\frac{a+b}{a(\lambda+1)}x_1, \frac{\lambda(a+b)}{b(\lambda+1)}y_1\right)$$

∴ $x_1=\frac{a+b}{a(\lambda+1)}x_1$ and $\frac{\lambda(a+b)}{b(\lambda+1)}y_1$

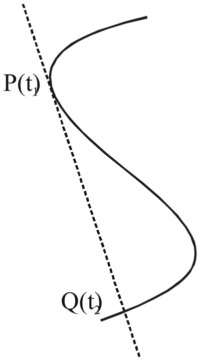
⇒ $a(\lambda+1)=a+b$ and $b(\lambda+1)=\lambda(a+b)$

⇒ $\lambda=\frac{b}{a}$ and $b=\lambda a$, i.e., $\lambda=\frac{b}{a}$

∴ P divides AB in the constant ratio b : a.

Problem 8: Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.

Solution: Let the tangent to the curve at $P(t_1)$, i.e. $(4t_1^2 + 3, 8t_1^3 - 1)$ be normal to the curve at $Q(t_2)$, i.e. $(4t_2^2 + 3, 8t_2^3 - 1)$.



The equation of the tangent at $P(t_1)$ is

$$y-(8t_1^3-1)=\left(\frac{dy}{dx}\right)_{t_1}.\{x-(4t_1^2+3)\}$$

or $y-(8t_1^3-1)=\left(\frac{dy/dt}{dx/dt}\right)_{t_1}.\{x=(4t_{12}+3)\}$

or $y-(8t_1^3-1)=\frac{24t_1^2}{8t_1}.\{x-(4t_1^2+3)\}$

or $y-(8t_1^3-1)=3t_1\{x-(4t_1^2+3)\}$ ...(1)

Clearly, slope of the tangent at 't_1'

= slope of the normal at 't_2'

and (1) passes through the point 't_2'.

∴ $\left(\frac{dy}{dx}\right)_{t_1}=\frac{-1}{\left(\frac{dy}{dx}\right)_{t_2}}$, .i.e., $3t_1=\frac{-1}{3t_2}$ ...(2)

and $(8t_2^3-1)-(8t_1^3-1)=3t_1\{(4t_2^2+3)+(4t_1^2+3)\}$

⇒ $8(t_2^3-t_1^3)=3t_1 . 4(t_2^2-t_1^2)$

⇒ $2(t_2^2+t_2t_1+t_1^2)\,3t_1(t_2+t_1)$

⇒ $2t_2^2+t_1t_2+t_1^2$, i.e., $2\left(\frac{-1}{9t_1}\right)^2=\frac{-1}{9}+t_1^2$

{using (2)}

⇒ $2=-9t_1^2+81t_1^4$

∴ $81t_1^4-9t_1^2-2=0$

or $(9t_1^2+1)(9t_1^2-2)=0$

∴ $9t_1^2-2=0$; ∴ $t_1=\pm\frac{\sqrt{2}}{3}$

∴ Putting in (1), the equations of the required lines are

$$y-\left(8.\frac{\pm2\sqrt{2}}{27}-1\right)=3\left(\pm\frac{\sqrt{2}}{3}\right)\left\{x-\left(4.\frac{2}{9}+3\right)\right\}$$

or $y\mp\frac{16\sqrt{2}}{27}+1\pm\sqrt{2}\left(x-\frac{35}{9}\right)$

or $27(y+1)\mp16\sqrt{2}=\pm\sqrt{2}(27x-105)$.

Problem 9: Consider the curve $y = \tan^{-1}x$ and a point $A\left(1, \frac{\pi}{4}\right)$ on it. If the variable point $P_i\,(x_i, y_i)$ moves on the curve for i = 1, 2, 3, ... n ($n \in N$) such that $y_r = \sum_{m=1}^{r} \tan^{-1}\left(\frac{1}{2m^2}\right)$ and B(x, y) be the limiting position of variable point P_n as $n \to \infty$, then find the value of reciprocal of the slope of AB.

Solution: $y = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\left(\frac{1}{2m^2}\right)$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\left\{\frac{2}{1+(2m+1)(2m-1)}\right\}$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \{\tan^{-1}(2m+1) - \tan^{-1}(2m-1)\}$$

$$= \lim_{n\to\infty} \{(\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3)$$
$$+ (\tan^{-1}7 - \tan^{-1}5)\ldots.. + \tan^{-1}(2n+1) - \tan^{-1}(2n-1)\}$$

$$= \lim_{n\to\infty} \{\tan^{-1}(2n+1) - \tan^{-1}1\} \to \frac{\pi}{4}$$

$\because$ $B \to \left(1, \frac{\pi}{4}\right)$ i.e. coordinates of B approach, towards those of 'A'.

$\therefore$ Chord AB approaches to be the tangent to y = f(x) at A

$\therefore$ $(\text{slope of AB})^{-1} = \left[\frac{d}{dx}\tan^{-1}x\right]^{-1}_{\text{at } x=1}$

$$= (1 + x^2)_{x=1} = 2.$$

Problem 10: If the normals drawn to the curve $y = x^2 - x + 1$ at the points A, B and C on the curve are concurrent at the point P(7/2, 9/2) then compute the sum of the slopes of the three normals . Also find their equations and the coordinates of the feet of the normals onto the curve.

Solution: Slope of the normal $m = \frac{1}{2x_1 - 1}$

$\Rightarrow$ $x_1 = \frac{m-1}{2m}$ $\Rightarrow$ $y_1 = \frac{3m^2+1}{4m^2}$;

The equation of the normal in terms of slope of the normal is $y = mx + \frac{5m^2 - 2m^3 + 1}{4m^2}$.

It passes through (7/2, 9/2)

$\Rightarrow$ $12m^3 - 13m^2 + 1 = 0$

$\Rightarrow$ sum of the slopes of the three normals = 13/12 .

Also $(m-1)(3m-1)(4m+1) = 0$

$\Rightarrow$ $m_1 = 1$; $m_2 = 1/3$; $m_3 = -1/4$

$\Rightarrow$ The equation of normals are $x - y + 1 = 0$; $x - 3y + 10 = 0$ and $2x + 8y - 43 = 0$

The feet of normals are A (0, 1) ; B (–1, 3);
C (5/2, 19/4)

Problem 11: Show that the curves

$$\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1 \text{ and } \frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$$

intersect orthogonally.

Solution: Given $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ ...(1)

and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$ ...(2)

Subtracting (2) from (1), we get

$$x^2\left(\frac{1}{a^2+k_1} + \frac{1}{a^2+k_2}\right) + y^2\left(\frac{1}{b^2+k_1} + \frac{1}{b^2+k_2}\right) = 0$$

$$\Rightarrow x^2\left(\frac{k_2-k_1}{(a^2+k_1)(a^2+k_2)}\right) + y^2\left(\frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)}\right) = 0$$

$$\therefore \quad \frac{x^2}{y^2} = -\frac{(x^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_1)} \quad \ldots(3)$$

Now from (1), $\frac{2x}{(a^2+k_1)} + \frac{2y}{(b^2+k_1)}\frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{x(b^2+k_1)}{y(a^2+k_1)} = m_1 \text{ (say)}$$

Similarly from (2),

$$\frac{dy}{dx} = -\frac{x(b^2+k_2)}{y(a^2+k_2)} = m_2 \text{ (say)}$$

$$m_1\, m_2 = \frac{x^2(b^2+k_1)(b^2+k_2)}{y(a^2+k_1)(a^2+k_2)} \quad \text{[From (3)]}$$

Hence given curves intersect orthogonally.

Problem 12: Prove that the curves y = f(x), f(x) > 0 and y = f(x) sin x, where f(x) is a differentiable function, have common tangent at common points.

Solution: The points of intersection of two curves are given by f(x) = f(x) sin x

$\Rightarrow$ sin x = 1 $\Rightarrow$ $x = (4n+1)\pi/2$

The slope of the tangent to y = f(x)
at $x = (4n+1)\pi/2$ is $f'((4n+1)\pi/2)$

Slope of the tangent to $y = f(x) \sin x$ is

$f'(x) \sin x + \cos x\, f(x)$.

So the slope of tangent at $x = (4n + 1)\pi/2$

is $f'((4n + 1)\pi/2).\ 1 + 0 = f'((4n + 1)\pi/2)$.

Here the slopes of the tangents are same at common points. Hence the two curves have the same tangent at their points of intersection.

Problem 13: Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal.

Solution: We need to prove that

$$\left(\frac{y}{(dy/dx)}\right)^2 = k \cdot y\left(\frac{dy}{dx}\right)$$

or $$y = k\left(\frac{dy}{dx}\right)^3$$

Now $2by\dfrac{dy}{dx} = 3(x + a)^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3(x+a)^2}{2by}$$

Hence, $$\frac{y}{(dy/dx)^3} = \frac{y\,8b^3y^3}{2y(x+a)^6}$$

$$= \frac{8b^3}{2y}\left(\frac{y^2}{(x+a)^3}\right)^2 = \frac{8b^3}{27}\cdot\frac{1}{b^2} = \frac{8b}{27}.$$

Problem 14: For the curve $y = a\, \ell n\, (x^2 - a^2)$ show that sum of length of tangent and the subtangent at any point is proportional to coordinates of point of tangency.

Solution: Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{2ax_1}{x_1^2 - a^2}$$

Length of tangent + subtangent

$$= y_1\sqrt{1 + \frac{1}{m^2}} + \frac{y_1}{m}$$

$$= y_1\sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= y_1\frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= \frac{y_1(x_1^2 + a^2)}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= \frac{y_1(x_1^2)}{2ax_1} = \frac{x_1\,y_1}{2a}.$$

Things to Remember

1. If f is differentiable at x, we define dy, called the differential of y, by the formula $dy = f'(x)dx$
2. The approximate value of a function f at x in the small neighbourhood of the point x_0 is given by
   $f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)$
3. The absolute error $\Delta f = f(x + \Delta x) - f(x)$

   The relative error is $\dfrac{\Delta f}{f} \approx \dfrac{df}{f}$

   The percentage error is $100\left(\dfrac{\Delta f}{f}\right)\%$
4. The tangent line to the graph of f at the point P(a, f(a)) is
   - (i) the line on P with slope $f'(a)$ if $f'(a)$ exists ,
   - (ii) the line $x = a$, if $\lim\limits_{x\to a}\left|\dfrac{f(x) - f(a)}{x - a}\right| = \infty$.

   $y - f(a) = f'(a)\,(x - a)$
   is an equation of the tangent line to the graph of f at the point P(a, f(a)).

   $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$ is the equation of normal.
5. The intercept that the tangent cuts off from the x-axis is $x - y\dfrac{dx}{dy}$.

   The intercept that the tangent cuts off from the y–axis is $y - x\dfrac{dy}{dx}$.
6. (i) If the tangent at any point on the curve is equally inclined to both the axes, then $dy/dx = \pm 1$.
   - (ii) If the tangent at any point on the curve cuts both the axes at equal distances from origin, then $dy/dx = \pm 1$ or it passes through the origin.
   - (iii) If the tangent at any point makes equal intercepts on the coordinate axes, then $dy/dx = -1$ or it passes through the origin.
7. The equation of tangent to the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is
   $axx_1 + h(xy_1 + yx_1) + byy_1 + g(x+x_1) + f(y+y_1) + c = 0$.
   Also, the equation of normal at (x_1, y_1) is
   $$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

8. Some common parametric coordinates :

(a) $x^{2/3}+y^{2/3}=a^{2/3} \Rightarrow x = a\cos^3\theta,\ y = a\sin^3\theta$.

(b) $\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow x = a\cos^4\theta,\ y = a\sin^4\theta$.

(c) $\dfrac{x^n}{a^n}+\dfrac{y^n}{b^n}=1$

$\Rightarrow x = a(\sin\theta)^{2/n},\ y = b(\sin\theta)^{2/n}$.

(d) $c^2(x^2+y^2)=x^2y^2 \Rightarrow x = c\sec\theta$ and $y = c\,\text{cosec}\theta$.

(e) $ay^2 = x^3, \Rightarrow x = at^2$ and $y = at^3$.

9. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.

10. Let two curves be $y = f(x)$ and $y = g(x)$ and let the point of intersection be $P(x_1, y_1)$.

and let $m_1 = \dfrac{dy}{dx} = f'(x_1)$ and $m_2 = \dfrac{dy}{dx} = g'(x_1)$

If angle between the two cuves is θ, then

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{f'(x_1)-g'(x_1)}{1+f'(x_1)g'(x_1)}\right|$$

11. (i) Condition for orthogonal intersection of two curves: If two curves cut each other orthogonally (at right angles), then $\theta = \dfrac{\pi}{2}$.

$\therefore m_1m_2 = -1$

(ii) Condition for two curves to touch each other: If the two curves touch each other, then $\theta = 0$.

$\therefore \tan\theta = 0 \Rightarrow m_1 = m_2$

12. The shortest distance between two non-intersecting curves is found along the common normal to the two curves. In fact, if the two curves also have the largest distance between them, then it is also found along the common normal to the two curves.

13. (i) Length of Tangent $= |y_1\,\text{cosec}\theta| = \left|y_1\sqrt{1+\left(\dfrac{dx}{dy}\right)_P^2}\right|$

(ii) Length of Normal $= |y_1\sec\theta| = \left|y_1\sqrt{1+\left(\dfrac{dy}{dx}\right)_P^2}\right|$

(iii) Subtangent $= |y_1\cot\theta| = \left|y_1\left(\dfrac{dx}{dy}\right)_P\right|$

(iv) Subnormal $= |y_1\tan\theta| = \left|y_1\left(\dfrac{dy}{dx}\right)_P\right|$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. Let $y = f(x)$ be an even function. If $f'(2) = -\sqrt{3}$ then the inclination of the tangent to the curve $y = f(x)$ at $x = -2$ with x-axis is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$

(C) $\dfrac{2\pi}{3}$ (D) $\dfrac{5\pi}{6}$

2. The equation of normal to $x + y = x^y$, where it cuts x-axis, is given by :

(A) $x + y = 1$ (B) $x - y - 1 = 0$

(C) $x - y + 1 = 0$ (D) None of these

3. A particle moving on a curve has the position at time t given by $x = f'(t)\sin t + f''(t)\cos t$, $y = f'(t)\cos t - f''(t)\sin t$, where f is a thrice differentiable function. Then the velocity of the particle at time t is

(A) $f'(t) + f''(t)$ (B) $f'(t) - f'''(t)$

(C) $f'(t) + f'''(t)$ (D) $f'(t) - f''(t)$

4. Let x be the length of one of the equal sides of an isosceles triangle, and let θ be the angle between them. If x is increasing at the rate (1/12)m/h, and θ is increasing at the rate of $\pi/180$ radians/h, then the rate in m^2/hr at which the area of the triangle is increasing when x =12m and $\theta = \pi/4$, is

(A) $2^{1/2}\left(1+\dfrac{2\pi}{5}\right)$ (B) $\dfrac{73}{2}\sqrt{2}$

(C) $\dfrac{3^{1/2}}{2}+\dfrac{\pi}{5}$ (D) $2^{1/2}\left(\dfrac{1}{2}+\dfrac{\pi}{5}\right)$

5. The number of tangents to the curve $x^{3/2} + y^{3/2} = 2a^{3/2}$, $a > 0$, which are equally inclined to the axes, is

(A) 2 (B) 1

(C) 0 (D) 4

6. Let $f(x) = \begin{cases} -x^2, & \text{for } x < 0 \\ x^2+8, & \text{for } x \ge 0 \end{cases}$. Then the x-intercept of the line that is tangent to the graph of f(x) at $x = -2$ is

(A) zero
(B) –1
(C) –3
(D) –4

7. The angle between the normals drawn to the curves $x^2 - y^2 = 8$ and $9x^2 + 25y^2 = 225$ at the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ is

(A) 0
(B) $\frac{\pi}{2}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{4}$

8. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y-axes, respectively, then the value of a^2b is

(A) $27c^3$
(B) $\frac{4}{27}c^3$
(C) $\frac{27}{4}c^3$
(D) $\frac{4}{9}c^3$

9. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin, intersect at an angle equal to

(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

10. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant, a normal is drawn. The normal intersects the y-axis at the point

(0, b). If $\lim\limits_{a \to 0} b = \frac{1}{2}$, then n equals

(A) 1
(B) 3
(C) 2
(D) 4

11. If the tangent at any point on the curve $x^4 + y^4 = a^4$ cuts off intercepts 'p' and 'q' on the coordinate axes, then the value of $p^{-4/3} + q^{-4/3}$ is

(A) $a^{-4/3}$
(B) $a^{-1/2}$
(C) $a^{1/2}$
(D) None

12. A flu epidemic hits Pune. Health officers estimate that the number of persons sick with the flu at time t (measured in days from the beginning of the epidemic) is estimated by $P(t) = 60t^2 - t^3$, $0 \le t \le 40$. At what time t is the flu spreading at the rate of 900 people per day ?

(A) t = 10 and 30
(B) t = 10 only
(C) t = 38
(D) t = 25 and 32

13. The minimum distance between the curve $y^2 = x^3$ and the curve $9x^2 + 9y^2 - 30y + 9 = 0$ is

(A) $\frac{\sqrt{13}}{3} + 1$
(B) $\left|\frac{\sqrt{13} - 4}{3}\right|$
(C) $\sqrt{13} + 1$
(D) $\sqrt{13} - 1$

14. Let the tangent to the cubic curve $x^3 + y^3 = a^3$ at $P(x_1, y_1)$ meet curve again at Q(h, k). Put $A = \frac{h}{x_1}$, $B = \frac{k}{y_1}$, $A \neq B$ then $x_1^{\,3}$ must be equal to

(A) $\frac{a^3(1-B)}{A-B}$
(B) $\frac{a^3(1-B)}{A-B}$
(C) $\frac{a^3(1-B)}{A+B}$
(D) None of these

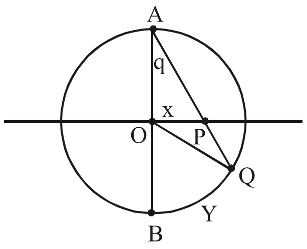
15. The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1 cm the altitude is 6 cm. When the radius is 6 cm, the volume is increasing at the rate of 1 cu cm/sec. When the radius is 36 cm, the volume is increasing at a rate of n cu cm/sec. The value of 'n' is equal to

(A) 12
(B) 22
(C) 30
(D) 33

16. Tangent and normal to the curve $y = 2 \sin x + \sin 2x$ are drawn at $P\left(x = \frac{\pi}{3}\right)$. The area of the quadrilateral formed by the tangent, the normal and the coordinate axes is

(A) $\frac{\pi\sqrt{3}}{2}$
(B) $\frac{\pi}{2}$
(C) $\frac{\sqrt{3}}{2}$
(D) None

17. The circle shown in figure has radius of 10 cm. If point P moves towards right with a speed of 5cm/sec, then the rate at which the length of arc BQ is increasing at ;the instant when ∠BAQ is 45°



(A) 10cm/sec
(B) $\frac{5}{\sqrt{2}}$ cm/sec
(C) $\frac{10}{\sqrt{2}}$ cm/sec
(D) 5cm/sec

18. Minimum distance between two points P and Q, where P lies on the parabola $y^2 - x + 2 = 0$ and Q lies on the parabola $x^2 - y + 2 = 0$ is

(A) $7\sqrt{2}$ unit
(B) 4 unit
(C) $\frac{7}{2\sqrt{2}}$ unit
(D) none of these

19. If normal drawn at any point P of parabola $y^2 = 4x$, meets the curve again at Q, then the least distance of Q from origin is equal to

(A) $6\sqrt{3}$ unit (B) $4\sqrt{6}$ unit

(C) $9\sqrt{6}$ unit (D) none of these

20. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

(A) –1 (B) 3

(C) –3 (D) 1

21. The point at which the normal to the curve $y = \ln x$ is perpendicular to the chord joining the points (1, 0) and (e, 1), is

(A) $[e - 1, \ln(e + 1)]$ (B) $[e - 1, \ln(e - 1)]$

(C) $[e + 1, \ln(e + 1)]$ (D) None of these

22. If $px^2 + qx + r = 0$, $p, q, r \in R$ has no real zero and the line $y + 2 = 0$ is tangent to $f(x) = px^2 + qx + r$ then

(A) $p + q + r > 0$ (B) $p - q + r > 0$

(C) $r < 0$ (D) None of these

23. The number of points with integral coordinates where tangent exists in the curve $y = \sin^{-1} 2x\sqrt{1-x^2}$ is

(A) 0 (B) 1

(C) 3 (D) None

24. The number of other points at which the tangent at origin to the curve $y = \tan^{-1}\dfrac{2x}{1-x^2}$ meets the curve is

(A) 0 (B) 2

(C) 3 (D) None

25. If the normal to $y = f(x)$ at $x = 0$ is $3x - y + 3 = 0$, then the value of $x^2\{f(x^2) - 5f(4x^2) + 4f(7x^2)\}^{-1}$ is

(A) –1/6 (B) 1/3

(C) 1/6 (D) –1/3

26. The equation of the common tangent to the curve $y = 6 - x - x^2$ and $xy = x + 3$ is

(A) $3x + y = 7$ (B) $3x - y = 7$

(C) $3x + y + 7 = 0$ (D) None of these

27. The tangent to the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$ at the point whose coordinates are

(A) (–1, 5) (B) (1, 2)

(C) (2, 7) (D) (3, 16)

28. The number of tangents to the curve, $y^2 - 2x^2 - 4y + 8 = 0$ that pass through (1, 2), is

(A) 3 (B) 1

(C) 2 (D) 6

29. The curve $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$ touches the line $\dfrac{x}{a} + \dfrac{y}{b} = 2$ at the point (a, b) for n =

(A) 1 (B) 2

(C) 3 (D) all non-zero values of n

30. If the curves $y = 1 - ax^2$ and $y = x^2$ are orthogonal then a is equal to

(A) 3 (B) 1

(C) 1/3 (D) None

31. The minimum slope of the curve $y = x^3 + 3x^2 + 3x - 1$ is

(A) 0 (B) – 2

(C) – 1 (D) None

32. Total number of parallel tangents of $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$ is

(A) 2 (B) 3

(C) 4 (D) None

33. The curves $x^2 - 4y^2 + c = 0$ and $y^2 = 4x$ will intersect orthogonally for

(A) $c \in (0, 16)$ (B) $c \in (-3, 4)$

(C) $c \in (3, 4)$ (D) None of these

34. Two points with abscissa $x_1 = a$, $x_2 = 3a$, $a \neq 0$ are given on the parabola $y = x^2$. A secant is drawn through these points. The coordinates of the point on the parabola where the tangent is parallel to the secant is

(A) $(2a, 4a^2)$ (B) $(-2a, 4a^2)$

(C) (a, a^2) (D) $(3a, 9a^2)$

35. The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ drawn at the point $x = 0$ is -

(A) $1/\sqrt{5}$ (B) $2/\sqrt{5}$

(C) $-1/\sqrt{5}$ (D) $2/\sqrt{3}$

36. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

(A) $\pi/6$ (B) $\pi/4$

(C) $\pi/3$ (D) $\pi/2$

37. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. The value of dx/dt when x = 3 is

(A) 4 (B) 9/2

(C) $3\sqrt{3}/2$ (D) None

38. A point is moving along the curve $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate, is

(A) (–3, 3) (B) $(-\infty, \infty)$

(C) (–1, 1) (D) $(-\infty, -3) \cup (3, \infty)$

39. If $y = x \ln x + \sin(\pi \ln x)$ $\forall x \in [e, e^2]$, then which of the following lines is a tangent to curve for some $x \in [e, e^2]$ (for some c')

(A) e(y – 2x) + (x – y) = c'

(B) e (y + 2x) + (x – y) = c'

(C) $e(y + 2x) - (x + y) = c'$
(D) nothing can be said

40. ℓ_1 and ℓ_2 are the side lengths of two variable squares S_1 and S_2 respectively. If $\ell_1 = \ell_2 + \ell_2^3 + 6$ then rate of change of the area of S_2 with respect to rate of change of the area of S_1 and $\ell_2 = 1$ is equal to
(A) 3/4
(B) 4/3
(C) 3/2
(D) None

41. If the tangent at (1, 1) on $y^2 = x(2 - x^2)$ meets the curve again at P, the P is
(A) (4, 4)
(B) (–2, 3)
(C) (9/4, 3/8)
(D) none of these

42. If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is a normal also the curve $x^3 - y^2 = 0$, then $9m^2$ is
(A) 1
(B) 2
(C) 3
(D) none

43. If the curves $y = 1 - ax^2$ and $y = x^2$ are orthogonal then a is equal to
(A) 3
(B) 1
(C) 1/3
(D) none

44. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$ then
(A) $|b| > \frac{1}{2\sqrt{2}}$
(B) $|b| < \frac{1}{2\sqrt{2}}$
(C) $|b| > \frac{1}{\sqrt{2}}$
(D) $|b| < \frac{1}{\sqrt{2}}$

45. The minimum distance between the curve $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is equal to
(A) $\sqrt{21}$
(B) $\sqrt{26} - \sqrt{5}$
(C) $\sqrt{21} - \sqrt{5}$
(D) $\sqrt{28} - \sqrt{5}$

46. Tangent drawn to $y = ax^2 + bx + c$ at (5, 4) is parallel to x-axis. If $a \in [2, 4]$ then maximum value of c is
(A) 54
(B) 56
(C) 104
(D) 106

47. The set of values of 'a' for which $|\sin 2x| - |x| - a = 0$ does not have solution is given by
(A) $\left(\frac{3\sqrt{3} - \pi}{6}, \infty\right)$
(B) $\left(\frac{3\sqrt{3} + \pi}{6}, \infty\right)$
(C) $(1, \infty)$
(D) None of these

48. A particle describes an ellipse whose semiaxes are 4 m.and 3 m. with a constant speed of 1 mt/sec. The velocity of the foot of the perpendicular from the particle on the major axis, when the particle is a distance of 1 meter from the major axis is equal to
(A) 2/11 m/s
(B) 11/2 m/s
(C) $\sqrt{(2/11)}$ m/s
(D) none of these

49. An edge of a variable cube is increasing at the rate of 3 cm/sec. When the edge is 10 cm. long, the volume of the cube is increasing at the rate
(A) 27 cm³/sec
(B) 90 cm²/sec
(C) 900 cm³/sec
(D) none of these

50. Two points with abscissa $x_1 = a$, $x_2 = 3a$, $a \neq 0$ are given on the parabola $y = x^2$. A secant is drawn through these points. The co-ordinates of the point on the parabola where the tangent is parallel to the secant is
(A) $(2a, 4a^2)$
(B) $(-2a, 4a^2)$
(C) (a, a^2)
(D) $(3a, 9a^2)$

51. If $f(x) = x - \sin^2 x - a$, then $f(x) = 0$ has
(A) two positive roots if $a > 0$
(B) one positive root if $a < 0$
(C) two positive roots if $a < 0$
(D) one positive root if $a > 0$

52. The curves $x^2 - 4y^2 + c = 0$ and $y^2 = 4x$ will intersect orthogonally for
(A) $c \in (0, 16)$
(B) $c \in (-3, 4)$
(C) $c \in (3, 4)$
(D) None of these

53. Let $y = f(x)$ be a curve whose parametric equation is $x = t^2 + t + 1$, $y = t^2 - t + 1$; where $t > 0$. Total number of tangents that can be drawn to this curve from (1, 1) is equal to
(A) 1
(B) 2
(C) 3
(D) None

54. The value of x_2 for which value of the expression $(x_1 - x_2)^2 + \left(x_1 - 4 - \frac{x_2^2}{4}\right)^2$ is least, where $x_1, x_2 \in R^+$, is
(A) 16
(B) 2
(C) 4
(D) $\sqrt{2}$

55. Let $x = \cos^3 t$ and $y = \sin^2 t$ defines a curve where t is a parameter. This curve passes through the point $\left(\frac{1}{8}, \frac{3}{4}\right)$ at some $t = t_0 \in \left(0, \frac{\pi}{2}\right)$. The slope of the curve at that point is
(A) $\frac{4}{3}$
(B) $-\frac{4}{3}$
(C) $-\frac{4}{3\sqrt{3}}$
(D) $-\frac{\sqrt{3}}{3}$

56. The equal sides of an isosceles triangle with fixed base 'b' are decreasing at the rate of 3 cm/sec. The rate at which its area is decreasing when the two equal sides are equal to the base, is
(A) $\frac{b}{\sqrt{3}}$ sq. cm/sec
(B) $b\sqrt{3}$ sq. cm/sec
(C) 3b sq. cm/sec
(D) $\sqrt{3b}$ sq. cm/sec

57. If the curves $y = x^3 + ax$ and $y = bx^2 + c$ pass through the point $(-1, 0)$ and have a common tangent line at this point then the value of $(a + b + c^2)$ is

(A) 0
(B) 1
(C) – 3
(D) – 1

58. $x + y - 9 = 0$ is a normal to the parabola $y^2 - 2y - 8x + 17 = 0$ at

(A) (5, 4)
(B) (4, 5)
(C) (4, – 5)
(D) (– 4, 5)

59. A line L is perpendicular to the curve $y = \frac{x^2}{4} - 2$ at its point P and passes through $(1, -1)$. The coordinates of the point P are

(A) (2, –1)
(B) (6, 7)
(C) (0, –2)
(D) (4, 2)

60. Let f be a differentiable function with $f(2) = 3$ and $f'(2) = 5$, and let g be the function defined by $g(x) = x\,f(x)$. y-intercept of the tangent line to the graph of 'g' at point with abscissa 2, is

(A) 20
(B) 8
(C) – 20
(D) – 18

61. The equation of the line tangent to the curve $x \sin y + y \sin x = \pi$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, is

(A) $x + y = \pi$
(B) $x - y = 0$
(C) $2x - y = \pi/2$
(D) none

62. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(A) (1, 0)
(B) (1/e, 0)
(C) (e, 0)
(D) (2e, 0)

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

63. The point on the curve $xy^2 = 1$, which is nearest to the origin is

(A) $(2^{1/3}, 2^{1/6})$
(B) $(2^{-1/3}, 2^{1/6})$
(C) $(2^{-1/3}, -2^{1/6})$
(D) $(-2^{-1/3}, 2^{1/6})$

64. The value of parameter a so that the line $(3 - a)x + ay + (a^2 - 1) = 0$ is normal to the curve $xy = 1$, may lie in the interval

(A) $(-\infty, 0)$
(B) $(1, 3)$
(C) $(0, 3)$
(D) $(3, \infty)$

65. Line joining the point $(0, -3)$ and $(-5, 2)$ is a tangent to the curve $y = \frac{ax}{1+x}$ then

(A) $a = 1 \pm \sqrt{3}$
(B) $a = 2 + 2\sqrt{3}$
(C) $a = -1 \pm \sqrt{3}$
(D) $a = 2 - 2\sqrt{3}$

66. The points on the curve $y = x\sqrt{1-x^2}$, $-1 \le x \le 1$ at which the tangent line is vertical are

(A) (–1, 0)
(B) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$
(C) (1, 0)
(D) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

67. Which of the follwoing is an intersection of the curves $y = 2x^2 - x$ and $y = 2 - x^3$?

(A) $\tan^{-1}\frac{3}{109}$
(B) $\tan^{-1}\frac{1}{8}$
(C) $\tan^{-1}\frac{1}{4}$
(D) $\tan^{-1}\frac{3}{4}$

68. Coordinates of the point on the graph of the function $f(x) = \frac{x-1}{2-x}$ where the tangent lines are perpendicular to the lines $8x + 2y = 1$, are

(A) (1, 0)
(B) (3, – 2)
(C) $\left(0, -\frac{1}{2}\right)$
(D) $\left(4, -\frac{3}{2}\right)$

69. A nursery sells plants after 6 year of growth. Two seedlings A and B are planted each of height 5 inches whose growth rates are $\frac{dh_A}{dt} = 0.5t + 2$ and $\frac{dh_B}{dt} = t + 1$ where heights h_A and h_B are in cms and t is the time in years. Then

(A) the height of the plants are equal at t = 3 (in years)
(B) the height of the plants are equal at t = 4 (in years)
(C) when the plants are sold, their heights are 26 cms and 29 cms
(D) none of these

70. Coordinates of the point on the curve $x^2 + 2xy + 2y^2 = 45$ at which the slope of the tangent is – 2, is

(A) $(3\sqrt{5}, 0)$
(B) (–9, 3)
(C) (–3, 1)
(D) (9, – 3)

71. Which of the following is an angle of intersection of the curves $y = 2x^2 - x$ and $y = 2 - x^3$?

(A) $\tan^{-1}\frac{3}{109}$
(B) $\tan^{-1}\frac{1}{8}$
(C) $\tan^{-1}\frac{1}{4}$
(D) $\tan^{-1}\frac{3}{4}$

72. The points on the curve $f(x) = \dfrac{x}{1-x^2}$ where the tangent is inclined at an angle of $\dfrac{\pi}{4}$ to x-axis, are

(A) $(0, 0)$ (B) $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$

(C) $\left(-2, \dfrac{2}{3}\right)$ (D) $\left(-\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$

73. Let the parabolas $y = x(c - x)$ and $y = x^2 + ax + b$ touch each other at the point $(1, 0)$, then

(A) $a + b + c = 0$ (B) $a + b = 2$

(C) $b - c = 1$ (D) $a + c = -2$

74. The angle between the tangent at any point P and the line joining P to the origin, where P is a point on the curve $\ell n(x^2 + y^2) = c\tan^{-1}\dfrac{y}{x}$, c is a constant, is

(A) independent of x and y

(B) dependent on c

(C) independent of c but dependent on x

(D) none of these

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

75. **Assertion (A) :** For the curve $\alpha y^2 = (x + \beta)^3$, square of the subtangent varies as the subnormal.

Reason (R) : Subnormal = |(ordinate)(slope of the tangent to the curve)| and subtangent = |(ordinate) (reciprocal of slope of the tangent to the curve)|

76. **Assertion (A) :** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ meets the curve again at $x = -1$.

Reason (R) : When the equation of a tangent is solved with the curve, repeated roots are obtained at point of tangency.

77. **Assertion (A) :** $y = f(x)$ is a parabola, having its axis parallel to y-axis. If the line $y = x$ touches this parabola at $x = 1$, then $f''(1) + f'(0)$ is equal to 1

Reason (R) : If $f(x) = ax^2 + bx + c$, then $ax^2 + bx + c = x$ should have $x = 1$ as a repeated root.

78. **Assertion (A) :** If at any point on a curve the subtangent and subnormal are equal, then the length of the normal is equal to $\sqrt{2}$ ordinate.

Reason (R) : If subtangent is equal to subnormal, then $\dfrac{dy}{dx} = \pm 1$.

79. **Assertion (A) :** The curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point with abscissa $\dfrac{3}{4}$.

Reason (R) : Let the circle be $(x - 4)^2 + y^2 = c^2$: for tangency, $\dfrac{dy}{dx}$ must be equal for both the curves $2y\dfrac{dy}{dx} = 3x^2 = -2x + 8$

80. **Assertion (A) :** The points of contact of the vertical tangents to $x = 2 - 3\sin\theta$, $y = 3 + 2\cos\theta$ are $(-1, 3)$ and $(5, 3)$

Reason (R) : For vertical tangent, $\dfrac{dx}{d\theta} = 0$

Comprehension - 1

The sales S of a new product after it has been on the market for t years is $S = 30(1 - e^{kt})$

81. The value of the constant k if 5 units have been sold after 1 year is

(A) $\ell n\ 6/5$ (B) $\ell n\ 5/6$

(C) $\ell n\ 4/5$ (D) None

82. The number of units which will saturate the market is

(A) 30 (B) 31

(C) 150 (D) None

83. The number of units sold after 5 years is

(A) $30(1 - (4/5)^5)$ (B) $30(1 - (2/3)^5)$

(C) $30(1 - (5/6)^5)$ (D) None of these

Comprehension - 2

Consider the two quadratic polynomials

$$C_a : y = \frac{x^2}{4} - ax + a^2 + a - 2 \quad \text{and}$$

$$C : y = 2 - \frac{x^2}{4}$$

84. If the origin lies between the zeroes of the polynomial C_a then the number of integral value(s) of 'a' is

(A) 1 (B) 2

(C) 3 (D) more than 3

85. If 'a' varies then the equation of the locus of the vertex of C_a, is

(A) $x - 2y - 4 = 0$ (B) $2x - y - 4 = 0$

(C) $x - 2y + 4 = 0$ (D) $2x + y - 4 = 0$

86. For a = 3, if the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are common tangents to the graph of C_a and C then the value of $(m_1 + m_2)$ is equal to

(A) – 6 (B) – 3

(C) 1/2 (D) none

Comprehension - 3

P and Q be any two points on the curve $ay^2 = x^3$ such that PQ always passes through a fixed point $R(at^2, at^3)$

87. If P(at_1^2, at_1^3) and Q(at_2^2, at_2^3) are arbitrary points on the curve then

(A) $t = \frac{t_1 t_2}{t_1 + t_2}$ (B) $t = \frac{t_1 - t_2}{t_1 + t_2}$

(C) $t = -\frac{t_1 t_2}{t_1 + t_2}$ (D) None of these

88. If tangent at P and Q intersect at (x, y) then $t_1^2 + t_2^2 + t_1 t_2$ must be equal to

(A) $\frac{2x}{a}$ (B) $\frac{3x}{a}$

(C) $-\frac{3x}{a}$ (D) None

89. The locus of point of intersection of tangents at P and Q is the curve

(A) $(3tx + 2y)^2 = 2at^3y$

(B) $(3tx + 2y)^2 = -2at^3y$

(C) $(3tx + 2y)^2 = 3a^3t^3y$

(D) $(3tx + 2y)^2 = -3a^2t^3y$

Comprehension - 4

For a person at rest, the rate of air in take v(ℓ/s) during a respiratory cycle v = 0.85 sin πt/3, where the person has a respiratory cycle for which the rate of air intake is v = 1.75 sin πt/2.

90. The volume of air inhaled during one cycle at rest is

(A) 5.1/π (B) 2.55/π

(C) 0 (D) None

91. The difference in frequencies of the respiratory cycle during rest and exercise is

(A) 1/24 (B) 1/6

(C) 1/12 (D) None

92. The increase in the person's long capacity due to exercising is

(A) 0.95/π (B) 1.9/π

(C) 0 (D) None

Comprehension - 5

The parametric equation of given curve are x = a (2 cost + cos 2t), y = a (2 sin t – sin 2t).

93. The equation of normal at any point 't' is

(A) $x \cos\left(\frac{t}{2}\right) + y \sin\left(\frac{t}{2}\right) = 3a \cos\left(\frac{3t}{2}\right)$

(B) $x \cos\left(\frac{t}{2}\right) - y \sin\left(\frac{t}{2}\right) = 3a \cos\left(\frac{3t}{2}\right)$

(C) $x \cos\left(\frac{t}{2}\right) + y \sin\left(\frac{t}{2}\right) = 3a \sin\left(\frac{3t}{2}\right)$

(D) $x \cos\left(\frac{t}{2}\right) - y \sin\left(\frac{t}{2}\right) = 3a \sin\left(\frac{3t}{2}\right)$

94. The subnormal at any point 't' is

(A) |y tan t| (B) |y cot t|

(C) |y cot(t/2)| (D) |y tan (t/2)|

95. If length of perpendiculars from origin on tangent and normal at 't' are p and p_1, respectively, then the value of $9p^2 + p_1^2$ is equal to

(A) $9a^2$

(B) $9a^2 \sin^2(t/2)$

(C) $9a^2 \cos^2\left(\frac{3t}{2}\right)$

(D) a^2

MATCH THE COLUMNS FOR JEE ADVANCED

96.

	Column I	Column II
(A)	The sides of a triangle vary slightly in such a way that its circumradius remains constant, if $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = \lvert m \rvert$, then the values of m is	(P) 1
(B)	The subtangent to the curve $x^2y^2 = 16$ at the point (–2, 2) is |k|, then the value of k is	(Q)–1
(C)	The curve $y = 2e^{2x}$ intersects the y–axis at an angle $\cot^{-1}\lvert(8n–4)/3\rvert$, then the value of n is	(R) 2
(D)	The area of a triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is |2t + 1|/6 sq. units, then the value of t is	(S) –2

97.

	Column I	Column II
(A)	A circular plate is expanded by heat from radius 6 cm to 6.06cm. Approximate increase in the area is	(P) 5

(B)	If an edge of a cube increases by 2%, then percentage increase in the volume is	(Q) 0.72π
(C)	If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease being nonzero)	(R) 6
(D)	The rate of increase in the area of an equilateral triangle of side 30 cm, when each side increases at the rate of 0.1 cm/s is	(S) $\frac{3\sqrt{3}}{2}$

98.

	Column - I	Column - II
(A)	The area of the quadrilateral formed by the tangents from the point (4, 0) to the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ and the pair of radii through the points of contact of the tangents is	(P) 1
(B)	The number of points at which the function $f(x) = \frac{1}{\ell n\,|x|}$ is discontinuous is	(Q) 3
(C)	Let $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$ if $f(5) = 2$ and $f'(0) = 3$, then $f'(5) =$	(R) 6
(D)	If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3) =$	(S) 4

99.

	Column - I	Column - II
(A)	The slope of the curve $2y^2 = ax^2 + b$ at (1, –1) is –1, then	(P) $a - b = 2$
(B)	If (a, b) be the point on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes, then	(Q) $a - b = 7/2$
(C)	If the tangent at any point (1, 2) on the curve $y = ax^2 = bx + \frac{7}{2}$ be parallel to the normal at (–2, 2) on the curve $y = x^2 + 6x + 10$, then	(R) $a - b = 4/3$ (S) $a + \|b\| = \frac{20}{3}$
(D)	If a is the number of horizontal tangents and b is the number of vertical tangents to the curve $y^2 - 3xy + 2 = 0$, then	(T) $a + b = 2$

100.

	Column-I	Column-II
(A)	If the curves $y = 1 - \cos x$, $x \in (-\pi, \pi)$ and $y = \frac{\sqrt{3}}{2}\|x\| + a$ touch each other then the number of other possible values of a is equal to	(P) 0
(B)	If two curves $y^2 = 4a(x - b_1)$, and $x^2 = 4a(y - b_2)$, where a is a positive constant number and b_1 and b_2 are variables, touch each other then their point of contact lies on $xy = ka^2$, where k is equal to	(Q) 2
(C)	The point on othe parabola $y^2 = 4x$, which is nearest to the circle $x^2 + (y - 12)^2 = 1$ has the ordinate equal to	(R) 3
(D)	The ordinate of the point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is parallel to y-axis is/are	(S) 4 (T) –2

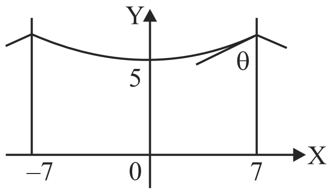
Review Exercises for JEE Advanced

1. A runner sprints around a circular track of radius 100 m at a constant speed of 7m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200m?

2. It is estimated that t years from now, the circulation of a local newspaper can be modeled by the formula
$C(t) = 100t^2 + 400t + 50t\,\ell nt$
(a) Find an expression for the rate at which the circulation will be changing with respect to time, t years from now.
(b) At what rate will the circulation be changing with respect to time 5 years from now ?
(c) By how much will the circulation actually change during the sixth year ?

3. A particle is moving along the curve whose equation is $\frac{xy^3}{1+y^2} = \frac{8}{5}$. Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point (1, 2)?
(a) At what rate is the y-coordinate of the point changing at that instant ?
(b) Is the particle rising or falling at that instant ?

4. A telephone line hangs between two poles 14 m apart in the shape of the curve
$y = 10(e^{\frac{x}{20}} + e^{\frac{-x}{20}}) - 15$, where x and y are measured in meters.
(a) Find the slope of this curve where it meets the right pole.
(b) Find the angle θ between the line and the pole.



5. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

6. Suppose that the position functions of two particles, P_1 and P_2, in motion along the same line are
$s_1 = \frac{1}{2}t^2 - t + 3$ and $s_2 = -\frac{1}{4}t^2 + t + 1$
respectively, for $t \geq 0$.
(a) Prove that P_1 and P_2 do not collide.
(b) How close can P_2 and P_2 get to one another ?
(c) During what intervals of time are they moving in opposite direction ?

7. If p(x) is the total value of the production when there are x workers in a plant, then the average productivity of the workforce at the plant is
$A(x) = \frac{p(x)}{x}$
(a) Find A'(x). Why does the company want to hire more workers if A'(x) > 0 ?
(b) Show that A'(x) > 0 if p'(x) is greater than the average productivity.

8. If the tangent to the curve xy + ax + by = 0 at (1, 1) is inclined at an angle $\tan^{-1}2$ with x-axis then find a and b.

9. Find the equation of a tangent and normal at x = 0 if they exist on the curve $y = x^{1/3}(1 - \cos x)$.

10. Find the equation of tangent and normal to the
curve $f(x) = \begin{cases} X - 2 & \text{if } x < 1 \\ x^2 - x - 1 & \text{if } x \geq 1 \end{cases}$ at x =1 if it exists.

11. If $lx + my = 1$ is a normal to the parabola $y^2 = 4ax$, prove that $al^3 + 2alm^2 = m^2$.

12. Show that the values of x for which the tangents to the curves y = xcos x, y = (sin x)/x are parallel to the axis of x, are roots of cot x = x, tan x = x respectively.

13. Find the equation of tangent to the curve
$y = \sin^{-1}\frac{2x}{1+x^2}$ at $x = \sqrt{3}$.

14. Show that there are exactly two tangent lines to the graph of $y = (x + 1)^3$ which pass through the origin, and find their equations.

15. Prove that in the ellipse x = a cos (θ + a), y = b cos (θ + β) the tangent at the point θ_1 will be parallel to the radius drawn to the point θ_2 if $\theta_1 - \theta_2 = \pm \pi/2$.

16. Prove that if b > 2a, three real normal can be drawn from the origin to the parabola $x^2 = 4a(y + b)$.

17. Find the intercepts made on the coordinate axes by the normal at any point of the rectangular hyperbola $x^2 - y^2 = a^2$; and prove that the difference of their squares is constant.

18. Prove that the equation of the tangent to the hyperbola x = kt, y = k/t is $x + t^2y = 2kt$.

19. The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

20. Two variable curves $y^2 = 4a(x - \lambda)$ and $x^2 = 4a(y - \mu)$ where a is a positive constant and λ, μ are parameters, touch each other. Find the locus of the point of contact.

21. Prove that the segment of the normal to the curve $x = 2a\sin t + a\sin t\cos^2 t$; $y = -a\cos^3 t$ contained between the coordinate axes is equal to 2a .

22. If $x\cos\alpha + y\sin\alpha = p$ touches the curve $(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1$, then prove that;
$(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$.

23. Find the condition for the line y = mx to cut at right angles the conic $x^2 + 4xy + 4y^2 = 1$. Hence find the equation of the axes of the conic.

24. Show that the curves $x^3 - 3xy^2 = -2$ and $3x^2y - y^3 = 2$ cut each other orthogonally.

25. Find the angle of intersection of the curves

$$y = \frac{x^2}{1+x^2} \text{ and } y = 2^{-x}.$$

26. Prove that the parabolas $x^2 = ay$ and $y^2 = 2ax$ intersect on the curve $x^3 + y^3 = 3axy$ and find the angles between each pair at the points of intersection.

27. At what point M (x,y) of the graph of the function $y = e^{-|x|}$ should a tangent be drawn for the area of the triangle bounded by that tangent and the coordinate axes be the greatest?

28. Prove that the curve $y = x^4 + 3x^2 + 2x$ does not meet the straight line $y = 2x - 1$ and find the distance between their nearest points.

29. A curve is defined parametrically by the equations $x = t^2$ and $y = t^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P and Q. If the locus of the point of intersection of the tangents at P and Q is $ay^2 = bx - 1$, then find the value of $(a + b)$.

30. If the tangent at the point (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2) then show that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.

31. Show that the distance from the origin of the normal at any point of the curve

$$x = a\,e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right) \;\&\; y = a\,e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$$

is twice the distance of the tangent at the point from the origin .

32. For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

33. Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

34. (i) Find the points on the curve $x = ye^y$ where the tangent line is vertical $(dx/dy = 0)$.

(ii) Is there a point on the curve where the tangent line is horizontal ?

(iii) Show that $x \to 0$ and $dy/dx \to -\infty$ as $y \to -\infty$

(iv) Show that $y \to \infty$ and $dy/dx \to 0$ as $x \to \infty$.

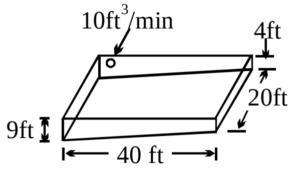
35. Prove that the curve $y = x^5 + 2x$ has no horizontal tangents. What is the smallest slope that a line tangent to this curve can have ?

Target Exercises for JEE Advanced

1. A swimming pool is 40 feet long, 20 feet wide, 4 feet deep at the shallow end, and 9 feet deep at the deep end (see figure). Water is being pumped into the pool at 10 cubic feet per minute and there is 4 feet of water at the deep end

(i) What percentage of the pool is filled ?

(ii) At what rate is the water level rising?

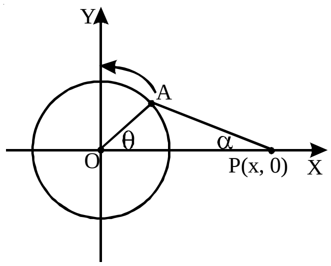


2. The figure shows a rotating wheel with radius 40 cm and connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.

(a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$,

(b) Express the distance $x = OP$ in terms of θ.

(c) Find an expression for the velocity of the pin P in terms of θ.



3. Show that the angle between the tangent at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\ell n\,(x^2 + y^2) = c\tan^{-1}(y/x)$ where c is constant.

4. On the graph of the function $y = \frac{3}{\sqrt{2}}\,x \ln x$, where $x \in [e^{-1.5}, \infty)$, find the point M(x, y) such that the segment of the tangent to the graph of the function at that point, intercepted between the point M and the y-axis, is the shortest.

5. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 at Q_2. Show that

$$\frac{PQ_1}{PP_1}+\frac{PQ_2}{PP_2}=1.$$

6. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

7. A lattice point in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centres. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

8. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. If we pour the liquid into the container at a rate of 2 cm³/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container ?

9. Prove that the portion of the tangent to the curve

$$x=\sqrt{a^2-y^2}+\frac{a}{2}\log\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}}$$

between the point of contact and the x-axis is of length a.

10. (a) Consider the curve with equation

$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$.

At how many points does this curve have horizontal tangents? Find the x-coordinates of these points.

(b) Find equations of the tangent lines at the points (0, 1) and (0, 2).

11. Find the length of the perpendicular dropped from the origin to the tangent to the curve

$2x = a(3\cos t + \cos 3t)$, $2y = a(3\sin t + \sin 3t)$.

Show that $4r^2 = 3p^2 + 4a^2$, where r is the polar radius of the given point and p is the length of the mentioned perpendicular.

12. Prove that the equation of the tangent at any point $(4m^2, 8m^3)$ of the semi-cubical parabola $x^3 - y^2 = 0$ is $y = 3mx - 4m^3$ and show that it meets the curve again at $(m^2, -m^3)$, where it is normal if $9m^2 = 2$.

13. The tangent at any point on the curve $x^3 + y^3 = 2a^3$ cuts off length p and q on the coordinate axes. Show that $p^{-3/2} + q^{-3/2} = 2^{-1/2}a^{-3/2}$.

14. Prove that the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at the point (a, b), whatever be the value of n.

15. Show that in the curve $y = a\ln(x^2 - a^2)$ the sum of the length of tangent and the subtangent varies as the product of the coordinates of the point.

16. Prove that in the ellipse $x^2/a^2 + y^2/b^2 = 1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent.

17. Show that for any point $P(x_0, y_0)$ of the hyperbola $x^2 - y^2 = a^2$ the segment of the normal from the point P to the point of intersection with the abscissa is equal to the radius vector of the point P.

18. Prove that the equation of the tangent at a point (x_1, y_1) on the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ is $\frac{x_1^{m-1}x}{a^m}+\frac{y_1^{m-1}x}{b^m}=1$

19. Prove that the equation of the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ may be written in the form $x\sin\phi - y\cos\phi + a\cos 2\phi = 0$.

20. Through the point (h, k) tangents are drawn to the curve $Ax^3 + By^3 = 1$. Prove that the points of contact lie on a conic.

21. The tangent to the curve $y = x - x^3$ at a point P meets the curve again at Q. Prove that one point of trisection of PQ lies on the y axis. Find the locus of the other point of trisection.

22. If the tangent at a point P on the curve $y = x^2 - x^3$ meets if again at Q, show that the locus of the middle point of PQ is $28x^3 - 28x^2 + 9x + y - 1 = 0$.

23. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y=\frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find the value of a.

24. Find all real values of 'a' for which $x^3 - 4x \ge |x - a|$ has atleast one negative solution.

25. A curve is given by $x = a(2\cos t + \cos 2t)$, $y = a(2\sin t - \sin 2t)$. Prove that (i) the tangent at P(t) meets the curve again in points Q(–t/2) and R $\left(\pi-\frac{t}{2}\right)$ and (ii) that the tangents at Q and R are at right angles and intersect on the circle $x^2 + y^2 = a^2$.

26. Prove that any tangent of $y = ax^3, bx^2 + cx + d$, $a \neq 0$, intersects the curve again.

27. Find the equation of tangents to the curves $y = x^3 - x - 1$ and $y = 3x^2 - 4x + 1$ which are parallel to each other

28. If the normal to the curve y = f(x) at x = 0 be given by the equation $3x - y + 3 = 0$, then find the value of $\lim_{x\to 0} x^2\{f(x^2) - 5f(4x^2) + 4f(7x^2)\}^{-1}$.

29. Find the least distance between any two points of the curves $y = \log_2 x$ and $y = 2^x$.

30. Prove that the segment of the tangent to the curve

$$y = \frac{a}{2}\,\ell n\,\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$$

contained between the y-axis and the point of tangency has a constant length .

31. Find the shortest distance between the curves $y^2 = x^3$ and $9x^2 + 9y^2 - 30y + 16 = 0$.

32. Find the equation of tangents to the curves $y = x^3 - x - 1$ and $y = 3x^2 - 4x + 1$ which are parallel to each other.

33. Find the equations of tangent of the curve $x = a\cos\theta$, $y = \sqrt{2}\ a\cos\theta\sin\theta/2$ at origin.

34. If in a curve the normal varies as the cube of the ordinate, find the subtangent and the subnormal.

35. If p be the length of the perpendicular drawn from the origin upon the tangent to the curve. $x = a\cos\theta + a\,\theta\sin\theta$, $y = a\sin\theta - a\,\theta\cos\theta$ at the point θ, then prove that:

(i) $p \propto \theta$

(ii) $p \propto \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$

Previous Year's Questions (JEE Advanced)

A. FILL IN THE BLANKS:

1. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H =........... and V = ...**[IIT-1994]**

B. MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER:

2. The normal to the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point 'θ' is such that

(A) it makes a constant angle with the x-axis

(B) it passes through the origin

(C) it is at a constant distance from the origin

(D) none of these **[IIT - 1983]**

3. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point **[IIT - 1992]**

(A) (1, 1) (B) at no point

(C) (0, 1) (D) (1, 0)

4. Two particles A and B are dropped from the height of 5m and 20m respectively. Then the ratio of time taken by A to that taken by B, to reach the ground is

(A) 1 : 4 (B) 2 : 1

(C) 1 : 2 (D) 1 : 1 **[IIT - 1993]**

5. If $y = 4x - 5$ is tangent to curve $y^2 = px^3 + q$ at (2, 3), then **[IIT - 1994]**

(A) $p = 2, q = 7$ (B) $p = -2, q = 7$

(C) $p = -2, q = -7$ (D) None of these

6. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $3\pi/4$ with the positive x-axis, then $f'(3)$ **[IIT - 2000]**

(A) -1 (B) $-\frac{3}{4}$

(C) $\frac{4}{3}$ (D) 1

7. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is **[IIT - 2001]**

(A) -1 (B) 3

(C) -3 (D) 1

8. The point(s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical, is (are) **[IIT - 2002]**

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$

(C) $\left(\sqrt{\frac{11}{3}}, 1\right)$ (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

9. Which one of the following curves cut the parabola $y^2 = 4ax$ at right angles ? **[IIT - 1994]**

(A) $x^2 + y^2 = a^2$ (B) $y = e^{-x/2a}$

(C) $y = ax$ (D) $x^2 = 4ay$

C. SUBJECTIVE PROBLEMS:

10. Find all the tangents to the curve $y = \cos(x + y), -2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$. **[IIT - 1985]**

11. What normal to the curve $y = x^2$ forms the shortest chord ? **[IIT - 1992]**

12. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2). **[IIT - 2005]**

D. COMPREHENSION **[IIT - 2007]**

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in R

Consider $f(x) = ke^x - x$ for all real x where k is real constant.

13. The line y = x meets $y = ke^x$ for $k \le 0$ at

(A) no point (B) one point

(C) two point (D) more that two points

14. The positive value of k for which $ke^x - x = 0$ has only one root is

(A) 1/e (B) 1

(C) e (D) $\log_a 2$

15. For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$ (B) $\left(\frac{1}{e}, 1\right)$

(C) $\left(\frac{1}{e}, \infty\right)$ (D) (0, 1)

ANSWERS

CONCEPT PROBLEMS—A

1. $15\sqrt{3}$

2. 60π cm²/sec

3. 2 unit/s at (3, 6) and – 2unit/s at (3, – 6)

4. At the points (3, 16/3) and (– 3, – 16/3).

5. $\frac{8}{3}$ cm/sec

6. –24 ft/s

7. t = 4s

8. $\frac{\sqrt{2}}{\pi}$ cm/sec.

9. 140 ft/min.

13. $5\sqrt{2}$ cm/s

15. 7.2 miles/hr

16. (a) 4.75 A (b) 5A

PRACTICE PROBLEMS—A

18. $\frac{200\pi r^3}{(r+5)^2}$

19. $\frac{\sqrt{2}}{4\pi}$

20. 4.5 cm/s; away

22. 58.75 ft/s

23. (a) 20 person per month

(b) 0.39% per month

24. (a) 0.035 lux/s; (b) 7.15m.

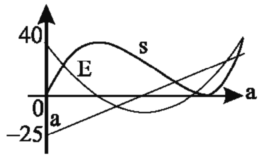
25. 10π cu. inch/sec

26. $dV/dh = 7.5\pi$ ft³/ft; $dr/dt = 1/(15\pi)$ ft/sec

27. n = 33

28. (a) 6t – 24; –6 m/s²

(b)



(c) Speeding up when 2 < t < 4 or t > 6; slowing down when $0 \le t < 2$ or 4 < t < 6

29. –1.6 cm/min.

30. $\frac{10}{3}$ cm/min

31. 0.3 m²/s

32. $1650/\sqrt{31} \approx 296$ km/h

33. $-10\sqrt{3}$ units/s

34. 0.637 ft/s

35. The revenue will be rising at $34,000/yr.

36. velocity, $\frac{dx}{dt} = \frac{12\pi x \sin\theta}{2\cos\theta - x}$; acceleration,

$\frac{d^2x}{dt^2} = \frac{144\pi^2 x}{(2\cos\theta - x)^3}\left[2\sin^2\theta \cos\theta + (2\cos\theta - x)\left(1 + \frac{1}{2} x\cos\theta\right)\right]$

CONCEPT PROBLEMS—B

1. 1.18

2. $\frac{377}{75}$

3. 0.03 or 3%

4. (a) 270 cm³, 0.01, 1%

(b) 36 cm³, $0.00\overline{6}$, $0.\overline{6}\%$

5. (a) $84/\pi \approx 27$ cm²; $\frac{1}{84} \approx 0.012$

(b) $1764/\pi^2 \approx 179$ cm³; $\frac{1}{56} \approx 0.018$

CONCEPT PROBLEMS—C

1. (1, 0)

3. x – 1 = 0 ; y = 0

4. $2 \pm \sqrt{3}$

5. Yes

6. (–1, –2)

7. $x = n\pi \pm (\pi/4)$, $n \in I$

8. No

9. 2 (1, 1, 0)

10. $-\frac{2}{\pi}$

12. y = 2x + 2

13. (–2, 21), (1, –6)

15. $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$

16. The tangent is x + 2y = 4a; the normal is y = 2x – 3a.

18. There are no such tangents.

19. Yes, y + 16 = – (x – 3) is tangent at (3, – 16)

PRACTICE PROBLEMS—B

21. (a) $4x + y - 1 = 0$ (b) $y = 0$

22. (i) $\frac{2}{9}$ (ii) $0, \frac{2}{3}$

23. $(\sqrt{2}, 0), \left(-\frac{2\sqrt{3}}{9}, \frac{\sqrt{15}}{9}\right)$ and $\left(-\frac{2\sqrt{3}}{9}, -\frac{\sqrt{15}}{9}\right)$

24. $(1/2, -15/32)$

25. $5x + 6y - 13 = 0, 6x - 5y + 21 = 0$

27. $0, \pm\frac{1}{4}$

28. where $x = \pm\frac{a}{\sqrt{2}}$

29. Horizontal tangents at (–4, 0), (–4, 10) Vertical tangents at (0, 5), (–8, 5)

30. Yes

31. $2x - y + 1 = 0$

33. 3.75

34. $\frac{c^4}{\sqrt{x^6 + y^6}}$

35. Tangents are $Y = \pm\frac{3\sqrt{3}}{8}x - \frac{a}{8}$.

Normals are $Y = \pm\frac{8\sqrt{3}}{9}x + \frac{41}{36}a$.

36. The inequality $5x^4 + 8 > 0$ holds for any $x \in R$.

CONCEPT PROBLEMS—D

1. $y = 0$; $(\pi + 4)x + (\pi - 4)y - \frac{\pi^2\sqrt{2}}{4} = 0$

2. $-\frac{a}{2b}$

3. $y = \frac{x_0}{2a}\left(x - \frac{x_0}{2}\right)$; $y - y_0 = -\frac{2a}{x_0}(x - x_0)$

4. 0

6. $\pi n, \frac{\pi}{8} + \frac{\pi n}{4}, n \in I$

7. (8, 0), (0, 0)

8. 2

9. $y + x = 0, 4x + 25y = 0$

PRACTICE PROBLEMS—C

11. $y = 2, x = 1$

12. (i) $t = \pi/2 + \alpha$; (ii) $t = \pi/6 + \alpha/3$

13. $-2x_0$

15. ± 30

16. $n = 1$

17. $1, -3, \frac{1}{9}(-13 \pm \sqrt{610})$

18. (0, 2)

19. (3, – 1)

20. (3, – 15), 21/2, 21

21. $y = x + 4, y = -x + 4$

22. y-intercept occurs when $y = -Ac^2$

23. $36y + 29x - 23 = 0$

24. (a) $ax = \pm by$, (b) $x = 0$ and $y = 0$

(c) $ax = \pm y\sqrt{b^2 - a^2}$

CONCEPT PROBLEMS—E

1. (i) $\pi/2$ (ii) $\pi/2$

2. $\tan^{-1}(2\sqrt{2})$ or $\sec^{-1}(3)$

3. 45° ; $\tan^{-1}2$

4. 40° 36'

7. 90°

PRACTICE PROBLEMS—D

8. (i) $\theta_1 = 0, \theta_2 = \tan^{-1}\frac{18}{31}$;

(ii) $\tan^{-1}\frac{8}{15}$

9. (i) The curves intersect at two points at the angles $\alpha_1 = \alpha_2 = \tan^{-1}\frac{41}{2}$;

(ii) The curves intersect at three points at the angles $\alpha_1 = \alpha_2 = \pi/6$ and $\alpha_3 = 0$.

12. 6

13. $\tan^{-1}(2)$

14. $n^2 - 1$

PRACTICE PROBLEMS—E

2. $x = 0, y = 0$

3. $y = 2x - 1$; $y = 4x - 4$.

4. $(-\infty, 0) \cup \left\{\frac{1}{3}\right\}$

5. $k = \frac{e^2}{4}$

6. $\frac{4}{\sqrt{3}}$.

10. The curves intersect at the points where $\cos ax = 1$. Since at these points $\sin ax = 0$, $y'_2 = f'(x)\cos ax - f(x)\, a \sin ax = f'(x) = y'_1$ i.e the curves are tangent.

11. The values required are the values of a for which $y = ax$ touches $y = \sin x$.

CONCEPT PROBLEMS—F

1. $S_T = S_N = 2$; $T = N = 2\sqrt{2}$

4. 24

6. $\sqrt{80}$

PRACTICE PROBLEMS—F

9. $T = a\sqrt{2}$, $N = a\sqrt{2}$, $S_T = a$, $S_N = a$

10. $\left|\frac{y}{\cos t}\right|, \left|\frac{y}{\sin t}\right|$, $|y \tan t|$ and $|y \cot t|$.

16. $cy + ax = c^2$

OBJECTIVE EXERCISES

1. B **2.** B **3.** C
4. D **5.** B **6.** B
7. B **8.** C **9.** D
10. C **11.** A **12.** A
13. B **14.** A **15.** D
16. A **17.** D **18.** C
19. B **20.** C **21.** B
22. C **23.** B **24.** A
25. D **26.** A **27.** B
28. C **29.** D **30.** C
31. A **32.** D **33.** D
34. A **35.** B **36.** D
37. A **38.** A **39.** D
40. D **41.** C **42.** B
43. C **44.** B **45.** C
46. C **47.** D **48.** C
49. C **50.** A **51.** D
52. D **53.** D **54.** B
55. B **56.** B **57.** D
58. B **59.** A **60.** C
61. A **62.** D **63.** BC
64. AD **65.** BD **66.** AC
67. ABD **68.** CD **69.** BC
70. BD **71.** ABD **72.** ABD
73. ACD **74.** AB **75.** A
76. C **77.** A **78.** A
79. D **80.** C **81.** B
82. A **83.** C **84.** B
85. A **86.** B **87.** C
88. B **89.** B **90.** A
91. C **92.** B **93.** B
94. D **95.** A
96. (A)–(PQ) ; (B)–(RS) ; (C)–(RQ) ; (D)–(PS)
97. (A)–(Q) ; (B)–(R) ; (C)–(P) ; (D)–(S)
98. (A)–(R) ; (B)–(Q) ; (C)–(R) ; (D)–(P)
99. (A)–(P) ; (B)–(RS) ; (C)–(Q) ; (D)–(T)
100. (A)–(Q) ; (B)–(S) ; (C)–(S) ; (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

1. $7\sqrt{15}/4 \approx 6.78$ m/s

2. (a) $200t + 50\ell nt + 450$ newspapers per year.
(b) 1,530 newspapers per year
(c) 1,635 newspapers

3. (a) $-\frac{60}{7}$ units per second
(b) falling

4. (a) 0.36 (b) 70.3°

5. 4π sq. cm

6. (b) $\frac{2}{3}$ unit, (c) $0 \le t < 1$ and $t > 2$.

7. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.

8. $a = 1, b = -2$ **9.** $y = 0; x = 0$

10. $x - y - 2 = 0; x + y = 0$

13. $y - \frac{\pi}{3} = \cot^{-1}2.\left(x-3\right)$ **14.** $y = 0, 4y = 27x$.

19. $(\pm\sqrt{3}, 0)$. **20.** $xy = 4a^2$

23. $2m^2 - 3m - 2 = 0$; $y = 2x, y = -\frac{1}{2}x$

25. $\tan^{-1}\left(\frac{2(1-\ell n2)}{4-\ell n2}\right)$ **26.** $\tan^{-1}2^{4/3}, \tan^{-1}2^{-2/3}$

27. $M_1(1, e^{-1}), M_2(-1, e^{-1})$ **28.** $\frac{1}{\sqrt{5}}$

29. 7 **32.** $k = \pm 3$

33. Yes ; at $x = 2n\pi$, $n \in I$

34. (i) $(-1/e, -1)$ (ii) No **35.** 2

TARGET EXERCISES for JEE ADVANCED

1. (i) 24.6% (ii) $\frac{1}{64}$ ft/min

2. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s

(b) $40\cos\theta + \sqrt{8+\cos^2\theta}$) cm

(c) $-480\pi\sin\theta\,(1 + \cos\theta/\sqrt{8+\cos^2\theta}$) cm/s

4. $M\left(e^{-4/3}, -\sqrt{2e^{-4/3}}\right)$

6. $(1, -2), (-1, 0)$

7. $\sqrt{29}/58$

8. $2 + \frac{375}{128}\pi \approx 11.204$ cm³/min

10. Eight ; $1 \mp \sqrt{3}/3$ (b) $y = -x + 1$, $y = \frac{1}{3}x + 2$

11. $p = 2a\cos t$. **21.** $y = x - 5x^3$

23. $a = 1$ **24.** $\frac{-10}{3}\sqrt{\frac{5}{3}} \le a \le 2$

27. $\frac{p-1+a\ln a}{2a}$ **28.** $-1/3$

29. $\sqrt{2}\left(\frac{\ell n(\ell n2)+1}{\ell n2}\right)$ **31.** $\frac{\sqrt{13}}{3} - 1$

32. $y = kx \mp 2\left(\frac{k+1}{3}\right)^{3/2}$

$y = kx - \frac{(k^2 + 8x + 2)}{12}$, $k \geq 1$

33. 1

34. If normal = y^3/b^2, subtangent = $b^2 y\sqrt{y^4 - b^4}$ subnormal = $y\sqrt{y^4 - b^4} / b^2$.

35.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. ϕ, $\{(1, 1)\}$ **2.** C

3. D **4.** C

5. D **6.** D

7. C **8.** D

9. BD

10. $2x + 4y - \pi = 0$, $2x + 4y + 3\pi = 0$

11. $x + \sqrt{2}\,y = \sqrt{2}$ or $x - \sqrt{2}\,y = -\sqrt{2}$

12. $y = 2$ **13.** B

14. A **15.** A

HINTS & SOLUTIONS

Objective Exercises Solutions

SINGLE CORRECT ANSWER TYPE

1. As $f(x)$ is even function

$\Rightarrow \quad f'(-2)=\sqrt{3}$

Hence, B is correct

2. $x+y=x^{y}$ cuts x -axis at (1,0).

Now, $1+y'=x^{y}\left(\frac{y}{x}+y'\ln x\right)$

$y'=-1$

Slope of normal = 1

eqn of normal

$y=x-1$

Hence, B is correct

3. $x=f'(t)\sin t+f''(t)\cos t$

$$\frac{dx}{dx}=f''(t)\sin t+f'(t)\cos t-f''(t)\sin t+f'''(t)\cos t$$

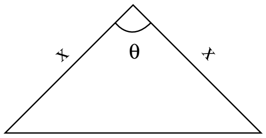
$$\frac{dx}{dt}=f'(t)\cos t+f'''(t)\cos t$$

$$\& \frac{dy}{dt}=f'(t)\sin t+f'''(t)\sin t$$

$$\frac{dy}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}}=f'(t)+f'''(t)$$

Hence, C is correct

(4)



$$A=\frac{1}{2}x^{2}\sin\theta$$

$$\frac{dA}{dt}=x\sin\theta\frac{dx}{dt}+\frac{x^{2}}{2}\cos\theta\frac{d\theta}{dt}$$

$$=12\times\frac{1}{\sqrt{2}}\times\frac{1}{12}+\frac{144}{2}\times\frac{1}{\sqrt{2}}\times\frac{\pi}{180}$$

$$=2^{1/2}\left(\frac{1}{2}+\frac{\pi}{5}\right)$$

Hence, D is correct

5. $x^{3/2}+y^{3/2}=2a^{3/2}$

$$\frac{3}{2}x^{1/2}+\frac{3}{2}y^{1/2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\sqrt{\frac{x}{y}}=-1$$

$y=x$ (I)

Putting in eqn

$x^{3/2}=a^{3/2}$

$\Rightarrow x=a,\ \ y=a$

Hence, B is correct

6. $f'(x)=\begin{cases}-2x, & x<0\\ 2x, & x\geq 0\end{cases}$

eqn of tangent → $y-y_1=2x_1(x-x_1),\ x\geq 0$

$y-y_2=-2x_2(x-x_2)\ \ x<0$

7. For

$x^{2}-y^{2}=8$

$x-yy_1=0$

Slope of normal $=-\frac{y_1}{x_1}=\frac{3}{5}$

For $9x^{2}+25y^{2}=225$

$$9x+25y\frac{dy}{dx}=0$$

Slope of normal $=\frac{25y_1}{9x_1}=\frac{-5}{3}$

Now as $m_1m_2=-1$

Hence, B is correct

8. $x^{2}dy/dx+2xy=0$

$$\frac{dy}{dx}=-\frac{2y_1}{x_1}$$

$$y-y_1=\frac{-2y_1}{x_1}(x-x_1).$$

$$a=\frac{3}{2}x_1,\quad b=3y_1$$

Now, $a^{2}b=\frac{27}{4}C^{3}$

Hence, C is correct

9. For

$y^3 - x^2y + 5y - 2x = 0$

$3y^2y_1 - x^2y_1 - 2xy + 5y_1 - 2 = 0$

at $(0,0)$, $y_1 = \frac{2}{5}$

For $x^4 - x^3y^2 + 5x + 2y = 0$

$4x^3 - 3x^2y^2 - 2x^3yy_1 + 5 + 2y_1 = 0$

at $(0,0)$, $y_1 = -\frac{5}{2}$

As $m_1 m_2 = -1$

Hence, D is correct

10. $y = x^n$

$y_1 = nx^{n-1}$

Slope of normal $= \frac{-1}{nx^{n-1}}$

eqn of Normal $\Rightarrow$

$y - y_1 = \frac{-1}{nx_1^{n-1}}(x - x_1)$

$b = y_1 + \frac{x_1}{nx_1^{n-1}}$

$b = a^n + \frac{1}{na^{n-2}}$

As $\lim\limits_{a\to 0} b = \frac{1}{2} \Rightarrow n = 2$

Hence, C is correct

11. $x^4 + y^4 = a^4$

$x^3 + y^3\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^3}{y^3}$

eqn of tangent →

$y - y_1 = -\frac{x_1^3}{y_1^3}(x - x_1)$

$p = \frac{y_1^4}{x_1^3} + x_1 = \frac{x_1^4 + y_1^4}{x_1^3} = \frac{a^4}{x_1^3}$

$q = y_1 + \frac{x_1^4}{y_1^3} = \frac{a^4}{y_1^3}$

Now, $p^{-4/3} + q^{-4/3} = a^{-4/3}$

Hence, A is correct

12. $P(t) = 60t^2 - t^3$

$P'(t) = 120t - 3t^2 = 900$

$\Rightarrow t^2 - 40t + 300 = 0$

$t = 10, 30$

Hence, A is correct

13. $x^2 + y^2 - \frac{10}{3}y + 1 = 0$

$x^2 + \left(y - \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2$ (I)

$y^2 = x^3$

$2yy_1 = 3x^2$

$y_1 = \frac{3x^2}{2y}$

Normal eqn $y - y_1 = -\frac{2y_1}{3x_1^2}(x - x_1)$

as it passes through $\left(0, \frac{5}{3}\right)$

$\frac{5}{3} - y_1 = \frac{2y_1}{3x_1}$

$5 - 3y_1 = 2y^{1/3}$

$5 - 9y^3 - 225y + 135y^2 = 8y$

$9y^3 + 233y - 135y^2 - 25 = 0$

14. $x^3 + y^3 = a^3$

$x^2 + y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^2}{y^2}$

Eqn of tangent $y - y_1 = \frac{-x_1^2}{y_1^2}(y - x_1)$

$y_1^2y - y_1^3 = -x_1^2x + x_1^3$

$y_1^2y + x_1^2x = a^3$ (I)

Now, $y_1^2k + x_1^2h = a^3$

$\frac{k}{y_1x_1^3} + \frac{h}{x_1y_1^3} = \frac{a^3}{x_1^3y_1^3}$

$$\frac{B}{x_1^3}+\frac{A}{y_1^3}=\frac{a^3}{x_1^3\left(a^3-x_1^3\right)}$$

$$\frac{B}{x_1^3}+\frac{A}{a^3-x_1^3}=\frac{a^3}{x_1^3\left(a^3-x_1^3\right)}$$

$$a^3B-x_1^3B+x_1^3A=a^3$$

$$x_1^3=\frac{a^3(1-B)}{A-B}$$

Hence, A is correct

15. $h=ar+b$

$$\frac{dh}{dt}=a\frac{dr}{dt}\Rightarrow a=3,\ b=3$$

$$v=3\pi r^2(r+1)$$

$$\frac{dv}{dt}=3\pi\left(3r^2\frac{dr}{dt}+2r\frac{dr}{dt}\right)$$

$$1=3\pi\left(3-6^2+12\right)\frac{dr}{dt}$$

$$\frac{dr}{dt}=\frac{1}{360\pi}$$

Now, when $r=36$

$$\frac{dv}{dt}=3\pi\left(3\cdot(36)^2+72\right)\times\frac{1}{360\pi}$$
$$=33$$

Hence, D is correct

16. At $x=\frac{\pi}{3},\ y=\frac{3\sqrt{3}}{2}$

$$\frac{dy}{dx}=2\cos x+2\cos 2x$$

$$\frac{dy}{dx}=0$$

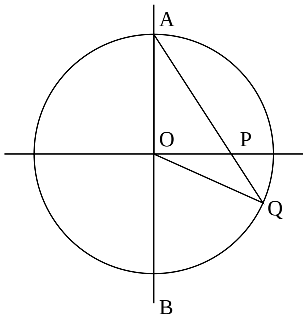
Tangent $\Rightarrow y=\frac{3\sqrt{3}}{2}$

Normal $\Rightarrow x=\frac{\pi}{3}$

$$\text{Area}=\frac{\pi}{3}\times\frac{3\sqrt{3}}{2}=\frac{\sqrt{3}\pi}{2}$$

Hence, A is correct

17.



$r=OA=OQ=10$

Let $\angle BOQ=\theta$

$\angle OQP=\theta/2$, $\quad \angle OPQ=\frac{\pi}{2}+\frac{\theta}{2}$

In ΔOPQ,

$$\frac{OP}{\sin\theta/2}=\frac{10}{\cos\theta/2}$$

$OP=10\tan\theta/2$

$$\frac{d(0p)}{dt}=\frac{10}{2}\sec^2\theta_2\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt}=\cos^2\theta/2$$

Now, $\text{arc }B\theta=\frac{\theta}{360^\circ}\times 2\pi r$

$l=\theta r$

$$\frac{dl}{dt}=10\frac{d\theta}{dt}=10\cos^2\theta/2$$

As $\angle BA\theta=45^\circ,\ \theta=\pi/2$

$$\frac{dl}{dt}=\frac{10}{2}=5\,\text{cm/sec}$$

Hence, D is correct

18. $y^2-x+2=0$

$$2y\frac{dy}{dx}-1=0$$

$\frac{dy}{dx}=\frac{1}{2y_1}$, Slope of normal $\rightarrow -2y_1$

Now, $x^2-y+2=0$

$$2x-\frac{dy}{dx}=0$$

$\frac{dy}{dx}=2x$, slope of normal $=\frac{-1}{2x_2}$

where $P \rightarrow (x_1, y_1)$ & $\theta \rightarrow (x_2, y_2)$

As both the curve are inverse of each other. Hence, they are mirror image about $y = x$

$\Rightarrow$ Slope of normal $= -1$

$$\Rightarrow y_1 = \frac{1}{2},\ x_1 = \frac{9}{4}$$

$$x_2 = \frac{1}{2},\ y_1 = \frac{9}{4}$$

Shortest distance (PQ) $= \dfrac{7}{2\sqrt{2}}$

Hence, C is correct

19. Let P be $(t^2, 2t)$

$$y^2 = 4x$$

$$2y\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{1}{t},$$

If *Q* is a pt where normal meets the parabola again with t_1 as parameter

$$t_1 = -t - \frac{2}{t}$$

Coordinate of $Q \rightarrow \left(\left(t + \frac{2}{t}\right)^2, -2t - \frac{4}{t}\right)$

Distance $= \sqrt{\left(t + \frac{2}{t}\right)^4 + 4\left(t + \frac{2}{t}\right)^2}$

Distance $= \left(t + \frac{2}{t}\right)\sqrt{\left(t + \frac{2}{t}\right)^2 + 4}$

Min Distance $= 2\sqrt{2}\ \sqrt{12} = 4\sqrt{6}$

Hence, B is correct

20. $y = x^2 + bx - b$, (1,1) lies on curve

$$\frac{dy}{dx} = 2x + b = b + 2$$

eqn of tangent $\Rightarrow y - 1 = (b+2)(x-1)$

x int $\Rightarrow 1 - \dfrac{1}{b+2}$

y int $\Rightarrow 1 - (b+2)$

Area of $\Delta = \dfrac{1}{2}\dfrac{(b+1)^2}{(b+2)}$

$$\Rightarrow (b+1)^2 = -4b - 8$$

$$\Rightarrow b^2 + 6b + 9 = 0$$

$$(b+3)^2 = 0$$

$$\Rightarrow b = -3$$

Hence, C is correct

21. $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

Slope of normal $\Rightarrow -x_1$

Slope of 1 chord $= \dfrac{1}{e-1}$

$$\Rightarrow \frac{x_1}{e-1} = 1$$

$$\Rightarrow \quad x_1 = e - 1$$

$$y_1 = \ln(e-1)$$

Hence, B is correct.

22. If $y = -2$ is a tangent

$\Rightarrow px^2 + qx + r + 2 = 0$ will have only one real root

$$\Rightarrow q^2 - 4p(r+2) = 0$$

$$= q^2 - 4pr - 8p = 0$$

$$\Rightarrow p < 0 \text{ as} \left(q^2 - 4pr < 0\right)$$

$$\Rightarrow r < 0$$

Hence, C is correct

23. $$y = \sin^{-1} 2x\sqrt{1-x^2} = \begin{cases} 2\sin^{-1} x & |x| \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & x > \dfrac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1} x\right) & x < \dfrac{-1}{\sqrt{2}} \end{cases}$$

$x = 0,\ y = 0$ is only integral point where the function has a unique tangent

Hence, B is correct

24. $y = \tan^{-1}\dfrac{2x}{1-x^2} = 2\tan^{-1} x$

$$\frac{dy}{dx} = \frac{2}{1+x^2} = 2$$

$y = 2x$ is the eqn of tangent

Now, if tangent intersects the curve again,

$$\Rightarrow \tan^{-1} x = x$$

$\Rightarrow x = \tan x$ has no pt of intersection if $x \in (0,1)$

Hence, A is correct

25. slope of normal $\Rightarrow \; 3x - y + 3 = 0$

$x = 0 \;\&\; y = 3$

Pt of normal $= (0,3)$

$$\frac{dy}{dx} = \frac{-1}{3} = f'(0)$$

$$\lim_{x\to 0} \frac{x^2}{f\left(x^2\right) + 4f\left(7x^2\right) - 5f\left(4x^2\right)}$$

$$\lim_{x\to 0} \frac{2x}{x\left[2f'\left(x^2\right) + 56f'\left(7x^2\right) - 40f'\left(4x^2\right)\right]}$$

$$= \frac{2}{-6} = \frac{-1}{3}$$

Hence, D is correct

26. $y = 6 - x - x^2$

$$\frac{dy}{dx} = -1 - 2x$$

eqn of tangent

$$y - 6 + x_1 + x_1^2 = -\left(1 + 2x_1\right)\left(x - x_1\right)$$

$xy = x + 3$

$$x\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} = \frac{1 - y_2}{x}$$

eqn of tangent $\rightarrow$

$$y - \frac{x_2 + 3}{x_2} = \frac{1 - \dfrac{x_2 + 3}{x_2}}{x_2}\left(x - x_2\right)$$

$$x_2 y - \left(x_2 + 3\right) = \frac{-3}{x_2}\left(x - x_2\right)$$

Comparing the two eqns

$$\frac{1}{x_2} = \frac{-\left(1 + 2x_1\right)}{-\dfrac{3}{x_2}} = \frac{x_1\left(1 + 2x_1\right) + 6 - x_1 - x_1^2}{3 + x_2 + 3}$$

$$\frac{1}{x_2} = \frac{x_2\left(1 + 2x_1\right)}{3} \quad \& \quad \frac{1}{x_2} = \frac{x_1^2 + 6}{x_2 + 6}$$

$$3 = x_2^2 + 2x_1x_2^2 \quad \& \quad x_2 + 6 = x_1^2x_2 + 6x_2$$

$$3 = \left(1 + 2x_1\right)x_2^2 \quad \& \quad x_2 = \frac{6}{5 - x_1^2}$$

$$\Rightarrow \quad 3 = \frac{\left(1 + 2x_1\right)36}{\left(5 - x_1^2\right)^2}$$

$$\Rightarrow \left(5 - x_1^2\right)^2 = 12\left(1 + 2x_1\right)$$

Let $y = mx + c$ be the common tangent

$mx + c = 6 - x - x^2$

$\Rightarrow x^2 + (m+1)x + (c-6) = 0$

$D = 0$

$(m+1)^2 - 4(c-6) = 0$ (I)

For $xy = x + 3$

$x(mx + c) = x + 3$

$mx^2 + (c-1)x - 3 = 0$

$D = 0$

$(c-1)^2 + 12m = 0$ (II)

Using (I) & (II)

$(m+1)^2 + 24 = 4c$

$$c - 1 = \frac{(m+1)^2 + 20}{4}$$

$$\Rightarrow \left(\frac{(m+1)^2 + 20}{4}\right)^2 + 12m = 0$$

$$\left((m+1)^2 + 20\right)^2 + 192m = 0$$

Upon solving, $m = -3$

$\Rightarrow c = 7$

eqn of tangent $= y = -3\,x + 7$

Hence, A is correct

27. $y = 2x^2 - x + 1$

$$\frac{dy}{dx} = 4x_1 - 1$$

$4x_1 - 1 = 3$

$x_1 = 1$

$y_1 = 2$

Hence, B is correct

28. $y^2-2x^2-4y+8=0$

$2y\frac{dy}{dx}-4x-4\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{2x}{y-2}$ (I)

Now, $y^2-2x^2-4y+8=0$

$(y-2)^2-2\left(x^2-2\right)=0$ (II)

eqn of tangent

$y-y_1=\frac{2x_1}{y_1-2}\left(x-x_1\right)$

$-\left(2-y_1\right)^2=2x_1\left(1-x_1\right)$

Using (II)

$-2\left(x_1{}^2-2\right)=2x_1-2x_1^2$

$2x_1=4$

$x_1=2$ & $x_1=0$ (Horizontal tangent)

Hence, C is correct

29. $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

$\frac{n}{a}\left(\frac{x}{a}\right)^{n-1}+\frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{b^n x^{n-1}}{a^n y^{n-1}}=-\frac{b}{a}$

Hence, D is correct

30. $y=1-ax^2$

$\frac{dy}{dx}=-2ax_1$

& $y=x^2$

$\frac{dy}{dx}=2x_1$

If curves are orthogonal, $m_1m_2=-1$

$+4ax^2=1$ (I)

Now $1-ax^2=x^2\Rightarrow\frac{1}{1+a}=x^2$

Using eq (1), we get

$\frac{4a}{1+a}=1$

$a=\frac{1}{3}$

Hence, C is correct

31. $y=x^3+3x^2+3x-1$

$\frac{dy}{dx}=3x^2+6x+3=3(x^2+2x+1)^2$

$=3(x+1)^2$

Min slope of curve is 0

Hence, A is correct

32. $f_1'(x)=2x-1$ & $f_2'(x)=3x^2-2x-2$

$\Rightarrow 2x_1-1=3x_2^2-2x_2-2$

$\Rightarrow 3x_2^2-2x_2-2x_1-1=0$

For x_2 be real,

$D\geq 0$

$4-4\left(2x_1+1\right)(3)$

$4-24x_1-12$

$\Rightarrow -\left(24x_1+8\right)$

There can be infinite such values of x_1

Hence, D is correct

33. $x^2-4y^2+c=0$

$2x-8y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{x}{4y}$

& $y^2=4x$

$\frac{dy}{dx}=\frac{2}{y}$

For orthogonal, $m_1m_2=-1$

$\frac{x}{2y^2}=-1$

$\Rightarrow y^2=4\left(-2y^2\right)\Rightarrow y=0$

So, no values of c exist

Hence, D is correct

34. Slope of secant $=\frac{9a^2-a^2}{3a-a}=4a$

$\frac{dy}{dx}=2x=4a$

$x=2a$

$y=4a^2$

Hence, A is correct

35. $y = e^{2x} + x^2$

$$\frac{dy}{dx} = 2e^{2x} + 2x$$

$$\text{at } \ x = 0, \ \ \frac{dy}{dx} = 2$$

$$-\frac{dx}{dy} = -\frac{1}{2}$$

$$\text{eqn of normal} \rightarrow y - 1 = -\frac{1}{2}x$$

$$\Rightarrow 2y + x - 2 = 0$$

$$\text{Distance from } (0,0) = \frac{2}{\sqrt{5}}$$

Hence, B is correct

36. $y^3 - x^2y + 5y - 2x = 0$

$$3y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy + \frac{5dy}{dx} - 2 = 0$$

$$\frac{dy}{dx} = \frac{2(xy+1)}{3y^2 - x^2 + 5} = \frac{2}{5} \quad \text{at } \ (0,0)$$

For $x^4 - x^3y^2 + 5x + 2y = 0$

$$4x^3 - 3x^2y^2 - 2x^3y\frac{dy}{dx} + 5 + \frac{2dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4x^3 - 3x^2y^2 + 5}{2x^3y - 2} = \frac{5}{-2} \ \text{ at } (0,0)$$

Hence, D is correct

37. $l^2 = x^2 + y^2$ (I)

$$l\frac{dl}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$y = x^{3/2}$ (II)

$$\frac{dy}{dt} = \frac{3}{2}x^{1/2}\frac{dx}{dt}$$

Using (1) & (2)

$$11\sqrt{x^2 + x^3} = x\frac{dx}{dt} + \frac{3}{2}x^{3/2}x^{1/2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{66}{3 + \frac{27}{2}} = \frac{66 \times 2}{33} = 4$$

Hence, A is correct

38. $y^3 = 27x$

$$3y^2\frac{dy}{dx} = 27\frac{dx}{dt}$$

$$\frac{3y^2}{27} < 1$$

$$y^2 < 9 \Rightarrow y \in (-3, 3)$$

Hence, A is correct

39. $y = x\ln x + \sin(\pi \ln x)$

$$\frac{dy}{dx} = 1 + \ln x + \cos(\pi \ln x)\frac{\pi}{x}$$

at $x = e$

$$\frac{dy}{dx} = 2 - \frac{\pi}{e} = \frac{2e - \pi}{e}$$

at $x = e^2$

$$\frac{dy}{dx} = 3 + \frac{\pi}{e^2} = \frac{3e^2 + \pi}{e^2}$$

Hence, D is correct

40. $l_1 = l_2 + l_2^3 + 6$

$$\frac{dl_1}{dt} = \frac{dl_2}{dt} + 3l_2^2\frac{dl_2}{dt} \qquad \text{(I)}$$

$$\frac{dS_2}{dS_1} = \frac{dS_2}{dt} \times \frac{dt}{dS_1}$$

$$= \frac{2l_2 dl_2 / dt}{2l_1 dl_1 / dt}$$

$$= \frac{1}{4} \times \frac{1}{8}$$

Hence, D is correct

41. $y^2 = x(2 - x^2)$

$$2y\frac{dy}{dx} = 2 - 3x^2$$

$$\frac{dy}{dx} = \frac{-1}{2} \qquad \text{at } \ (1,1)$$

eqn of tangent

$$y - 1 = \frac{-1}{2}(x - 1)$$

$$2y + x - 3 = 0$$

Solving with curve

$$\left(\frac{3-x}{2}\right)^2 = 2x - x^3$$

$$4x^3 + x^2 - 14x + 9 = 0$$

42. $x^3 - y^2 = 0$

$3x^2 - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3x^2}{2y} = \frac{3\times 16m^4}{2\times 8m^3} = 3m$

Slope of normal $= \frac{-1}{3m_1}$ at point m_1

$y - 8m^3 = 3m\left(x - 4m^2\right) \rightarrow$ eqn of tangent

$y - 8m_1^3 = \frac{-1}{3m_1}\left(x - 4m_1^2\right) \rightarrow$ eqn of normal.

$\Rightarrow \quad \frac{3m}{-1/3m_1} = 1$

$\Rightarrow 9mm_1 = -1$ (1)

Now, $\frac{8m^3 - 12m^3}{-4m^3} = 8m_1^3 + \frac{4}{3}m_1$

$= \frac{-8}{(9m)^3} - \frac{4}{27m}$

$\Rightarrow \quad +4m^6 = \frac{8}{9^3} + \frac{4}{27}m^2$

$= \frac{8 + 108m^2}{9^3}$

43. $y = 1 - ax^2$

$\frac{dy}{dx} = -2ax$

& $y = x^2$

$\frac{dy}{dx} = 2x$

For orthogonal $\Rightarrow 4ax^2 = 1$ (1)

Now, $1 - ax^2 = x^2$

$x^2 = \frac{1}{1+a}$

Using eq (1)

$\frac{4a}{1+a} = 1$

$a = \frac{1}{3}$

Hence, C is correct

44. Let $y = mx + \frac{1}{m}$ be tangent to $y^2 = 4x$

eqn of normal at (x_1, y_1) to $x^2 = 4$ by is

$y - y_1 = -\frac{2b}{x_1}(x - x_1)$

$\Rightarrow y = -\frac{2b}{x_1}x + \frac{x_1^2}{4b} + 2b$

Comparing two eqn

$m = -\frac{2b}{x_1}$ (I)

$\frac{x_1^2}{4b} + 2b = \frac{1}{m}$ (II)

Using (I) & (II)

$2bm^2 - m + b = 0$

For real values of m, $D > 0$

$1 - 8b^2 > 0$

$|D| < \frac{1}{2\sqrt{2}}$

Hence, B is correct

45. eqn of normal

$y = mx - 2m - m^3$

It passes through $(6, 0)$

$m^3 - 4m = 0$

$m\left(m^2 - 4\right) = 0$

$m = 0, \pm 2$

Pts are $\left(am^2 - 2am\right)$

$(0,0), (4,-4), (4, 4)$

Minimum distance $= \sqrt{21} - \sqrt{5}$

Hence, C is correct

46. $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b = 10a + b = 0$ (at 5, 4) (I)

Now, $4 = 25a + 5b + c$

using (I)

$4 = 25a + 5(-10a) + c$

$c = 4 + 25a$

$\Rightarrow$ Maximum value of $c = 104$

Hence, C is correct

47. $|\sin 2x| = |x| - a$

$|x| - a > 1$ for no solution

$|x| > 1 + a$

Hence, D is correct

48. Ellipse $\rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\Rightarrow \frac{x}{8}\frac{dx}{dt} + \frac{2y}{9}\frac{dy}{dt} = 1$$

$$\Rightarrow \frac{4\sqrt{1-y^2/9}}{8}\frac{dx}{dt} + \frac{2y}{9}\times\frac{dy}{dx}\times\frac{dx}{dt} = 1$$

Now put $\frac{dy}{dx} = 1$ & $y = 1$

$$\Rightarrow \frac{4}{6\sqrt{2}}\frac{dx}{dt} + \frac{2}{9}\frac{dx}{dt} = 1$$

$$\Rightarrow \frac{dx}{dt}\left(\frac{\sqrt{2}}{3} + \frac{2}{9}\right) = 1$$

$$\Rightarrow \frac{dx}{dt} = \frac{9}{3\sqrt{2}+2}$$

49. $v = x^3$

$$\frac{dv}{dt} = 3x^2\frac{dx}{dt}$$

$$= 900$$

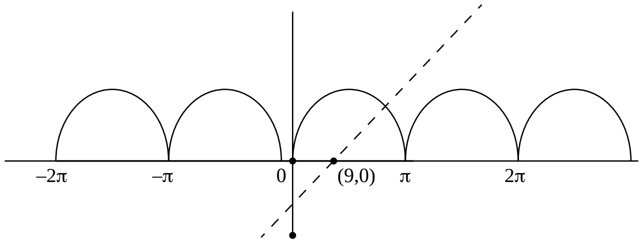
Hence, C is correct

50. Similar to Q.34

Hence, A is correct

51. $f(x) = x - \sin^2 x - a = 0$

$x - a = \sin^2 x$



Hence, D is correct

52. $x^2 - 4y^2 + c = 0$

$$2x - 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{4y}$$

$y^2 = 4x$

$$y\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{y}$$

For orthogonal, $\frac{x}{2y^2} = -1$ (I)

There is no x, y which will satisfy eq. (I)

Hence, D is correct

53. $x = t^2 + t + 1$

$$\frac{dx}{dt} = 2t + 1$$

$$y = t^2 - t + 1$$

$$\frac{dy}{dt} = 2t - 1$$

$$\frac{dy}{dx} = \frac{2t-1}{2t+1}$$

$$\Rightarrow y - \left(t^2 - t + 1\right) = \frac{2t-1}{2t+1}\left(x - \left(t^2 + t + 1\right)\right)$$

$$\Rightarrow \left(t - t^2\right) = \left(\frac{2t-1}{2t+1}\right)\left(-\left(t^2 + t\right)\right)$$

$$\Rightarrow 2t^2 + t - 2t^3 - t^2 = t^2 + t - 2t^3 - 2t^2$$

$$\Rightarrow 2t^2 = 0$$

$$\Rightarrow t = 0$$

Hence, D is correct

54. $y = \left(x_1 - x_2\right)^2 + \left(x_1 - 4 - \frac{x_2^2}{4}\right)^2$

Min value of *y* is shortest distance between

$y = x - 4$ & $y = \frac{x^2}{4}$

slope of normal $= -1$

$$\frac{dy}{dx} = \frac{x}{2}$$

$$-\frac{dx}{dy} = \frac{-2}{x} = -1$$

$$x = 2$$

Hence, B is correct

55. $x = \cos^3 t$

$$\frac{dx}{dt} = -3\cos^2 t \sin t$$

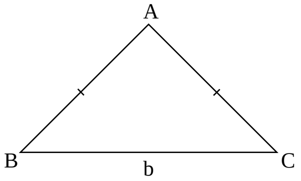
$$y=\sin^2 t$$

$$\frac{dy}{dt}=\sin 2t$$

$$\frac{dy}{dx}=\frac{2\sin t\cos t}{-3\cos^2 t\sin t}=\frac{-2}{3}\sec t=\frac{-4}{3}$$

Hence, B is correct

56.



$$A=\frac{1}{2}\,b\sqrt{x^2-\frac{b^2}{4}}$$

$$\frac{dA}{dt}=\frac{b}{2}\;\frac{\times 2x}{\times 2\sqrt{x^2-b^2/4}}\frac{dx}{dt}$$

$$=\sqrt{3}b$$

Hence, B is correct

57. $y=x^3+ax$

$$\frac{dy}{dx}=3x^2+a$$

$$\frac{dy}{dx}=3+a$$

$$y=bx^2+c$$

$$\frac{dy}{dx}=2bx=-2b$$

$\Rightarrow 3+a=-2b$

$\Rightarrow 3+a+2b=0$ (I)

As $-1-a=0$

$\Rightarrow a=-1$

$b+c=0$

$b=-1$

$c=1$

Then, $(a+b+c^2)=-1-1+1=-1$

Hence, D is correct

58. $y^2-2y-8x+17=0$

$$2y\frac{dy}{dx}-2\frac{dy}{dx}-8=0$$

$$\frac{dy}{dx}=\frac{y}{y-1}=1$$

$\Rightarrow y=5,\quad x=4$

Hence, B is correct

59. $y=\dfrac{x^2}{4}-2$

$$\frac{dy}{dx}=\frac{x}{2}$$

$$\left(y-\frac{x_1^2}{4}+2\right)=\frac{-2}{x}(x-x_1)$$

$$-1-\frac{x_1^2}{4}+2=\frac{-2}{x_1}(1-x_1)$$

$$\Rightarrow 2x_1-\frac{x_1 3}{4}=-2+2x_1$$

$x_1^3=8$

$x_1=2,\quad y=-1$

Hence, A is correct

60. $g(x)=x\,f(x)$

$g'(x)=f(x)+xf'(x)$

$g'(2)=13$

& $g(2)=6$

$y-6=13(x-2)\rightarrow$ eqn of tangent

For y int, $x=0$

$y=-20$

Hence, C is correct

61. $x\sin y+y\sin x=\pi$

$$\sin y+x\cos y\frac{dy}{dx}+\frac{dy}{dx}\sin x+y\cos x=0$$

$$\frac{dy}{dx}=\frac{-(\sin y+y\cos x)}{x\cos y+\sin x}=-1$$

eqn of tangent $\rightarrow y=-x+\pi$

Hence, A is correct

62. $y=-t+e^{at}=0$

$x=t+e^{at}$

$$\frac{dy}{dt}=-1+ae^{at}$$

$$\frac{dx}{dt}=1+ae^{at}$$

$$\frac{dy}{dx}=\frac{ae^{at}-1}{ae^{at}+1}=0 \qquad \text{(As touches x-axis)}$$

$\Rightarrow e^{at} = \frac{1}{a}$

$\Rightarrow at = -\ln a$

$t = -\frac{\ln a}{a}$

$x = -\frac{\ln a}{a} + \frac{1}{a} = \frac{\ln e/a}{a} = \frac{2}{a}$

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

63. $xy^2 = 1$

$y^2 + 2xy\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y}{2x}$

$-\frac{dx}{dy} = \frac{2x}{y} = \frac{2}{y^3}$

$y - y_1 = \frac{2}{y_1^3}(x - x_1)$

$+y_1^4 = 2x_1$

$y_1^6 = 2$

$y_1 = \pm 2^{1/6}$

$x_1 = \pm 2^{-1/3}$

Hence, B, C is correct

64. $xy = 1$

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

$-\frac{dx}{dy} = \frac{x_1}{y_1} = x_1^2$

$\Rightarrow x_1^2 = \frac{a-3}{a} \geq 0$

$a \in (-\infty, 0)\ U\ (3, \infty)$

Hence, A, D is correct

65. $y = \frac{ax}{1+x} = a - \frac{a}{1+x}$

$\frac{dy}{dx} = \frac{a}{(1+x)^2} = -1$

$a = -(1+x_1)^2$

eqn of tangent

$y - \frac{ax_1}{1+x_1} = -1(x - x_1)$

$y + x = x_1 + \frac{ax_1}{1+x_1}$

& $y - 3 = -x \Rightarrow y + x = 3$

$\Rightarrow x_1 + \frac{ax_1}{1+x_1} = 3$

$\Rightarrow x_1 - x_1(1+x_1) = 3$

66. $y = x\sqrt{1-x^2}$

$\frac{dy}{dx} = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

$\Rightarrow x = \pm 1$

Hence, A, C are correct

67. $y = 2x^2 - x,\quad y = 2 - x^3$

Pt of intersection are $2x^2 - x = 2 - x^3$

$\Rightarrow \quad 2(x^2 - 1) + x(x^2 - 1) = 0$

$\Rightarrow (x^2 - 1)(x + 2) = 0$

$\Rightarrow x = \pm 1, -2$

$\Rightarrow (1,1), (-1,3), (-2,10)$

For $y = 2x^2 - x$

$\frac{dy}{dx} = 4x - 1$

For $y = 2 - x^3$

$\frac{dy}{dx} = -3x^2$

At $(1,1)$

$\theta = \tan^{-1}\left(\frac{3+3}{1-9}\right) = \tan^{-1}\left(\frac{3}{-4}\right)$

At $(-1,3)$

$$\theta = \tan^{-1}\left(\frac{-5+3}{1+15}\right) = \tan^{-1}\left(\frac{1}{8}\right)$$

At $(-2;10)$

$$\theta = \tan^{-1}\left(\frac{-9+12}{1+108}\right) = \tan^{-1}\frac{3}{109}$$

Hence, A, B, D is correct

68. $f(x) = \dfrac{x-1}{2-x} = -1 + \dfrac{1}{2-x}$

$$f'(x) = +\frac{1}{(2-x)^2} = \frac{+1}{4}$$

$$\Rightarrow \quad x-2 = \pm 2$$

$x = 0$ or 4

$$\text{Pt} \to \left(0, \frac{-1}{2}\right), \quad \& \quad \left(4, \frac{-3}{2}\right)$$

Hence, C, D is correct

69. $\dfrac{dh_a}{dt} = 0.5t + 2$

$$\text{Integrating } = \frac{t^2}{4} + 2t + 5$$

$$\frac{dh_b}{dt} = t + 1$$

$$\text{Integrating, } h_b = \frac{t^2}{2} + t + 5$$

Hence, B, C are correct

70. $x^2 + 2xy + 2y^2 = 45$ (I)

$$x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(x+y)}{x+2y} = -2$$

$$\Rightarrow \quad x + y = 2x + 4y$$

$\Rightarrow x + 3y = 0$ (II)

Put eq (II) in eq (I)

$$9y^2 - 6y^2 + 2y^2 = 45$$

$$y = \pm 3$$

$$x = \pm 9$$

$$\text{Pt} \to (9,-3), \quad (-9,3)$$

Hence, B, D is correct

71. Similar to Q·67

Hence, A, B, D is correct

72. $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{\left(1-x^2\right) - x(-2x)}{\left(1-x^2\right)^2} = \frac{1+x^2}{\left(1-x^2\right)^2} = 1$$

$$1 + x^2 = 1 + x^4 - 2x^2$$

$$\Rightarrow \quad x^2\left(x^2 - 3\right) = 0$$

$$\Rightarrow x = 0, \pm\sqrt{3}$$

$$\text{Pt} \to (0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right), \left(-\sqrt{3}, \frac{\sqrt{3}}{2}\right)$$

73. For $y = x(c-x)$

$$\frac{dy}{dx} = c - 2x$$

For $y = x^2 + ax + b$

$$\frac{dy}{dx} = 2x + a$$

At (1, 0)

$$0 = (C-1) \Rightarrow C = 1$$

$$0 = 1 + a + b, \; 1 - 2 = 2 + a$$

$$\Rightarrow \quad a = -3$$

$$\Rightarrow b = 2$$

Hence, A, C, D is correct

74. $\ln\left(x^2 + y^2\right) = c\tan^{-1} y/x$

$$\frac{2x + 2y\,dy/dx}{x^2 + y^2} = c \times \frac{x^2}{x^2 + y^2} \times \frac{(x\,dy/dx - y)}{x^2}$$

$$2x + 2y\frac{dy}{dx} = c\left(x\frac{dy}{dx} - y\right)$$

$$\frac{dy}{dx} = \frac{2x + cy}{cx - 2y}$$

$$\tan\theta = \frac{\dfrac{2x+cy}{cx-2y} - \dfrac{y}{x}}{1 + \left(\dfrac{2x+cy}{cx-2y}\right)\dfrac{y}{x}}$$

$$= \frac{2x^2 + 2y^2}{c\left(x^2 + y^2\right)} = \frac{2}{c}$$

Hence, A, B is correct

75. $\alpha y^2 = (x+\beta)^3$

$$2\alpha y\frac{dy}{dx} = 3(x+\beta)^2$$

$$y\frac{dy}{dx} = \frac{3(x+\beta)^2}{2\alpha} = \text{Sub Normal}$$

$$y\frac{dx}{dy} = \frac{2\alpha y^2}{3(x+\beta)^2} = \text{Sub Tangent}$$

Square of sub tangent$=\dfrac{4\alpha^2(x+\beta)^4}{9}$

Hence, A is correct

76. $y = x^3 - x^2 - x + 2$

$$\frac{dy}{dx} = 3x^2 - 2x - 1 = 0 \text{ at } \quad x = 1$$

eqn of tangent

$y = 1$

solving with curve,

$$x^3 - x^2 - x + 1 = 0$$

$$(x-1)\left(x^2-1\right) = 0$$

$x = \pm 1$

Hence, C is correct

77. $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

At $x = 1, \quad y = 1$

$\Rightarrow a + b + c = 1$

$2a + b = 1$

$f''(1) + f'(0) = 2a + b = 1$

Hence, A is correct

78. $y\dfrac{dy}{dx} = y\dfrac{dx}{dy}$

$$\Rightarrow \frac{dy}{dx} = \pm 1$$

$$\text{Length of normal} = y_1\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{2}y$$

Hence, A is correct

79. $(x-4)^2 + y^2 = c^2$

$$2(x-4) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4-x}{y}$$

For $y^2 = x^3 + 1$

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\Rightarrow \frac{4-x}{y} = \frac{3x^2}{2y}$$

$$\Rightarrow 3x^2 + 2x - 8 = 0$$

$$\Rightarrow (3x-4)(x+2) = 0$$

$$x = \frac{4}{3} \text{ or } -2$$

Hence, D is correct

80. $x = 2 - 3\sin\theta, \quad y = 3 + 2\cos\theta$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1 \Rightarrow \text{eqn of ellipse}$$

end pts of major axis → (–1, 3) & (5, 3)

Hence, C is correct

Comprehension 1:

81. $s = 30(1-e^{kt})$

$$5 = 30(1-e^{k})$$

$$e^k = \frac{5}{6}$$

$$k = \ln(5/6)$$

Hence, B is correct

82. $S = 30(1-e^{kt})$

$$\frac{dS}{dt} = -30ke^{kt} > 0 \quad (\text{as } k < 0)$$

Max S when $t \to \infty$

Max $S = 30$

Hence, A is correct

83. $S = 30\left(1-e^{(\ln 5/6)5}\right) = 30\left(1-\left(\dfrac{5}{6}\right)^5\right)$

Hence, C is correct

Comprehension-2:

84. $f(0) < 0$

$\Rightarrow a^2 + a - 2 < 0$

$\Rightarrow a \in (-2, 1)$

Integral values of a are –1, 0.

Hence, B is correct.

85. $y = \left(\frac{x}{2} - a\right)^2 + a - 2$

$4(y - (a-2)) = (x - 2a)^2$

Vertex $\Rightarrow h = 2a,\ k = a - 2$

Locus of vertex $\Rightarrow y = \frac{x}{2} - 2$

$\Rightarrow 2y = x - 4$

Hence, A is correct

86. $y = \frac{x^2}{4} - 3x + 10$ (I)

For $y = 2 - \frac{x^2}{4}$

eqn of tangent

$y - 2 = mx + m^2$ (II)

Solving (II) & (I)

$mx + 2 + m^2 = \frac{x^2}{4} - 3x + 10$

$\Rightarrow \frac{x^2}{4} - (m+3)x + \left(8 - m^2\right) = 0$

$D = 0$

$(m+3)^2 - \left(8 - m^2\right) = 0$

$2m^2 + 6m + 1 = 0$

$m_1 + m_2 = -3$

Hence, B is correct.

Comprehension-3:

87. eqn of Pθ

$$y - at_1^3 = \frac{a\left(t_2^3 - t_1^3\right)}{a\left(t_2^2 - t_1^2\right)}\left(x - at_1^2\right)$$

$$y - at_1^3 = \frac{t_2^2 + t_1^2 + t_1t_2}{t_1 + t_2}\left(x - at_1^2\right)$$

It passes through $R(at^2, at^3)$

$$\Rightarrow t^3 - t_1^3 = \frac{t_1^2 + t_2^2 + t_1t_2}{t_1 + t_2}\left(t^2 - t_1^2\right)$$

$$\Rightarrow \left(t^2 + t_1^2 + tt_1\right)\left(t_1 + t_2\right) = \left(t + t_1\right)\left(t_1^2 + t_2^2 + t_1t_2\right)$$

$$\Rightarrow \quad t^2t_1 + t_1^3 + tt_1^2 + t^2t_2 + t_1^2t_2 + tt_1t_2 = tt_1^2 + tt_2^2$$

$$+tt_1t_2 + t_1^3 + t_1t_2^2 + t_1^2t_2$$

$$\Rightarrow \quad t^2t_1 + t^2t_2 = tt_2^2 + t_1t_2^2$$

$$\Rightarrow \left(t_1 + t_2\right)t^2 - t_2^2(t) - t_1t_2^2 = 0$$

$$\Rightarrow \quad t = \frac{t_2^2 \pm \sqrt{t_2^4 + 4\left(t_1 + t_2\right)t_1t_2^2}}{t_1 + t_2}$$

$$t = \frac{t_2^2 + t_2\sqrt{t_2^2 + 4t_1^2 + 4t_1t_2}}{2\left(t_1 + t_2\right)}$$

$$t_2 = \frac{t_2^2 \pm \left(t_2 + 2t_1\right)t_2}{2\left(t_1 + t_2\right)}$$

$$t = \frac{-t_1t_2}{t_1 + t_2}$$

Hence, C is correct

88. $ay^2 = x^3$

$2ay\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3a^2t^4}{2a^2t^3} = \frac{3}{2}t$

eqn of tangent

$y = at^3 = \frac{3}{2}t\left(x - at^2\right)$

$\Rightarrow 2y - 2at^3 = 3tx - 3at^3$

$2y = 3tx - at^3$

$\Rightarrow at^3 - (3x)t + 2y = 0$

$t_1 + t_2 + t_3 = 0$ (I)

$t_1t_2 + t_1t_3 + t_2t_3 = -\frac{3x}{a}$ (II)

Using (II) & (I)

$t_1t_2 + t_3\left(t_1 + t_2\right) = \frac{-3x}{a}$

$t_1t_2 - \left(t_1 + t_2\right)^2 = \frac{-3x}{a}$

$t_1^2 + t_2^2 + t_1t_2 = \frac{3x}{a}$

Hence, B is correct

89. eqn of tangent P & Q are

$$2y = 3t_1x - at_1^3$$
$$2y = 3t2x - at_2^3$$

Solving

$$3(t_1 - t_2)x + a(t_2^3 - t_1^3) = 0$$
$$3x = a(t_1^2 + t_2^2 + t_1t_2) \qquad \text{(I)}$$
$$2(t_2 - t_1)y = a(t_1t_2^3 - t_1^3t_2)$$
$$2y = at_1t_2(t_2 + t_1) \qquad \text{(II)}$$

from (I) & (II)

$$3tx + 2y = at(t_1^2 + t_2^2 + t_1t_2) + at_1t_2(t_2 + t_1)$$
$$= \frac{-at_1t_2}{t_1 + t_2}\left[t_1^2 + t_2^2 + t_1t_2 - (t_1 + t_2)^2\right]$$
$$= \frac{+a(t_1t_2)^2}{t_1 + t_2}$$
$$\Rightarrow (3tx + 2y)^2 = a^2t^2(t_1t_2)^2$$
$$= -\frac{2yt}{a}a^2t^2$$
$$= -2at^3y$$

Hence, B is correct

Comprehension - 4:

$v = 0.85\sin\left(\frac{\pi t}{3}\right)$ has a period of 6

90. Volume inhaled $= \int_0^6 0.85\sin\frac{\pi t}{3}dt$

$$= -0.85\cos\frac{\pi t}{3} \times \frac{3}{\pi}\int_0^6$$
$$= \frac{2.3}{\pi}(0.85) = \frac{5.1}{\pi}$$

Hence, A is correct

91. At exercise, respiratory cycle has a period of 4.

Hence difference in frequency is

$$\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Hence, C is correct

92. Volume inhaled while exercising $= \int_0^4 1.75\sin\frac{\pi t}{2}dt$

$$= -1.75\left(\cos\frac{\pi t}{2}\right) \times \frac{2}{\pi}$$
$$= \frac{7}{\pi}$$

Difference $= \frac{7}{\pi} - \frac{5.1}{\pi}$

$$= \frac{1.9}{\pi}$$

Hence, B is correct

Comprehension 5:

$x = a(2\cos t + \cos 2t), \qquad y = a(2\sin t - \sin 2t)$

$$\frac{dx}{dt} = a(-2\sin t - 2\sin 2t)$$
$$= -2a[\sin t + 2\sin 2t]$$
$$= -2a\sin t[1 + 2\cos t]$$
$$\frac{dy}{dt} = a[2\cos t - 2\cos 2t]$$
$$= -2a\left[2\cos^2 t - \cos t - 1\right]$$
$$= -2a(\cos t - 1)(2\cos t + 1)$$
$$\frac{dy}{dx} = \frac{(\cos t - 1)(2\cos t + 1)}{(\sin t)(1 + 2\cos t)} = -\tan t/2$$

slope of normal $= \cot t/2$.

93. eqn of normal

$$\sin t/2\, y - a\left[2\sin\frac{t}{2}\sin t - \sin 2t\sin\frac{t}{2}\right]$$
$$= x\cos\frac{t}{2} - a\left[2\cos t\cos\frac{t}{2} + \cos 2t\cos\frac{t}{2}\right]$$
$$\Rightarrow x\cos\frac{t}{2} - y\sin\frac{t}{2} = a\left[\begin{array}{l} 2\cos t\cos\frac{t}{2} - 2\sin t\sin\frac{t}{2} \\ +\cos 2t\cos\frac{t}{2} + \sin 2t\sin\frac{t}{2} \end{array}\right]$$
$$= a\left[2\cos\frac{3t}{2} + \cos\frac{3t}{2}\right]$$
$$= 3a\cos\frac{3t}{2}$$

Hence, B is correct

94. Subnormal $= y\frac{dy}{dx} = \left|y\tan\frac{t}{2}\right|$

Hence, D is correct

95. Eqn of tangent

$$\cos\frac{t}{2}y - a\left[2\cos\frac{t}{2}\sin t - \sin 2t\cos\frac{t}{2}\right] =$$

$$-\sin\frac{t}{2}x + a\left[2\sin\frac{t}{2}\cos t + \cos 2t\sin\frac{t}{2}\right]$$

$$\left(\cos\frac{t}{2}\right)y + \left(\sin\frac{t}{2}\right)x = a\begin{bmatrix}2\left[\sin\frac{t}{2}\cos t + \cos\frac{t}{2}\sin t\right]\\ +\cos 2t\sin\frac{t}{2} - \sin 2t\cos\frac{t}{2}\end{bmatrix}$$

$$= a\left[2\sin\frac{3t}{2} - \sin\frac{3t}{2}\right]$$

$$= a\sin\frac{3t}{2}.$$

so, $p = a\sin\frac{3t}{2},\ p_1 = 3a\cos\frac{3t}{2}.$

$9p^2 + p_1^2 = 9a^2$

Hence, A is correct

MATCH THE COLUMNS FOR JEE ADVANCED

96. A) As $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$\Rightarrow a = 2R\sin A$

$\Rightarrow \dfrac{da}{dA} = 2R\cos A$

$\Rightarrow \dfrac{da}{\cos A} = 2RdA$

Similarly, $\dfrac{db}{\cos B} = 2RdB, \dfrac{dc}{\cos c} = 2RdC$

$\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} + 1 =$

$2R(dA + dB + dC) + 1$ (I)

As $A + B + C = \pi$

$\Rightarrow dA + dB + dC = 0$

using eq (I)

$|m| = 1$

$\Rightarrow m = \pm 1$

B) $x^2y^2 = 16$

$2x^2y\dfrac{dy}{dx} + 2xy^2 = 0$

$\dfrac{dy}{dx} = -\dfrac{y}{x} = 1$

Subtangent $= y / dy / dx = 2 = |k|$

$\Rightarrow k = \pm 2$

C) $y = 2e^{2x}$

$\dfrac{dy}{dx} = 4e^{2x} = 4$ for y axis

$$\tan^{-1}4 = \frac{\pi}{2} - \cot^{-1}\left(\frac{8n-4}{3}\right) = \tan^{-1}\left|\left(\frac{8n+4}{3}\right)\right|$$

$12 = |8n - 4|$

$n = 2$ or -1

D) $x = e^{\sin y}$

$1 = e^{\sin y}\cos y\dfrac{dy}{dx}$

At $(1, 0) \Rightarrow \dfrac{dy}{dx} = 1$

eqn of Normal $\rightarrow y = -(x - 1)$

$y + x - 1 = 0$

Area of $\Delta = \dfrac{1}{2}\times 1\times 1 = \dfrac{1}{2}$

$\Rightarrow |2t + 1| = 3$

$2t + 1 = \pm 3 \Rightarrow t = 1$ or -2

A → (PQ), B → (RS), C → (RQ), D → (PS)

97. A) $A = \pi r^2$

$dA = 2\pi r dr$

Approximate increase in Area $= 0.72\pi$

B) $v = x^3$

$dv = 3x^2dx = 3(0.02)x^3$

% inc $= \dfrac{dv}{v}\times 100 = 6$

C) $y = \dfrac{x^2}{2} - 2x + 5$

$\dfrac{dy}{dt} = x\dfrac{dx}{dt} - 2\dfrac{dx}{dt}$

$\Rightarrow +3\dfrac{dx}{dt} = x\dfrac{dx}{dt} - 2\dfrac{dx}{dt}$

$\Rightarrow x = 5$

D) $A = \frac{\sqrt{3}}{4}x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt} = \frac{\sqrt{3}}{2}\times 30\times 0.1 = \frac{3\sqrt{3}}{2}$$

$A \to (Q), B \to (R), C \to (P), D \to (S)$

98. A) Area of ABCD $= 2Ar(\Delta ABC)$

$$= 2\times\frac{1}{2}\times BC\times AB$$

$$= 3\times 2 = 6$$

B) $f(x) = \frac{1}{\ln|x|}$ is discontinous at $x = 0, \pm 1$

C) $f'(x) = \lim\limits_{n\to 0}\frac{f(x+n)-f(x)}{n} = \lim\limits_{n\to 0} f(x)\left(\frac{f(n)-1}{n}\right)$

$= f(x)f'(0)$

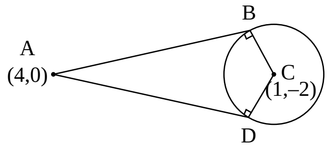
Put $x = 5$

$f'(5) = f(5)f'(0) = 6$

D) $\tan\frac{3\pi}{4} = -\frac{1}{f'(3)}$

$\Rightarrow f'(3) = 1$

$A \to (R),\ B \to (Q),\ C \to (R),\ D \to (P)$



99. A) $2y^2 = ax^2 + b$

$4y\frac{dy}{dx} = 2ax$

At$(1,-1)$, $\frac{dy}{dx} = \frac{-a}{2} = -1$

$\Rightarrow a = 2 \quad \& \quad 2 = 2 + b$

$\Rightarrow b = 0$

So, $a - b = 2$

B) $9y^2 = x^3$

$18y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{x^2}{6y} = \frac{a^2}{6b}$

$\frac{-dx}{dy} = \frac{-6b}{a^2} = \pm 1$

Now as $9b^2 = a^3$

$\Rightarrow \frac{-6a^{3/2}}{3a^2} = \pm 1$

$\Rightarrow a^{1/2} = \pm 2$

$a = 4 \quad \& \quad b = \pm\frac{8}{3}$

$a - b = \frac{4}{3}$ & $a + |b| = \frac{20}{3}$

C) $y = ax^2 + bx + \frac{7}{2}$

D) $y^2 - 3xy + 2 = 0$

$2y\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$

$\frac{dy}{dx} = \frac{3y}{2y - 3x}$

Horizontal tangent $\to y = 0$, but no value of x

Hence, zero tangents

$\Rightarrow a = 0$

Vertical tangent $\Rightarrow 2y = 3x$

$y^2 - 3y\left(\frac{2y}{3}\right) + 2 = 0$

$y^2 - 2y^2 + 2 = 0$

$y = \pm\sqrt{2}$

$x = \pm\frac{2\sqrt{3}}{3}$

$\Rightarrow b = 2.$

100. A) $y = 1 - \cos x$

$\frac{dy}{dx} = \sin x$

& $y = \frac{\sqrt{3}}{2}|x| + a.$

Now, $\sin x = \frac{\sqrt{3}}{2}$ for touching of two curves

Hence, there are two values of 'a'

B) For $y^2 = 4a(x - b_1)$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y_1}$ & $y_1^2 = 4a\left(x_1 - b_1\right)$

For $x^2 = 4a\left(y - b_2\right)$

$$\frac{dy}{dx} = \frac{x_1}{2a} \quad \& \quad x_1^2 = 4a\left(y_1 - b_2\right)$$

Now, $x_1 y_1 = 4a^2$

$\Rightarrow k = 4.$

C) Normal to parabola →

$y = mx - 2m - m^3$

2 passes through (0,12).

$$m^3 + 2m + 12 = 0$$

$$(m+2)\left(m^2 - 2m + 6\right) = 0$$

$$\Rightarrow m = -2.$$

Ordinate $= -2m = 4$

D) $y^3 + 3x^2 - 12y = 0$

$$3y^2\frac{dy}{dx} + 6x - 12\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

As tangent is parallel to y-axis, $y^2 = 4$

$$\Rightarrow y = \pm 2$$

At $y = -2$, there is no value of x.

Hence, ordinate is 2 only.

A → (θ), B → (S), C → (S), D → (Q).

Review Exercises Solution

1. Let position of runner is $(100\cos\theta, 100\sin\theta)$

$$d_1 = \sqrt{(100)^2(\cos\theta + 2)^2 + (100)^2\sin^2\theta}$$

$$d_1 = 100\sqrt{5 + 4\cos\theta}$$

$$\frac{dd_1}{dt} = \frac{100}{2\sqrt{5 + 4\cos\theta}} \times -4\sin\theta\frac{d\theta}{dt} \qquad \text{(I)}$$

$x = 100\cos\theta$, $y = 100\sin\theta$

$$\frac{dx}{dt} = -100\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt} = 100\cos\theta\frac{d\theta}{dt}$$

$$7 = \sqrt{(100)^2\left(\frac{d\theta}{dt}\right)^2}$$

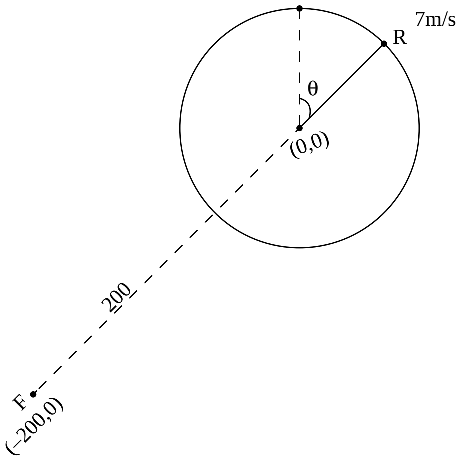
$$7 = 100\frac{d\theta}{dt} \qquad \text{(II)}$$

If $d_1 = 200$, $\cos\theta = \frac{-1}{4}$

Using (I)

$$\frac{dd_1}{dt} = \frac{100}{2\cdot 2} \times -4 \times -\frac{\sqrt{15}}{4} \times \frac{7}{100}$$

$$= \frac{7\sqrt{15}}{4}\text{ m/s}$$



2. a) $c(t) = 100t^2 + 400t + 50t\ln t$

$$\frac{d(ct)}{dt} = 200t + 400 + 50\ln t + 50$$

$$= 200t + 50\ln t + 450$$

b) Put $t = 5$

$$\frac{d(ct)}{dt} = 1450 + 50\ln 5 = 1530$$

c) $c(6) = 3600 + 2400 + 300\ln 6 = 6537.$

$c(5) = 2500 + 2000 + 250\ln 5 = 4902.$

Net change = 1635 newspaper

3. $\dfrac{xy^3}{1+y^2}=\dfrac{8}{5}$

$5xy^3=8+8y^2$

$5y^3\dfrac{dx}{dt}+15xy^2\dfrac{dy}{dt}=16y\dfrac{dy}{dt}$

a) At $(1,2), \dfrac{dx}{dt}=6$

$\Rightarrow 240+60\dfrac{dy}{dt}=32\dfrac{dy}{dt}$

$\Rightarrow \dfrac{dy}{dt}=\dfrac{-240}{28}=\dfrac{-60}{7}$ units/second

b) Falling

4. a) $y=10\left(e^{x/20}+e^{-x/20}\right)-15$

$\dfrac{dy}{dx}=10\left[\dfrac{1}{20}\left[e^{x/20}-e^{-x/20}\right]\right]$

$=0.35$ for $x=7$

b) $\alpha=19\cdot 7^\circ$

$\theta=70\cdot 3^\circ$

5. $A=\pi r^2$ As $\dfrac{\Delta r}{r}\times 100=2$

$\Delta r=0\cdot 2$

$dA=2\pi r dr$

$dr=0\cdot 2$

$dA=20\pi\times 0.2=4\pi$

6. $s_1=\dfrac{t^2}{2}-t+3,\quad s_2=\dfrac{-t^2}{4}+t+1$

a) To collide, $\dfrac{t^2}{2}-t+3=-\dfrac{t^2}{4}+t+1$

$\Rightarrow 2t^2-4t+12=-t^2+4t+4$

$\Rightarrow 3t^2-8t+8=0$

$D<0$

Hence, they will not collide.

b) Shortest distance between

$\dfrac{ds_1}{dt}=t_1-1\quad,\quad \dfrac{ds_2}{dt}=-\dfrac{t_2}{2}+1=\dfrac{2-t_2}{2}$

$2t_1+t_2=4$ (I)

$y-\dfrac{t_1^2}{2}+t_1-3=\dfrac{-1}{t_1-1}(x-t_1)$

$\left(y+\dfrac{t_2^2}{4}-t_2-1\right)=\dfrac{-2}{2-t_2}(x-t_2)$

$$1=\frac{2-t_2}{2(t_1-1)}=\frac{\dfrac{t_1}{t_1-1}+\dfrac{t_1^2}{2}-t_1+3}{\dfrac{2t_2}{2-t_2}-\dfrac{t_2^2}{4}+t_2+1}$$

7. a) $A(x)=\dfrac{p(x)}{x}$

$A'(x)=\dfrac{xp'(x)-p(x)}{x^2}$

If $A'(x)>0$, productivity per worker increases with increase in workers

b) $A'(x)=\dfrac{p'(x)-p(x)/x}{x}$

As x is always +ve, $A'(x)>0$ if $p'(x)\rightarrow\dfrac{p(x)}{x}=A(x)$

8. $xy+ax+by=0$

$x\dfrac{dy}{dx}+y+a+b\dfrac{dy}{dx}=0$

$\dfrac{dy}{dx}=-\dfrac{(a+y)}{(b+x)}=-\dfrac{(a+1)}{(b+1)}=2$

$\Rightarrow 2b+a+3=0$ (I)

$1+a+b=0$ (II)

Solving $b=-2$

$a=1$

9. $y=x^{1/3}(1-\cos x)$

$\dfrac{dy}{dx}=\dfrac{1}{3}x^{-2/3}(1-\cos x)+\sin x\, x^{1/3}$

$\dfrac{dy}{dx}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^{2/3}}=0$

Eqn of tangent $\Rightarrow y=0$

Eqn of normal $\Rightarrow x=0$

10. $f(x)=\begin{cases}x-1, & x<1\\ x^2-x-1, & x\ge 1\end{cases}$

$f'(x)=1$ at $x=1$

eqn of tangent $\Rightarrow y+1=1(x-1)$

$y=x-2$

eqn of normal $\Rightarrow$

$y=-x\Rightarrow x+y=0$

11. eqn of normal $\rightarrow y=m_1x-2am_1-am_1^3$

$m\ y=-l\ x+1$

Comparing two eqn

$$\frac{1}{m} = \frac{m_1}{-l} = \frac{-2am_1 - am_1^3}{1}$$

$$\frac{1}{m} = 2a\frac{l}{m} + \frac{al^3}{m^3}$$

$$\Rightarrow al^3 + 2alm^2 = m^2.$$

Hence Proved.

12. $y = x\cos x$

$$\frac{dy}{dx} = \cos x - x\sin x = 0$$

$$x = \cot x.$$

For $y = \dfrac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2} = 0$$

$$x = \tan x$$

13. $y = \sin^{-1}\dfrac{2x}{1+x^2} = \pi - 2\tan^{-1}x$

$$\frac{dy}{dx} = \frac{-2}{1+x^2} = -\frac{1}{2} \quad \text{at} \quad x = \sqrt{3}$$

$$y - \frac{\pi}{3} = \frac{-1}{2}(x - \sqrt{3})$$

14. $y = (x+1)^3$

$$\frac{dy}{dx} = 3(x+1)^2$$

eqn of tangent $\Rightarrow y - y_1 = 3(x_1+1)^2(x - x_1)$

As it passes through ((0,0)

$$y_1 = 3(x_1+1)^2 x_1$$

$$\Rightarrow (x_1+1)^3 = 3(x_1+1)^2 x_1$$

$$\Rightarrow (x_1+1)^2(x_1+1-3x_1) = 0$$

$$x_1 = -1 \text{ or } \frac{1}{2}$$

$$y_1 = 0 \text{ or } \frac{27}{8}$$

$$\Rightarrow y = 0 \quad \text{or} \quad y - \frac{27}{8} = 3\left(\frac{9}{4}\right)\left(x - \frac{1}{2}\right)$$

$$\Rightarrow y = 0 \quad \text{or} \quad 8y - 27 = 2.27x - 27$$

$$\Rightarrow y = 0 \quad \text{or} \quad 4y = 27x.$$

15. $x = a(\cos(\theta + a)$

$$\frac{dx}{d\theta} = -a\sin(\theta + a)$$

$$y = b\cos(\theta + \beta)$$

$$\frac{dy}{d\theta} = -b\sin(\theta + \beta)$$

$$\frac{dy}{dx} = \frac{b\sin(\theta_1 + \beta)}{a\sin(\theta_1 + a)} \qquad \text{(I)}$$

radius drawn to $\theta_2 = \dfrac{b\cos(\theta_2 + \beta)}{a\cos(\theta_2 + a)}$

Now, $\dfrac{b\sin(\theta_1+\beta)}{a\sin(\theta_1+a)} = \dfrac{b\cos(\theta_2+\beta)}{a\cos(\theta_2+a)}$

$$\Rightarrow \cos(\theta_2 \pm a)\sin(\theta_1+\beta) - \cos(\theta_2+\beta)\sin(\theta_1+a) = 0$$

$$\Rightarrow \theta_1 - \theta_2 = \pm\frac{\pi}{2}$$

16. $x^2 = 4a(y+b)$

$$2x = 4a\frac{dy}{dx}$$

$$\frac{-dx}{dy} = \frac{-2a}{x}$$

eqn of Normal $\rightarrow y - \left(\dfrac{x_1^2}{4a} - b\right) = \dfrac{-2a}{x_1}(x - x_1)$

$$x_1(x_1^2 - 4ab) = -8a^2x_1$$

$$\Rightarrow x_1(x_1^2 - 4ab + 8a^2) = 0$$

For three normals, $8a^2 - 4ab < 0$

$$4a(2a - b) < 0$$

$$\Rightarrow \quad 2a - b < 0$$

$$\Rightarrow b > 2a \text{ if } a > 0$$

17. $x^2 - y^2 = a^2$

$$x - y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x_1}{y_1}$$

$$-\frac{dx}{dy} = \frac{-y_1}{x_1}$$

$y - y_1 = \dfrac{-y_1}{x_1}(x - x_1) \rightarrow$ Eqn of normal

$x - \text{int} \Rightarrow 2x_1$

$y - \text{int} = 2y_1$

Difference of their square $= 4\left(x_1^2 - y_1^2\right) = 4a^2 =$ Constant

18. $x = kt$

$$\frac{dx}{dt} = k$$

$$y = \frac{k}{t}$$

$$\frac{dy}{dt} = \frac{-k}{t^2}$$

$$\frac{dy}{dx} = \frac{-1}{t^2}$$

$$y - \frac{k}{t} = \frac{-1}{t^2}(x - kt)$$

$$t^2 y + x = 2kt$$

19. $x^2 - xy + y^2 = 3$ crosses the x-axis at $x = \pm\sqrt{3}$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y} = 2 \text{ at } x = \pm\sqrt{3}$$

20. $y^2 = 4a(x - d)$

$$\frac{dy}{dx} = \frac{2a}{y_1}$$

$$y - y_1 = \frac{2a}{y_1}\left(x - x_1\right) \qquad \text{(I)}$$

$$x^2 = 4a(y - \mu)$$

$$\frac{x_1}{2a} = \frac{dy}{dx}$$

$$y - y_1 = \frac{x_1}{2a}\left(x - x_1\right) \qquad \text{(II)}$$

Comparing (I) & (II)

$$1 = \frac{4a^2}{x_1 y_1} = \frac{y_1 - \frac{2ax_1}{y_1}}{y_1 - \frac{x_1^2}{2a}}$$

$$\Rightarrow x_1 y_1 = 4a^2$$

21. $x = 2a\sin t + a\sin t\cos^2 t$

$$\frac{dx}{dt} = 2a\cos t + a\cos^3 t - a\sin t\sin 2t$$

$$y = -a\cos^3 t$$

$$\frac{dy}{dt} = +3a\cos^2 t\sin t$$

$$\frac{dy}{dx} = \frac{3\cos^2 t\sin t}{2\cos t + \cos^3 t - \sin t\sin 2t} = \frac{3\cos^2 t\sin t}{3\cos^3 t} = \tan t$$

eqn of normal

$y - y_1 = m\left(x - x_1\right)$, Where $m = -\cot t$

$$x\text{ int} = x_1 - \frac{y_1}{m}, \qquad y\text{ int} = y_1 - mx_1$$

$$x\text{ int} = 2a\sin t + a\sin t\cos^2 t + \frac{a\cos^3 t\sin t}{-\cos t} = 2a\sin t$$

$$y\text{ int} = -a\cos^3 t + 2a\cos t + a\cos^3 t = 2a\cos t$$

$$\text{Segment} = \sqrt{(x\text{ int})^2 + (y\text{ int})^2} = 2a$$

22. $\left(\frac{x}{a}\right)^{n/n-1} + \left(\frac{y}{b}\right)^{n/n-1} = 1$

$$\frac{1}{a}(x)^{1/n-1} + \frac{1}{b}(y)^{1/n-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b}{a}\left(\frac{x}{y}\right)^{1/n-1} = -\cot\alpha$$

$$P = y_1\sin\alpha + x_1\cos\alpha$$

eqn of tangent

$$y - y_1 = \frac{-b}{a}\left(\frac{x_1}{y_1}\right)^{1/n-1}\left(x - x_1\right)$$

$$\left(ay_1^{1/n-1}\right)y - a\left(y_1\right)^{n/n-1} = -b\left(x_1\right)^{1/n-1}x + bx_1^{n/n-1}$$

Comparing

$$\frac{\cos\alpha}{bx_1^{1/n-1}} = \frac{\sin\alpha}{ay_1^{1/n-1}} = \frac{p}{ay_1^{n/n-1} + bx_1^{n/n-1}}$$

$$\frac{a\cos\alpha}{x_1^{1/n-1}} = \frac{b\sin\alpha}{y_1^{1/n-1}} = \frac{p}{ay_1^{n/n-1} + bx_1^{n/(n-1)}}$$

$$\frac{(a\cos\alpha)^n + (b\sin\alpha)^n}{x_1^{n/n-1} + y_1^{n/n-1}} \qquad \frac{p^n}{\left(ay_1^{n/n-1} + bx_1^{n/(n-1)}\right)^n}$$

23. $x^2 + 4xy + 4y^2 - 1 = 0$

$$x + 2y + 2x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}=-\frac{(x+2y)}{2(x+2y)}=-\frac{1}{2}$$

$\Rightarrow m=2$

Differentiate wrt x Keeping y = constant

$$2x+4y=0$$

$$x+2y=0$$

24. $x^3-3xy^2+2=0$

$$3x^2-3y^2-6xy\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{x^2-y^2}{2xy}$$

$$3x^2y-y^3-2=0$$

$$6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\left(\frac{2xy}{x^2-y^2}\right)$$

As, $m_1m_2=-1$, Hence curves are orthogonal.

25. $y=1-\dfrac{1}{1+x^2}$

$$\frac{dy}{dx}=\frac{2x}{\left(1+x^2\right)^2}$$

& $y=2^{-x}$

$$\frac{dy}{dx}=-2^{-x}\ln 2$$

$$\tan\theta=\frac{\dfrac{2x}{\left(1+x^2\right)^2}+2^{-x}\ln 2}{1-\left(2^{-x}\right)\dfrac{2x}{\left(1+x^2\right)^2}\ln 2}$$

$$=\frac{2x+2^{-x}\left(1+x^2\right)^2\ln 2}{\left(1+x^2\right)^2-(2x)2^{-x}\ln 2} \qquad \text{(I)}$$

As $\dfrac{x^2}{1+x^2}=2^{-x}$

$\Rightarrow x=1$

using (I), $\tan\theta=\dfrac{2+2\ln 2}{4-\ln 2}$

26. $x^2=ay, y^2=2ax$

$$\Rightarrow\left(\frac{x^2}{a}\right)^2=2ax$$

$$\Rightarrow x^4-2a^3x$$

$$\Rightarrow x\left(x^3-2a^3\right)=0$$

$\Rightarrow x=0$ or $x=2^{1/3}a$

$\Rightarrow y=0$ or $y=+2^{4/3}a$

For $x^2=ay$

$$\frac{dy}{dx}=\frac{2x}{a}=2^{4/3}$$

For $y^2=2ax$

$$\frac{dy}{dx}=\frac{a}{y}=2^{-2/3}$$

For $x^3+y^3=3axy$.

$$x^2+y^2\frac{dy}{dx}=a\left(x\frac{dy}{dx}+y\right)$$

$$\frac{dy}{dx}=\frac{x^2-ay}{ax-y^2}=0$$

$\theta=\tan^{-1}\left(2^{4/3}\right)$ & $\tan^{-1}\left(2^{-2/3}\right)$

27. $y=\begin{cases}e^{-x}, & x\ge 0\\ e^{x}, & x<0\end{cases}$

Eqn of tangent $\Rightarrow y-y_1=m\left(x-x_1\right)$

$x-\text{int}=x_1-\dfrac{y_1}{m}$

$y-\text{int}=y_1-mx_1$

For $x\ge 0$

$$\text{Area}=\frac{1}{2}\left[x_1+\frac{e^{-x_1}}{e^{-x_1}}\right]\left[e^{-x_1}+e^{-x_1}x_1\right]$$

$$=\frac{1}{2}e^{-x_1}\left(x_1+1\right)^2$$

$$\frac{dA}{dx}=\frac{1}{2}e^{-x_1}(2)\left(x_1+1\right)-\frac{e^{-x_1}\left(x_1+1\right)^2}{2}$$

$$=\frac{e^{-x_1}\left(x_1+1\right)}{2}\left(2-x_1-1\right)$$

$$=\frac{e^{-x_1}(x_1+1)(1-x_1)}{2}$$

$\therefore$ Greatest area at $x_1=1$

As $f(x)$ is even function, thus, $x_1=-1$ will also be there.

Hence, $M\to\left(1,\frac{1}{e}\right)$ or $\left(-1,\frac{1}{e}\right)$

28. $y=x^4+3x^2+2x,\ y=2x-1$

$x^4+3x^2+2x=2x-1$

$x^4+3x^2+1=0$

As $D<0$, there is no such value of x

So these two functions will never intersect.

Now, $\frac{dy}{dx}=4x^3+6x+2$

$\Rightarrow 4x^3+6x+2=2$

$\Rightarrow\ 2x(2x+3)=0$

$\Rightarrow x=0$

$\Rightarrow y=0$

Min distance $=\frac{1}{\sqrt{5}}$

29. Let two lines be $y=mx,\ y=\frac{-x}{m}$

Point $P\to(m^2,m^3)$ & $Q\to\left(\frac{1}{m^2},\frac{-1}{m^3}\right)$

Now, $y^2=x^3$

$$\frac{dy}{dx}=\frac{3x^2}{2y}$$

Slope at $P=\frac{3m}{2}$ & $Q=\frac{-3}{2m}$

Equation of tangent at $P\Rightarrow 2y=3mx-m^3$

Equation of tangent at $Q\Rightarrow 2y=\frac{-3x}{m}+\frac{1}{m^3}$

Solving, $x=\frac{1}{3}\left(\left(m-\frac{1}{m}\right)^2+1\right)$

$y=-\frac{1}{2}\left(m-\frac{1}{m}\right)$

$\Rightarrow 4y^2=3x-1$

Hence $a+b=7$

30. $x^3+y^3=a^3$

$x^2+y^2\frac{dy}{dx}=0$

$$\frac{dy}{dx}=-\frac{x_1^2}{y_1^2}$$

Now, $-\frac{x_1^2}{y_1^2}=\frac{y_2-y_1}{x_2-x_1}$

$$\left(\frac{y_2^3-y_1^3}{x_2^3-x_1^3}\right)\times\frac{(x_1^2+x_1x_2+x_2^2)}{(y_1^2+y_1y_2+y_2^2)}=\frac{-x_1^2}{y_1^2}$$

$$-\frac{\left(x_1^2+x_1x_2+x_2^2\right)}{\left(y_1^2+y_1y_2+y_2^2\right)}=\frac{-x_1^2}{y_1^2}$$

$\Rightarrow x_1^2y_2^2-x_2^2y_1^2=x_1x_2y_1^2-x_1^2y_1y_2$

$\Rightarrow x_1y_2+x_2y_1=-x_1y_1$

$\Rightarrow\frac{x_2}{x_1}+\frac{y_2}{y_1}=-1$

31. $x=ae^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)$

$$\frac{dx}{d\theta}=ae^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)+ae^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)$$

$$=\frac{5}{2}ae^{\theta}\cos\frac{\theta}{2}$$

$y=ae^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$

$$\frac{dy}{d\theta}=ae^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)+ae^{\theta}\left(\frac{-1}{2}\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)$$

$$=-ae^{\theta}\frac{5}{2}\sin\frac{\theta}{2}$$

$$\frac{dy}{dx}=-\cot\frac{\theta}{2}$$

Eqn of tangent →

$$y-ae^{0}\left(\begin{matrix}\cos\frac{\theta}{2}\\ -2\sin\frac{\theta}{2}\end{matrix}\right)=-\cot\frac{\theta}{2}\left(x-ae^{\theta}\left(\begin{matrix}\sin\frac{\theta}{2}\\ +2\cos\frac{\theta}{2}\end{matrix}\right)\right)$$

$$\text{Distance from } (0,0) = \frac{ae^{\theta}\left(\begin{matrix}\cot\frac{\theta}{2}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)\\ +\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\end{matrix}\right)}{\operatorname{cosec}\frac{\theta}{2}}$$

$$= ae^{\theta}\left[\cos\frac{\theta}{2}\sin\frac{\theta}{2}+2\cos^2\frac{\theta}{2}+\sin\frac{\theta}{2}\cos\frac{\theta}{2}-2\sin^2\frac{\theta}{2}\right]$$

$$= ae^{\theta}[\sin\theta + 2\cos\theta]$$

Eqn of normal →

$$y - ae^{\theta}\left(\begin{matrix}\cos\frac{\theta}{2}\\ -2\sin\frac{\theta}{2}\end{matrix}\right) = \tan\frac{\theta}{2}\left(x - ae^{\theta}\left(\begin{matrix}\sin\frac{\theta}{2}\\ +2\cos\frac{\theta}{2}\end{matrix}\right)\right)$$

$$\text{Distance from } (0,0) = \frac{ae^{\theta}\left(\begin{matrix}\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\tan\frac{\theta}{2}-2\sin\frac{\theta}{2}\end{matrix}\right)}{\sec\frac{\theta}{2}}$$

$$= ae^{\theta}\left[\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= ae^{\theta}[\cos\theta - 2\sin\theta]$$

32. $y = x^3 + kx^2 + 3x - 4$

$\frac{dy}{dx} = 3x^2 + 2kx + 3$

$D = 0$

$4k^2 - 36 = 0$

$k = \pm 3$

33. $\frac{dy}{dx} = \frac{1}{2\sqrt{x_1}}$

$y - \sqrt{x_1} = \frac{1}{2\sqrt{x_1}}(x - x_1) \rightarrow$ eqn of tangent

Now, put $x = -1,\ y = 0$

$\Rightarrow -\sqrt{x_1} = \frac{-x_1}{2\sqrt{x_1}}$

$\Rightarrow 2x_1 = x_1$

$\Rightarrow x_1 = 0$

eqn of tangent $\Rightarrow x = 0$

34. i) $x = ye^y$

$\frac{dx}{dy} = e^y + ye^y$

$= 0$

$\Rightarrow y = -1,\quad x = \frac{1}{e}$

ii) No

iii) $\frac{dy}{dx} = \frac{1}{e^y(1+y)}$ & $x = ye^y$

As $x \to 0,\ y \to 0$ or $-\infty$

But if $y \to 0$, $\frac{dy}{dx}$ will not tend to $-\infty$

Hence, $y \to -\infty$

iv) $y \to \infty$, $\frac{dy}{dx}$ will tend to zero & x tends to ∞.

35. $y = x^5 + 2x$

$\frac{dy}{dx} = 5x^4 + 2 \neq 0$

Hence, no horizontal tangent.

smallest value of slope will exit at $x = 0$

$\Rightarrow \frac{dy}{dx} = 2$

Target Exercises Solution

1. Volume of water = $v = \frac{w \times l \times h}{2}$

Where h is height of water at its deepest point, l is the length of surface area, w is width.

$\Rightarrow w = 25$

As $\frac{l}{h} = \frac{40}{6} \Rightarrow l = \frac{20}{3}h$

$\Rightarrow V = \frac{250h^2}{3}$

$\Rightarrow \frac{dv}{dh} = \frac{500h}{3}$

$\Rightarrow \frac{dh}{dv} = \frac{3}{500h}$

Now $\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$= \frac{3}{500 \times 4} \times 10 = 0.015$ ft/min

$V = \frac{250}{3} \times 4 \times 4 = \frac{4000}{3}$ water is there

Total volume = 250 × 12 + 3 × 40 × 25

= 3000 + 3000 = 6000

% of water $= \frac{4000}{3 \times 6000} \times 100 = \frac{200}{9} = 22 \cdot 22\%$

2. We can assume that $\theta = 0$ at $t = 0$, so that $\theta = 12\pi t$

Position of

$A = (40\cos\theta, 40\sin\theta) = (40\cos(12\pi t), 40\sin(12\pi t))$

& $\sin\alpha = \frac{y}{1 \cdot 2} = \frac{40\sin\theta}{1 \cdot 2m} = \frac{\sin\theta}{3}$ rad/cm

(a) differentiating $\cos\alpha \frac{d\alpha}{dt} = 4\pi\cos\theta$

& when $\theta = \frac{\pi}{3}$

$\sin\theta = 3\sin\alpha = \frac{\sqrt{3}}{6} \Rightarrow \cos\alpha = \sqrt{\frac{11}{12}}$

& $\frac{d\alpha}{dt} = \frac{4\pi\cos\pi/3}{\cos\alpha} = \frac{4\pi\sqrt{3}}{\sqrt{11}} = 6 \cdot 56$ rad/s

(b) $|AP|^2 = |0A|^2 + |OP|^2 - 2|OA||OP|\cos\theta$

$\Rightarrow |OP| = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8}$

$= 40\cos\theta + 40\sqrt{\cos^2\theta + 8}$

$\Rightarrow 0P = 0$ at $\theta = 0$ & $80\sqrt{2}$ when $\theta = \frac{\pi}{2}$

(c) x-coordinate of P $\Rightarrow x = 40(\cos\theta + \sqrt{\cos^2\theta + 8})$

$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right)$

3. $\ln\left(x^2 + y^2\right) = c\tan^{-1}\frac{y}{x}$

Differentiating wrt x,

$\frac{2x + 2yy_1}{x^2 + y^2} = c\frac{1}{1 + \frac{y^2}{x^2}} \times \frac{xy_1 - y}{x^2}$

$\Rightarrow y_1 = \frac{2x + y}{x - 2y}$

Let point $P(h,k) \Rightarrow$ slope of tangent $= \frac{2h + k}{h - 2k}$

$$\tan\theta = \frac{\frac{2h+k}{h-2k} - \frac{k}{h}}{1 + \left(\frac{2h+k}{h-2k}\right)\frac{k}{h}}$$

$$= \frac{2h^2 + hk - hk + 2k^2}{h^2 - 2kh + 2hk + k^2} = 2$$

Hence proved.

4. $y = \frac{3}{\sqrt{2}} x \ln x$

$y' = \frac{3}{\sqrt{2}}(1 + \ln x)$

eqn of tangent at (x_1, y_1)

$y - y_1 = \frac{3}{\sqrt{2}}(1 + \ln x_1)(x - x_1)$

$\Rightarrow y - y_1 = \frac{3}{\sqrt{2}}(x - x_1) + \frac{3}{\sqrt{2}} x \ln x_1 - \frac{3}{\sqrt{2}} x_1 \ln x_1$

$\Rightarrow y = \frac{3}{\sqrt{2}}(1 + \ln x_1)x - \frac{3}{\sqrt{2}} x_1$

It will cut y axis at $\left(0, -\frac{3}{\sqrt{2}} x_1\right)$

Distance 'd' $= \sqrt{x_1^2 + \left(\frac{3}{\sqrt{2}} x_1 \ln x_1 + \frac{3}{\sqrt{2}} x_1\right)^2}$

$= x_1\sqrt{1 + \frac{9}{2}(1 + \ln x_1)^2}$

Now, $f(x) = x^2 + \frac{9}{2}x^2(1 + \ln x)^2$

$f'(x) = 2x + 9x(1 + \ln x)^2 + 9x(1 + \ln x)$

$\Rightarrow x\left(2 + 9 + 9\ln^2 x + 18\ln x + 9 + 9\ln x\right)$

$\Rightarrow x = 0$ or $9\ln^2 x + 27\ln x + 20 = 0$

$\Rightarrow \ln x = \frac{-27 \pm \sqrt{729 - 720}}{18}$

$= \frac{-27 + 3}{18}$ or $-\frac{30}{18}$

$\Rightarrow -4/3$ or $-5/3$

Shortest distance will occur at $x = e^{-4/3}$

$\Rightarrow y = \frac{3}{\sqrt{2}} \times -\frac{4}{3} e^{-4/3}$

$= -2\sqrt{2}e^{-4/3}$

5. Let pt $P_1\left(t_1, t_1^2\right) \& P_2\left(t_2, t_2^2\right) \& Q\left(t_3, t_3^2\right)$

Now, $y = x^2 \Rightarrow y' = 2x$

eqn of tangent at $t \Rightarrow y - t^2 = 2t(x - t) \Rightarrow y = 2tx - t^2$

Pt P⇒ point of intersection of T_1 & T_2 is soln of

$y = 2t_1x - t_1^2 \;\&\; y = 2t_2x - t_2{}^2$

$$\Rightarrow \text{ Point } P \equiv \left(\frac{t_1 + t_2}{2}, t_1t_2\right)$$

$$\text{Point } \theta_1 \equiv \left(\frac{t_1 + t_3}{2}, t, t_3\right) \quad \& \text{ Point } \theta_2 \equiv \left(\frac{t_2 + t_3}{2}, t_2t_3\right)$$

$$\text{Now } PQ_1 = \sqrt{\left(t_2 - t_3\right)^2 + t_1^2\left(t_2 - t_3\right)^2}$$

$$PP_1 = \sqrt{\frac{\left(t_2 - t_1\right)^2}{4} + t_1^2\left(t_2 - t_1\right)^2}$$

$$PQ_2 = \sqrt{\frac{(t_1 - t_3)^2}{4} + t_2^2(t_2 - t_3)^2}$$

$$PP_2 = \sqrt{\frac{\left(t_1 - t_2\right)^2}{4} + t_2^2\left(t_1 - t_2\right)^2}$$

$$\text{Now } \frac{PQ_1}{PP_1} = \sqrt{\frac{1 + 4t_1^2}{1 + 4t_1^2}} \times \frac{\left(t_2 - t_3\right)}{\left(t_2 - t_1\right)} = \left|\frac{t_2 - t_3}{t_2 - t_1}\right|$$

$$\frac{PQ_2}{PP_2} = \left|\frac{t_1 - t_3}{t_2 - t_3}\right|$$

$$\Rightarrow \frac{PQ_1}{PP_1} + \frac{PQ_2}{PP_2} = 1$$

Hence proved.

6. $y = x^4 - 2x^2 - x$

$\Rightarrow y' = 4x^3 - 4x - 1$

eqn of tangent at x = a is

$$y - \left(a^4 - 2a - a\right) = \left(4a^3 - 4a - 1\right)(x - a)$$

$$\Rightarrow y = \left(4a^3 - 4a - 1\right)x + \left(-3a^4 + 2a^2\right)$$

Similarly at x = b

$$y = \left(4b^3 - 4b - 1\right)x + \left(-3b^4 + 2b^2\right)$$

Comparing, $4a^3 - 4a - 1 = 4b^3 - 4b - 1 \;\&\; -3a^4 + 2a^2$

$= -3b^4 + 2b^2$

$\Rightarrow a^3 - b^3 = a - b$

$\Rightarrow 1 = a^2 + b^2 + ab$ .... (1)

or $3\left(a^4 - b^4\right) = 2\left(a^2 - b^2\right)$

$\Rightarrow 3\left(a^2 - b^2\right)\left(a^2 + b^2\right) = 2\left(a^2 - b^2\right) \Rightarrow a + b = 0$ ... (11)

Using (1) & (11)

$1 = a^2 - a^2 + a^2$

$\Rightarrow a = \pm 1$

$\Rightarrow b = \mp 1$ Thus, the points are (1, –2) & (–1, 0)

7. Max value of r occurs when there is a line with slope 2/5 which touches the circle centered at (3,1) & circle at (0,0) and (5,2)

To find p, solve $x^2 + y^2 = r^2, y = -\dfrac{5}{2}x$

$$\Rightarrow x = \frac{2}{\sqrt{25}}r, \qquad y = -\frac{5}{\sqrt{25}}r$$

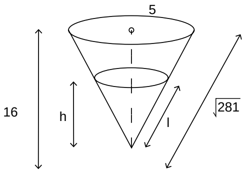
To find q, solve $(x - 3)^2 + (y - 1)^2 = r^2 \;\&\; y - 1 = -\dfrac{5}{2}(x - 3)$

$$\Rightarrow x = 3 - \frac{2}{\sqrt{29}}r, \quad y = 1 + \frac{5}{\sqrt{29}}r$$

$$\text{Slope of } PQ = \frac{2}{5} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \left(\frac{2}{\sqrt{29}}r\right)}$$

$$\Rightarrow r = \frac{\sqrt{29}}{58} \approx 0.093$$

8.



$$\text{As } \frac{r}{5} = \frac{h}{16}$$

$$\Rightarrow r = \frac{5h}{16}$$

$$\text{Volume of cone } = \frac{1}{3}\pi r^2 h$$

$$v = \frac{1}{3}\pi \times \frac{25}{256}h^3$$

$$\Rightarrow \frac{dv}{dt} = \frac{25\pi}{3 \times 256} \times 3h^2 \frac{dh}{dt} \text{ .... } (1)$$

$$\text{As } \frac{dv}{dt} = 2 - k\pi rl \text{ .... } (11)$$

equating (1) & (11) & h = 10, $-\dfrac{dh}{dt} = 0.3, r = \dfrac{25}{8}$

$$\frac{l}{\sqrt{281}} = \frac{10}{16}$$

we get,

$$k = \frac{256 + 375\pi}{250\pi\sqrt{281}}$$

Now, to maintain a certain height, $\frac{dv}{dt} = \pi krl$

$$\Rightarrow \quad k\pi rl = \frac{256+375\pi}{128} = 11.204\,cm^3/min$$

9. $x = \sqrt{a^2-y^2} + \frac{a}{2}\log\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}}$

Put $y = a\sin\theta \Rightarrow \frac{dy}{d\theta} = +a\cos\theta$

$$\Rightarrow x = a\cos\theta + \frac{a}{2}\log\left(\frac{1-\cos\theta}{1+\cos\theta}\right)$$

$$\Rightarrow x = a\cos\theta + a\log\tan\theta/2$$

$$\frac{dx}{d\theta} = -a\sin\theta + \frac{a}{\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\theta\sin\theta}{1-\sin^2\theta} = \tan\theta$$

eqn of target $= y - a\sin\theta = \tan\theta\left(x - \left(a\cos\theta + a\log\tan\frac{\theta}{2}\right)\right)$

$$\Rightarrow \quad y = x\tan\theta - a\tan\theta\log\tan\frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Put $y = 0$

$x = +a\log\tan\theta/2$

Portion of tangent $= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a$

Hence proved.

10. As $y\left(y^2-1\right)(y-2) = (x)(x-1)(x-2)$

(a) diff wrt x

$$y_1\left(y^2-1\right)(y-2) + 2y^2y_1(y-2) + y^2\left(y^2-1\right)y_1 = 3x^2-6x+2$$

$$\Rightarrow y_1\left[y^3-2y^2-y+2+2y^3-4y^2+y^4-y^2\right] = 3x^2-6x+2$$

$$y_1\left[y^4+3y^3-7y^2-y+2\right] = 3x^2-6x+2$$

For horizontal tangent, $y_1 = 0$

$$\Rightarrow 3x^2-6x+2 = 0$$

$$\Rightarrow x = \frac{6\pm2\sqrt{3}}{6} = 1\pm\frac{\sqrt{3}}{3}$$

For each value of x, there are four values of y on the curve. Hence, total no. of 8 horizontal tangents.

(b) $\frac{dy}{dx} = \frac{3x^2-6x+2}{y^4+3y^3-7y^2+2-y}$

For pt(0,1), $\frac{dy}{dx} = -1$

eq $\to y-1 = -x$

For pt(0,2), $\frac{dy}{dx} = \frac{1}{3}$

eqn $\Rightarrow y-2 = \frac{1}{3}x$

11. Let $\frac{a}{2} = k$

$$\Rightarrow x = k(3\cos t + \cos 3t)$$

$$\frac{dx}{dt} = k(-3\sin t - 3\sin 3t) = -3k(\sin t + \sin 3t)$$

$$\& \; y = k(3\sin t + \sin 3t)$$

$$\frac{dy}{dt} = k(3\cos t + 3\cos 3t) = 3k(\cos t + \cos 3t)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(\cos t + \cos 3t)}{\sin t + \sin 3t} = -\frac{2\cos 2t\cos t}{2\sin 2t\cos t}$$

$$= -\cot 2t$$

eqn of tangent at $t \Rightarrow$

$$y - k(3\sin t + \sin 3t) = -\cot 2t(x - k(3\cos t + \cos 3t))$$

Let length of $\perp$ from (0, 0) = p

$$p = \frac{k\cot 2t(3\cos t + \cos 3t) + k(3\sin t + \sin 3t)}{\operatorname{cosec} 2t}$$

$$= k\left[3\cos t\cos 2t + \cos 3t\cos 2t + \sin 2t(3\sin t + \sin 3t)\right]$$

$$= k\left[\cos 2t\left[4\cos^3 t\right] + \left(6\sin t - 4\sin^3 t\right)\sin 2t\right]$$

$$= k\left[4\cos^3 t\cos 2t + 4\sin^2 t\cos t\left(1+2\cos^2 t\right)\right]$$

$$= 4k\cos t$$

Now, $r^2 = k^2(3\cos t + \cos 3t)^2 + k^2(3\sin t + \sin 3t)^2$

$$r^2 = k^2\left[\left(4\cos^3 t\right)^2 + \left(6\sin t - 4\sin^3 t\right)^2\right]$$

$$= k^2\left[\left(4\cos^3 t\right)^2 + 4\sin^2 t(3-2\sin^2 t)^2\right]$$

$$= k^2\left[16\cos^6 t + 4\sin^2 t\left(1+2\cos^2 t\right)^2\right]$$

$$= k^2\left[16\cos^6 t + 4\sin^2 t + 16\sin^2 t\cos^4 t + 16\sin^2 t\cos^2 t\right]$$

$$= k^2\left[16\cos^4 t + 4\sin^2 t + 16\sin^2 t\cos^2 t\right]$$

$$= k^2\left[16\cos^2 t + 4\sin^2 t\right]$$

Now

$$4r^2 - 4a^2 = 4k^2\left(16\cos^2 t + 4\sin^2 t - 4\right)$$

$$= 48k^2\cos^2 t$$

$$\& \ 3p^2 = 3(4k\cos t)^2 = 48k^2\cos^2 t$$

Hence $4r^2 - 4a^2 = 3p^2$

$\Rightarrow 4r^2 = 3p^2 + 4a^2$ Hence proved.

12. As $x^3 - y^2 = 0$ .... (1)

$$3x^2 - 2y\frac{dy}{dx} = 0$$

slope of tangent at $P = \frac{3x^2}{2y} = 3m$

eqn $\rightarrow$ $y - 8m^3 = 3m\left(x - 4m^2\right)$

$\Rightarrow y = 3mx - 4m^3$ .... (2)

Solving (1) & (2)

$$x^3 - \left(3mx - 4m^3\right)^2 = 0$$

$$x^3 - \left(9m^2x^2 + 16m^6 - 24m^4x\right) = 0$$

Upon solving, $x = m^2$

$y = m^3$

Then $\theta \equiv \left(m^2, -m^3\right)$

Slope of tangent at $\theta = \frac{3(m^2)^2}{2(-m^3)} = -\frac{3m}{2}$

$$m_1m_2 = -1 \Rightarrow 3m\left(-\frac{3m}{2}\right) = 1$$

$$\Rightarrow 9m^2 = 2$$

Hence proved.

13. As $x^3 + y^3 = 2a^3$

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^2$$

eqn of tangent

$$y - y_1 = -\left(\frac{x_1}{y_1}\right)^2(x - x_1)$$

$$\Rightarrow y_1^2y - y_1^3 = -x_1^2x + x_1^3$$

$$\Rightarrow (y_1^2)y = -x_1^2x + 2a^3$$

Now $\rightarrow p = \frac{2a^3}{x_1^2}$, $q = \frac{2a^3}{y_1^2}$

Now, $p^{-3/2} + q^{-3/2} = \frac{x_1^3}{(2)^{3/2}a^{9/2}} + \frac{y_1^3}{2^{3/2}a^{9/2}}$

$$= \frac{2a^3}{2^{3/2}a^{9/2}} = 2^{-1/2}a^{-3/2}$$

Hence proved.

14. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$$\Rightarrow \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-n\left(\frac{x}{a}\right)^{n-1} \times \frac{1}{a}}{n\left(\frac{y}{b}\right)^{n-1} \times \frac{1}{b}} = -\frac{b}{a}\frac{\left(\frac{x}{a}\right)^{n-1}}{\left(\frac{y}{b}\right)^{n-1}}$$

Now, $-\frac{b}{a}\frac{(x/a)^{n-1}}{(y/a)^{n-1}} = -\frac{b}{a}$

$$\Rightarrow \left(\frac{x}{a}\right)^{n-1} = \left(\frac{y}{b}\right)^{n-1}$$

$\Rightarrow \frac{x}{a} = \frac{y}{b}$ is possible only for x = a & y = b

eqn of tangent $\Rightarrow y - b = -\frac{b}{a}(x - a)$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

Hence proved.

15. As $y = a\ln\left(x^2 - a^2\right)$

$\frac{dy}{dx} = \frac{2ax_1}{x_1^2 - a^2}$ at Pt $P(x_1, y_1)$

Now, $PT + TN = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} + y\frac{dx}{dy}$

$$= y_1\sqrt{1 + \left(\frac{x_1^2 - a^2}{2ax_1}\right)^2} + y_1\left(\frac{2ax_1}{x_1^2 - a^2}\right)^{-1}$$

$$= \frac{y_1\left(x_1^2 + a^2\right)}{2ax_1} + \frac{y_1\left(x_1^2 - a^2\right)}{2ax_1}$$

$$= \frac{x_1y_1}{a}$$

Hence proved.

16. Let point

$P = (a\cos\theta, b\sin\theta)$

As, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$

For Pt P, $\frac{dy}{dx}=-\frac{b}{a}\cot\theta$

Length of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= b\sin\theta\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}{a\sin\theta}$

$= \frac{b}{a}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ .... (1)

Eqn of tangent at P = $\frac{x\cos\theta}{a}+y\frac{\sin\theta}{b}=1$

$\perp$ distance from $(0,0)=\frac{ab}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$ .... (2)

From (1) & (2)

length of normal $\alpha\ \frac{1}{\perp\text{distance of tangent from origin}}$

Hence proved.

17. As $x^2-y^2=a^2$ .... (1)

Let P be (x_1, y_1) & 0 is origin

$OP=\sqrt{x_1^2+y_1^2}$

diff eq (1) wrt x,

$2x-2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{x_1}{y_1}$

slope of normal at $P=\frac{-y_1}{x_1}$

eqn of normal $\Rightarrow y-y_1=-\frac{y_1}{x_1}(x-x_1)$

as it passes through $(x_2, 0)$

$\Rightarrow -y_1=-\frac{y_1}{x_1}(x_2-x_1)$

$\Rightarrow -x_1y_1=-x_2y_1+x_1y_1$

$\Rightarrow 2x_1y_1=x_2y_1$

$\Rightarrow x_1=\frac{x_2}{2}$

Now point $P(x, y_1)$ & $s=(2x, 0)$

$PS=\sqrt{x_1^2+y_1^2}=op$

Hence proved.

18. As $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$

$\Rightarrow m\frac{x^{m-1}}{a^m}+m\frac{y^{m-1}}{b^m}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$

eqn of tangent at $(x_1, y_1)\Rightarrow$

$\Rightarrow y-y_1=\frac{-b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}(x-x_1)$

$a^my_1^{m-1}y-a^my_1^m=-b^mx_1^{m-1}x+b^mx_1^m$

$\Rightarrow \frac{x_1^{m-1}x}{a^m}+\frac{y_1^{m-1}y}{b^m}=1$

Hence proved.

19. As $x^{2/3}+y^{2/3}=a^{2/3}$

$\Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}$

slope of normal at $(x_1, y_1)=\frac{x_1^{1/3}}{y_1^{1/3}}$

eqn of normal $\Rightarrow y-y_1=\frac{x_1^{1/3}}{y_1^{1/3}}(x-x_1)$

$\Rightarrow x_1^{1/3}x-y_1^{1/3}y+y_1^{4/3}-x_1^{4/3}=0$

$\Rightarrow$ let $x_1^{1/3}=-a\cos\phi$

$\Rightarrow y_1^{1/3}=-a\sin\phi$

$a(x)\cos\phi-a\sin\phi y+\left(a^2\cos^2\phi-a^2\sin^2\phi\right)=0$

$\Rightarrow x\cos\phi-\sin\phi y+a(\cos 2\phi)=0$

Hence proved.

20. As $ax^3+by^3=1$

$\Rightarrow 3ax^2+3by^2\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{ax^2}{by^2}$

eqn of tangent at $pt(x_1, y_1)$

$$y - y_1 = -\frac{ax_1^2}{by_1^2}(x - x_1)$$

$\Rightarrow by_1^2 y + ax_1^2 x = 1$

$\Rightarrow$ As it passes through (h, k)

$\Rightarrow ax_1^2 h + by_1^2 k = 1$

$\Rightarrow \frac{x_1^2}{1/ah} + \frac{y_1^2}{1/bk} = 1$ represent a conic

Hence proved.

21. As $y = x - x^3$ .... (1)

$\Rightarrow \frac{dy}{dx} = 1 - 3x^2$

eqn of tangent at $P\left(x_1, x_1 - x_1^3\right) \Rightarrow$

$y - x_1 + x_1^3 = \left(1 - 3x_1^2\right)(x - x_1)$

$y = \left(1 - 3x_1^2\right)x + 2x_1^3$ .... (11)

Now, solving (1) & (11)

$x - x^3 = (1 - 3x_1^2)x + 2x_1^3$

$\Rightarrow x^3 - \left(3x_1^2\right)x + 2x_1^3 = 0$

$\Rightarrow (x - x_1)\left(x^2 + xx_1 - 2x_1^2\right) = 0$

$\Rightarrow x = x_1$ or $\frac{-x_1 \pm 3x_1}{2}$

$\Rightarrow x = x_1$ or $-2x_1$ or x_1

For $x = -2x_1, y = -2x_1 + 8x_1^3$

So, $Q \equiv \left(-2x_1, -2x_1 + 8x_1^3\right)$

Let R be the point of trisection

x coordinate of $R = \frac{2 \times x_1 - 2(x_1)}{3} = 0$

Hence, it lies on the y-axis

Other pt of trisection

$x = \frac{x_1 - 4x_1}{3}, \quad y = \frac{y_1 - 4x_1 + 16x_1^3}{3}$

$x = -x_1, \quad y = \frac{x_1 - x_1^3 - 4x_1 + 16x_1^3}{3}$

$x = -x_1 \;\; \& \; y = -x_1 + 5x_1^3$

eliminating x_1, we get

$y = x - 5x^3$ Ans

22. As $y = x^2 - x^3$ .... (1)

$\frac{dy}{dx} = 2x - 3x^2$

eqn of tangent at $P(x_1, y_1) \Rightarrow$

$y - x_1^2 + x_1^3 = \left(2x_1 - 3x_1^2\right)(x - x_1)$

$y = \left(2x_1 - 3x_1^2\right)x - x_1^2 + 2x_1^3$

Solving with eq (1)

$x^2 - x^3 = (2x_1 - 3x_1^2)x - x_1^2 + 2x_1^3$

$\Rightarrow x^3 - x^2 + (2x_1 - 3x_1^2)x - x_1^2 + 2x_1^3 = 0$

$\Rightarrow (x - x_1)\left(x^2 + (x_1 - 1)x + \left(x_1 - 2x_1^2\right)\right) = 0$

$\Rightarrow (x - x_1)(x - x_1)(x + 2x_1 - 1) = 0$

$\Rightarrow x = 1 - 2x_1$

Point $Q \equiv \left(1 - 2x_1, (1 - 2x_1)^2 - (1 - 2x_1)^3\right)$

Midpt of PQ be h, k

$\Rightarrow 2h = 1 - x_1 \quad \& \quad 2k = (1 - 2x_1)^2 - (1 - 2x_1)^3 + x_1^2 - x_1^3$

Replace x, by 1 − 2h

$\Rightarrow 2k = (1 - 2(1 - 2h))^2 - (1 - 2(1 - 2h))^3 + (1 - 2h)^2 - (1 - 2h)^3$

Simply & replace $h = x$ & $k = y$, we get

$28x^3 - 28x^2 + 9x + y - 1 = 0$

Hence proved.

23. Mid pt of chord $\equiv (2, -1)$.

As $y = \frac{1}{1 - x}$

$\frac{dy}{dx} = \frac{1}{(1 - x)^2}$

slope of tangent = 1

Eqn of chord for parabola is

$$\frac{y + y_1}{2} + a^2 xx_1 - 5a\left(\frac{x + x_1}{2}\right) + 4 = y_1 + a^2 x_1^2 - 5ax_1 + 4$$

Now, slope $= -\frac{a^2 x_1 + 5a/2}{1/2}$

$= -4a^2 + 5a$

so, $-4a^2 + 5a = 1$

$\Rightarrow 4a^2 - 5a + 1 = 0$

$\Rightarrow (a - 1)(4a - 1) = 0$

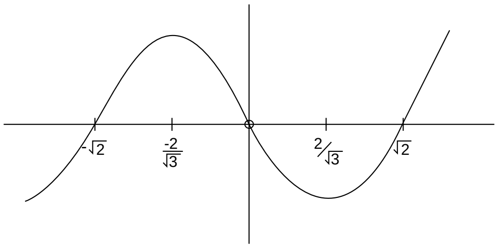
$\Rightarrow a = 1$ or $\frac{1}{4}$

For $a = \frac{1}{4}$, Point(2, –1) will lie outside the parabola.

Hence, a = 1 Ans

24. As $f(x) = x^3 - 4x = x\left(x^2 + 2\right)\left(x^2 - 2\right)$

$f'(x) = 3x^2 - 4$



For $x^3-4x \ge |x-a|$ will have atleast one negative solution

For $a>0$, $y=a-x$ is a tangent to the curve.

so, $3x^2-4=-1$

$x=\pm1$

Now for $x=-1,\quad y=3$

this fulfills $y=a-x$

$a=2$

$\Rightarrow$ a < 2 for line to intersect the graph.

For a < 0, y = x – a is a tangent to f(x)

$\Rightarrow 3x^2-4=1$

$\Rightarrow\ x^2=\frac{5}{3}\Rightarrow x=-\sqrt{\frac{5}{3}}$

For $x=-\frac{\sqrt5}{\sqrt3}, y=-\left(\frac{5}{3}\right)^{\frac{3}{2}}+\left(\frac{5}{3}\right)^{\frac{1}{2}}4$

$=-\frac{5}{3}\sqrt{\frac{5}{3}}+4\sqrt{\frac{5}{3}}=+\frac{7}{3}\sqrt{\frac{5}{3}}$

Now, it will lie on y = x – a

$\Rightarrow \frac{7}{3}\sqrt{\frac{5}{3}}=-\sqrt{\frac{5}{3}}-a$

$a=-\frac{10}{3}\frac{\sqrt5}{\sqrt3}$

$\Rightarrow a>\frac{-10}{3}\sqrt{\frac{5}{3}}$ for line to intersect the graph

Hence, $-\frac{10}{3}\sqrt{\frac{5}{3}}\le a\le 2\cdot$

25. $x=a(2\cos t+\cos 2t)$

$\frac{dx}{dt}=a(-2\sin t-2\sin 2t)=-2a\sin t(1+2\cos t)$

$y=a\left(2\sin t-\sin 2t\right)$

$\frac{dy}{dt}=a\left(2\cos t-2\cos 2t\right)$

$=-2a\left(2\cos^2 t-\cos t-1\right)\ =-2a(\cos t-1)(2\cos t+1)$

$\frac{dy}{dx}=\frac{(\cos t-1)(2\cos t+1)}{\sin t(1+2\cos t)}=-\tan\frac{t}{2}$

eqn of tangent

$y-a(2\sin t_1-\sin 2t_1)=-\tan\frac{t_1}{2}(x-a(2\cos\ t_1+\cos 2t_1)$ ...(1)

put x & y in eq (1)

$2\sin t-\sin 2t-2\sin t_1+\sin 2t_1$

$=-\tan\frac{t_1}{2}\left(2\cos t+\cos 2t-2\cos t_1-\cos 2t_1\right)$

$\Rightarrow 2\left(\sin t-\sin t_1\right)+\left(\sin 2t_1-\sin 2t\right)$

$=-\tan\frac{t_1}{2}\left[2\left(\cos\ t-\cos\ t_1\right)+(\cos 2t-\cos 2t_1)\right]$

$\Rightarrow 2\cos\frac{t+t_1}{2}\sin\frac{t-t_1}{2}+\sin\left(t-t_1\right)\cos\left(t_1+t\right)$

$=-\tan\frac{t_1}{2}\left[2\sin\left(\frac{t+t_1}{2}\right)\sin\left(\frac{t_1-t}{2}\right)+\sin(t+t_1)\sin(t_1-t)\right]$

$\Rightarrow \cos\left(\frac{t+t_1}{2}\right)-\cos\left(\frac{t-t_1}{2}\right)\cos\left(t_1+t\right)$

$=-\tan\frac{t_1}{2}\left[-\sin\left(\frac{t+t_1}{2}\right)-\sin\left(t+t_1\right)\cos\left(\frac{t-t_1}{2}\right)\right]$

$\Rightarrow \cos\left(\frac{t+t_1}{2}\right)\cos\frac{t_1}{2}-\cos\frac{t_1}{2}\cos\left(\frac{t-t_1}{2}\right)\cos\left(t+t_1\right)$

$=\sin\frac{t_1}{2}\sin\frac{t+t_1}{2}+\sin\frac{t_1}{2}\sin\left(t+t_1\right)\cos\left(\frac{t-t_1}{2}\right)$

$\Rightarrow\ \cos\left(\frac{t+t_1+t_1}{2}\right)=\cos\left(\frac{t-t_1}{2}\right)\left[\cos\left(t+t_1-\frac{t_1}{2}\right)\right]$

$\Rightarrow \cos\left(\frac{t+2t_1}{2}\right)=\cos\left(\frac{t-t_1}{2}\right)\cos\left(\frac{2t+t_1}{2}\right)$

$\Rightarrow 2\cos\left(\frac{t+2t_1}{2}\right)=\cos\left(\frac{3t}{2}\right)+\cos\left(\frac{t+2t_1}{2}\right)$

$\Rightarrow \cos\left(\frac{t+2t_1}{2}\right)=\cos\frac{3t}{2}$

$\Rightarrow \frac{t+2t_1}{2}=\frac{-3t}{2}\Rightarrow t=-t_{1/2}$

or $\frac{t+2t_1}{2}=2\pi-\frac{3t}{2}$

$2t=2\pi-t_1$

$t=\pi-\frac{t_1}{2}$

So point Q is $\left(-\frac{t_1}{2}\right)$ point R is $\left(\pi-\frac{t_1}{2}\right)$

(ii) $\frac{dy}{dx}$ at $Q=-\tan\left(\frac{-t_1}{4}\right)=\tan\frac{t_1}{4}$

$\frac{dy}{dx}$ at R $=-\tan\left(\frac{\pi-\frac{t_1}{2}}{2}\right)=-\cot\frac{t_1}{4}$

$\Rightarrow$ tangents are at right angle eqn of tangent at θ ⇒

$$y-a\left(-2\sin\frac{t_1}{2}+\sin t_1\right)=\tan\frac{t_1}{4}\left(x-a\left(\begin{matrix}2\cos\frac{t_1}{2}\\+\cos t_1\end{matrix}\right)\right)\text{.......}\quad(11)$$

eqn of tangent at R ⇒

$$y-a\left(2\sin\frac{t_1}{2}+\sin t_1\right)=\tan\frac{t_1}{4}\left(x-a\left(\begin{matrix}-2\cos\frac{t_1}{2}\\+\cos t_1\end{matrix}\right)\right)\text{.......}\quad(111)$$

Solving (11) & (111) & eliminating t_1, we get

$x^2+y^2=a^2$

Hence proved.

26. $y = ax^3 + bx^2 + cx + d$ .... (1)

$$\frac{dy}{dx}=3ax^2+2bx+c$$

eqn of tangent at x = h,

$$y-\left(ah^3+bh^2+ch+d\right)=\left(3ah^2+2bh+c\right)(x-h)\text{ .....}\quad(11)$$

equating (1) & (11)

$$ax^3+bx^2+cx+d-\left(ah^3+bh^2+ch+d\right)$$

$$=(3ah^2+2bh+c)x-3ah^3-2bh^2-ch$$

$$\Rightarrow\quad ax^3+bx^2-\left(3ah^2+2bh\right)x-2ah^3+bh^2=0$$

$$\Rightarrow(x-h)(ax^2+(ah+b)x-(2ah^2+bh))=0$$

$$\Rightarrow(x-h)(x-h)(ax+(2ah+b))=0$$

$$\Rightarrow x=h\text{ or }-\frac{(2ah+b)}{a}$$

Hence proved.

27. $y=x^3-x-1$

$$\frac{dy}{dx}=3x^2-1$$

slope of tangent at $x_1=3x_1^2-1$

As $y=3x^2-4x+1$

$$\frac{dy}{dx}=6x-4$$

Slope of tangent at $x_2=6x_2-4$.

if tangents are $||^{el}$, $3x_1^2-1=6x_2-4$

Which can have infinite solutions.

Hence, infinite tangents can be drawn.

28. As $y=f(x)$

slope of normal at $x=0=\frac{-1}{f'(0)}=+3$

$$\Rightarrow f'(0)=\frac{-1}{3}\text{ ....}\quad(1)$$

Now, $\lim\limits_{x\to0}\frac{x^2}{f\left(x^2\right)-5f\left(4x^2\right)+4f\left(7x^2\right)}$

$$\Rightarrow\lim_{x\to0}\frac{2x}{2xf'\left(x^2\right)-40xf'\left(4x^2\right)+56xf'\left(7x^2\right)}$$

$$\lim_{x\to0}\frac{1}{f'\left(x^2\right)-20f'\left(4x^2\right)+28f'\left(7x^2\right)}$$

$$=\frac{1}{\frac{-1}{3}+\frac{20}{3}-\frac{28}{3}}=-\frac{1}{3}$$

29. $c_1\Rightarrow y=\log_2 x$

$c_2\Rightarrow y=2^x$

As both curves are inverse to each other.

Hence, symmetrical about y = x line

Let P be any pt on $y=\log_2 x$

$$\frac{dy}{dx}=\frac{1}{(\ln 2)x}$$

& slope of y = x is 1

$$\Rightarrow\frac{1}{x_1\ln 2}=1\Rightarrow x_1=\frac{1}{\ln 2}$$

$$\Rightarrow y_1=\log_2(1/\ln 2)=-\log_2\ln 2$$

Distance of P from y = x is

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\ln 2}+\log_2\ln 2\right)$$

shortest distance $=\sqrt{2}\left[\frac{1}{\ln 2}+\frac{\log\log 2}{\ln 2}\right]$

$$=\sqrt{2}\left[\frac{1+\log\log 2}{\log 2}\right]$$

30. $y=\frac{a}{2}\ln\left(\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right)-\sqrt{a^2-x^2}$

Put $x=a\sin\theta$

$$\frac{dx}{d\theta}=a\cos\theta$$

$$y=a\ln\cot\frac{\theta}{2}-a\cos\theta$$

$$\frac{dy}{d\theta}=a\left(\frac{-\operatorname{cosec}^2\theta/2}{2\cot\theta/2}+\sin\theta\right)\quad=a\left(\frac{-1}{\sin\theta}+\sin\theta\right)$$

$$=-\frac{a\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{dx}=-\cot\theta$$

eqn of tangent

$$y-\left(a\ln\cot\frac{\theta}{2}-a\cos\theta\right)=-\cot\theta(x-a\sin\theta)$$

Put x = 0

$$y=a\ln\cot\frac{\theta}{2}$$

Distance $=\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}=a=\text{constant}$

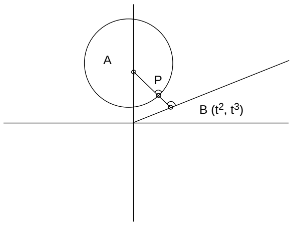
Hence proved.

31. $c_1\to y^2=x^3$

$c_2\to 9x^2+9y^2-30y+16=0$

$$\Rightarrow x^2+y^2-\frac{10}{3}y+\frac{16}{9}=0$$

centre $\rightarrow\left(0,\frac{5}{3}\right)$, $r=\sqrt{\frac{25}{9}-\frac{16}{9}}=1$



Now $AP+PB=AB$

$PB=AB-AP$

$=AB-1$

$$\Rightarrow PB=\sqrt{t^4+\left(t^3-\frac{5}{3}\right)^2}-1 \quad \dots (1)$$

$$\text{Let } f(t)=t^4+\left(t^3-\frac{5}{3}\right)^2$$

$$f'(t)=4t^3-6t^2\left(t^3-\frac{5}{3}\right)$$

For $f'(t)=0, t=0$ or 1

For $t=1, f(t)$ will be minimum

$$\Rightarrow PB=\sqrt{1+\left(1-\frac{5}{3}\right)^2}-1$$

$$=\frac{\sqrt{13}}{3}-1$$ Ans

32. $y=x^3-x-1$ ………………………………………… (1)

$$\frac{dy}{dx}=3x^2-1$$

& $y=3x^2-4x+1$ ………………………………………… (11)

$$\frac{dy}{dx}=6x-4$$

if tangents have same slope $\Rightarrow 3x_1^2-1=6x_2-4$

$$\Rightarrow 6x_2=3\left(x_1^2+1\right)$$

$$\Rightarrow x_2=\frac{x_1^2+1}{2}$$

Now, eqn of tangent for eq (1),

$$y-x_1^3+x_1+1=\left(3x_1^2-1\right)\left(x-x_1\right)$$

$$y=\left(3x_1^2-1\right)x-2x_1^3-1$$

& for eq (11)

$$y-3x_2^2+4x_2+1=\left(6x_2-4\right)\left(x-x_2\right)$$

Let $3x_1^2-1=k$

$$\Rightarrow x_1^2=\frac{k+1}{3}$$

$$\Rightarrow x_2=\frac{1}{2}\left(\frac{k+1}{3}+1\right)=\frac{k+4}{6}$$

$$\Rightarrow y=k\left(x-\frac{(k+4)}{6}\right)+3\left(\frac{k+4}{6}\right)^2+4\left(\frac{k+4}{6}\right)+1$$

$$y=kx-\frac{k(k+4)}{6}+3\left(\frac{k+4}{6}\right)^2+\frac{4(k+4)}{6}+1$$

$$y=kx-\frac{\left(k^2+8k+2\right)}{12}$$ Ans

33. $x=a\cos\theta$

$$\frac{dx}{d\theta}=-a\sin\theta$$

As $y=\sqrt{2}a\cos\theta\sin\frac{\theta}{2}$

$$\frac{dy}{d\theta}=\sqrt{2}a\left(-\sin\theta\sin\frac{\theta}{2}+\frac{1}{2}\cos\theta\cos\frac{\theta}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{\sqrt{2}\left(\sin\theta\sin\frac{\theta}{2}-\frac{1}{2}\cos\theta\cos\frac{\theta}{2}\right)}{\sin\theta}$$

For origin $\Rightarrow \theta=\frac{\pi}{2}$

$$\Rightarrow\frac{dy}{dx}\text{ at }(0,0)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=1$$

$\Rightarrow$ eqn of tangent $y=x$

34. Normal length $=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=ky^3$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=k^2y^4$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^2=k^2y^4-1$$

$$\text{Sub tangent}=\frac{y}{dy/dx}=\frac{y}{\pm\sqrt{k^2y^4-1}}$$

$$\text{Sub normal}=y\frac{dy}{dx}=y\sqrt{k^2y^4-1}$$

35. $x=a\cos\theta+a\,\theta\sin\theta$

$$\frac{dx}{d\theta}=-a\sin\theta+a\sin\theta+a\,\theta\cos\theta=a\,\theta\cos\theta$$

$$y=a\sin\theta-a\,\theta\cos\theta$$

$$\frac{dy}{d\theta}=a\cos\theta-a\cos\theta+a\,\theta\sin\theta=a\,\theta\sin\theta$$

$$\frac{dy}{dx}=\tan\theta$$

eqn of tangent $\Rightarrow y-a\sin\theta+a\,\theta\cos\theta$

$=\tan\theta\,(x-a\cos\theta-a\,\theta\sin\theta)$

$\Rightarrow y=\tan\theta\,x-a\,\theta\,(\cos\theta+\sin\theta\tan\theta)$

$\Rightarrow y=\tan\theta\,x-a\,\theta\sec\theta$

i) Now $p=\frac{a\,\theta\sec\theta}{\sqrt{1+\tan^2\theta}}=a\,\theta$

$\Rightarrow p\propto\theta$ Hence proved.

ii) $\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{a^2\theta^2\cos^2\theta+a^2\theta^2\sin^2\theta}$

$=a\,\theta\propto\theta$

$\Rightarrow p\propto\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}$ Hence proved.

Previous Years Questions Solutions

1. $y^3 - 3xy + 2 = 0$

$$3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3y}{3\left(y^2 - x\right)}$$

H = 0 points

v = 1 point

2. $x = a(\cos\theta + \theta\sin\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

$$\frac{dy}{dx} = \tan\theta$$

eqn of normal $\longrightarrow y - (a(\sin\theta - \theta\cos\theta) =$

$-\cot\theta(x - a(\cos\theta + \theta\sin\theta)$

Distance from origin

3. $y - e^{xy} + x = 0$

$$\frac{dy}{dx} - e^{xy}\left(x\frac{dy}{dx} + y\right) + 1 = 0$$

$$\frac{dy}{dx} = \frac{e^{xy}y - 1}{1 - xe^{xy}}$$

$1 - xe^{xy} = 0$

$\Rightarrow \quad x = 1, \ y = 0$

Hence, D is correct

4. $s = \frac{1}{2}gt^2$

$$\text{so, } \frac{s_1}{s_2} = \left(\frac{t_1}{t_2}\right)^2$$

$$\Rightarrow \quad \frac{20}{5} = \left(\frac{T_1}{T_2}\right)^2$$

$\Rightarrow T_1 : T_2 = 2:1$

Hence, C is correct

5. $y^2 = px^3 + q$

$$2y\frac{dy}{dx} = 3px^2$$

$$\frac{dy}{dx} = \frac{3px^2}{2y} = 2p = 4$$

$p = 2$

As $g = 8p + q$

$\Rightarrow q = -7$

Hence, D is correct

6. $f'(3) = -\left(\tan\frac{3\pi}{4}\right)^{-1} = +1$

Hence, D is correct

7. Similar to Q.20 (Single correct answer type)

Hence, C is correct

8. $y^3 + 3x^2 - 12y = 0$

$$3y^2\frac{dy}{dx} + 6x - 12\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{4 - y^2}$$

For vertical tangent, $y^2 = 4$

$\Rightarrow y = \pm 2$

For $y = 2, \quad x = \pm\frac{4}{\sqrt{3}}$

Hence, D is correct

9. Check for orthogonality $\Rightarrow m_1m_2 = -1$

Hence, BD is correct

10. $y = \cos(x + y)$

$$\frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\frac{\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2}$$

$\Rightarrow \sin(x + y) = 1$

$$y - \cos(x + y) = -\frac{1}{2}(x - x_1)$$

either $x = \pi/2$ & $y = 0$ or $x = -3\pi/2$ & $y = 0$

$$y = -\frac{1}{2}\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad y = \frac{-1}{2}\left(x + 3\frac{\pi}{2}\right)$$

$2y+x=\frac{\pi}{2}$ or $2y+x=-\frac{3\pi}{2}$

$4\,y+2\,x-\pi=0 \qquad 4\,y+2\,x+3\pi=0$

11. For $y=x^2$

$\frac{dy}{dx}=2x_1 \Rightarrow -\frac{dx}{dy}=\frac{-1}{2a}$ at $\left(a,\,a^2\right)$

eqn of normal $\Rightarrow y-a^2=\frac{-1}{2a}(x-a)$

Solving with curve,

we have $a,-a-\frac{1}{2a}$

$PQ^2=\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2$

$=\left(x_1-x_2\right)^2\left[1+\left(x_1+x_2\right)^2\right]$

$=\frac{\left(4a^2+1\right)^3}{16a^4}$

$PQ=\frac{\left(4a^2+1\right)^{3/2}}{4a^2}$

$\frac{dPQ}{dt}=0$ at $a=\pm\frac{1}{\sqrt{2}}$

Hence equation are $x+\sqrt{2}y=\sqrt{2}$ or $x-\sqrt{2}y=-\sqrt{2}$

12. If $\left|f(x_1)-f\left(x_2\right)\right|<\left(x_1-x_2\right)$

Put $x_1=x_2+h$

$\lim\limits_{h\to0}\left|\frac{f\left(x_1\right)-f\left(x_2\right)}{x_1-x_2}\right|<\lim\limits_{h\to0}\left|x_1-x_2\right|$

$f'(x)<0$

$\Rightarrow f'(x)=0$

Eqn of tangent $\Rightarrow y=2$

13. Line $y = x$ intersect the curve $y = ke^x$ for $k\le0$ only at one point

Hence, B is correct

14. $f(x)=ke^x-x$

$f'(x)=ke^x-1=0$

$\Rightarrow x=-\ln k \qquad \& \qquad f^{11}(x)=ke^x>0$

$f(-\ln k)=1+\ln k$

for one root of $f(x)=0$

$1+\ln k=0$

$K=\frac{1}{e}$

Hence, A is correct

15. For two distinct roots $1+\ln k<0$

$k<\frac{1}{e}$

Hence, A is correct

6

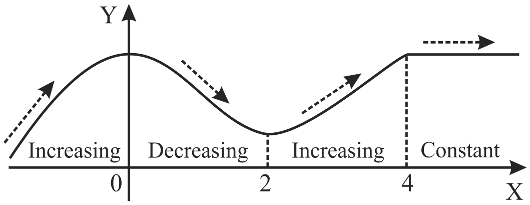
MONOTONICITY

6.1 Definitions

Introduction

Monotonicity is the study of increasing–decreasing behaviour of a function. The terms increasing, decreasing, and constant are used to describe the behaviour of a function over an interval as we travel left to right along its graph.

For example, the function graphed in the figure can be described as increasing on the interval (–∞, 0), decreasing on the interval (0, 2), increasing again on the interval (2, 4), and constant on the interval (4, ∞).



Monotonicity about a Point

Let a function f be defined on an open interval containing the point x = a. We have a set of four standard terms to describe the increasing–decreasing behaviour of the function in a sufficiently small neighbourhood around x = a. They are as follows:

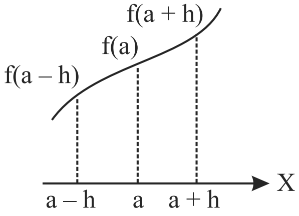
(i) Strictly increasing
(ii) Strictly decreasing
(iii) Non-decreasing
(iv) Non-increasing

If a function follows any of the four conditions, it is said to be monotonic about x = a, otherwise, it is said to be non-monotonic.

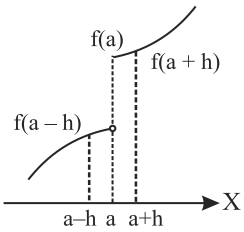
(i) Strictly Increasing

A function f(x) is said to be strictly increasing about the point x = a if f(a – h) < f(a) < f(a + h), where h is a small positive arbitrary number.

Consider the graph of a function in the neigh-bourhood of the point x = a as shown in the figure:



We notice that f(a – h) is less than f(a) while f(a + h) is greater than f(a) for any h in a small neighbourhood around x = a. Then, we say that the function is strictly increasing about x = a. A similar situation is found in the graph of the following figure, which is discontinuous at x = a.

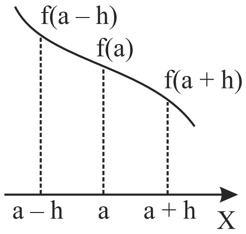


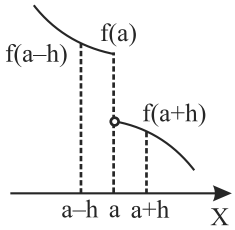
For example, the functions f(x) = e^x, f(x) = 2x + 1, and $f(x)=\begin{cases}1-x^2, & x<0,\\ 2+x^2, & x\ge 0\end{cases}$ are strictly increasing about x = 0.

(ii) Strictly Decreasing

A function f(x) is said to be strictly decreasing about the point x = a if f(a – h) > f(a) > f(a + h), where h is a small positive arbitrary number.

Consider the graph of a function in the neighbourhood of the point x = a as shown in the figures below:



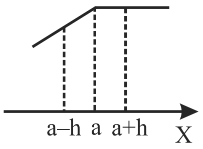


Each of these functions are strictly decreasing about x = a.

(iii) Non-Decreasing

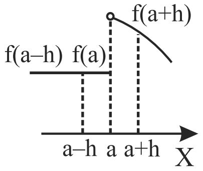
A function f(x) is said to be non-decreasing about the point x = a if f(a – h) ≤ f(a) ≤ f(a + h), where h is a small positive arbitrary number.

Consider the graph of a function in the neighbourhood of the point x = a as shown in the figure:



We observe that in the given figure,

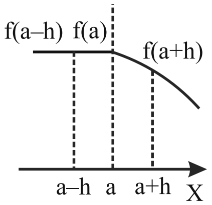
f(a – h) < f(a) = f(a + h), hence, we say that the function is non-decreasing at x = a. The function shown below is also non-decreasing at x = a.

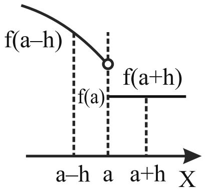


(iv) Non-Increasing

A function f(x) is said to be non-increasing about the point x = a if f(a – h) ≥ f(a) ≥ f(a + h), where h is a small positive arbitrary number.

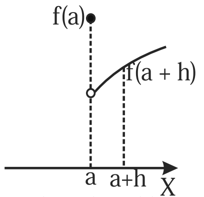
The functions shown below are non-increasing at x = a.





Note:

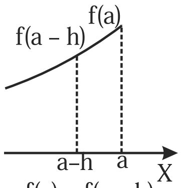
1. If f(x) is constant in the neighbourhood of the point x = a then it said to either non-decreasing or non-increasing.
2. If x = a is an endpoint then we use the appropriate one-sided inequality to test monotonicity of f(x) at x = a. For example, if x = a is the left endpoint, we check as shown below:



f(a + h) < f(a)

strictly decreasing at x = a

If x = a is the right endpoint, we check as follows:



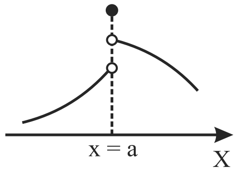
f(a) > f(a – h)

strictly increasing at x = a

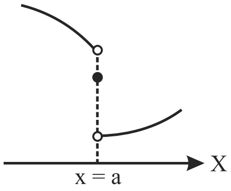
3. It should be noted that we can talk of monotonicity of f at x = a only if x = a lies in the domain of f, without any consideration of continuity or differentiability of f at x = a.

Example 1: Which of the following function is strictly increasing, strictly decreasing, non-increasing, non-decreasing or neither increasing nor decreasing (non-monotonous) at x = a.

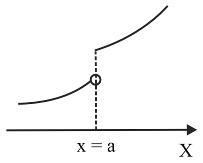
(i)



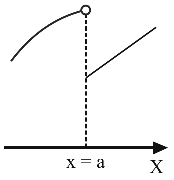
(ii)



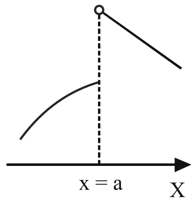
(iii)



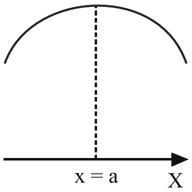
(iv)



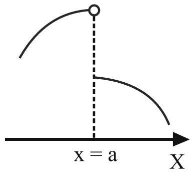
(v)



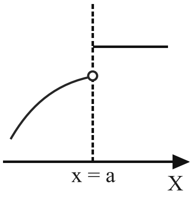
(vi)



(vii)



(viii)



Solution:

(i) Neither increasing nor decreasing as $f(a-h) < f(a)$ and $f(a) > f(a+h)$

(ii) Strictly decreasing as $f(a-h) > f(a) > f(a+h)$

(iii) Strictly increasing as $f(a-h) < f(a) < f(a+h)$

(iv) Neither increasing nor decreasing as $f(a-h) > f(a)$ and $f(a) < f(a+h)$

(v) Strictly increasing

(vi) Neither increasing nor decreasing as $f(a-h) < f(a)$ and $f(a) > f(a+h)$

(vii) Strictly decreasing

(viii) Non-decreasing as $f(a-h) < f(a)$ and $f(a) = f(a+h)$.

Example 2: Examine the behaviour of the function $f(x) = \dfrac{1}{1+x}$ at the point $x = 0$.

Solution: We have $f(x) = \dfrac{1}{1+x}$.

$f(0) = 1$

$$f(0-h) = \frac{1}{1-h} > 1$$

$$f(0+h) = \frac{1}{1+h} < 1$$

Since, $f(0-h) > f(0) > f(0+h)$, where h is a small positive arbitrary number, f is strictly decreasing at $x = 0$.

Test for Finding Monotonicity at a Point

Sufficient Conditions for Monotonicity at a Point

Let a function f be differentiable at $x = a$.

(i) If $f'(a) > 0$ then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a) < 0$ then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a) = 0$ then we need to examine the signs of $f'(a-h)$ and $f'(a+h)$.

   (a) If $f'(a-h) > 0$ and $f'(a+h) > 0$ then $f(x)$ is strictly increasing at $x = a$.

   (b) If $f'(a-h) < 0$ and $f'(a+h) < 0$ then $f(x)$ is strictly decreasing at $x = a$.

   (c) If $f'(a-h)$ and $f'(a+h)$ have opposite signs then $f(x)$ is neither increasing nor decreasing (non-monotonous) at $x = a$.

(iv) If none of the above conditions are followed, then the function needs more investigation, which shall be discussed later.

Example 3: Examine the behaviour of the function $f(x) = x^3 - 3x + 2$ at the points $x = 0, 1, 2$.

Solution: $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

At the point $x = 0$, $f'(0) = -3 < 0$

$\Rightarrow$ $f(x)$ is decreasing at $x = 0$.

At the point $x = 1$, $f'(1) = 0$

But, $f'(1-h)$ = negative and $f'(1+h)$ = positive

$\Rightarrow$ $f(x)$ is neither increasing nor decreasing at $x = 1$.

At the point $x = 2$, $f'(2) = 9 > 0$

$\Rightarrow$ $f(x)$ is increasing at $x = 2$.

Example 4: Let $f(x) = \begin{cases} (x-1)e^x + 1, & x < 0 \\ -(1+x^{4/3}), & x \geq 0 \end{cases}$.

Investigate the behaviour of the function at $x = 0$.

Solution: We have

$$f'(x) = \begin{cases} xe^x, & x < 0 \\ -\dfrac{4}{3}x^{1/3}, & x > 0 \end{cases}$$

At $x = 0$, $f'(0^-) = 0$ and $f'(0^+) = 0$.

But $f'(0-h) < 0$ and $f'(0+h) < 0$

Hence $f(x)$ is strictly decreasing at $x = 0$.

Example 5: Test the function $f(x) = 1 - (x-2)^{3/5}$ for monotonicity at $x = 2$.

Solution: We have $f'(x) = -\dfrac{2}{5}(x-2)^{-2/5}$.

$f'(2^-) = -\infty$ and $f'(2^+) = -\infty$

$f(x)$ is continuous at $x = 2$.

The function has an infinite(negative) derivative at $x = 2$. This implies that $f(x)$ is strictly decreasing at $x = 2$.

Test for Finding Monotonicity at an Endpoint

If $x = a$ is an endpoint then we use the sign of the appropriate one-sided derivative to test the monotonocity of $f(x)$ at $x = a$. Assume that the function f is differentiable at $x = a$.

If $x = a$ is the left endpoint, we check as follows:

(i) If $f'(a^+) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a^+) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a^+) = 0$, but $f'(a+h) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(iv) If $f'(a^+) = 0$, but $f'(a+h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

If $x = a$ is the right endpoint, we check as follows:

(i) If $f'(a^-) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a^-) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a^-) = 0$, but $f'(a-h) > 0$, then f(x) is strictly increasing at x = a.

(iv) If $f'(a^-) = 0$, but $f'(a-h) < 0$, then f(x) is strictly decreasing at x = a.

For example, consider the function $f(x) = (x-1)^{3/2}$.

x = 1 is the left endpoint of the domain $[1, \infty)$.

$$f'(x) = \frac{3}{2}(x-1)^{1/2}$$

$$f'(1^+) = 0, \text{ but } f'(1+h) = \frac{3}{2}h^{1/2} > 0.$$

Hence, f(x) is strictly increasing at x = 1.

Example 6: Let $f(x) = \begin{cases} x^2 + 2x, & -2 \le x < 0 \\ \sin^{-1} x, & 0 \le x \le 1 \end{cases}$.

Investigate the behaviour of the function at x = –2, 0 and 1.

Solution: We have

$$f'(x) = \begin{cases} 2x+2, & -2 \le x < 0 \\ \dfrac{1}{\sqrt{1-x^2}}, & 0 < x < 1 \end{cases}$$

At x = –2, $f'(-2^+) < 0$. Hence f(x) is strictly decreasing at x = –2.

At x = 0, $f'(0^-) = 2 > 0$, $f'(0^+) = 1 > 0$.

Hence f(x) is strictly increasing at x = 0.

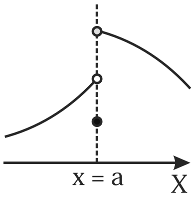
At x = 1, $f'(1^-) = \infty$. f is continuous at x = 1.

The infinite(positive) derivative implies that, f(x) is strictly increasing at x = 1.

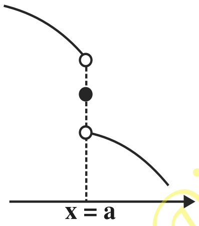
Concept Problems A

1. For each of the following graph comment whether f(x) is increasing or decreasing or neither increasing nor decreasing at x = a.

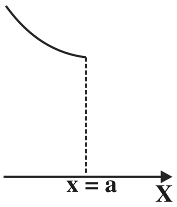
(i)



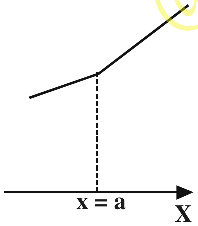
(ii)



(iii)

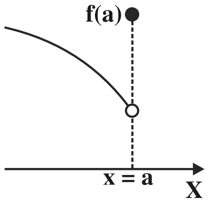


(iv)

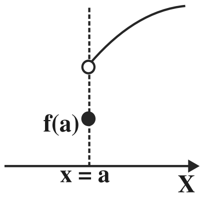


2. Consider the following graphs of functions which have x = a as an endpoint. Find the monotonicity at x = a.

(i)



(ii)



3. Let $f(x) = x^3 - 3x^2 + 3x + 4$. Comment on the monotonic behaviour of f(x) at (i) x = 0 (ii) x = 1.

4. Find out the behaviour of the function y = x – ln x at the points $x_1 = 1/2$, $x_2 = 2$, $x_3 = e$ and $x_4 = 1$, and show that if the given function increases at the point x = a > 0, then it decreases at the point $x = \frac{1}{a}$.

5. Find the behaviour of the functions at x = 0:

(i) $y = x^5 - x^3$ (ii) $y = |\ln(x+1)|$ (iii) $y = 1 - x^{4/5}$

6. Show that the function $y = \ln(x^2 + 2x - 3)$ increases at the point $x_1 = 2$ and decreases at the point $x_2 = -4$.

7. Draw the graph of function $f(x) = \begin{cases} x & 0 \le x < 1 \\ [x] & 1 \le x \le 2 \end{cases}$.

Comment on the monotonic behaviour of f(x) at x = 0, 1 and 2.

8. Test the behaviour of f(x) = x{x} at x = 0, where {.} represents fractional part function.

9. Comment on the monotonic behaviour of the following functions at x = 1.

(i) $f(x) = x^2(x-2)^2$ (ii) $f(x) = x\ln x$

(iii) f(x) = sin x + cos x.

6.2 Monotonicity over an interval

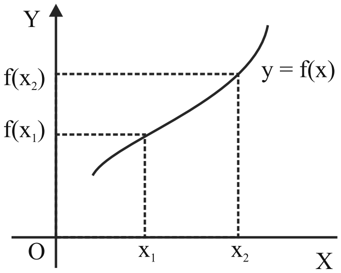
Let a function f be defined on a domain D. If a function follows any of the four conditions given below, it is said to be monotonic in D, otherwise, it is said to be non-monotonic in D.

(i) Strictly Increasing Function

The function f(x) is said to be strictly increasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) < f(x_2)$ holds true.

It means that there is a definite increase in the value of f(x) with an increase in the value of x (See the figure given below). In other words, if the graph of a function is rising on an interval (and never flattens out on that interval), we say that f is strictly increasing on that interval. Note that a function is strictly increasing in an interval (a, b) if it is strictly increasing at every point within the interval.



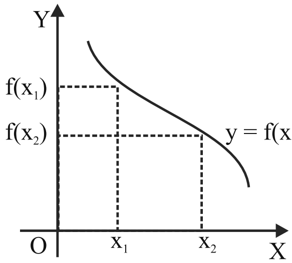
Note: In different texts, strictly increasing function is named differently. For instance, it is called as monotonically increasing, strictly monotonically increasing, steadily increasing, increasing, etc.

(ii) Strictly Decreasing Function

The function f(x) is said to be strictly decreasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) > f(x_2)$ holds true.

It means that there is a definite decrease in the value of f(x) with an increase in the value of x (See the figure given below).

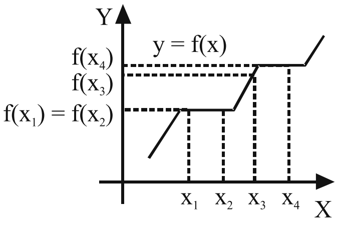


(iii) Non-Decreasing Function

The function f(x) is said to be non-decreasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) \le f(x_2)$ holds true.

It means that the value of f(x) never decreases with an increase in the value of x. (See the figure given below).



In the above figure we find that for $x_1 < x_2 < x_3 < x_4$ we have $f(x_1) = f(x_2) < f(x_3) < f(x_4)$. Such a function is called non-decreasing.

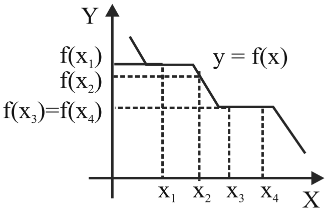
Note: In some texts, non-decreasing function is named as increasing.

(iv) Non-Increasing Function

The function f(x) is said to be non-increasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) \ge f(x_2)$ holds true.

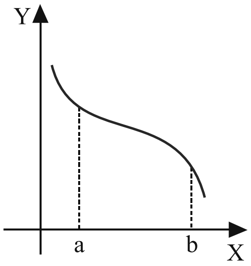
It means that the value of f(x) never increases with an increase in the value of x. (See the figure given below).



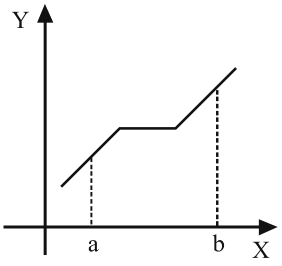
In the above figure we find that for $x_1 < x_2 < x_3 < x_4$ we have $f(x_1) > f(x_2) > f(x_3) = f(x_4)$. Such a function is called non-increasing.

Note: A function which is constant over an interval, is said to be either non-decreasing or non-increasing.

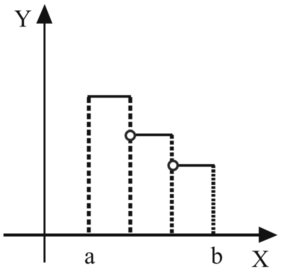
The following figures show the behaviour of some functions monotonic in the interval [a, b].



Strictly decreasing in [a, b]

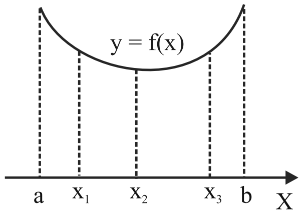


Non-decreasing in [a, b]



Non-increasing in [a, b]

Now consider the following function in the interval [a, b].

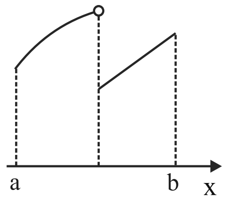


Here, we find that for $x_1 < x_2 < x_3$, $f(x_1) > f(x_2) < f(x_3)$.

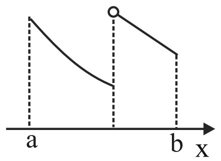
So, the order between function's value at two different points is not maintained throughout the interval. Hence, the function is not monotonic in the interval [a, b].

Note that the following functions are not monotonic in the interval [a, b].

(i)

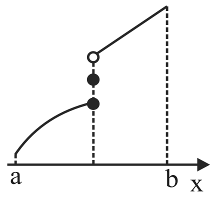


(ii)

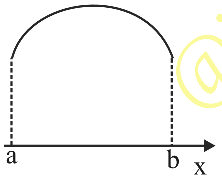


Example 1: Which of the following function is strictly increasing, strictly decreasing, non-increasing, non-decreasing or neither increasing nor decreasing (non-monotonous) in the interval [a, b]?

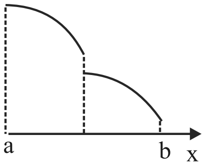
(i)



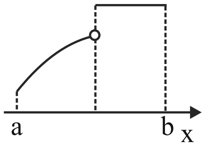
(ii)



(iii)



(iv)



Solution:

(i) Strictly increasing as $f(x_1) < f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

(ii) Neither increasing nor decreasing.

(iii) Strictly decreasing as $f(x_1) > f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

(iv) Non-decreasing as $f(x_1) \le f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

Test for Finding Monotonicity Over an Interval

Let a function f be defined and continuous on a certain interval (a, b) and have a derivative everywhere in the interval (a, b) except possibly at a finite number of points. We can predict the behaviour of f in the interval by studying the sign of its derivative $f'(x)$ over the interval.

Necessary Conditions for Monotonicity

(i) If a differentiable function f(x) increases in an interval its derivative $f'(x)$ is nonnegative: $f'(x) \ge 0$.

(ii) If a differentiable function f(x) decreases in an interval its derivative $f'(x)$ is non-positive: $f'(x) \le 0$.

(iii) If a differentiable function f(x) does not vary in an interval (i.e. is equal to a constant) its derivative is identically equal to zero : $f'(x) = 0$.

Theorem 1 If a differentiable function f(x) is strictly increasing on the interval (a, b), then $f'(x) \ge 0$ for any x in the interval (a, b).

Proof According to the definition of a function strictly increasing on (a, b), if $x > x_0$, then $f(x) > f(x_0)$. and if $x < x_0$, then $f(x) < f(x_0)$.

Consequently, for any x_0 and x in (a, b), $x \ne x_0$, the inequality $\frac{f(x) - f(x_0)}{x - x_0} > 0$ holds true.

Since f(x) is differentiable on (a, b), proceeding to the limit in the last inequality as $x \to x_0$, we get

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \ge 0$$

The theorem is proved.

Theorem 2 If a differentiable function f(x) is strictly decreasing of the interval (a, b), then $f'(x) \le 0$ for any x in the interval (a, b).

Proof Since f(x) is a strictly decreasing function, the function F(x) = –f(x) is a strictly increasing one, and therefore, by Theorem 1, $F'(x_0) = -f'(x_0) \ge 0$ for any $x_0 \in (a, b)$. Hence, it follows that $f'(x_0) \le 0$ for any $x_0 \in (a, b)$. The theorem is proved.

Theorem 3 If a function f(x) is constant in the interval (a, b), then $f'(x) = 0$ for any x in the interval (a, b).

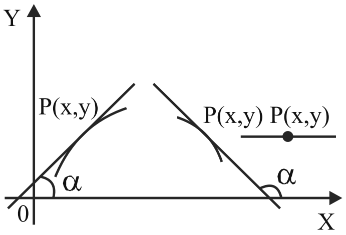
Proof If f(x) is constant its derivative is known to be equal to zero : $f'(x) = 0$.

Note: The foregoing theorems express the following geometric fact. If on an interval (a, b) a function f(x) is strictly increasing, then as the variable point P(x,y) traces the graph of the function from left to right, i.e. as the abscissa increases, the value of the function moves upward along y-axis.

Then, the tangent to the curve y = f(x) at each point on this interval forms an acute angle α with the x-axis (or, at some points, the tangent line is horizontal); the tangent of this angle is nonnegative:

$f'(x) = \tan\alpha \ge 0$.

If the function f(x) is strictly decreasing on the interval (a, b), then the angle of inclination of the tangent line forms on obtuse angle (or, at some points, the tangent line is horizontal) the tangent of this angle is nonpositive.



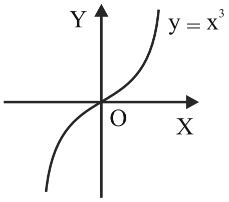
If the variable point P(x, y) moves on a horizontal line then, the tangent is also horizontal and hence,

$f'(x) = \tan 0 = 0$.

It should be stressed that the derivative of a strictly increasing or strictly decreasing function may vanish at some separate points.

For instance, the cubic function $f(x) = x^3$ increases strictly throughout the x-axis, but its derivative $f'(x) = 3x^2$ turns into zero at the point $x = 0$, although at all the other points it is positive. The function is strictly increasing at $x = 0$ since larger values of x have correspondingly larger values of f. We can see from the figure that

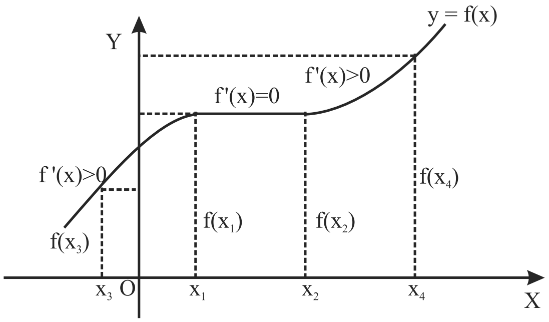
$$f(0 - h) < f(0) \text{ and } f(0 + h) > f(0).$$



Geometrically, this means that the tangent to the graph of a strictly increasing or strictly decreasing function may be parallel to x-axis at some points.

Theorem 4 For the function f(x) differentiable on an interval I, not to decrease (not to increase) on that interval, it is necessary and sufficient that $\forall\ x \in I$ the inequality $f'(x) \geq 0$ $(f'(x) \leq 0)$ be satisfied.

Consider the graph of a non-decreasing function.



It is clear from the figure that

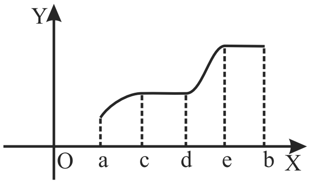
for $x_3 < x < x_1$ we have $f(x) < f(x_1)$ and $f'(x) > 0$

for $x_1 < x < x_2$ we have $f(x_1) = f(x_2)$ and $f'(x) = 0$

for $x_2 < x < x_4$ we have $f(x) > f(x_2)$ and $f'(x) > 0$

Combining all cases, we say that for any $x \in (x_3, x_4)$ we have $f(x_1) \leq f(x_2)$ and $f'(x) \geq 0$.

Now, see the figure shown below:



Here also $f'(x) \geq 0$ for all $x \in (a, b)$, but note that in this case, equality of $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Here $f'(x)$ becomes identically zero on two subintervals and hence the given function cannot be assumed to be strictly increasing in (a, b). We say that f(x) is non-decreasing in (a, b).

For example, $f(x) = x^3 + |x|^3$, $x \in R$ is not a strictly increasing function, rather, it is a non-decreasing function.

Thus, if $f'(x) \geq 0$ in an interval I, with $f'(x) = 0$ on one or more subintervals of I, then f(x) is said to be a non-decreasing function in the interval I.

But, if $f'(x)$ vanishes at a countable number of isolated points, provided it be elsewhere uniformly positive, then f(x) will strictly increase.

Example 2: Prove that $f(x) = x - \sin x$ is a strictly increasing function.

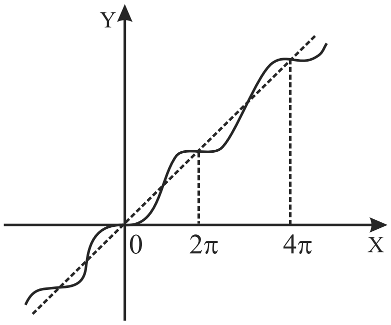
Solution: Given $f(x) = x - \sin x$

$\Rightarrow\ f'(x) = 1 - \cos x$

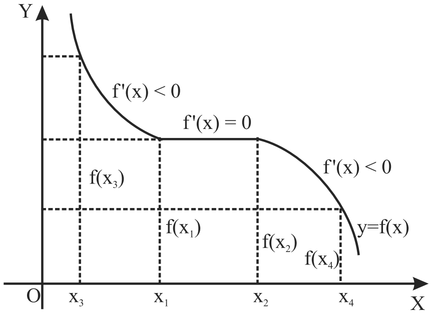
We have $f'(x) \geq 0$.

$f'(x) = 0$ at $x = 0, \pm 2\pi, \pm 4\pi, \ldots\ldots$

Now, $f'(x) > 0$ everywhere except at $x = 0, \pm 2\pi, \pm 4\pi, \ldots\ldots$ but all these points are discrete (separated) and do not form an interval. Hence we can conclude that f(x) is strictly increasing for all x. In the figure, we see that the graph passes through the point $(2n\pi, 2n\pi)$, $n \in I$, by increasing continuously, across a horizontal tangent.



Now, consider the graph of a non-increasing function.



It is clear from the figure that

for $x_3 < x < x_1$ we have $f(x) > f(x_1)$ and $f'(x) < 0$

for $x_1 < x < x_2$ we have $f(x_1) = f(x_2)$ and $f'(x) = 0$

for $x_2 < x < x_4$ we have $f(x) < f(x_2)$ and $f'(x) < 0$

Combining all cases, we say that for any $x \in (x_3, x_4)$ we have $f(x_1) \geq f(x_2)$ and $f'(x) \leq 0$.

Thus, if $f'(x) \leq 0$ in an interval I, with $f'(x) = 0$ on one or more subintervals of I, then f(x) is said to be a non-increasing function in the interval I.

But, if $f'(x)$ vanishes at a finite number of isolated points, and is otherwise negative, then f(x) will strictly decrease.

It is now clear that, in an interval of monotonicity of a differentiable function its derivative cannot change sign to the opposite.

The above results allow us to judge upon the sign of the derivative of a monotonous differentiable function in a given interval by its increase or decrease in this interval. But when we begin to investigate a given function, its behaviour is usually not known, and therefore, it is much more important to establish the converse of the above results, which enables us to study the character of the variation of a function in a given interval by reducing the problem to the simpler question of determining the sign of its derivative.

Sufficient Conditions for Monotonicity

Theorem Let f(x) be a differentiable function on the interval (a, b). Then:

(i) If the derivative $f'(x)$ is everywhere positive (i.e. $f'(x) > 0$) in the interval (a, b), then the function f(x) is strictly increasing in the interval (a, b).

(ii) If the derivative $f'(x)$ is everywhere negative (i.e. $f'(x) < 0$) in the interval (a, b), then the function f(x) is strictly decreasing in the interval (a, b).

(iii) If the derivative $f'(x)$ is everywhere equal to zero in the interval (a, b), then the function f(x) does not vary in the interval (a, b) (i.e. it is constant).

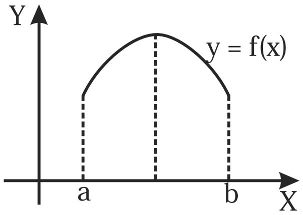
It should be borne in mind that the conditions of the theorem are sufficient, but not necessary, for a function to increase (or decrease). There are cases when a function f can increase at a point x, but the derivative $f'(x)$ is not positive there. Consider, for instance, the function $f(x) = x^3$ at x = 0.

Study Tip

If a function is such that $f'(x) \geq 0$ for all $x \in (a, b)$ where $f'(x) = 0$ at discrete points in (a, b), then f(x) is strictly increasing in (a, b).

Note: By $f'(x) = 0$ at discrete points, we mean that the points where $f'(x)$ becomes 0 do not form an interval. That is, they are separated from each other.

A function which is increasing as well as decreasing in an interval, is said to be non-monotonic, provided it is not a constant function. If the function is differentiable, then it must change the sign of its derivative somewhere in the interval.



A graph of such a function is shown here.

Example 3: Prove that f(x) = 2x – cosx is a strictly increasing function.

Solution: Given f(x) = 2x – cos x

$\Rightarrow \; f'(x) = 2 + \sin x$

We have $f'(x) > 0$ for all x.

Hence, the function f(x) is strictly increasing for all x.

Example 4: Prove that $f(x) = -x - \cot^{-1}x$ is a strictly decreasing function.

Solution: Given $f(x) = -x - \cot^{-1}x$

$$\Rightarrow \; f'(x) = -1 + \frac{1}{1+x^2} = -\frac{x^2}{1+x^2}$$

We have $f'(x) \leq 0$ for all x.

$f'(x) = 0$ at x = 0 only.

Now, $f'(x) < 0$ everywhere except at x = 0. Hence we can conclude that f(x) is strictly decreasing for all x.

Example 5: Is the function sin (cos x) increasing or decreasing on the interval (π, 3π/2) ?

Solution: Let f(x) = sin (cos x)

$\Rightarrow \; f'(x) = \cos(\cos x) \cdot (-\sin x)$

On the interval (π, 3π/2), cos x lies in (–1, 0). Hence, cos (cos x) > 0.

(– sin x) > 0 on the interval (π, 3π/2).

Thus, we have $f'(x) > 0$ for all x in (π, 3π/2).

Hence, the function f(x) is strictly increasing for all x.

Example 6: Find the behaviour of the function

$$f(x) = \begin{cases} e^x, & x < 0, \\ \ln(1+x)+1, & x \geq 0 \end{cases} \quad \text{for } x \in \mathbb{R}.$$

Solution: Given $f(x) = \begin{cases} e^x, & x < 0, \\ \ln(1+x)+1, & x \geq 0 \end{cases}$

f(x) is continuous for all x.

$$f'(x) = \begin{cases} e^x, & x < 0, \\ \dfrac{1}{1+x}, & x > 0 \end{cases}$$

Here, $f'(0^-) = 1 = f'(0^+)$

We have $f'(x) > 0$ for all $x \in R$.
Hence, we can conclude that f(x) is a strictly increasing function for all x.

Example 7: Find the behaviour of the function

$$f(x) = \begin{cases} 1, & x < 0, \\ x^3 + 1, & x \geq 0 \end{cases} \text{ for } x \in R.$$

Solution: Given $f(x) = \begin{cases} 1, & x < 0, \\ x^3 + 1, & x \geq 0 \end{cases}$

f(x) is continuous for all x.

$$f'(x) = \begin{cases} 0, & x < 0, \\ 3x^2, & x > 0 \end{cases}$$

Here, $f'(0^-) = 0 = f'(0^+)$
We have $f'(x) \geq 0$ for all $x \in R$.
But, $f'(x) = 0$ in the interval $x \in (-\infty, 0]$.
Hence, we can conclude that f(x) is a non-decreasing function for all x.

Example 8: Find the least value of k for which the function $x^2 + kx + 1$ is a strictly increasing function in the interval $1 \leq x \leq 2$.

Solution: Let $f(x) = x^2 + kx + 1$
For f(x) to be strictly increasing, $f'(x) \geq 0$ in the interval $1 \leq x \leq 2$.
$\Rightarrow \quad 2x + k \geq 0 \quad \Rightarrow \quad k \geq -2x$
Here k must be greater than or equal to the largest value of $-2x$ found in the interval [1, 2]
i.e. $k \geq -2$
Hence, the least value of k is -2.

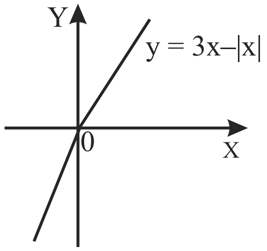
Example 9: For what values of b, the function $f(x) = \sin x - bx + c$ decreases strictly for all $x \in R$?

Solution: Here $f(x) = \sin x - bx + c$
$\therefore \quad f'(x) = \cos x - b$
f(x) will decrease for all $x \in R$ if $f'(x) \leq 0$
or $\quad \cos x - b \leq 0$, i.e., $\cos x \leq b$ for all $x \in R$.
$\therefore \quad b \geq$ the greatest value of cos x
$\quad b \geq 1$
Thus, $b \in [1, \infty)$.
Note that when b = 1, $f'(x) = \cos x - 1 \leq 0$
Here f(x) = 0 at $x = 2n\pi$, which are a set of discrete points, not forming an interval. Hence, f(x) decreases strictly for all $x \in R$.

Monotonicity at Points where f'(x) does not Exist

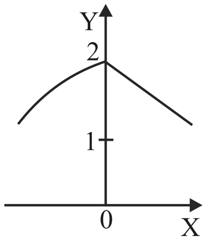
We have investigated the case where a function has a derivative at all points on some interval. Now what about those points at which there is no derivative? The following examples will help in identifying the behaviour of such a function.

(i) Let $f(x) = 3x - |x|$
The given function has no derivative at the point x = 0, as it is a corner point.



For all $x \neq 0$, $f'(x) > 0$. The function is continuous for all x with $f'(0 - h) > 0$ and $f'(0 + h) > 0$.
Hence, f(x) is a strictly increasing function for all x.

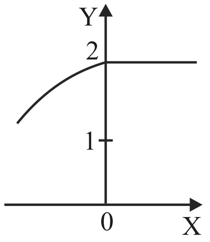
(ii) Now, consider the function $f(x) = \begin{cases} 2 - x^2, & x < 0, \\ 2 - x, & x \geq 0 \end{cases}$.



$$f'(x) = \begin{cases} -2x, & x < 0, \\ -1, & x > 0 \end{cases}$$

Here, $f'(0 - h) > 0$, $f'(0^-) = 0$, $f'(0^+) = -1$.
We notice that $f'(x)$ changes sign about x = 0.
Thus, f(x) is not strictly increasing function at x = 0. In fact, the function is non-monotonous.

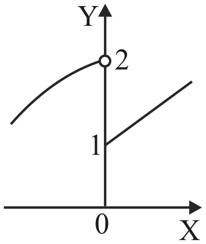
(iii) Now, consider the function $f(x) = \begin{cases} 2 - x^2, & x < 0, \\ 2, & x \geq 0 \end{cases}$.



$$f'(x) = \begin{cases} e^{-x}, & x < 0, \\ 0, & x > 0 \end{cases}$$

We notice that $f'(x) \geq 0$ for all $x \neq 0$.
Here, $f'(0^-) = 1$, $f'(0^+) = 0$, $f'(0 + h) = 0$.
In fact, $f'(x) = 0$ for all $x > 0$.
Thus, f(x) is a non-decreasing function.

(iv) Now, consider the function $f(x) = \begin{cases} 2 - x^2, & x < 0, \\ x + 1, & x \geq 0 \end{cases}$.



$$f'(x) = \begin{cases} -2x, & x < 0, \\ 1, & x > 0 \end{cases}$$

For all $x \neq 0$, $f'(x) > 0$. The function is discontinuous at $x = 0$. We can say that f(x) is a strictly increasing function for $x \in (-\infty, 0)$ and for $x \in (0, \infty)$.

We need to check the monotonicity of the function at $x = 0$ using basic definition. We can see from the figure that $f(0 - h) > f(0)$ and $f(0) < f(0 + h)$. This means that f(x) is not increasing function at $x = 0$.

Hence, f(x) is not a strictly increasing function for all x.

We can now understand that in the case of continuous functions, the sign of the derivative in the neighbourhood of the point is adequate in determining the monotonicity of the function. If the derivative maintains the same sign across the point, the function is monotonous.

However, in the case of discontinuous functions, the sign of the derivative in the neighbourhood of the point is inadequate in determining the monotonicity of the function. We need to apply the basic definition of monotonicity.

Test for finding Monotonicity at Points where $f'(x)$ does not Exist

Consider a continuous function f(x) whose derivative $f'(x)$ does not exist at $x = c$ but exists in the neighbourhood of c.

(i) If $f'(c^-) > 0$ and $f'(c^+) > 0$, then f(x) is strictly increasing at $x = c$.

(ii) If $f'(c - h) > 0$, $f'(c^-) \geqq 0$, $f'(c^+) \geqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at $x = c$.

(iii) If $f'(c^-) < 0$ and $f'(c^+) < 0$, then f(x) is strictly decreasing at $x = c$.

(iv) If $f'(c - h) > 0$, $f'(c^-) \leqq 0$, $f'(c^+) \leqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at $x = c$.

Here, $\geqq$ implies that either greater than or equal to holds.

Now, consider a function f(x) which is discontinuous at $x = c$ but its derivative exists in the neighbourhood of c.

(i) If $f'(c - h) > 0$, $f(c^-) \leqq f(c)$, $f(c) \leqq f(c^+)$, $f'(c + h) > 0$, then f(x) is strictly increasing at $x = c$.

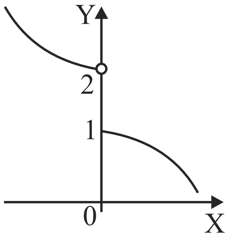
(ii) If $f'(c - h) < 0$, $f(c^-) \geqq f(c)$, $f(c) \geqq f(c^+)$, $f'(c + h) < 0$, then f(x) is strictly decreasing at $x = c$.

Example 10: Find the behaviour of the function

$f(x) = \begin{cases} x^2 e^{-x} + 2, & x < 0, \\ 1 - x^2, & x \geq 0 \end{cases}$ for $x \in R$.

Solution: Given $f(x) = \begin{cases} x^2 e^{-x} + 2, & x < 0, \\ 1 - x^2, & x \geq 0 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} (2x - x^2)e^{-x}, & x < 0, \\ -2x, & x > 0 \end{cases}$$



For all $x \neq 0$, $f'(x) < 0$.

The function is discontinuous at $x = 0$.

Here, $f'(0 - h) < 0$, $f'(0 + h) < 0$,

$f(0) = 1$, $f(0^-) = 2 > f(0)$, and $f(0) = f(0^+)$.

The value of the function is falling across the point $x = 0$.

Hence, f(x) is strictly decreasing at $x = 0$ and finally it is strictly decreasing for all $x \in R$.

Example 11: Let $f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ x^2 e^x, & x \geq 0 \end{cases}$.

Investigate the behaviour of the function for $x \in R$.

Solution: We have

$$f'(x) = \begin{cases} 3x^2 + 2x + 10, & x < 0, \\ x(x + 2)e^x, & x > 0 \end{cases}$$

For $x < 0$, $f'(x)$ is a quadratic expression whose discriminant is positive, with coefficient of x^2 positive. Hence it is positive.

Clearly $f'(x) > 0$ for $x > 0$.

Thus, $f'(x) > 0$ for all $x \neq 0$.

Now consider the point $x = 0$. f(x) is continuous but non-differentiable at $x = 0$.

Here, $f'(0^-) = 10 > 0$, $f'(0^+) = 0$, $f'(0 + h) > 0$.

This means that f is strictly increasing at $x = 0$.

Finally, f(x) is a strictly increasing function for $x \in R$.

Example 12: Show that $f(x) = \begin{cases} x^2 + 2x, & -1 \leq x < 0 \\ 3x, & 0 \leq x \leq 1 \end{cases}$

is strictly increasing in [–1, 1], but

$$g(x) = \begin{cases} x^2 + 2x, & -1 \leq x < 0 \\ 3x, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

is not strictly increasing in [–1, 1].

Solution: We have $f'(x) = \begin{cases} 2x+2, & -1 \le x < 0 \\ 3, & 0 < x \le 1 \end{cases}$.

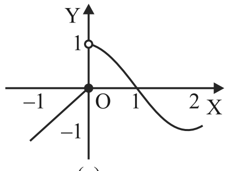
For all $x \ne 0$, $f'(x) > 0$. f is continuous at x = 0
and , $f'(0^-) > 0$ and $f'(0^+) > 0$.
Hence f(x) is strictly increasing at x = 0.
At x = – 1, $f'(-1^+) = 0$ and $f'(-1 + h) > 0$,
At x = 1, $f'(1^-) > 0$.
Hence f(x) is strictly increasing at both the endpoints.
Finally, f(x) is strictly increasing in [–1, 1].

Now, $g'(x) = \begin{cases} 2x+2, & -1 \le x < 0 \\ 3, & 0 < x < 1 \end{cases}$

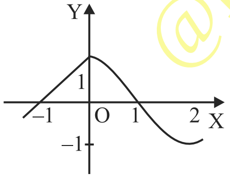
For all $x \ne 0, 1$, $g'(x) > 0$.
At x = 0 and – 1, g(x) is strictly increasing like f(x).
At x = 1, g(x) is discontinuous
where $g(1^-) = 3$ while g(1) = 1.
Hence, g(x) is not strictly increasing at x = 1.
Thus, g(x) is not strictly increasing in [–1, 1].
However, g(x) is strictly increasing in [–1, 1).

Example 13: Let $f(x) = \begin{cases} \cos\dfrac{\pi x}{2}, & x > 0 \\ x + a, & x \le 0 \end{cases}$. Find the values of a if f(x) is monotonous at x = 0.

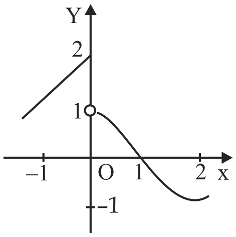
Solution: We draw the graph of f with different values of a.



(a)



(b)



(c)

(a) Clearly, f(0 – h) < f(0) < f(0 + h).
Hence f(x) is strictly increasing at x = 0.
For this case, f(0) should be less than the R.H.L. of f at x = 0 ⇒ a < 1.

(b) In this case f(0 – h) < f(0) > f(0 + h) and hence f(x) is non-monotonous at x = 0.

(c) Here also f(0 – h) < f(0) > f(0 + h) and hence f(x) is non-monotonous at x = 0.
Hence f(x) is monotonous at x = 0 for a < 1.

Example 14: Find the behaviour of the function

$$f(x) = \begin{cases} x, & x \le 0, \\ x\sin\dfrac{1}{x}, & x \ge 0 \end{cases} \text{ at } x = 0.$$

Solution: Clearly f is continuous at x = 0.

We have $f'(x) = \begin{cases} 1, & x < 0, \\ \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}, & x > 0 \end{cases}$

Here, $f'(0^-) = 1 > 0$ but $f'(0^+)$ does not exist.
Further, f′(0+h) changes sign in the right neighbourhood since $f'(x) > 0$ at the points $x = \dfrac{1}{(2n+1)\pi}$ $(n \in N)$, and

$f'(x) < 0$ at the points $x = \dfrac{1}{2n\pi}$ $(n \in N)$,

Hence, f(x) is non-monotonous at x = 0.

Example 15: Prove that the function

$$f(x) = \begin{cases} x + x^2\sin(2/x) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases},$$

increases at the point x = 0 but does not increase on any interval $(-\varepsilon, \varepsilon)$, $(\varepsilon > 0$ is an arbitrary number).

Solution: $f'(x) = \begin{cases} 1 + 2x\sin(2/x) - 2\cos(2/x) & \text{for } x \ne 0, \\ 1 & \text{for } x = 0 \end{cases}$

Since $f'(0) = 1 > 0$, it follows that the function f(x) increases at the point x = 0.
If the function f(x) were increasing on an interval $(-\varepsilon, \varepsilon)$, then the condition $f'(x) \ge 0$ would be satisfied $\forall\ x \in (-\varepsilon, \varepsilon)$. We shall show that this is not so.
Let $x_n = 1/(n\pi)$ (n is a natural number).
It is evident that $\forall\ \varepsilon > 0$, there exists n such that $1/(n\pi) < \varepsilon$, i.e. $x_n \in (-\varepsilon, \varepsilon)$. Substituting $x = x_n = 1/(n\pi)$ into the expression for $f'(x)$ when $x \ne 0$,
we get $f'(x_n) = -1 < 0$. This proves that the function f(x) is not increasing on any interval $(-\varepsilon, \varepsilon)$.

Example 16: Let the function $g : R \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ be given by $g(t) = \dfrac{\pi}{2} - 2\cot^{-1}(3^{-t})$. Prove that g is odd and is strictly decreasing in $(-\infty, \infty)$.

Solution: We have $g(t) = \dfrac{\pi}{2} - 2\cot^{-1}(3^{-t})$

$$\therefore\quad g(-t) = \frac{\pi}{2} - 2\cot^{-1}(3^{t}) = \frac{\pi}{2} - 2\tan^{-1}(3^{-t})$$

$$\left(\text{As } \cot^{-1}x = \tan^{-1}\frac{1}{x},\ x > 0\right)$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{2} - \cot^{-1}(3^{-t})\right) = -\frac{\pi}{2} + 2\cot^{-1}(3^{-t})$$

$$\left(\text{As } \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2},\ \forall\ x \in R\right)$$

$= -g(t)$

Hence, $g(-t) = -g(t) \Rightarrow g$ is an odd function.

Also $\quad g'(t) = \dfrac{-2.3^{-t} \cdot \ln 3}{1 + (3^{-t})^2}$.

$\therefore \quad g'(t) < 0,\ \forall\ t \in R.$

$\Rightarrow$ g is strictly decreasing in $(-\infty, \infty)$.

Example 17: Find the set of all values of ‘a’ for which

$f(x) = \left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^5 - 3x + \ln 5$ decreases for all x.

Solution: $f(x) = \left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^5 - 3x + \ln 5$

$\therefore \quad f'(x) = 5\left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^4 - 3$

Since f(x) decreases for all x, $\quad f'(x) \le 0$

$\Rightarrow \quad 5\left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^4 - 3 \le 0$

$\Rightarrow \quad 5\left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right) \le \left(\dfrac{3}{5x^4}\right)$

The L.H.S. should be less than or equal to the least value of R.H.S.

$\Rightarrow \quad 5\left(\dfrac{\sqrt{(a+4)}}{(1-a)} - 1\right) \le 0$

$\Rightarrow \quad \dfrac{\sqrt{(a+4)}}{(1-a)} \le 1 \qquad ...(1)$

It is clear that $a + 4 \ge 0$

Case I: If $1 - a > 0$ i.e. $a < 1$

then $\sqrt{(a+4)} \le (1-a)$

On squaring, we get $a + 4 \le a^2 - 2a + 1$

$\Rightarrow \quad a^2 - 3a - 3 \ge 0$

$\Rightarrow \quad a \in \left(-\infty, \dfrac{3-\sqrt{21}}{2}\right] \cup \left[\dfrac{3+\sqrt{21}}{2}, \infty\right)$ but $-4 \le a < 1$

$\therefore \quad a \in \left[-4, \dfrac{3-\sqrt{21}}{2}\right] \qquad ..(2)$

Case II: If $1 - a < 0 \Rightarrow$ i.e. $a > 1 \qquad ...(3)$

$\Rightarrow \quad \sqrt{(a+4)} \ge (1-a)$ which is always true for $a > 1$ since R.H.S. is negative.

Combining (2) and (3), we get

$a \in \left[-4, \dfrac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$.

Example 18: Prove that the function $f(x) = \dfrac{\ln x}{x}$ is strictly decreasing in (e, ∞). Hence, prove that

$303^{202} < 202^{303}$.

Solution: We have $f(x) = \dfrac{\ln x}{x}$, $x > 0$.

Then $f'(x) = \dfrac{1 - \ln x}{x^2} < 0,\ \forall\ x > e$

$\Rightarrow \quad f(x)$ strictly decreases in (e, ∞).

When a function is strictly decreasing, and $x_1 < x_2$ then $f(x_1) > f(x_2)$

Thus, we have $f(303) < f(202)$ as $303 > 202$.

$\Rightarrow \quad \dfrac{\ln(303)}{303} < \dfrac{\ln(202)}{202}$

$\Rightarrow \quad 202 \ln(303) < 303 \ln(202)$

$\Rightarrow \quad 303^{202} < 202^{303}$

which is the desired result.

Example 19: If $f(x) = x^3 - x^2 + 100x + 2000$, then prove that

(i) $f(1000) < f(1001)$

(ii) $f\left(\dfrac{1}{2000}\right) > f\left(\dfrac{1}{2001}\right)$

(iii) $f(x-1) > f(x-2)$

(iv) $f(2x-3) > f(x)$ for $x > 3$.

Solution: $f(x) = x^3 - x^2 + 100x + 2000$

$f'(x) = 3x^2 - 2x + 100 > 0\ \forall\ x \in R$

$\therefore \quad f(x)$ is strictly increasing $\forall\ x \in R$.

Since $1000 < 1001$ and $f(x)$ is strictly increasing $\forall$ x, we have $f(1000) < f(1001)$.

Also $\quad \dfrac{1}{2000} > \dfrac{1}{2001}$

$\therefore \quad f\left(\dfrac{1}{2000}\right) > f\left(\dfrac{1}{2001}\right)$

We have $f(x-1) > f(x-2)$ as $x - 1 > x - 2$ for $\forall$ x

and $f(2x-3) > f(x)$ for such x for which $2x - 3 > x$

i.e. $x > 3$.

Example 20: Let f(x) and g(x) be two continuous function defined from $R \rightarrow R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\ x_1 > x_2$, then find the solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$.

Solution: Obviously, f is an increasing function and g is a decreasing function.

Hence $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

$\Rightarrow \quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$ as f is increasing

$\Rightarrow \quad \alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing

$\Rightarrow$ $\alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow$ $\alpha \in (1, 4)$.

Example 21: Let $f(x) = 1 - x - x^3$.
Find all real values of x satisfying the inequality,

$$1 - f(x) - f^3(x) > f(1 - 5x).$$

Solution: $f(x) = 1 - x - x^3$

$\Rightarrow$ $f'(x) = -1 - 3x^2$ which is negative $\forall\ x \in R$

$\Rightarrow$ f is strictly decreasing.

$f[f(x)] = 1 - f(x) - f^3(x)$

$\therefore$ $f[f(x)] > f(1 - 5x)]$

Since, f (x) is strictly decreasing

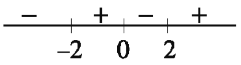
$f(x_1) > f(x_2) \Rightarrow x_1 < x_2$

$\therefore$ $f(x) < 1 - 5x$

$1 - x - x^3 < 1 - 5x$

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$\therefore$ $x \in (-2, 0) \cup (2, \infty)$.

Example 22: Prove that $(1 + x^{-1})^{1+x}$ is a strictly decreasing function for x > 0 with limit e as $x \to \infty$.

Solution: Let $f(x) = (1 + 1/x)^{1+x}$.

Then $f'(x) = f(x).g(x)$, where $g(x) = \ln(1 + 1/x) - 1/x$.

Since f(x) is positive, we must prove $g(x) < 0$ for $x > 0$.

Now $g'(x) = 1/(x^2(x + 1)) > 0$ for $x > 0$,

so g(x) is strictly increasing.

Since $g(x) \to 0$ as $x \to \infty$, we have $g(x) < 0$ for $x > 0$.

The simplest way to find $\lim\limits_{x \to \infty} f(x)$ is to write

$f(x) = (1 + 1/x)(1 + 1/x)^x$.

Clearly $(1 + 1/x) \to 1$, and $(1 + 1/x)^x \to e$.

Example 23: If $f(x) = \dfrac{x^2}{2 - 2\cos x}$ and $g(x) = \dfrac{x^2}{6x - 6\sin x}$ where $0 < x < 1$, then show that f is a strictly increasing and g is a strictly decreasing function.

Solution: $f'(x) = \dfrac{1}{2}\left[\dfrac{(1 - \cos x)\,2x - x^2 \sin x}{(1 - \cos x)^2}\right]$

Now consider the numerator as

$p(x) = 2(1 - \cos x) - x \sin x$

$$= 4\sin^2\frac{x}{2} - 2x\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2x\sin\frac{x}{2}\cos\frac{x}{2}\left[\frac{\tan\frac{x}{2}}{\frac{x}{2}} - 1\right] > 0$$

$\Rightarrow f'(x) > 0 \Rightarrow$ f is strictly increasing.

Now, $g'(x) = \dfrac{1}{6}\left[\dfrac{(x - \sin x)\,2x - x^2(1 - \cos x)}{(x - \sin x)^2}\right]$

Again, consider the numerator as

$q(x) = x - 2\sin x + x\cos x$

$$= 2x\cos^2\frac{x}{2} - 4\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2x\cos^2\frac{x}{2}\left[1 - \frac{\tan\frac{x}{2}}{\frac{x}{2}}\right] < 0$$

$\Rightarrow g'(x) < 0 \Rightarrow$ g is strictly decreasing.

Example 24: Find possible values of a such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is strictly increasing for $x \in R$.

Solution: $f(x) = e^{2x} - (a + 1)e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)e^x + 2$

Now, $2e^{2x} - (a + 1)e^x + 2 \ge 0$ for all $x \in R$

$\Rightarrow$ $2\left(e^x + \dfrac{1}{e^x}\right) - (a + 1) \ge 0$ for all $x \in R$

$(a + 1) \le 2\left(e^x + \dfrac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow$ $a + 1 \le 4$ $\left(\because e^x + \dfrac{1}{e^x} \text{ has the least value } 2\right)$

$\Rightarrow$ $a \le 3$.

Alternative:

$2e^{2x} - (a + 1)\,ex + 2 \ge 0$ for all $x \in R$

Putting $e^x = t$ where $t \in (0, \infty)$

$2t^2 - (a + 1)t + 2 \ge 0$ for all $t \in (0, \infty)$

Hence either

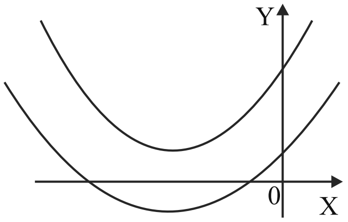
(i) $D \le 0$

$\Rightarrow$ $(a + 1)^2 - 4 \le 0$

$\Rightarrow$ $(a + 5)(a - 3) \le 0$

$\Rightarrow$ $a \in [-5, 3]$

or (ii) both roots are negative



We solve $D \ge 0$, $-\dfrac{b}{2a} < 0$ and $f(0) \ge 0$

$D \ge 0 \Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$

$-\dfrac{b}{2a} < 0 \Rightarrow \dfrac{a+1}{4} < 0 \Rightarrow a < -1$

$f(0) \ge 0 \Rightarrow 2 \ge 0 \Rightarrow a \in R$

Hence, $a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

Concept Problems B

1. Find out whether each of the following statements is true:

(i) "If a function increases at a point x_0, then it has a positive derivative at the point".

(ii) "If the function f(x) differentiable at a point x_0 increases at that point, then $f'(x_0) > 0$".

2. Find out whether the following statement is true: "If the function f(x) differentiable on an interval X increases on that interval, then $f'(x) > 0 \ \forall \ x \in X$".

3. Let the function f(x) be defined in a neighbourhood of every point of the set X. Find out whether each of the following statements is true:

(i) "If f(x) increases on the set X, then it increases at every point $x_0 \in X$".

(ii) "If f(x) increases at every point $x_0 \in X$, then it increases on the set X". (Consider the function $f(x) = -1/x$).

4. Prove that if a function increases at every point of an open interval, then it increases on that interval. Will the statement remain true if we replace the interval by an arbitary set?

5. Suppose that f is an increasing function on [a, b] and that x_0 is a number in (a, b). Prove that if f is differentiable at x_0, then $f'(x_0) \geq 0$.

6. Show that if f(x) is strictly decreasing on an interval I where it is differentiable, then $f'(x) \leq 0$ for all x in I.

7. Prove that the following functions are strictly increasing.

(i) $f(x) = \cot^{-1}x + x$

(ii) $f(x) = \ln(1 + x) - \dfrac{2x}{2+x}$

8. Is the function cos (sin t) increasing or decreasing on the closed interval [–π/2, 0]?

9. (i) Show that g(x) = 1/x decreases on every interval in its domain.

(ii) If the conclusion in (a) is really true, how do you explain the fact that g(1) = 1 is actually greater than g(–1) = – 1?

10. Find the monotonicity of the following functions for $x \in R$.

(i) $f(x) = \begin{cases} 2x, & x < 0, \\ 3x+5, & x \geq 0 \end{cases}$

(ii) $f(x) = \begin{cases} 2x^3+3, & x \neq 0, \\ 4, & x = 0 \end{cases}$

11. Let $f(x) = \begin{cases} x^3, & x \leq 0 \\ 2\sin 2x, & 0 < x \leq a \end{cases}$.

Find the largest value of a so that f(x) is a monotonous function.

12. Find the behaviour of the function

$f(x) = \begin{cases} 2x, & x < 0 \\ 2\cos x, & x \geq 0 \end{cases}$ at x = 0.

13. Find the behaviour of the function

$f(x) = \begin{cases} x^2, & x < 0 \\ -1, & x = 0 \\ 2\sin(x-\pi), & x > 0 \end{cases}$ at x = 0.

14. Suppose an odd function is known to be increasing on the interval x > 0. What can be said of its behaviour on the interval x < 0?

15. Suppose an even function is known to be increasing on the interval x < 0. What can be said of its behaviour on the interval x > 0?

16. Find the value of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x.

17. If $f(x) = \dfrac{2x}{1-x^2}$ strictly increasing in its domain?

18. Show that the function $y = \dfrac{x^2-1}{x}$ increases in any interval not containing the point x = 0

19. Find the values of k for which the function $f(x) = (k-1)x + k^2 - 3$, $x \in (-\infty, \infty)$ is

(i) strictly increasing (ii) strictly decreasing.

Practice Problems A

20. Prove that $f(x) = \dfrac{3}{2}x - \sin^2 x$ increases for $x \in R$.

21. Show that $f(x) = \dfrac{x}{\sqrt{a^2+x^2}}$ is an increasing function of x.

22. Show that $g(x) = \dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$ is a decreasing function of x.

23. Prove that the function tan (cost) is decreasing on the closed interval [0, π/2].

24. Is the function cos (sin (cos t)) increasing or decreasing on the closed interval [π/2, π] ?

25. Prove that the function $f(x) = (1+x)^{3/2} - \dfrac{3}{2}x - 1$ is strictly increasing on (0, ∞).

26. Prove that for $a \in \left(0, \frac{20}{9}\right)$ the function $f(x) = x^5 + a(x^3 + x) + 1$ is invertible.

27. For which values of the constant k is the function $7x + k \sin 2x$ always increasing?

28. For what values of a is the function $f(x) = x^3 - ax$ strictly increasing for all x ?

29. If $a^2 - 3b + 15 < 0$, then show that $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is an increasing function for all x.

30. If $f(x) = e^x(x^2 - x + 2) - (x^2 + x + 2)$, prove that when x is positive, f(x) increases as x increases.

31. If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is strictly decreasing $\forall\, x \in R$, find 'a'.

32. Prove that $f(x) = \frac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is strictly increasing.

33. Show that the function $f(x) = \frac{x}{\sqrt{1+x}} - \ell n(1 + x)$ is an increasing function when $x > -1$.

34. Find the set of all real values of μ so that the function $f(x) = (\mu + 1)x^3 + 2x^2 + 3\mu x - 7$ is

(i) strictly increasing (ii) strictly decreasing.

35. Prove that the function

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin\frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is not monotonic in any interval containing the origin.

36. Show that the derivative

$f'(x) = \frac{1}{2} + 2x \sin\frac{1}{x} - \cos\frac{1}{x}$, $(x \neq 0)$, is equal to $\frac{3}{2}$ at the points $x = \frac{1}{(2n+1)\pi}$ $(n = 0, \pm 1, \pm 2, \ldots)$, and to $-\frac{1}{2}$ at the points $x = \frac{1}{2n\pi}$, i.e. the derivative changes sign in any vicinity of the origin.

37. Suppose that f is increasing on every closed interval [a, b] provided that $2 \le a < b$. Prove that f is increasing on the unbounded open interval $(2, \infty)$.

38. Show that $y = \tan^{-1}x - x$ decreases everywhere and hence deduce that $\tan^{-1}1 + 1 > \tan^{-1}2$.

39. For what values of a is the function, $f(x) = \left(\frac{a^2-1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$ monotonous.

6.3 Critical Point

We understand that not all functions are monotonous in their domain. In general, a function increases and decreases in different parts of its domain. Suppose a function f defined in (a, b), increases in (a, c) and then decreases in (c, b), we are now interested in knowing what must have happened at x = c and how do we get c?

To answer these questions, we should first define the term 'critical points'. We have seen in the previous section that the sign of the derivative helps in determining the behaviour of a function. These points play a crucial role in finding the sign of the derivative of a function.

Definition. A critical point of a function f is an interior point c in the domain of f such the either $f'(c) = 0$ or $f'(c)$ does not exist.

Sometimes we will want to distinguish critical numbers at which $f'(x) = 0$ from those at which f is not differentiable. We will call a point on the graph of f at which $f'(x) = 0$, a **stationary point** of f.

The stationary points of f are the x-intercepts of the graph of f′.

Note:

(i) If x = c is a critical point of the function f, then it is also a critical point of the function $g(x) = f(x) + k$, where k is a constant.

(ii) If x = c is a critical point of the function f, then x = c + k is a critical point of the function $g(x) = f(x - k)$, where k is a constant.

For example, x = 0 is a critical point of $f(x) = x^2$ and x = 1 is a critical point of $g(x) = (x - 1)^2$.

Remark In some texts, if $f'(c) = 0$ or $f'(c)$ does not exist, then x = c is called as critical number and (c, f(c)) is called as the critical point.

Example 1: Find the critical points of the function $f(x) = x^{3/5}(4 - x)$.

Solution: $f'(x) = \frac{3}{5}x^{-2/5}(4 - x) + x^{3/5}(-1)$

$$= \frac{3(4-x)}{5x^{2/5}} - x^{3/5}$$

$$= \frac{3(4-x)-5x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}.$$

Therefore, $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when x = 0.

Thus, the critical points of f(x) are 0 and $\frac{3}{2}$.

Example 2: Find the critical points for the function

$$f(x) = \frac{e^x}{x-2}$$

Solution: $f'(x) = \dfrac{(x-2)e^x - e^x(1)}{(x-2)^2} = \dfrac{e^x(x-3)}{(x-2)^2}$

The derivative is not defined at x = 2, but f is not defined at 2 either, so x = 2 is not a critical point. The critical points are found by solving $f'(x) = 0$:

$$\frac{e^x(x-3)}{(x-2)^2} = 0 \Rightarrow x = 3.$$

So, x = 3 is the only critical point.

Example 3: Find the critical points of $f(x) = (x-2)^{2/3}(2x+1)$.

Solution: Given, $f(x) = (x-2)^{2/3}(2x+1)$

$\Rightarrow \quad f'(x) = \dfrac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3}\cdot 2$

or $\quad f'(x) = 2\left[\dfrac{(2x+1)}{3(x-2)^{1/3}} + (x-2)^{2/3}\right]$

Clearly, $f'(x)$ does not exist at x = 2, so, x = 2 is a critical point. Other critical points are given by, $f'(x) = 0$

i.e., $2\left[\dfrac{(2x+1)+3(x-2)}{(x-2)^{1/3}}\right] = 0$

$\Rightarrow \quad 5x - 5 = 0 \Rightarrow x = 1$

Hence, x = 1 and x = 2 are two critical points of f(x).

Example 4: Find all possible values of the parameter b for each of which the function,

$f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$

is strictly decreasing throughout the number line and has no critical points.

Solution: We have $f'(x)$

$= 2\cos 2x + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= 2(1 - 2\sin^2 x) + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= -4[\sin^2 x - 2(b+2)\sin x + (b^2 + 4b + 1)]$

For f to be strictly decreasing with no critical points $f'(x) < 0 \ \forall\ x \in R$

Now, $D = 4(b+2)^2 - 4(b^2 + 4b + 1)$

$= 12$ which is always positive.

Let $\sin x = y$ where $y \in [-1, 1]$

and $g(y) = y^2 - 2(b+2)y + (b^2 + 4b + 1)$

We have to find those values of 'b' for which

$g(y) > 0$ for all $y \in [-1, 1]$

The conditions are

(i) $g(-1) > 0$ and $-\dfrac{b}{2a} < -1$, or

(ii) $g(1) > 0$ and $-\dfrac{b}{2a} > 1$

The condition (i) gives

$1 + 2(b+2) + b^2 + 4b + 1 > 0$

$b^2 + 6b + 6 > 0 \qquad \text{...(1)}$

and $\dfrac{2(b+2)}{2} < -1$ or $b < -3 \qquad \text{...(2)}$

(1) and (2) $\Rightarrow b < -(3 + \sqrt{3})$

Similarly, the condition (ii) gives $b > \sqrt{3} - 1$

Hence, $b \in (-\infty, -(3 + \sqrt{3})) \cup (\sqrt{3} - 1, \infty)$.

Practice Problems B

1. Prove that $f(x) = \dfrac{x^5}{e^x - 1}$ has two stationary points.
2. Find the stationary points of $f(x) = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$.
3. Find the critical points of the function:
   - (i) $f(t) = 3t^4 + 4t^3 - 6t^2$
   - (ii) $f(x) = x^{4/5}(x-4)^2$
   - (iii) $f(\theta) = 2\cos\theta + \sin^2\theta$
4. Find the critical points of the function:
   - (i) $y = x + \cos^{-1} x + 1$
   - (ii) $y = x\tan^{-1} x$
   - (iii) $y = e^{|x|} - 2x + 1$
5. Define f(x) to be the distance from x to the nearest integer. What are the critical points of f ?
6. Find the critical points of the function:
   - (i) $y = \sqrt{x^2 - 6x + 15}$ (ii) $y = (x+2)\sqrt{x-1}$.
7. Find the critical points of the function:
   - (i) $f(x) = e^3 - \sqrt{4x^2 - 12x + 9} - 4\sin^2\dfrac{x}{2}$
   - (ii) $f(x) = \sin^2 3x + 3\sqrt{x^2 - 4x + 4} + \cos 1$
8. Find the critical points of the function:
   - (i) $y = 3\sin x + 2(x-1)$
   - (ii) $y = \cos 2x + ax - \sqrt{3}$
9. Find the critical points of the function $y = 2\sin^2\dfrac{x}{6} + \sin\dfrac{x}{3} - \dfrac{x}{3}$. whose coordinates satisfy the inequality $x^2 - 10 < -19.5x$.

10. Find all the values of a for which the function f(x) does not posses critical points where

$$f(x) = (4a - 3)(x + \ln 5) + 2(a - 7)\cot\frac{x}{2}\sin^2\frac{x}{2}.$$

6.4 Intervals of Monotonicity

We now wish to find the various intervals in which a non-monotonous function f increases or decreases. For this purpose we first find the domain of the function f. Suppose that the function is defined in [a, b]. We then find the critical points, say $x_1, x_2, \ldots, x_{n-1}, x_n$.

Next we split the whole interval [a, b], by means of the points x_i, into the subintervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, b)$. In each subinterval, the derivative has constant sign. We assume that the derivative is continuous in each subinterval, and hence, if it changed the sign, $f'(x)$ would turn into zero at some new point, which is impossible since all such points have already been found as critical points (they serve as the end points of the subintervals).

Thus, these subintervals are the intervals of monotonicity of the function.

Now we must determine the sign of the derivative in each subinterval. The sign of the derivative in each subinterval can be determined by computing the value of the function $f'(x)$ at an arbitrary point of every subinterval.

If the derivative is represented as a product of a number of factors it is sufficient to determine the signs of these factors without computing their values since these signs specify the sign of the derivative. The sign of the derivative specifies the character of variation of the function in each interval of monotonicity, that is its increase or decrease.

For example, take the function

$$y = 2/3x^3 - 2x^2 - 6x + 3.$$

$$y' = 2x^2 - 4x - 6$$

Solving $y' = 0$ we get the critical points $x = -1$ and 3.

Note that y is a differentiable function for all x.

To check that in the interval $-1 < x < 3$, whether it decreases, it is sufficient to verify that its derivative

$y' = 2x^2 - 4x - 6$ is negative for $-1 < x < 3$.

The latter does in fact take place since

$y' = 2(x + 1)(x - 3)$, and the factor $(x + 1)$ is positive for all the values of x in this interval while the factor $(x - 3)$ is negative.

Caution

It was noticed earlier that a function may be strictly monotonous even when its derivative became zero at several discrete points or the derivative didnot exist. This implies that all critical points may not be instrumental in changing the monotonic behaviour of the function. This means that $f'(x)$ need not change sign at each critical point.

For instance, the function

$$f(x) = (x^2 - 2x + 2)e^x \text{ whose } f'(x) = x^2e^x$$

has a critical point x = 0, but the function does change its increasing behaviour at x = 0 since its derivative maintains positive sign across the point x = 0. Here, f(x) is strictly increasing for $x \in (-\infty, \infty)$.

Steps for Finding Intervals of Monotonicity

Let us now formulate the rule for finding the intervals of monotonicity of a function:

1. Compute the derivative $f'(x)$ of a given function f(x), and then find the points at which $f'(x)$ equals zero or does not exist at all. These points are the critical points for the function f(x).
2. Using the critical points, separate the domain of definition of the function f(x) into several intervals on each of which the derivative $f'(x)$ retains its sign. These intervals will be the intervals of monotonicity.
3. Investigate the sign of $f'(x)$ on each of the found intervals. If on a certain interval $f'(x) > 0$, then the function f(x) increases on this interval, and if $f'(x) < 0$, then f(x) decreases on this interval.

Example 1: Test the function $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^2$ for increase or decrease.

Solution: First, we find the derivative : $f'(x) = x^4 - x^2$

Next, we determine the critical points:

$f'(x)$ exist for all x and $f'(x) = 0$ when $x^4 - x^2 = 0$, we find the points $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, at which the derivative $f'(x)$ vanishes.

Since $f'(x)$ can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, $f'(x)$ has no discontinuities), the derivative in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ retains its sign; for this reason, the function under investigation is monotonic in each of these intervals.

To determine in which of the indicated intervals the function increases and in which it decreases, one has to determine the sign of the derivative in each of the intervals.

To determine what the sign of $f'(x)$ is in the interval $(-\infty, -1)$, it is sufficient to determine the sign of $f'(x)$ at some point of the interval; for example, taking $x = -2$, we get $f'(-2) = 12 > 0$; hence, $f'(x) > 0$ in the interval $(-\infty, -1)$ and the function in this interval increases.

Similarly, we find that $f'(x) < 0$ in the interval $(-1, 0)$ (as a check, we can take $x = -\frac{1}{2}$), $f'(x) < 0$ in the interval $(0, 1)$ (here, we can use $x = 1/2$ and $f'(x) > 0$ in the interval $(1, \infty)$.

Thus, the function increases in the interval $(-\infty, -1)$, decreases in the interval $(-1, 1)$ and again increases in the interval $(1, \infty)$.

Example 2: Determine where the function $f(x) = x^3 - 3x^2 - 9x + 1$ is strictly increasing and where it is strictly decreasing.

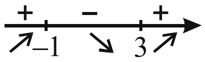
Solution: First, we find the derivative:

$$f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$$

Next, we determine the critical points:

$f'(x)$ exist for all x and $f'(x) = 0$ at $x = -1$ and $x = 3$.

These critical points divide the x-axis into three parts, and we select a typical number from each of these intervals. For example, we select –2, 0 and 4, and evaluate the derivative at these numbers, and mark each interval as increasing (↑) or decreasing(↓), according to whether the derivative is positive or negative, respectively.



Thus, the function increases in the intervals $(-\infty, -1)$ and $(3, \infty)$ and decreases in the interval $(-1, 1)$.

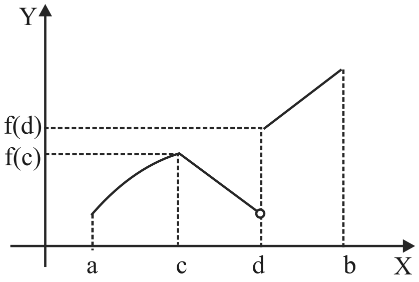
Caution

While writing the intervals of increase or decrease it is not advisable to use the union symbol '∪', unless due care has been taken. Suppose that in the above example we write the intervals of increase as

$(-\infty, -1) \cup (3, \infty)$, then by definition it means that if $x_1, x_2 \in (-\infty, -1) \cup (3, \infty)$ where $x_1 < x_2$ then $f(x_1) < f(x_2)$ for all such x_1, x_2. This is not true since x_1 can belong to $(-\infty, -1)$ and x_2 can belong to $(3, \infty)$ and we have not checked that the maximum value of f(x) obtained in $(-\infty, -1]$ is whether less than or equal to the minimum value of f(x) obtained in $[3, \infty)$. In this function it is surely not so.

In case of discontinuous functions there is a chance for this to happen.

For instance, see the function y = f(x) graphed below:



Here, the function increases in the intervals (a, c), (d, b) and we may proceed to write that it increases in (a, c) ∪ (d, b) because we in this function we have $f(c) \le f(d)$.

Example 3: Find whether the function $f(x) = xe^{-3x}$, increases or decreases.

Solution: We find the derivative

$$f'(x) = e^{-3x} - 3xe^{-3x} = e^{-3x}(1 - 3x).$$

The derivative $f'(x)$ exists everywhere and vanishes at the point 1/3. The point $x = 1/3$ divides the number line into two intervals, $(-\infty, 1/3)$ and $(1/3, \infty)$.

Since the function e^{-3x} is always positive, the sign the derivative is decided by the second factor. Consequently $f'(x) > 0$ on the interval $(-\infty, 1/3)$ and $f'(x) < 0$ on the interval $(1/3, \infty)$.

Hence, the function f(x) increases on the interval $(-\infty, 1/3)$ and decreases on the interval $(1/3, \infty)$.

Example 4: Find the intervals of monotonicity of $f(x) = (2^x - 1)(2^x - 2)^2$.

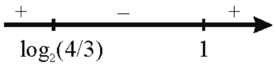
Solution: $f(x) = (2^x - 1)(2^x - 2)^2$

$$\Rightarrow \quad f'(x) = 2^x\log2(2^x - 2)^2 + 2(2^x - 2)\log2(2^x - 1)$$
$$= 2^x\log2(2^x - 2)[(2^x - 2) + 2(2^x - 1)]$$
$$= 2^x\log2(2^x - 2)[3.2^x - 4]$$

$f'(x) = 0 \Rightarrow \quad 2^x - 2 = 0 \qquad$ or, $\quad 3.2^x - 4$

$\Rightarrow \quad x = 1 \qquad \Rightarrow \quad x = \log_2(4/3)$

Sign scheme of $f'(x)$



Thus, f(x) is increasing in $(-\infty, \log_2(4/3))$ and $(1, \infty)$ and decreasing in $(\log_2(4/3), 1)$.

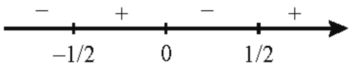
Example 5: Find the intervals of monotonicity of the following functions:

(i) $f(x) = 2x^2 - \ln|x|$ (ii) $f(x) = \dfrac{x^3}{x^4 + 27}$.

Solution: (i) We have $f(x) = 2x^2 - \ln|x|$

and $f'(x) = 4x - \frac{1}{x} = \frac{4\left(x+\frac{1}{x}\right)\left(x-\frac{1}{2}\right)}{x}$

Now, from the sign scheme for $f'(x)$, we have



⇒ f(x) strictly decreases in (–∞, –1/2)
strictly increases in (–1/2, 0)
strictly decreases in (0, 1/2)
strictly increases in (–1/2, ∞)

Finally, f(x) increases in $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \infty\right)$

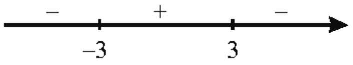
and decreases in $\left(-\infty, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)$.

(ii) We have $f(x) = \frac{x^3}{x^4+27}$ and

$$f'(x) = \frac{(x^4+27)(3x^2) - x^3(4x^3)}{(x^4+27)^2} = \frac{-x^2(x^4-81)}{(x^4+27)^2}$$

$$= \frac{x^2(x^2+9)(x+3)(x-3)}{(x^4+27)^2}$$

Now, from the sign scheme for $f'(x)$, we have



⇒ f(x) strictly decreases in (–∞, –3)
strictly increases in (–3, 3)
strictly decreases in (3, ∞).

Finally, f(x) increases in (–3, 3)
and decreases in (–∞, –3), (3, ∞).

Example 6: Determine the intervals of increase and decrease of the function $y = \frac{1}{x+2}$.

Solution: Here, $x = -2$ is a discontinuity of the function and $y' = \frac{1}{(x+2)^2} < 0$ for $x \neq -2$.

Hence, the function y decreases in the intervals
$-\infty < x < -2$ and $-2 < x < \infty$.

Note that the function decreases in two separate intervals and we cannot say that the function is decreasing in its domain, since $f(-2^-) = -\infty$ and $f(-2^+) = \infty$. In fact the values of the function on the left of $x = -2$ are smaller than those on the right of –2.

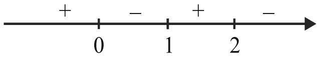
Example 7: Find the intervals of monotonicity of the function $f(x) = \frac{|x-1|}{x^2}$.

Solution: We have $f(x) = \frac{1-x}{x^2}$, $x < 1$

$= \frac{1-x}{x^2}$, $x \geq 1$

and $f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$, $x < 1$

$= \frac{2-x}{x^3}$, $x > 1$



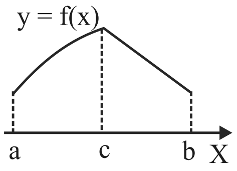
Now, from the sign scheme for $f'(x)$, we see that f(x) strictly increases in (–∞, 0), (1, 2) and strictly decreases in (0, 1), (2, ∞).

Remark It is a convention that we write the intervals of monotonicity using open intervals but, ideally the use of closed intervals is more informative, particularly in discontinuous functions. In the case of continuous functions defined on a close interval, the open intervals of monotonicity can be easily replaced by closed intervals. However, in the case of discontinuous functions due care must be taken in using closed brackets.

For instance, consider the following functions. Using open brackets we would write in each of the functions that f(x) increases in the interval (a, c) and decreases in the interval (c, b).

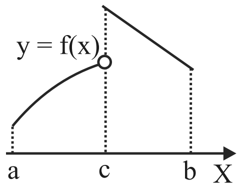
Now, we write the intervals of monotonicity using closed brackets.

(i)



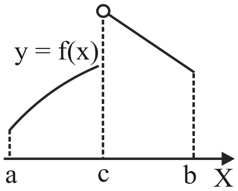
f(x) increases in the interval [a, c] and decreases in the interval [c, b].

(ii)



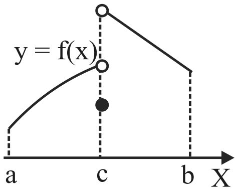
f(x) increases in the interval [a, c] and decreases in the interval [c, b].

(iii)



f(x) increases in the interval [a, c] and decreases in the interval (c, b].

(iv)



f(x) increases in the interval [a, c) and decreases in the interval (c, b].

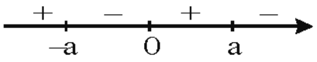
Example 8: Find the intervals of increase of the function $f(x) = x^2\ e^{-x^2/a^2}$, $a > 0$

Solution: $f'(x) = 2x\,e^{-x^2/a^2} + x^2 e^{-x^2/a^2} \times \frac{-2x}{a^2}$

$$= 2x\,e^{-x^2/a^2}\left(1 - \frac{x^2}{a^2}\right) = \frac{2e^{-x^2/a^2}}{a^2}x(a^2 - x^2)$$

= positive no. $[-1\,(x + a)\,(x - 0)\,(x - a)]$

Sign scheme of $f'(x)$



f(x) is increasing in $(-\infty, -a]$ and $[0, a]$.

Example 9: Find the intervals of monotonicity of the function $f(x) = x + \frac{4}{x^2}$.

Solution: The function f is undefined at x = 0, but continuous elsewhere. Evidently $f'(x) = 1 - \frac{8}{x^3}$, and $f'(x) = 0$ if and only if, x = 2. Thus, 2 is the only critical point. Therefore, f is strictly monotonic in each of the intervals $(-\infty, 0)$, $(0, 2]$, $[2, \infty]$.

Since $f'(-1) > 0$, $f'(1) < 0$, $f'(3) > 0$, we have:

f is strictly increasing in $(-\infty, 0)$, strictly decreasing in $(0, 2]$, and strictly increasing in $[2, \infty)$.

Note:

(i) If f is increasing on the intervals (a, c] and [c, b), then f is increasing on (a, b).

(ii) If f is decreasing on the intervals (a, c] and [c, b), then f is decreasing on (a, b).

Remark Suppose that the function is defined in (a, b). If it increases in two consecutive intervals, say (a, c) and (c, b) then can we always write that it increases in (a, b)? The answer is no. This can be surely done if the function is continuous at x = c. However, if it is discontinuous at x = c, then we can join the intervals only when $f(c^-) \le f(c) \le f(c^+)$.

Similarly, if the function decreases in two consecutive intervals then we can join the intervals only when $f(c^-) \ge f(c) \ge f(c^+)$.

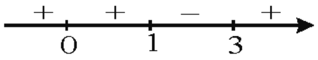
For example, consider the function

$f(x) = x^5 - 5x^4 + 5x^3 + 1$.

$f'(x) = 5x^2(x - 1)(x - 3)$

The critical points are x = 0, 1 and 3.

Sign scheme of $f'(x)$:



The function increases in $(-\infty, 0)$, $(0, 1)$ and $(3, \infty)$.

Since, f is continuous at x = 0, we may prefer to write that f increases in $(-\infty, 1)$ and $(3, \infty)$.

Example 10: Show that $f(x) = \begin{cases} x^2 + 2x, & -1 < x < 0 \\ 3x + 1, & 0 \le x < 1 \end{cases}$

is strictly increasing in (–1, 1), but

$g(x) = \begin{cases} x^2 + 2x, & -1 < x < 0 \\ 3x - 1, & 0 \le x < 1 \end{cases}$ is not strictly increasing in (–1, 1).

Solution: We have $f'(x) = \begin{cases} 2x + 2, & -1 < x < 0 \\ 3, & 0 < x \le 1 \end{cases}$

For all $x \ne 0$, $f'(x) > 0$.

Thus, f(x) is increasing in (– 1, 0) and (0, 1).

At x = 0, f is discontinuous,

however $f(0^-) = 0$, $f(0) = 1$ and $f(0^+) = 1$.

Because of the non-decreasing order of these quantities, we can proceed to say that f(x) is increasing in (– 1, 1).

Quite similar to f, g(x) is decreasing in (– 1, 0) and (0, 1). At x = 0, g is also discontinuous.

But $g(0^-) = 0$, $g(0) = 1$ and $g(0^+) = -1$.

Since, the non-decreasing order of these quantities, is not maintained, g(x) cannot be said to increase in (– 1, 1).

Example 11: Find the intervals of increase and decrease of the function $f(x) = \frac{x}{\ln x}$.

Solution: The function $f(x) = \frac{x}{\ln x}$ is defined for $x \in (0, 1) \cup (1, \infty)$.

Now, $f'(x) = \frac{\ln x - 1}{(\ln x)^2}$.

Let $f'(x) = 0 \Rightarrow \ln x - 1 = 0 \Rightarrow x = e$

If follows that $f'(x) > 0$ if $x > e$ and $f'(x) < 0$ if $x < e$.

Consequently, f(x) decreases on the intervals (0, 1) and (1, e) and increases on the interval (e, ∞).

Note that here we cannot join the intervals of decrease (0, 1) and (1, e) because f is undefined at x = 1.

Example 12: A function f(x) is given by the equation, $x^2 f'(x) + 2x\,f(x) - x + 1 = 0$ $(x \neq 0)$. If f(1) = 0, then find the intervals of monotonicity of f.

Solution: We have $x^2 f'(x) + 2x\,f(x) - x + 1 = 0$

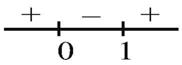
$$\Rightarrow \frac{d}{dx}[x^2 y] = x - 1 \text{ where } y = f(x)$$

$$\Rightarrow x^2 y = \int (x-1)\,dx \quad \Rightarrow \quad x^2 y = \frac{x^2}{2} - x + c$$

Now f(1) = 0 $\Rightarrow$ $c = \frac{1}{2}$.

This gives $y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$.

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^3} = \frac{x-1}{x^3}$$



Considering the sign of $\frac{dy}{dx}$, we find that f(x) increases in (– ∞, 0) and (1, ∞) while it decreases in (0, 1).

Example 13: Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\, x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the intervals in which g(x) is increasing and decreasing.

Solution: Here,

$$f'(\sin x) < 0 \text{ and } f''(\sin x) > 0,\ \forall\, x \in \left(0, \frac{\pi}{2}\right) \qquad ...(1)$$

and $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow$ $g'(x) = f'(\sin x)\cdot\cos x + f'(\cos x)(-\sin x)$

$\Rightarrow$ $g''(x) = \{-f'(\sin x)\cdot\sin x + f''(\sin x)\cos^2 x\}$
$- \{f'(\cos x)\cdot\cos x - f''(\cos x)\sin^2 x\}$ ...(2)

From (1), we have $f'(\cos x) < 0$ and $f''(\cos x) > 0$

$$\forall\, x \in \left(0, \frac{\pi}{2}\right) \qquad ...(3)$$

$\therefore$ Using (1), (2) and (3), we have,

$$g''(x) = \underbrace{\{-f'(\sin x)\cdot\cos x\}}_{\text{positive}} + \underbrace{\{f''(\sin x)\cdot\cos^2 x\}}_{\text{positive}}$$
$$+ \underbrace{\{f''(\cos x)\cdot\sin^2 x\}}_{\text{positive}} + \underbrace{\{-f'(\cos x)\cdot\cos x\}}_{\text{positive}}$$

$$\Rightarrow g''(x) > 0\ \forall\, x \in \left(0, \frac{\pi}{2}\right) \qquad ...(4)$$

$\Rightarrow$ $g'(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

Now putting $g'(x) = 0$, we have

$f'(\sin x)\cdot\cos x - f'(\cos x)\sin x = 0$

$\Rightarrow$ $x = \frac{\pi}{4}$ [by trial and error]

Since $g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$, it has atmost one root.

Hence $x = \frac{\pi}{4}$ is the only critical point of g(x).

Also $g'(x) < 0$, when $x \in \left(0, \frac{\pi}{4}\right)$

$g'(x) > 0$, when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

$\therefore$ g(x) is decreasing when $x \in \left(0, \frac{\pi}{4}\right)$

g(x) is increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Example 14: Find the intervals of increase of g(x), where $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall\, x \in R$, given that $f''(x) > 0\ \forall\, x \in R$.

Solution: $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x2)$

$$\Rightarrow g'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2)$$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\} \qquad ...(1)$$

But given that $f''(x) > 0 \Rightarrow f'(x)$ is increasing for all $x \in R$.

Case I: Let $\frac{x^2}{2} > (6 - x^2) \Rightarrow x^2 > 4$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty)$

$\therefore$ $f'\left(\frac{x^2}{2}\right) > f'(6 - x^2)$

$\Rightarrow$ $f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) > 0$ for $x \in (-\infty, -2)$ and $(2, \infty)$...(2)

From (1) and (2), $g'(x) > 0$ for $x \in (2, \infty)$
and $g'(x) < 0$ for $x \in (-\infty, -2)$.

Case II: Let $\frac{x^2}{2} < (6 - x^2) \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$\therefore$ $f'\left(\frac{x^2}{2}\right) < f'(6 - x^2)$

$\Rightarrow$ $f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) < 0$ for $x \in (-2, 2)$ ...(3)

From (1) and (3), $g'(x) < 0$ for $x \in (0, 2)$.
and $g'(x) > 0$ for $x \in (-2, 0)$

Combining both cases, g(x) is increasing in

$x \in (-2, 0)$ and $(2, \infty)$.

Application of Monotonicity in Isolation of Roots

Suppose that

(i) f is continuous on [a, b] and differentiable on (a, b),

(ii) f(a) and f(b) have opposite signs,

(iii) $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b),

then f has exactly one root between a and b.

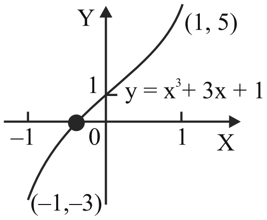
It cannot have more than one root because it is either increasing on [a, b] or decreasing on [a, b]. Yet it has atleast one root, by the Intermediate Value Theorem.

For example, $f(x) = x^3 + 3x + 1$ has exactly one zero on [–1, 1] because

(i) f is differentiable on [–1, 1],

(ii) $f(-1) = -3$ and $f(1) = 5$ have opposite signs, and

(iii) $f'(x) = 3x^2 + 3 > 0$ for all x in [–1, 1].



Consider another example. Let us take the equation

$$f(x) = x^3 + 1.1x^2 + 0.9x - 1.4 = 0$$

Since $f'(x) = 3x^2 + 2.2x + 0.9 > 0$ for all the values of x, the function f(x) is strictly increasing, and hence its graph cuts the x-axis only once.

Besides, $f(0) = -1.4$ and $f(1) = 1.6$, which means that there is a single real root located within the interval [0, 1].

Let us compute $f(0.5) = -0.55$ and then $f(0.7) = 0.112$. This shows that [0.5, 0.7] is a reduced interval of isolation of the sought-for root.

Example 15: Show that the equation $x^5 - 3x - 1 = 0$ has a unique root in [1, 2].

Solution: Consider the function

$$f(x) = x^5 - 3x - 1,\ x \in [1, 2]$$

and $f'(x) = 5x^4 - 3 > 0\ \forall\ x \in (1, 2)$

$\Rightarrow$ f(x) is strictly increasing in (1, 2).

Also, we have

$$f(1) = 1 - 3 - 1 = -3$$

and $f(2) = 32 - 6 - 1 = 25$

Hence y = f(x) will cut the x-axis exactly once in [1, 2]

i.e. f(x) will have a unique root in [1, 2].

Example 16: Find the values of a, if the equation

$$x - \sin x = a \text{ has a unique root in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Solution: Consider the function

$$f(x) = x - \sin x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then $f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) > 0$

$$\forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$\Rightarrow$ f(x) is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Also, we have $f\left(\frac{-\pi}{2}\right) = -\frac{\pi}{2} + 1 - a$

and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a.$

The curve y = f(x) will cuts the x-axis exactly once, if $f\left(\frac{-\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is positive or zero.

i.e. $-\frac{\pi}{2} + 1 - a \le 0$ and $\frac{\pi}{2} - 1 - a \ge 0$

i.e. $a \ge -\frac{\pi}{2} + 1$ and $a \le \frac{\pi}{2} - 1$

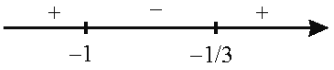
Hence, we have $a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$.

Example 17: Let $f(x) = x^3 + 2x^2 + x + 5$. Show that f(x) has only one real root α such that $[\alpha] = -3$.

Solution: We have $f(x) = x^3 + 2x^2 + x + 5,\ x \in R$

and $f'(x) = 3x^2 + 4x + 1 = (x + 1)(3x + 1),\ x \in R$

Sign scheme of $f'(x)$



f(x) strictly increases in $(-\infty, -1)$

strictly decreases in $(-1, -1/3)$

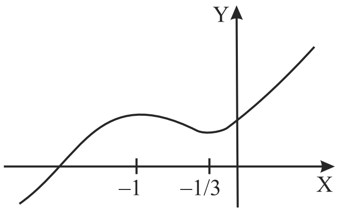
strictly increases in $(-1/3, \infty)$

Also, we have

$$f(-1) = -1 + 2 - 1 + 5 = 5$$

and $f\left(\frac{-1}{3}\right) = \frac{-1}{27} + \frac{2}{9} - \frac{1}{3} + 5 = 5 - \frac{4}{27} = 4.85$

The graph of f(x) (see figure) shows that f(x) cuts the x-axis only once.



Now, we have $f(-3) = -27 + 12 - 3 + 5 = -13$

and $f(-2) = -8 + 8 - 2 + 5 = 3$,
which are of opposite signs. This proves that the curve cuts the x-axis somewhere between –2 and –3.

$\Rightarrow$ $f(x) = 0$ has only one real root α lying between – 2 and –3. Hence, $[\alpha] = -3$.

Concept Problems C

1. Find the intervals of monotonicity of the following functions :
(i) $f(x) = ax^2 + bx + c$ $(a > 0)$,
(ii) $f(x) = x^3 + 3x^2 + 3x$,
(iii) $f(x) = \frac{2x}{1+x^2}$
(iv) $f(x) = x + \sin x$,
(v) $f(x) = x + 2 \sin x$
(vi) $f(x) = \sin(\pi/x)$,
(vii) $f(x) = x^2 2^{-x}$
(viii) $f(x) = x^n e^{-x}$ $(n > 0, x \geq 0)$.

2. Determine the intervals of monotonicity of the following functions:
(i) $f(x) = 3x^4 - 6x^2 + 4$ (ii) $f(x) = \frac{x}{x^2+1}$
(iii) $f(x) = \frac{x^3}{1-x}$

3. Find the intervals of the increase of the function $y = \frac{x+2}{x^2-1}$.

4. Find the intervals of decrease of the function $y = \sqrt{2x^2 - x + 1}$.

5. Show that the equation $xe^x = 2$ has only one positive root found in the interval (0, 1).

6. Show that the function $f(x) = xe^x - 2$ increases and has opposite signs at the endpoints of the interval (0, 1).

7. (i) Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.
(ii) How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

8. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on [0, 1].

Practice Problems C

9. Find the intervals of monotonicity of the given functions.
(i) $y = (x - 2)^5 (2x + 1)^4$
(ii) $y = x^2 e^{-x}$
(iii) $y = \frac{x}{\ln x}$
(iv) $y = x - 2 \sin x$ $(0 \leq x \leq 2\pi)$

10. Find the intervals of monotonicity of the following functions (make use of closed bracket wherever possible):
(i) $f(x) = -x^3 + 6x^2 - 9x - 2$
(ii) $f(x) = x + \frac{1}{x+1}$
(iii) $f(x) = x \,.\, e^{x-x^2}$
(iv) $f(x) = x - \cos x$

11. Find the intervals of increase of the function
(i) $y = |x| - \cos 2x$
(ii) $y = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

12. Determine the intervals of increase and decrease of the function $y = 2e^{x^2-4x}$.

13. Determine the intervals in which the function $f(x) = x^2 \ln 27 - 6x \ln 27 + (3x^2 - 18x + 24) \times \ln(x^2 - 6x + 8)$ is strictly decreasing.

14. Find the critical points of the function $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \sin 2a \sin 6a + \sqrt{\ln(2a - a^2)}$.
Does f(x) decrease or increase at the point $x = 1/2$?

15. Show that $f(x) = x^7 + x^5 + x^3 + x + 1$ has precisely one real zero. How can this result be generalized to polynomials without even powers?

16. Show that the equation $x^2 = x \sin x + \cos x$ holds for exactly two real values of x.

17. Show that the equation $e^{ax} = bx$, where a and b are positive, has two real roots, one, or none, according as $b > ae$, $b = ae$, or $b < ae$.

18. Show that the equation $e^x = 1 + x$ has no real root except $x = 0$, and that $e^x = 1 + x + x^2/2$ has one real root.

19. Show that the equation $3 \tan x + x^2 = 2$ has exactly one solution in the interval $[0, \pi/4]$.

20. Consider the function $f(x) = x^3 + ax^2 + c$. Show that if $a < 0$ and $c > 0$, then f has exactly one negative root.

6.5 Monotonicity in Parametric Functions

Let the function y = f(x) be given by

$x = \phi(t)$

$y = \psi(t)$

One way is to eliminate t to find the function y in terms of x and then proceed as usual.

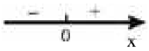
Another way is to continue with t. We first find the critical point in terms of t and then investigate the sign of $\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$.

Example 1: Find the intervals of monotonicity of the function y = f(x) given by x = ln t, $y = (t-1)^2$.

Solution:

Method 1 : We have $t = e^x \Rightarrow y = (e^x - 1)^2$

$$\frac{dy}{dx} = 2(e^x - 1)\, e^x$$



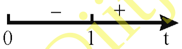
f(x) is increasing for x ∈ (0, ∞) and it is decreasing for x ∈ (– ∞, 0).

Example 2: $x = \ln t,\ y = (t-1)^2$, are defined for t > 0.

$$\frac{dy}{dx} = \frac{2(t-1)}{1/t} = 2t\,(t-1)$$

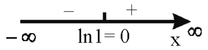
Critical point : t = 1

Sign of $\frac{dy}{dx}$ based on t :



Since, x = lnt increases with t, the order of sign of the derivative is maintained on the x number line.

Sign of $\frac{dy}{dx}$ based on x:



f(x) is increasing for x ∈ (0, ∞) and it is decreasing for x ∈ (– ∞, 0).

Example 3: Find the intervals of monotonicity of the function y = f(x) given by $x = 1 + 2^{-t}$, $y = 2t - t^2$, t ∈ R.

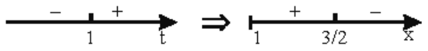
Solution: $\frac{dy}{dx} = \frac{2-2t}{-2^{-t}\ln 2} = \frac{2(t-1)}{2^{-t}\ln 2}$

Critical point : t = 1

Since, $x = 1 - 2^{-t}$ decreases with t, the order of sign of the derivative is reversed on the x number line.

Note that x > 1.

Sign of $\frac{dy}{dx}$



f(x) is increasing for x ∈ (1, 3/2) and f(x) is decreasing for x ∈ (3/2, ∞).

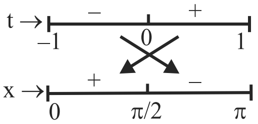
Example 4: Find the intervals of monotonicity of the function y = f(x) given by

$$x = \cos^{-1} t,\ \ y = \ln(4 - t^2),\ -1 \le t \le 1.$$

Solution: $\frac{dy}{dx} = \frac{2t\sqrt{1-t^2}}{(4-t^2)}$

Critical point : t = 0

Sign of $\frac{dy}{dx}$



f(x) is increasing for x ∈ (0, π/2) and f(x) is decreasing for x ∈ (π/2, π).

6.6 Algebra of Monotonous Functions

Negative

If f(x) is a strictly increasing function then its negative g(x) = – f(x) is a strictly decreasing function and vice–versa.

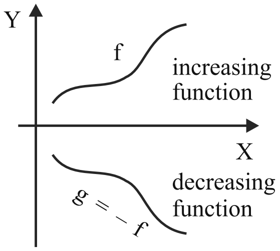
Assuming f to be differentiable, g′(x) = – f ′(x)

Since f ′(x) > 0, g′(x) < 0

⇒ g is a strictly decreasing function

f(x) = $\tan^{-1}x$ is strictly increasing.

∴ g(x) = – $\tan^{-1}x$ is strictly decreasing.



In short, –(an increasing function)
= a decreasing function

i.e. – I = D

Similarly, – D = I

Reciprocal

The reciprocal of a nonzero strictly increasing function is a strictly decreasing function and vice-versa.

In short, $\frac{1}{\text{an increasing function}}$ = a decreasing function

i.e. (i) $\frac{1}{I} = D$ (ii) $\frac{1}{D} = I$

Example 1: Find the intervals of monotonicity of

$$g(x) = \frac{1}{4x^3 - 9x^2 + 6x}$$

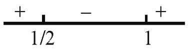
Solution: Let $f(x) = 4x^3 - 9x^2 + 6x$

g(x) = 1/f(x)

Hence, when f increases then g decreases and vice-versa. So we first find the monotonicity of f(x).

$f'(x) = 12x^2 - 18x + 6$

$\Rightarrow$ $6(2x^2 - 3x + 1) = 6(2x - 1)(x - 1)$



f is strictly increasing in $(-\infty, 1/2)$, $(1, \infty)$ and strictly decreasing in $(1/2, 1)$.

and $y = 1/x$ is strictly decreasing in $(-\infty, 0)$, $(0, \infty)$

$4x^3 - 9x^2 + 6x = 0 \Rightarrow x(4x^2 - 9x + 6) = 0 \Rightarrow x = 0.$

For $x < 0$, $f(x) = x(4x^2 - 9x + 6) < 0$

$\therefore$ $g(x) = 1/f(x)$ is strictly decreasing in $(-\infty, 0)$.

For $x > 0$, $f(x) = x(4x^2 - 9x + 6) > 0$

$\therefore$ $g(x)$ is strictly decreasing in $(0, 1/2)$ and $(1, \infty)$ and it is strictly increasing in $(1/2, 1)$.

Sum

If f and g are strictly increasing functions then $h(x) = f(x) + g(x)$ is also a strictly increasing function.

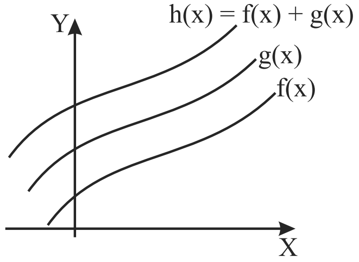
Assuming f and g to be differentiable,

$h'(x) = f'(x) + g'(x)$

Since, f and g are strictly increasing $f'(x)$ and $g'(x)$ are positive.

$\Rightarrow$ $f'(x) + g'(x)$ is positive

$\Rightarrow$ $h(x) = f(x) + g(x)$ is strictly increasing



In short, a strictly increasing function + a strictly increasing function = a strictly increasing function

i.e. (i) I + I = I

Similarly (ii) D + D = D

Note that we cannot say anything about I + D.

For example, $f(x) = \sqrt{x}$ and $g(x) = \ln x$ are strictly increasing, hence $y = \sqrt{x} + \ln x$ is also strictly increasing.

We find that $y = \sqrt{3-x} + \cos^{-1}\left(\frac{x-1}{2}\right)$ is strictly decreasing because $y = \sqrt{3-x}$ and $y = \cos^{-1}\left(\frac{x-1}{2}\right)$ are strictly decreasing.

Difference

Monotonicity of the difference of two function can be predicted using (1) and (3)

Consider $y = \ln(x + \sqrt{x^2+1}) - \cot^{-1}x$

increasing – decreasing

$\Rightarrow$ increasing + (–decreasing)

$\Rightarrow$ increasing + increasing $\Rightarrow$ increasing

Hence, the function is strictly increasing.

I – I = I + (–I) = I + D = cannot say anything

I – D = I + (–D) = I + I = increasing

D – I = D + (–I) = D + D = decreasing

D – D = D + (–D) = D + I = cannot say anything.

Product

Consider $h(x) = f(x) \times g(x)$

Case I: Both the function f and g involved in the product are positive. Further if, both f and g are strictly increasing then $h(x) = f(x) \times g(x)$ is also strictly increasing.

We have $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

Here, all the terms in the R.H.S. are positive under the conditions given above. Hence, h(x) is strictly increasing.

In short, I × I = I

D × D = D

I × D = cannot say anything

Case II: If f is strictly increasing and takes negative values and g is strictly decreasing and takes positive values then

$h(x) = f(x) \times g(x)$ is strictly increasing.

$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

Under the given conditions the R.H.S. becomes positive and hence the function h is strictly increasing.

Division

Monotonicity of division of two functions can be predicted by using reciprocal and product.

Assuming that both the functions I and D take positive values, we have

$$\frac{I}{D} = I \times \frac{1}{D} = I \times I = I.$$

Composition

If $y = g(u)$ and $u = f(x)$, then $y = (gof)(x) = g(f(x))$

and $\frac{dy}{dx} = (g(f(x)))' = g'(f(x)).f'(x).$

(i) If f is strictly increasing in [a, b] and g is strictly increasing in [f(a), f(b)], then gof is strictly increasing in [a, b].
For example, consider $y = \tan^{-1}(e^x)$.
$\because$ $\tan^{-1}x$ is strictly increasing for all x and e^x is strictly increasing for all x, so the composite function $\tan^{-1}(e^x)$ is strictly increasing for all x.

(ii) If f is strictly decreasing in [a, b] and g is strictly decreasing in [f(b), f(a)], then gof is strictly increasing in [a, b].
For example, $y = \cot^{-1}(\log_{1/2}x)$ is strictly increasing because $\log_{1/2}x$ is strictly decreasing for all x and $\cot^{-1}x$ is also strictly decreasing for all x.

(iii) If f is strictly increasing in [a, b] and g is strictly decreasing in [f(a), f(b)], then gof is strictly decreasing in [a, b].

For example, $y = \cos(\sin^{-1}x)$ is strictly decreasing in [0, 1], because $\sin^{-1}x$ is strictly increasing in [0, 1] and cos x is strictly decreasing in [0, π/2].

(iv) If f is strictly decreasing in [a, b] and g is strictly increasing in [f(b), f(a)], then gof is strictly decreasing in [a, b].
For example, $y = \ln(\cot^{-1}x)$ is strictly decreasing for all x because $\cot^{-1}x$ is strictly decreasing for all x and ln x is strictly increasing for all x > 0.

In short,

(i) I(I)= I (ii) I(D) = D
(iii) D(I) = D (iv) D(D) = I

Example 2: Find the monotonicity of

$$h(x) = e^{(\sin x + \cos x)},\ x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Solution: Let f(x) = sin x + cos x and $g(x) = e^x$

Then h(x) = g(f(x))

$$f(x) = \sqrt{2}\sin(x + \pi/4)$$

$$f'(x) = \sqrt{2}\cos(x + \pi/4) < 0 \text{ for } x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

Hence f is strictly decreasing while g is strictly increasing. Using the above results, h(x) is strictly decreasing in

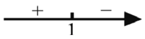
$$\left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Example 3: Find the intervals of monotonicity of

$$y = \sqrt{2x - x^2}.$$

Solution: Let $f(x) = 2x - x^2$ and $g(x) = \sqrt{x}$.

$\Rightarrow$ $f'(x) = 2 - 2x$



f is strictly increasing in (– ∞, 1) and strictly decreasing in (1, ∞).

Domain of the given function : $2x - x^2 \geq 0$

$\Rightarrow$ $0 \leq x \leq 2$

g is strictly increasing for x ≥ 0.

So, for 0 < x < 1, f is strictly increasing and for 0 < x < 1, g is strictly increasing. Hence gof is strictly increasing in (0, 1).

Similarly, for 1 < x < 2, f is strictly decreasing and for 0 < x < 1, g is strictly increasing. Hence gof is strictly decreasing in (1, 2).

Example 4: Given $f(x) = \tan^{-1}(\sin x + \cos x)^3$, find intervals of increase in (0, 2π).

Solution: Since $\tan^{-1}(\sin x + \cos x)^3$ and x^3 are both increasing functions, f(x) is an increasing function when sin x + cos x is an increasing function.

Let g(x) = sin x + cos x, x ∈ (0, 2π).

We should have g'(x) = cos x – sin x > 0

$$\Rightarrow \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0 \quad \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$$

then $0 < x + \frac{\pi}{4} < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x + \frac{\pi}{4} < 2\pi$

$$\Rightarrow -\frac{\pi}{4} < x < \frac{\pi}{4} \text{ and } \frac{5\pi}{4} < x < \frac{7\pi}{4}$$

Hence, the intervals of increase of f(x) are

$$\left(0, \frac{\pi}{4}\right) \text{ and } \left(\frac{5\pi}{4}, 2\pi\right).$$

Example 5: Find the intervals in which the function $f(x) = \sin(\ell nx) - \cos(\ell nx)$ increases.

Solution: Since ln x is an increasing function, f(x) increases when sin x – cos x increases.

Let g(x) = sin x – cos x

$$g'(x) = \cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

$$g'(x) > 0 \text{ when } 2n\pi - \frac{\pi}{2} < x - \frac{\pi}{4} < 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow 2n\pi - \frac{\pi}{4} < 2n\pi + \frac{2\pi}{4},\ n \in I$$

Now the intervals of increase of f(x) are given by:

$$2n\pi - \frac{\pi}{4} < \ln x < 2n\pi + \frac{3\pi}{4}$$

$$e^{2n\pi - \frac{\pi}{4}} < x < e^{2n\pi + \frac{3\pi}{4}},\ n \in I.$$

Alternative: $f(x) = \sin(\ell nx) - \cos(\ell nx)$

$$= \sqrt{2}\sin\left(\ell nx - \frac{\pi}{4}\right),\ x > 0.$$

$$\therefore\ f'(x) = \frac{\sqrt{2}}{x}\cos\left(\ell nx - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{2} + \ell nx - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} + \ell nx\right) > 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + \ell nx\right) > 0 \qquad \text{(since } x > 0\text{)}$$

$$\Rightarrow 2n\pi < \frac{\pi}{4} + \ell nx < 2n\pi + \pi,\ n \in I$$

$$\Rightarrow 2n\pi - \frac{\pi}{4} < \ell nx < 2n\pi + \frac{3\pi}{4},\ n \in I$$

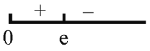
$$\Rightarrow e^{2n\pi - \frac{\pi}{4}} < x < e^{2n\pi + \frac{3\pi}{4}},\ n \in I.$$

Example 6: Find the interval of monotonicity of

$$y = \ln\left(\frac{\ln x}{x}\right).$$

Solution: Let $f(x) = \frac{\ln x}{x}$ and $g(x) = \ln x$

$$f'(x) = \frac{1-\ln x}{x^2}$$



f is strictly increasing in $(0, e)$ and strictly decreasing in (e, ∞). g is increasing for $x > 0$.

Domain of the given function : $\frac{\ln x}{x} > 0 \Rightarrow x > 1$.

So, for $1 < x < e$, f is strictly increasing and for $0 < x < 1/e$, g is strictly increasing. Hence gof is strictly increasing in $(1, e)$. Similarly, for $e < x < \infty$, f is strictly decreasing and for $1 < x < \infty$, g is strictly increasing. Hence gof is strictly decreasing in (e, ∞).

6.7 Proving Inequalities

Greatest and Least Values of a Function

If a continuous function $y = f(x)$ is strictly increasing in the closed interval $[a, b]$ then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in $[a, b]$. (See Figure - 1)

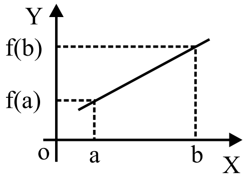


Figure - 1

If $f(x)$ is strictly decreasing in $[a, b]$ then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in $[a, b]$.(See figure - 2)

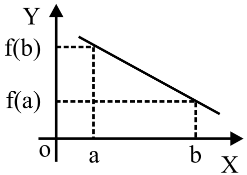


Figure - 2

However if $f(x)$ is non-monotonic in $[a, b]$ and is continuous then the greatest and least values of $f(x)$ in $[a, b]$ are found where $f'(x) = 0$ or $f'(x)$ does not exist or at the endpoints. (See figure - 3)

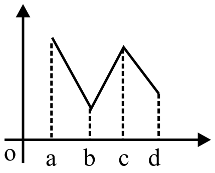


Figure - 3

Example 1: Show that

$$0 < x \sin x - \frac{1}{2}\sin^2 x < \frac{(\pi-1)}{2} \quad \forall\ x \in \left(0, \frac{\pi}{2}\right).$$

Solution: Let $f(x) = x \sin x - \frac{1}{2}\sin^2 x$

$\Rightarrow \quad f'(x) = x \cos x + \sin x - \sin x \cos x$

$\qquad = \sin x(1 - \cos x) + x \cos x$

For $x \in (0, \pi/2)$, $\sin x > 0$, $1 - \cos x > 0$, $\cos x > 0$

$\Rightarrow \quad f'(x) > 0\ \forall\ x \in (0, \pi/2)$.

f is strictly increasing in $(0, \pi/2)$.

$\Rightarrow$ The range of f(x) is $\left(\lim\limits_{x\to0} f(x),\ \lim\limits_{x\to\pi/2} f(x)\right)$

$$\equiv \left(0, \frac{\pi-1}{2}\right)$$

$$\Rightarrow \quad 0 < x \sin x - \frac{1}{2}\sin^2 x < \frac{\pi-1}{2}.$$

Example 2: If $1/6 < x < 5/6$, then prove that

$$\frac{1}{2} < 3\left(x + \frac{1}{2\pi} - \frac{\sin \pi x}{\pi}\right) < \frac{5}{2}.$$

Solution: Consider $f(x) = 3\left(x + \frac{1}{2\pi} - \frac{\sin \pi x}{\pi}\right)$.

Now, $f'(x) = 3(1 - \cos \pi x) > 0\ \forall\ x \in (1/6, 5/6)$

Applying the increasing function f on the inequality $\frac{1}{6} < x < \frac{5}{6}$, we get $f\left(\frac{1}{6}\right) < f(x) < f\left(\frac{5}{6}\right)$

i.e. $\frac{1}{2} < f(x) < 5/2$

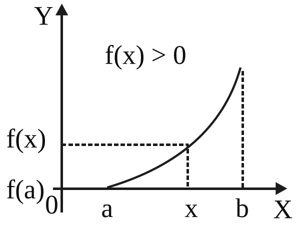
Hence, the statement is proved.

General Approach in Proving Inequalities

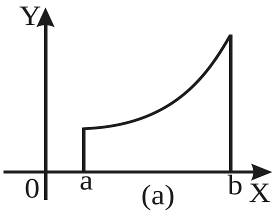
The methods of the investigation of the behaviour of functions can be applied to proving inequalities.

If $f'(a) > 0$ then $f(x) < f(a)$ for all values of x less than a but sufficiently near to a, and $f(x) > f(a)$ for all values of x greater than a but sufficiently near to a.

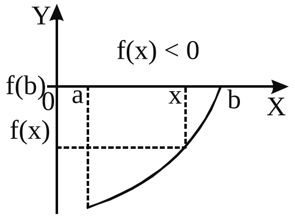
If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) where $f'(x) > 0$ for all $x \in (a, b)$ and $f(a) \geq 0$ then $f(x)$ is positive throughout the interval $(a, b]$.



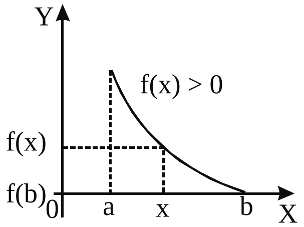
If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) where $f'(x) < 0$ for all $x \in (a, b)$ and $f(a) \leq 0$ then $f(x)$ is negative throughout the interval $(a, b]$.



If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) > 0$ for all $x \in (a, b)$ and $f(b) \leq 0$ then f(x) is negative throughout the interval [a, b).



If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) < 0$ for all $x \in (a, b)$ and $f(b) \geq 0$ then f(x) is positive throughout the interval [a, b).



Note: If f(x) is discontinuous at the endpoints a or b then one sided limits $f(a^+)$ or $f(b^-)$ are used in place of f(a) or f(b) for understanding the sign of the function.

Thus, to prove $f(x) \geq g(x)\ \forall\ x \geq a$,
we assume $h(x) = f(x) - g(x)$, and
find $h'(x) = f'(x) - g'(x)$
If $h'(x) \geq 0\ \forall\ x \geq a$,
h is an increasing function for $x \geq a$.
Thus, $h(x) \geq h(a)$.
If $h(a) \geq 0$, then $h(x) \geq 0\ \forall\ x \geq a$
i.e. the given inequality is established.

Note: If the sign of $h'(x)$ is not obvious, then to determine its sign we assume $p(x) = h'(x)$ and apply the above procedure on p(x).

Example 3: Prove the inequality

$$\ln(1 + x) > x - \frac{x^2}{2}\quad \forall\ x \in (0,\ \infty).$$

Solution: Consider the function

$$f(x) = \ln(1 + x) - x + \frac{x^2}{2},\ x \in (0, \infty)$$

Then $f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0\quad \forall\ x \in (0, \infty)$

$\Rightarrow$ f(x) is strictly increasing in $(0, \infty)$.

$\Rightarrow$ $f(x) > f(0)\quad \forall\ x \in (0,\ \infty)$
$\Rightarrow$ $f(x) > 0$

i.e. $\ln(1 + x) - x + \frac{x^2}{2} > 0,\quad x \in (0, \infty)$

$\Rightarrow$ $\ln(1 + x) > x - \frac{x^2}{2}\quad \forall\ x \in (0,\ \infty)$

which is the desired result.

Example 4: For $x \in (0, \pi/2)$, prove that $\sin x < x < \tan x$.

Solution: Let $f(x) = x - \sin x$
$\Rightarrow$ $f'(x) = 1 - \cos x$
$f'(x) > 0$ for $x \in (0, \pi/2)$
$\Rightarrow$ f(x) is strictly increasing for $x \in (0, \pi/2)$
$\Rightarrow$ $f(x) > f(0)\quad \Rightarrow x - \sin x > 0$
$\Rightarrow$ $x > \sin x$
Similarly consider another function, $g(x) = x - \tan x$
$\Rightarrow$ $g'(x) = 1 - \sec^2 x$
$g'(x) < 0$ for $x \in (0, \pi/2)$
$\Rightarrow$ g(x) is strictly decreasing for $x \in (0, \pi/2)$
Hence, $g(x) < g(0) \Rightarrow x - \tan x < 0$
$\Rightarrow$ $x < \tan x$
Thus, $\sin x < x < \tan x$ for $x \in (0, \pi/2)$.

Example 5:

(i) Find the order relation between x and $\tan^{-1}x$.
(ii) Show that $\ln(1 + x) < x$ for all $x > 0$

Solution:

(i) Let $f(x) = x - \tan^{-1}x$.

$\Rightarrow$ $f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0\ \forall x \in R$

Thus f(x) is a strictly increasing function.

Now, $f(0) = 0 \Rightarrow f(x) < 0,\ \forall\ x \in (-\infty, 0)$ and

$f(x) \geq 0,\ x \in [0, \infty)$

$\Rightarrow$ $x < \tan^{-1}x,\ x \in (-\infty, 0)$

and $x \geq \tan^{-1}x,\ x \in [0, \infty)$.

(ii) Let us assume $f(x) = \ln(1 + x) - x$.

$\Rightarrow$ $f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$

Clearly, $f'(x) < 0\ \forall\ x \in (0, \infty)$.

Hence f(x) is strictly decreasing for $x > 0$.

Moreover f(0) = 0, hence f(x) < 0 for x > 0.

$\Rightarrow$ $\ln(1 + x) - x < 0$

$\Rightarrow$ $\ln(1 + x) < x$ for all x > 0.

Example 6: Prove the inequality $2\sin x + \tan x \geq 3x$ for $x \in [0, \pi/2)$.

Solution: Consider $f(x) = 2 \sin x + \tan x - 3x$

$$f'(x) = 2 \cos x + \sec^2 x - 3$$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$$= \frac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x}$$

Hence $f'(0) > 0$ for $x \in (0, \pi/2)$.

$\Rightarrow$ f(x) is strictly increasing for $x \in (0, \pi/2)$.

$\therefore$ $f(x) > f(0)$ for $x \in (0, \pi/2)$

Since f(0) = 0, $f(x) \geq 0$ for $x \in [0, \pi/2)$.

Hence $f(x) \geq 0$ for $x \in [0, \pi/2)$.

$\Rightarrow$ $2\sin x + \tan x \geq 3x$ for $x \in [0, \pi/2)$.

Example 7: For $x \in (0, \pi/2)$, prove that

$$\sin x > x - \frac{x^3}{6}.$$

Solution: Let $f(x) = \sin x - x + \frac{x^3}{6}$.

$$\Rightarrow f'(x) = \cos x - 1 + \frac{x^2}{2}$$

It is difficult to decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic behaviour of $f'(x)$

$$f''(x) = x - \sin x$$

Since $f''(x) > 0$ for $x \in (0, \pi/2)$, we conclude that

$f'(x)$ is strictly increasing for $x \in (0, \pi/2)$.

$\Rightarrow$ $f'(x) > f(0)$ $\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ f(x) is strictly increasing for $x \in (0, \pi/2)$

$\Rightarrow$ $f(x) > f(0)$ for $x \in (0, \pi/2)$

$$\Rightarrow \sin x - x + \frac{x^3}{6} > 0$$

$$\Rightarrow \sin x > x - \frac{x^3}{6}.$$

Example 8: Establish the inequality

$$\frac{2}{2x+1} < \ell n\left(1+\frac{1}{x}\right) < \frac{1}{x} \text{ for } x > 0.$$

Solution: Consider $f(x) = \frac{2}{2x+1} - \ell n\left(1+\frac{1}{x}\right)$

$$f'(x) = \frac{-4}{(2x+1)^2} - \frac{1}{1+(1/x)} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x(x+1)} - \frac{4}{(2x+1)^2} = \frac{1}{x(x+1)(2x+1)^2}$$

which is always positive for x > 0

Hence f(x) is strictly increasing for x > 0.

$$\therefore \; f(x) < \lim_{x\to\infty} f(x)$$

$$\Rightarrow f(x) < \lim_{x\to\infty} \left\{\frac{2}{2x+1} - \ln\left(1+\frac{1}{x}\right)\right\} = 0.$$

So, f(x) < 0

$$\Rightarrow \frac{2}{2x+1} < \ell n\left(1+\frac{1}{x}\right)$$

Similarly, consider $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x}$.

$$g'(x) = \frac{1}{x^2} - \frac{1}{x(x+1)} = \frac{1}{x^2(x+1)}$$

$g'(x)$ is strictly increasing for x > 0.

So, $g(x) < \lim\limits_{x\to\infty} g(x)$

$\Rightarrow$ g(x) < 0.

Example 9: Prove that, $2x \sec x + x > 3 \tan x$ for $0 < x < \pi/2$.

Solution: $f(x) = 2x \sec x + x - 3 \tan x$

$f'(x) = 2 \sec x + 2x \sec x \tan x + 1 - 3 \sec^2 x$

$= \sec^2 x \,[2 \cos x + 2x \sin x + \cos^2 x - 3]$

Consider $g(x) = 2 \cos x + 2x \sin x + \cos^2 x - 3$

$g'(x) = -2 \sin x + 2x \cos x + 2 \sin x - 2 \sin x \cos x$

$= 2 \cos x (x - \sin x) > 0$ for $x \in (0, \pi/2)$.

Hence, $g(x) > g(0)$ $\Rightarrow$ $g(x) > 0$.

Now, $f'(x) > 0$

Hence, $f(x) > f(0)$ $\Rightarrow$ $f(x) > 0$

Thus, $2x \sec x + x > 3 \tan x$ for $0 < x < \pi/2$.

Example 10: Show that

$$1 + x \ln\left(x + \sqrt{x^2+1}\right) \geq \sqrt{1+x^2} \text{ for all } x \geq 0.$$

Solution: Let $f(x) = 1 + x \ln\left(x + \sqrt{x^2+1}\right) - \sqrt{1+x^2}$

$$\therefore \; f'(x) = \frac{x}{[x+\sqrt{x^2+1}]} \cdot \left[1+\frac{1.(2x)}{2\sqrt{x^2+1}}\right] + \ln(x+\sqrt{x^2+1}).1 - \frac{1(2x)}{2\sqrt{1+x^2}}$$

$$\Rightarrow f'(x) = \frac{x}{(x+\sqrt{x^2+1})}\left[\frac{\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right] + \ln(x+\sqrt{x^2+1}) - \frac{x}{\sqrt{1+x^2}}$$

$\Rightarrow\ f'(x) = \dfrac{x}{\sqrt{x^2+1}} + \ln(x+\sqrt{x^2+1}) - \dfrac{x}{\sqrt{1+x^2}}$

$\Rightarrow\ f'(x) = \ln\ (x+\sqrt{x^2+1})$

We have $x \geq 0$, so, $\ln\ (x+\sqrt{x^2+1})\ \geq 0$

$\Rightarrow\ f'(x) \geq 0 \ \Rightarrow$ f(x) is increasing for $x \geq 0$

$\Rightarrow\ f(x) \geq f(0)$

$\Rightarrow\ 1 + x \ln\ (x+\sqrt{x^2+1})\ -\ \sqrt{1+x^2}\ \geq 1 + 0 - \sqrt{1}$

$\Rightarrow\ 1 + x \ln (x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for $x \geq 0$.

Example 11: Examine which is greater sin x tan x or x^2. Hence evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \dfrac{\pi}{2}\right)$

Solution: Let $f(x) = \sin x \,.\, \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x$

$\Rightarrow\ f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow\ f''(x) = \cos x + \cos x \sec^2 x$
$\quad + 2 \sec^2 x \sin x \tan x - 2$

$\Rightarrow\ f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\quad \cos x + \sec x - 2 = (\sqrt{\cos x} - \sqrt{\sec x})^2$

and $2 \sec^2 x \tan x \,.\, \sin x > 0$ because $x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow\ f''(x) > 0 \ \Rightarrow\ f'(x)$ is strictly increasing.

Hence $\quad f'(x) > f'(0)$

$\Rightarrow\ f'(x) > 0 \ \Rightarrow$ f(x) is strictly increasing

$\Rightarrow\ f(x) > 0 \ \Rightarrow\ \sin x \tan x - x^2 > 0$

Hence, $\sin x \tan x - x^2$

$\Rightarrow\ \dfrac{\sin x \tan x}{x^2} > 1$

$\Rightarrow\ \lim\limits_{x\to 0}\left[\dfrac{\sin x \tan x}{x^2}\right] = 1.$

Example 12: Prove the inequalities.

$\dfrac{2}{\pi} x < \sin x < x$ for $0 < x < \dfrac{\pi}{2}$.

Solution: We introduce the function

$f(x) = \dfrac{\sin x}{x}, \quad x \neq 0$

Its derivative $f'(x) = \dfrac{\cos x}{x^2}(x - \tan x)$

is negative in the interval $\left(0, \dfrac{\pi}{2}\right)$ since $x < \tan x$.

Thus, f(x) is a decreasing function in that interval.

$f(0^+) = 1$ and $f(\dfrac{\pi}{2}) = \dfrac{\sin\dfrac{\pi}{2}}{\dfrac{\pi}{2}} = \dfrac{2}{\pi}$

Hence, for $0 < x < \dfrac{\pi}{2}$, the values of the expression $\dfrac{\sin x}{x}$ are less than the value 1 of the function reached at the point x = 0 and exceed the value of the function attained at the point $x = \dfrac{\pi}{2}$.

Thus, $f(\dfrac{\pi}{2}) < f(x) < f(0^+)$

$\Rightarrow\ \dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1.$

Example 13: If P(1) = 0 and $\dfrac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$, then prove that P(x) > 0 for all x > 1.

Solution: Here, $\dfrac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$

$\Rightarrow\ e^{-x}\dfrac{dP(x)}{dx} > P(x)e^{-x}$, for all $x \geq 1$ since $e^{-x} > 0$

$\Rightarrow\ e^{-x}\dfrac{dP(x)}{dx} - P(x)e^{-x} > 0$, for all $x \geq 1$

$\Rightarrow\ \dfrac{d}{dx}\{P(x)e^{-x}\} > 0$, for all $x \geq 1$

$\Rightarrow\ P(x)e^{-x}$ is an increasing function for all $x \geq 1$

$\Rightarrow\ P(x)e^{-x} > P(1)e^{-1}$ for all x > 1

$\Rightarrow\ P(x)e^{-x} > 0$ for all x > 1 {since P(1) = 0}

Thus, P(x) > 0 for all x > 1 {since $e^{-x} > 0$ for all x}.

Inequalities Based on Composite Functions

Example 14: For $x \in (0, \pi/2)$, prove that $\cos(\sin x) > \sin(\cos x)$.

Solution: Let f(x) = x – sin x

$\therefore\ f'(x) = 1 - \cos x > 0,\ x \in (0, \pi/2)$.

Hence f(x) in increasing in $(0, \pi/2)$

Then f(x) > f(0)

or $\quad x - \sin x > 0 \ \Rightarrow \quad x > \sin x$ ...(1)

Again $x \in (0, \pi/2)$and $0 < \cos x < 1$, therefore using (1)

$\cos x > \sin(\cos x)$ ...(2)

Now using (1) and the fact that cosx is decreasing in $(0, \pi/2)$

$\therefore\ \cos x < \cos(\sin x)$ ...(3)

From (2) and (3) we get

$\sin(\cos x) < \cos x < \cos(\sin x)$

Hence, sin(cos x) < cos(sin x).

Example 15: Prove that

$\sin^2\theta < \theta < \sin(\sin\theta)$ for $\theta \in (0, \pi/2)$

Solution: We have to prove

$\sin^2\theta < \theta\sin(\sin\theta)$ or $\dfrac{\sin\theta}{\theta} < \dfrac{\sin(\sin\theta)}{\sin\theta}$

Now let $f(x) = \frac{\sin x}{x}$, $x \in (0, \pi/2)$

$\therefore\ f'(x) = \frac{(x\cos x - \sin x)}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$

which is negative since $x - \tan x < 0$ in $x \in (0, \pi/2)$.

$\Rightarrow$ f(x) is a decreasing function

$\because$ $\sin\theta < \theta$ for $x \in (0, \pi/2)$ we have

$f(\sin\theta) > f(\theta)$

$\Rightarrow \frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta}$

Hence $\sin^2\theta < \theta \sin(\sin\theta)$ for $x \in (0, \pi/2)$.

Example 16: Prove that sin 1 > cos (sin 1). Also show that the equation sin (cos (sin x)) = cos (sin (cos x)) has only one solution in $x \in (0, \pi/2)$.

Solution: $\sin 1 > \cos(\sin 1)$ if $\cos\left(\frac{\pi}{2} - 1\right) > \cos(\sin 1)$

i.e. if $\frac{\pi}{2} - 1 < \sin 1$

i.e. if $\sin 1 > \left(\frac{\pi - 2}{2}\right)$ ...(1)

We have $\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}} > \left(\frac{\pi - 2}{2}\right)$.

Hence (1) is true $\Rightarrow \sin 1 > \cos(\sin 1)$.

Now let $f(x) = \sin(\cos(\sin x)) - \cos(\sin(\cos x))$

$f'(x) = -\cos(\cos(\sin x))\sin(\sin x)\cos x$

$\quad - \sin(\sin(\cos x))\cos(\cos x) . \sin x$

$\Rightarrow f'(x) < 0\ \forall\ x \in (0, \pi/2)$

$\Rightarrow$ f(x) is decreasing in $(0, \pi/2)$

and $f(0) = \sin 1 - \cos(\sin 1) > 0$

$f\left(\frac{\pi}{2}\right) = \sin(\cos(1)) - 1 > 0$

Since f(0) is positive and $f\left(\frac{\pi}{2}\right)$ is negative.

$f(x) = 0$ has one solution in $x \in (0, \pi/2)$.

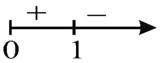
Example 17: Show that $\ln x < x\ \forall\ x > 0$. Hence, prove that $\ln(\cos\theta) < \cos(\ln\theta)$ where $e^{-\pi/2} < \theta < \frac{\pi}{2}$.

Solution: Consider the function

$f(x) = \ln x - x,\ x > 0$

Then $f'(x) = \frac{1}{x} - 1 = \frac{1 - x}{x}$

Sign of f′(x)



$\Rightarrow$ f(x) is strictly increasing in (0, 1),

and strictly decreasing in $(1, \infty)$.

$\Rightarrow$ f(x) has greatest value at x = 1.

$\Rightarrow f(x) \le f(1) = -1 < 0$

i.e. $\ln x < x$.

Now, we have

$e^{-\pi/2} < \theta < \frac{\pi}{2}$

i.e. $0 < \theta < \frac{\pi}{2}$

i.e. $0 < \cos\theta < 1$

i.e. $\ln(\cos\theta) < 0$ $[\because \ln x < 0\ \forall\ x \in (0, 1)]$ ...(1)

Also, we have

$e^{-\pi/2} < \theta < \frac{\pi}{2}$

i.e. $\frac{-\pi}{2} < \ln\theta < \ln\frac{\pi}{2}$ $[\because \ln x$ is increasing in $(0, \infty)]$

i.e. $\frac{-\pi}{2} < \ln\theta < \frac{\pi}{2}$ [using $\ln x < x$] ...(2)

i.e. $0 < \cos(\ln\theta) < 1$

From results (1) and (2), we can infer that

$\ln(\cos\theta) < \cos(\ln\theta)$.

Example 18: Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise prove that $\sin(\tan x) \ge x$ for all $x \in \left[0, \frac{\pi}{4}\right]$

Solution: Let $f(x) = \sin(\tan x) - x$

Then, $f'(x) = \cos(\tan x) \cdot \sec^2 x - 1$

$\Rightarrow f'(x) = \cos(\tan x)(1 + \tan^2 x) - 1$

Using the relation $2(1 - \cos x) < x^2$ i.e. $\cos x > 1 - \frac{x^2}{2}$

We have $\cos(\tan x) > 1 - \frac{\tan^2 x}{2}$

Hence, $f'(x) > \left(1 - \frac{\tan^2 x}{2}\right)(1 + \tan^2 x) - 1$

$f'(x) > 1/2 \tan^2 x (1 - \tan^2 x)$

$\Rightarrow f'(x) \ge 0,\ \forall\ x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow$ f(x) is an increasing function for all $x \in \left[0, \frac{\pi}{4}\right]$

$\therefore$ $f(x) \ge f(0)$, for all $x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \sin(\tan x) - x > \sin(\tan 0) - 0$

$\Rightarrow \sin(\tan x) \ge x$ for all $x \in \left[0, \frac{\pi}{4}\right]$.

Comparision of Constants

Example 19: Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

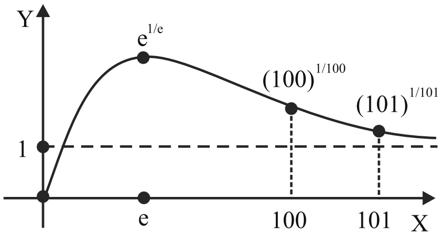
Solution: Assume $f(x) = x^{1/x}$ and let us examine monotonic behaviour of f(x)

$f'(x) = x^{1/x} . \left(\frac{1 - \ell n x}{x^2}\right)$

$f'(x) > 0 \Rightarrow x \in (0,e)$

and $f'(x) < 0 \Rightarrow x \in (e,\infty)$

Hence, f(x) is strictly dereasing for $x \geq e$



and since $e < 100 < 101$

$\Rightarrow f(100) > f(101)$

$\Rightarrow (100)^{1/100} > (101)^{1/101}$.

Example 20: Prove that

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

Solution: We have to prove

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

or $$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e}\right)^2+1}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{e^2+1}}$$

Now let $f(x) = (\tan^{-1}x)^2 + \frac{2}{\sqrt{x^2+1}}$ ...(1)

$$\therefore \quad f'(x) = \frac{2\tan^{-1}x}{(1+x^2)} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{(1+x^2)}\left\{\tan^{-1}x - \frac{x}{\sqrt{(x^2+1)}}\right\}$$ ...(2)

To find sign of $f'(x)$ we consider

$$g(x) = \tan^{-1}x - \frac{x}{\sqrt{(x^2+1)}}$$

$$\therefore \quad g'(x) = \frac{1}{(1+x^2)}\left\{1 - \frac{1}{\sqrt{(x^2+1)}}\right\} > 0$$

$\Rightarrow g'(x) > 0$

$\therefore$ g(x) is an increasing function

Thus, $g(x) > g(0) \quad \Rightarrow g(x) > 0$ ...(3)

Using (2) and (3), $f'(x) > 0$

$\Rightarrow$ f(x) is increasing function

Since $\frac{1}{e} < e$, we have $f\left(\frac{1}{e}\right) < f(e)$

$$\Rightarrow \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e^2}+1\right)}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

Hence, $\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}$.

Example 21: Using the function

$f(x) = 2x - \tan^{-1}x - \ln\left(x + \sqrt{1+x^2}\right)$,

prove that $\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) < \frac{4}{\sqrt{3}} - \frac{\pi}{6}$.

Solution: $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0 \; \forall x \in R$

and equality holds at $x = 0$ only.

So, f(x) is increasing in $(-\infty, \infty)$.

Since $\frac{\pi}{6} < \frac{\pi}{3}$ and f is an increasing function we have

$$f\left(\frac{\pi}{6}\right) < f\left(\frac{\pi}{3}\right).$$

Hence, $\frac{2}{\sqrt{3}} - \frac{\pi}{6} - \ln\sqrt{3} < 2\sqrt{3} - \frac{\pi}{3} - \ln\left(\sqrt{3}+2\right)$.

Thus, $\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) < \frac{4}{\sqrt{3}} - \frac{\pi}{6}$.

Example 22: Prove that for $0 \leq p \leq 1$ and for any positive a and b the inequality $(a + b)^p \leq a^p + b^p$ is valid.

Solution: By dividing both sides of the inequality by b^p we get

$$\left(\frac{a}{b}+1\right)^p \leq \left(\frac{a}{b}\right)^p + 1$$

or $(1 + x)^p \leq 1 + x^p$, ...(1)

where $x = \frac{a}{b}$.

Let us show that the inequality (1) holds true at any positive x. Consider the function

$f(x) = 1 + x^p - (1 + x)^p$, $x \geq 0$.

The derivative of this function

$$f'(x) = px^{p-1} - p(1+x)^{p-1} = p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right]$$

is positive everywhere, since, by hypothesis, $1 - p \geq 0$ and $x > 0$.

Hence, the function increases in the interval $[0, \infty)$, i.e. $f(x) = 1 + x^p - (1 + x)^p > f(0) = 0$, Thus, $1 + x^p > (1 + x)^p$, which completes the proof.

If we put $p = 1/n$, then we obtain

$$\sqrt[n]{a+b} \le \sqrt[n]{a} + \sqrt[n]{b},\ (n \ge 1).$$

Inequalities Based on Non-Monotonous Functions

Example 23: Prove that the function

$$f(x) = -2x^3 + 21x^2 - 60x + 41$$

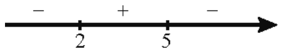
is strictly positive in the interval $(-\infty, 1)$.

Solution: $f(x) = -2x^3 + 21x^2 - 60x + 41$

$\therefore$ $f'(x) = -6x^2 + 42x - 60$

$= -6(x^2 - 7x + 10) = -6(x - 5)(x - 2)$.

The sign scheme for $f'(x)$, $x \in R$ is as follows:



$\therefore$ For $x \in (-\infty,1)$ $f(x)$ is strictly decreasing.

So when $x \in (-\infty, 1)$, $f(x) > f(1)$.

We have $f(1) = -1 + 21 - 60 + 41 = 0$.

$\therefore$ For $x \in (-\infty,1)$ we have $f(x) > 0$.

$\therefore$ $f(x)$ is strictly positive in the interval $(-\infty, 1)$.

Example 24: Prove $1 + \cot x \le \cot \frac{x}{2}$ $\forall$ $x \in (0, \pi)$.

Solution: Consider the function

$$f(x) = \cot\left(\frac{x}{2}\right) - 1 - \cot x,\ x \in (0, \pi).$$

Then $f'(x) = \frac{-1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) + \operatorname{cosec}^2 x$

$$= \frac{1}{\sin^2 x} - \frac{1}{2\sin^2(x/2)}$$

$$= \frac{1}{2\sin^2(x/2)}\left[\frac{1}{2\cos^2(x/2)} - 1\right]$$

$$= \frac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)}$$

$$= \frac{-\cos x}{\sin^2 x} < 0\ \forall\ x \in (0, \pi/2)$$

$$> 0\ \forall\ x \in (\pi/2, \pi).$$

$\Rightarrow$ $f(x)$ strictly decreases in $(0, \pi/2)$ and strictly increases in $(\pi/2, \pi)$.

$\Rightarrow$ $f(x)$ has least value at $x = \pi/2$.

$\Rightarrow$ $f(x) \ge f(\pi/2) = 0$

i.e. $\cot\left(\frac{x}{2}\right) \ge 1 + \cot x$

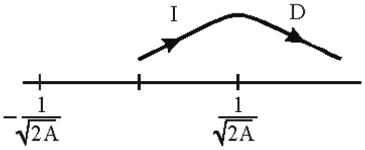
which proves the desired result.

Example 25: Find the smallest positive constant A such that $\ln x \le Ax^2$ for all $x > 0$.

Solution: Note that $A > 0$

Consider the function $f(x) = \ln x - Ax^2$

$$f'(x) = \frac{1}{x} - 2Ax = \frac{1 - 2Ax^2}{x}$$



$\Rightarrow$ $f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2A}}$

and $f'(x) < 0$ for $x > \frac{1}{\sqrt{2A}}$

$\therefore$ $\ln x < Ax^2$ for $x > 0$

This must also hold good for $x = \frac{1}{\sqrt{2A}}$

$\therefore$ $\ln \frac{1}{\sqrt{2A}} < A \cdot \frac{1}{2A}$

$\Rightarrow$ $-\ln \sqrt{2A} < \frac{1}{2}$ $\Rightarrow$ $\ln \sqrt{2A} > -\frac{1}{2}$

$$\sqrt{2A} = e^{-\frac{1}{2}} \Rightarrow A > \frac{1}{2e}$$

$\therefore$ The least value of $A = \frac{1}{2e}$.

Example 26: Find the least natural number a for which

$x + ax^{-2} > 2$ $\forall$ $x \in (0, \infty)$.

Solution: Let $f(x) = x + ax^{-2}$

$f'(x) = 1 - 2ax^3 = 0 \Rightarrow x = (2a)^{-1/3}$

$f''(x) = 6ax^4 > 0$ $\forall x \in (0, \infty)$ (as a is a natural number)

Thus, $(2a)^{1/3} + a(2a)^{-2/3} > 2$

$\Rightarrow$ $a > \frac{32}{27}$ $\Rightarrow$ least natural number $a = 2$.

***Alternative*:**

$x + ax^{-2} > 2 \Rightarrow x^2 - 2x^2 + a > 0$

Let $f(x) = x^3 - 2x^2 + a$

Since $f(x) > 0$ $\forall x \in (0, \infty)$ $\Rightarrow$ $\min f(x) > 0$

For minimum $f(x)$, $f'(x) = 3x^2 - 4x = 0$ $\Rightarrow$ $x = 0, 4/3$

$f(4/3) > 0 \Rightarrow a > \frac{32}{27}$.

The least natural number a for which $x + ax^{-2} > 2$ is 2.

Example 27: If $ax^2 + \frac{b}{x} \geq c \ \forall \ x \in R^+$ where a, b, c are positive constants then prove that $27ab^2 \geq 4c^3$.

Solution: Let $f(x) = ax^2 + \frac{b}{x} - c$

$\therefore \ f'(x) = 2ax - \frac{b}{x^2}$

If $f'(x) = 0$ then $x = \left(\frac{b}{2a}\right)^{1/3}$, which is a positive critical point.

We can find that the least value of f(x) occurs at $x = \left(\frac{b}{2a}\right)^{1/3}$.

Since $ax^2 + \frac{b}{x} \geq c \ \forall \ x \in R^+$ we should have $f\left(\frac{b}{2a}\right)^{1/3} \geq 0$.

$$\Rightarrow \quad a\left\{\frac{b}{2a}\right\}^{2/3} + \frac{b}{\left\{\frac{b}{2a}\right\}^{1/3}} \geq c$$

$$\Rightarrow \quad a\left(\frac{b}{2a}\right) + b \geq c\left(\frac{b}{2a}\right)^{1/3}$$

$$\Rightarrow \quad \left(\frac{3b}{2}\right)^3 \geq \frac{b}{2a}.c^3 \quad \Rightarrow \quad \frac{27b^3}{8} \geq \frac{b}{2a}.c^3$$

or $\quad 27b^2a \geq 4c^3 \qquad \Rightarrow 27ab^2 \geq 4c^3$.

Concept Problems D

1. Prove that $f(x) = \sin(\cos x)$ in $(0, \pi/2)$ is strictly decreasing and $g(x) = \cos(\cos x)$ in $(0, \pi/2)$ is strictly increasing.
2. Let f and g be strictly increasing functions on the interval [a, b].
   (i) If $f(x) > 0$ and $g(x) > 0$ on [a, b] show that the product fg is also strictly increasing on [a, b]
   (ii) If $f(x) < 0$ and $g(x) < 0$ on [a, b], is fg strictly increasing, strictly decreasing, or neither? Explain.
3. Suppose that f is continuous on [a, b] and that c is an interior point of the interval. Show that if $f'(x) \leq 0$ on [a, c) and $f'(x) \geq 0$ on (c, b], then f(x) in never less than f(c) on [a, b].
4. Suppose that $f'(x) \geq 0$ on (a, b) and $f'(c) > 0$ for some c. Prove that $f(b) > f(a)$.
5. Suppose $f'(x) \geq 0$ on (a, b) and $f(a) = f(b)$. Prove that f is constant.
6. Suppose $f'(x) \geq g'(x)$ on (a, b) and $f(a) = g(a)$. Prove that $f(x) \geq g(x)$ on [a, b]. Further, if $f(b) = g(b)$, then prove that $f = g$.
7. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.
   (i) Show that $f(x) \geq 1$ for all x.
   (ii) Must $f'(1) = 0$? Explain.
8. Suppose that f is differentiable on [a, b] and that $f(b) < f(a)$. Can you then say anything about the values of $f'(x)$ on [a, b]?
9. Prove that the following functions
   (i) $y = e^x - 1 - x$,
   (ii) $y = e^{-x} - 1 + x$,
   (iii) $y = 1 - x^2/2 + x^3/3 - (1 + x)\,e^{-x}$ are positive and increase steadily for positive x.
10. Prove that $2x > 3 \sin x - x \cos x$ for all $x \in (0, \pi/2)$.

Practice Problems D

11. (i) Suppose f is strictly increasing for $x > 0$. Let n be a positive integer and define $g(x) = x^n f(x)$ for $x > 0$. If $f(x) > 0$ on an interval [a, b] with $a > 0$, show that g is strictly increasing on the same interval.
   (ii) Let $g(x) = x^n \sin x$ for $0 < x < \frac{\pi}{2}$ where n is a positive integer. Explain why g is strictly increasing on this interval.
12. Prove the inequality
   $2\sqrt{x} > 3 - \frac{1}{x}, \ x > 1$
13. Prove that $\sin x + \tan x > 2x$, $0 < x < \pi/2$
14. Prove the validity of the following inequalities:
   (i) $x - \frac{x^2}{2} < \ln(1 + x) < x$ for $x > 0$,
   (ii) $\ln(1 + x) > \frac{x}{1+x}$ for $x > 0$,
15. Prove $\sin x < x < 1 - \cos x + \sin x$ for $0 < x < \frac{1}{2}\pi$
16. Prove that the expression $(x - 1)\,e^x + 1$ is positive for all positive value of x.
17. Show that $2x \tan^{-1} x \geq \ln(1 + x^2)$ for all $x \in R$.

18. Show that $e^x - 1 > (1 + x).\ln(1 + x)$, if $x > 0$.

19. Show that $\tan^{-1}x > \dfrac{x}{1+\dfrac{x^2}{3}}$, if $x \in (0, \infty)$.

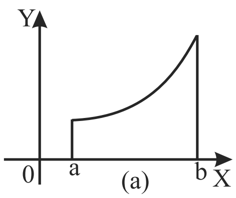
20. Prove that cos x lies between $1-\dfrac{1}{2}x^2$ and $1-\dfrac{1}{2}x^2+\dfrac{1}{24}x^4$

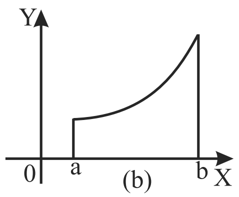
21. Prove that, if $x^2 < 1$, ln (1 + x) lies between $x-\dfrac{1}{2}x^2$ and $x-\dfrac{1}{2}x^2+\dfrac{1}{3}x^3$.

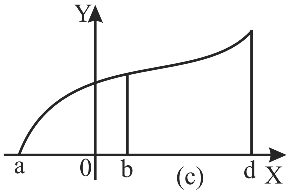
22. Find the set of values of x for which $ln(1 + x) > \dfrac{x}{1+x}$

6.8 Concavity and Point of Inflection

The figure represents the graphs of three functions each of which increases on the interval [a, b], but the difference in their behaviour is obvious.







In (a) the graph of the function is "bending downward"; in (b) it is "bending upward"; in (c) it is bending upward on the interval (a, c) and downward on the interval (c, b). From the geometrical point of view, the meaning of the expressions "bending downward" and "bending upward" is quite clear. Let us now attach a strict mathematical sense to these expressions and give a criterion for finding out in which direction the graph of a function is bending.

A curve is said to be concave up at a point P when in the immediate neighbourhood of P it lies wholly above the tangent at P. Similarly, it is said to be concave down when in the immediate neighbourhood of P it lies wholly below the tangent at P.

An arc of a curve is said to be concave up or down if it lies entirely on one side of the tangent drawn through any point of the arc.

It is of course supposed here that tangent can be drawn at each point of the arc, that is, there are neither corner points nor cusps on it.

The definition of concavity is demonstrated in the figures below. Figure 1 represents a curve which is concave up and Figure 2 represents a concave down curve. The curve in Figure 3 is neither concave up nor concave down, since it does not lie on one side of the tangent passing through the point P.

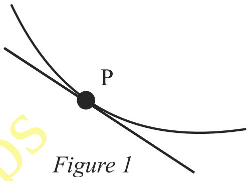


Figure 1

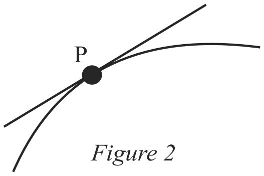


Figure 2

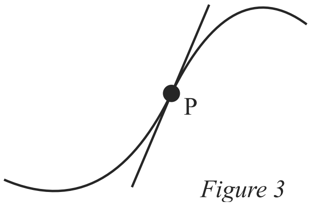


Figure 3

Point of Inflection

A point of a curve separating its concave up arc from a concave down arc is termed as point of inflection.

At a point of inflection, the tangent intersects the curve. In the vicinity of such a point the curve lies on both sides of its tangent drawn through that point.

Definition. Let f be a function and let c be a number. Assume that there are numbers a and b such that a < c < b and

(i) f is continuous on the open interval (a, b),
(ii) f has a tangent at the point (c, f(c)),
(iii) f is concave up in the interval (a, b) and concave down in the interval (c, b), or vice versa.

Then the point (c, f(c)) is called an inflection point or point of inflection. The number a is called an inflection number.

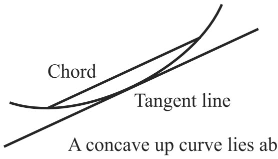
The point P in Figure 3 is a point of inflection.

We also speak about concavity of curves in a different way : an arc concave up (concave down) is said to be convex down (convex up).

Finally, a concave down (i.e. convex up) arc is sometimes briefly called convex while a concave up (i.e. convex down) arc is simply referred to as concave. In what follows we shall use the first variant of the terminology.

Alternative Definition

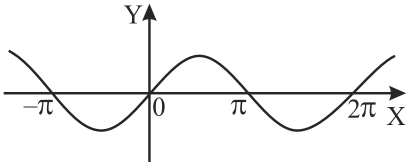
The graph of the function y = f(x) is said to be concave up on the interval (a, b) if it lies below all chords joining two points of the curve. See figure.



A concave up curve lies above its tangent lines and below its chords

Similarly, the graph of the function y = f(x) is said to be concave down on the interval (a, b) if it lies above all chords joining two points of the curve.

The curve y = sin x is concave up for
$x \in (2k\pi, (2k + 1)\pi)$, $k \in I$, and
concave down for $x \in ((2k - 1)\pi, 2k\pi)$, $k \in I$.



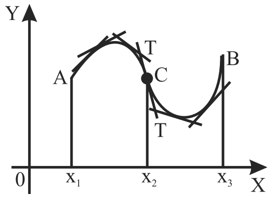
Testing a Curve for Concavity

It turns out that the criteria for determining the type of concavity (up or down) bring into play the second derivative – and it is the sign of the second derivative which is the discriminating device.

We consider a curve y = f(x) where the function f(x) is continuous together with its first and second derivatives f′(x) and f″(x). There is a simple relationship between the properties of the second derivative f″(x) and the concavity of the curve y = f(x) which we shall establish without a rigorous proof by resorting only to some simple geometrical considerations. These considerations are based on the following proposition.

On an interval of increase of the first derivative the graph of the function is concave up and on the interval of decrease of the first derivative it is concave down.

For, if the derivative f′(x) increases, the slope of the tangent line to the curve also increases as the variable point of the graph traces it from left to right; then the tangent line turns as if it "lifted up" the curve preventing it from descending below the tangent. Therefore, the corresponding arc must lie above its every tangent and thus is concave. This situation is clearly demonstrated by the example of the arc CB of the curve depicted above.



In the figure arc CB is concave up while the arc AC is concave down.

On the contrary, if the derivative f′(x) decreases the slope of the tangent decreases. The tangent turns as if it "pressed down" the curve preventing it from lifting above the tangent. Therefore the arc must lie below its every tangent and thus is concave down.

Sufficient Condition for Concavity

Now we can take advantage of the basic theorem establishing the connection between the character of variation of a function and the sign of its derivative; as a function under consideration we take f′(x) whose derivative is (f′(x))′ = f″(x). If f″(x) > 0 the derivative f′(x) increases and f″(x) < 0 it decreases. We thus arrive at the following theorem.

Theorem If the second derivative f″(x) is everywhere positive within an interval the arc of the curve y = f(x) corresponding to that interval is concave up. If the second derivative f″(x) is everywhere negative in an interval, the corresponding arc of the curve y = f(x) is concave down.

The student should know the following mnemonic rule (the '**rain rule**"): if the graph of the function on an interval is concave up, then y″ > 0; if the graph of the function is concave down, then y" < 0. Writing these inequality in the form $\underset{0}{\underset{\vee}{y''}}$, $\underset{0}{\underset{\wedge}{y''}}$ we note that the signs of the inequalities correspond to the directions of concavity of the curve (∨ upward, that is, "holds water", and ∧ downward, that is, "spills water").

In an interval where f″(x) is positive, the function f′(x) is increasing, and so the function f is concave upward. However, if a function is concave upward f″(x) is not necessarily positive. For instance, $y = x^4$ is concave upward over any interval, since the derivative $4x^3$ is increasing. The second derivative $12x^2$ is not always positive; at x = 0 it is 0.

It should be noted that if the second derivative f″(x) has constant sign, for instance, if it is positive, everywhere except at some separate points where it vanishes, the function f′(x) remains increasing and the corresponding arc of the graph of the function y = f(x) is concave up.

In our foregoing investigation we supposed that the function f(x) in question was twice differentiable throughout the interval in question. If this condition is violated it is necessary to investigate f′(x) and f″(x) in the vicinity of those separate point at which the derivatives donot exist. At these points also the concavity of the graph of the function may change. An example of this kind is the graph of the function $y = \sqrt[3]{x}$, for which the point (0, 0) is a point of inflection.

Hyper-Critical Point

In general, a function is concave up and concave down in different parts of its domain. Suppose a function f defined in (a, b), is concave up in (a, c) and then concave down in (c, b), we are now interested in knowing what must have happened at x = c and how do we get c? To answer these questions, we should first define the term 'hyper-critical points' or critical points of the second kind or second-order critical points.

A hyper-critical point of a function f is a number c in the domain of f such the either $f''(c) = 0$ or $f''(c)$ does not exist.

Steps for Finding Intervals of Concavity

1. Compute the second derivative $f'(x)$ of a given function f(x), and then find the hyper-critical points i.e. points at which $f''(x)$ equals zero or does not exist .
2. Using the hyper-critical points, separate the domain of definition of the function f(x) into several intervals on each of which the derivative $f''(x)$ retains its sign. These intervals will be the intervals of concavity.
3. Investigate the sign of $f''(x)$ on each of the found intervals. If on a certain interval $f''(x) > 0$, then the function f(x) is concave up on this interval, and if $f''(x) < 0$, then f(x) is concave down on this interval.

Example 1: Find the intervals of concavity of the graph of the function $y = x^5 + 5x - 6$.

Solution: We have $f'(x) = 5x^4 + 5$, $f''(x) = 20x^3$.

$f''(x)$ exists for all x. Now $f''(x) = 0$ at $x = 0$.

Hence, $x = 0$ is a hyper-critical point of the function.

If $x < 0$, then $f''(x) < 0$ and the curve is concave down and if $x > 0$, then $f''(x) > 0$ and the curve is concave up.

Thus we see that the curve is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

Example 2: Find the intervals in which the curve $y = x \sin(\ln x)$, $x > 0$ is concave up or concave down.

Solution: We find the derivatives:

$$y' = \sin(\ln x) + \cos(\ln x),$$

$$y'' = \frac{1}{x}[\cos(\ln x) - \sin(\ln x)] = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} - \ln x\right).$$

The second derivative vanishes at the points

$$x_k = e^{\pi/4 + k\pi},\ k = 0, \pm 1, \pm 2, \ldots.$$

The function $\sin(\pi/4 - \ln x)$, and together with it y'', changes sign when passing through each point x_k.

In the intervals $(e^{2k\pi - 3\pi/4}, e^{2k\pi + \pi/4})$ the curve is concave up, and in the intervals $(e^{2k\pi + \pi/4}, e^{2k\pi + 5\pi/4})$ it is concave down.

Method for Finding Point of Inflection

An important characteristic of a curve are the points separating its concave up and concave down arcs.

Suppose the graph of a function f has a tangent line (possibly vertical) at the point P(c, f(c)) and that the graph is concave up on one side of P and concave down on the other side. Then P is called an inflection point of the graph.

The concavity of the graph of f will change only at points where $f''(x) = 0$ or $f''(x)$ does not exist–that is, at the hyper-critical points.

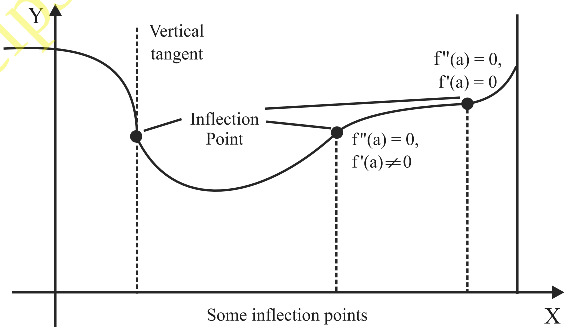
If $x = c$ is a hyper-critical point and the inequalities $f''(c - h) < 0, f''(c + h) > 0$ (or inequalities $f''(c - h) > 0$, $f''(c + h) < 0$) hold for an arbitrary sufficiently small $h > 0$, then the point of the curve $y = f(x)$ with the abscissa $x = c$ is a point of inflection.

If $f''(c - h)$ and $f''(c + h)$ are of the same sign, then the point $x = c$ is not a point of inflection.

If the second derivative $f''(x)$ changes sign as x passes through $x = c$ (from the left to right) then c is a point of inflection, if it changes sign from – to + there is an interval of concave down on the left of the point c and interval of concave up on the right of it, and, conversely, if it changes sign from + to –, an interval of concave down follows an interval of concave up as x passes through c.

For example, the second derivative of the function $y = x^5 + x$ is $y'' = 20x^3$, and $y''(0) = 0$.

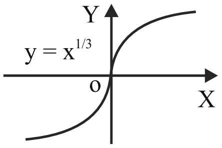
We have $y'' < 0$ for $x < 0$ and $y'' > 0$ for $x > 0$. Consequently, the origin is a point of inflection of the graph of the given function, the interval of concave down lying on the left of it and the interval of concave up on the right.



Note:

1. A continuous function f need not have an inflection point where $f''(x) = 0$. For instance, if $f(x) = x^4$, we have $f''(0) = 0$, but the graph of f is always concave up.
   Let us take the function $y = x^5 + 5x^4$. Here $y'' = 20x^2(x + 3)$, and $y'' = 0$ for $x = -3$ and for $x = 0$. As x passes through the point $x = -3$, the second derivative changes sign, and thus $x = -3$ is a point of inflection.
   When x passes through the point $x = 0$, the second derivative retains constant sign, and therefore, the origin is not a point of inflection; the graph of the given function is concave up on both sides of the origin.
2. If $x = c$ is a point of inflection of a curve $y = f(x)$ and at this point there exists the second derivative $f''(c)$, then $f''(c)$ is necessarily equal to zero ($f''(c) = 0$).
3. The point (–1, 0) in $y = (x - 1)^3$, being both a critical point and a point of inflection, is a point of **horizontal inflection**.
4. If a function f is such that the derivative f''' is continuous at $x = c$ and $f''(c) = 0$ while $f'''(c) \neq 0$, then the curve $y = f(x)$ has a point of inflection for $x = c$.

5. It should be noted that a point separating a concave up arc of a curve from a concave down one may be such that the tangent at that point is perpendicular to the x-axis i.e. a vertical tangent or such that the tangent does not exist.
This is demonstrated by the behaviour of the graph of the function $y = \sqrt[3]{x}$ in the vicinity of the origin. In such a case we speak of a **point of inflection with vertical tangent**.



6. A number c such that $f''(c)$ is not defined and the concavity of f changes at c will correspond to an inflection point if and only if f(c) is defined.

Example 3: Find the inflection points of the graph of the function $f(x) = x^4 - 4x^3 + x - 7$.

Solution: Since $f''(x) = 12x^2 - 24x = 12x(x - 2)$, we have $f''(x) = 0 \Rightarrow x = 0$ or $x = 2$.

Thus, the points (0, –7), (2, –21) are the only possible inflection points.

Now since $12x(x - 2) < 0$ for $0 < x < 2$ and $12x(x - 2) > 0$ for $x < 0$ it is clear that (0, –7) is in fact an inflection point.

Similarly, since $12x(x - 2) > 0$ for $x > 2$ and $12x(x - 2) < 0$ for $0 < x < 2$, the change in sign guarantees that (2, –2l) also is an inflection point.

Example 4: Find the points of inflection of the function $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

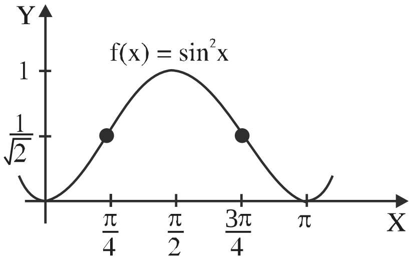
Solution: $f(x) = \sin^2 x$

$f'(x) = \sin 2x$

$f''(x) = 2 \cos 2x$

$$f''(0) = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Both these points are inflection points as sign of $f''(x)$ change about these points.



Example 5: Find the inflection points of the curve $y = (x - 5)^{5/3} + 2$.

Solution: We find $y' = \frac{5}{3}(x - 5)^{2/3}$,

$$y'' = \frac{10}{9\sqrt[3]{(x-5)}}.$$

The second derivative does not vanish for any value of x and does not exist at x = 5.
The tangent exists at x = 5.
Since $y''(5 - h) < 0$, $y''(5 + h) > 0$, the point x = 5 is the abscissa of the inflection point Thus, (5, 2) is the inflection point.

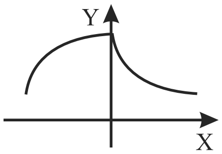
Example 6: Find the points of inflection of the graph of the function $y = \sqrt[3]{x + 2}$.

Solution: We have

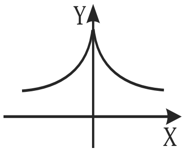
$$y'' = -\frac{2}{9}(x + 2)^{-\frac{5}{2}} = \frac{-2}{9\sqrt[3]{(x + 2)^5}} \qquad \text{...(1)}$$

It is obvious that y'' does not vanish anywhere.
Equating to zero the denominator of the fraction on the right of (1), we find that y'' does not exist for x = –2.
The tangent at this point is parallel to the y-axis, since the first derivative y' is infinite at x = –2.
Since $y'' > 0$ for $x < -2$ and $y'' < 0$ for $x > -2$, it follows that (–2, 0) is the point of inflection.

Note: We can draw a curve with a corner point separating its concave up and concave down arcs.



We shall not include corner points of this kind into the class of points of inflection. There may of course exist a corner point at which the character of concavity of the curve does not change. See the figure below.



Cusp

If $f'(x)$ approaches ∞ from one side of a point x = c and –∞ from the other side, then the function f is said to have a **cusp** at x = c.
Let $f(x) = 2x^{5/3} + 5x^{2/3}$.

$$f'(x) = 2\left(\frac{5}{3}\right) x^{2/3} + 5\left(\frac{2}{3}\right) x^{-1/3} = \frac{10}{3} x^{-1/3}(x + 1)$$

$$f''(x) = \frac{10}{3}\left(\frac{2}{3}\right)x^{-1/3} + \frac{10}{3}\left(-\frac{1}{3}\right)x^{-4/3}$$

$$= \frac{10}{9}\,x^{-4/3}(2x-1)$$

Note that the graph is concave down on both sides of $x = 0$ and that the slope $f'(x)$ decreases without bound to the left of $x = 0$ and increases without bound to the right. This means the graph changes direction abruptly at $x = 0$, and we have a cusp at the origin.

Example 7: Find the points of inflection of the curve $y = 2 - |x^5 - 1|$.

Solution: The given function can be written as:

$$y = \begin{cases} 2 - (x^5 - 1), & x \ge 1, \\ 2 + (x^5 - 1), & x < 1. \end{cases}$$

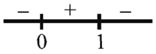
Therefore, $y' = \begin{cases} -5x^4, & x > 1, \\ 5x^4, & x < 1. \end{cases}$

At the point $x = 1$ there is no derivative.

Further, $y'' = \begin{cases} -20x^3, & x > 1, \\ 20x^3, & x < 1; \end{cases}$

$y'' = 0$ at the point $x = 0$. Hence we have to investigate three intervals : $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$.

Sign scheme of y"



The point (0, 1) is a point of inflection, the point (1, 2) being a corner point.

Example 8: What conditions must the coefficients a, b, c satisfy for the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection?

Solution: Find the second derivative:

$y'' = 12ax^2 + 6bx + 2c$

The curve has points of inflection if and only if the equation

$6ax^2 + 3bx + c = 0$

has different real root, i.e. when the discriminant

$9b^2 - 24ac > 0$, or $3b^2 - 8ac > 0$.

Example 9: Consider a curve C: $y = \cos^{-1}(2x - 1)$ and a straight line L : $2px - 4y + 2\pi - p = 0$

Find the set of values of 'p' for which the line L intersects the curve at three distinct points.

Solution: $y = \cos^{-1}(2x - 1)$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}} = -(x-x^2)^{-1/2}$$

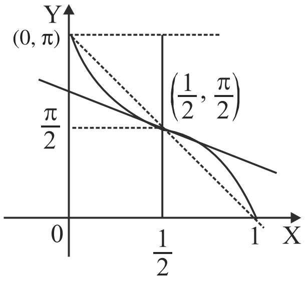
$$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{(1-2x)}{(x-x^2)^{3/2}} = 0 \Rightarrow x = \frac{1}{2}$$

The point $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ is a point of inflection of the curve is and it satisfies the line L.

The line L is always passing through point of inflection of the curve C.

Slope of the tangent to the curve C at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -2$$



As the slope decreases from –2, line cuts the curve at three distinct points and minimum slope of the line when it intersects the curve at three distinct points is

$$\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

$$\therefore \quad \frac{p}{2} \in [-\pi, -2) \Rightarrow p \in [-2\pi, -4)$$

Example 10: Prove the inequality

$$\sin x + 2x \ge \frac{3x(x+1)}{\pi} \quad \forall \; x \in (0, \pi/2).$$

Solution: $f(x) = \sin x + 2x$

$$g(x) = \frac{3(x^2 + x)}{\pi}$$

$f'(x) = \cos x + 2 > 0$; $f''(x) = -\sin x < 0$

Hence f is concave down and increasing.

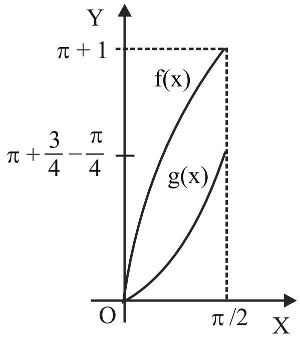
$$g'(x) = \frac{3}{\pi}(2x+1) \qquad g''(x) = \frac{3}{\pi}(2) > 0$$

$\Rightarrow$ g is concave up and increasing.

$$f\left(\frac{\pi}{2}\right) = \pi + 1$$

$$g\left(\frac{\pi}{2}\right) = 3\frac{\pi}{2\pi}\left(\frac{\pi}{2}+1\right) = \frac{3}{2}\left(\frac{\pi}{2}+1\right) = \frac{3\pi}{4} + \frac{3}{2}$$

$$= \pi + \frac{3}{4} - \frac{\pi}{4}.$$



From the graph, it is clear that f(x) ≥ g(x) ∀ x ∈ (0, π/2).

Example 11: Find equations of the tangent lines at the points of inflection of $y = f(x) = x^4 - 6x^3 + 12x^2 - 8x$.

Solution: $f(x) = 4x^3 - 18x^3 + 24x - 8$

$f''(x) = 12x^2 - 36x + 24 = 12(x - 1)(x - 2)$

$f'''(x) = 24x - 36 = 12(2x - 3)$

The possible points of inflection are at x = 1 and 2. Since $f'''(1) \neq 0$ and $f'''(2) \neq 0$, the points (1, – 1) and (2, 0) are points of inflection.

At (1, – 1), the slope of the tangent line is

$m = f'(1) = 2$, and its equation is

$y = y_1 = m(x - x_1)$ or $y + 1 = 2(x - 1)$

or $y = 2x - 3$

At (2, 0), the slope is $f'(2) = 0$, and the equation of the tangent line is $y = 0$.

Example 12: Let $f'(x) > 0$ and $f''(x) > 0$ where $x_1 < x_2$. Then show that $f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$.

Solution: Since f(x) > 0 and $f''(x) > 0$, the function is strictly increasing and concave up. A sample graph of f has been shown in the figure below.

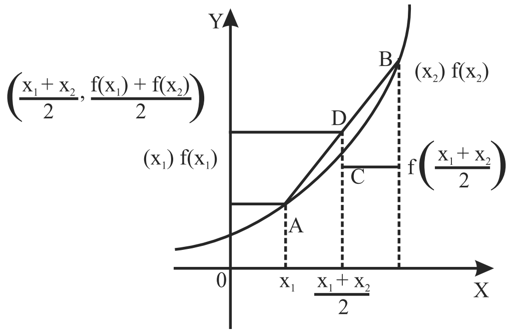
We know, $x_1 < \frac{x_1+x_2}{2} < x_2$

and $\left(\frac{x_1+x_2}{2}, \frac{f(x_1)+f(x_2)}{2}\right)$ is the midpoint of the chord joining A and B.

Since the graph is concave up

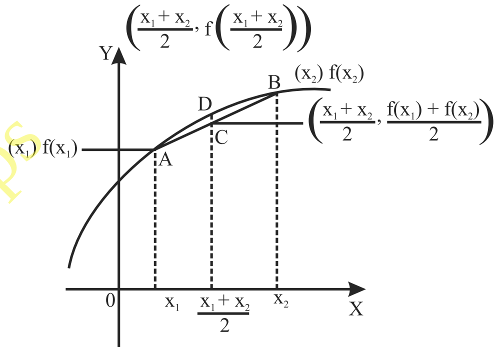
$$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$$

because ordinate of a point on the curve is less than that of a point on the chord at the same abscissa.


Y
$\left(\frac{x_1+x_2}{2}, \frac{f(x_1)+f(x_2)}{2}\right)$
B
$(x_2)\ f(x_2)$
D
$(x_1)\ f(x_1)$
C
$f\left(\frac{x_1+x_2}{2}\right)$
A
0
x_1
$\frac{x_1+x_2}{2}$
X

Similarly, for a concave down graph as shown in the figure below, we have

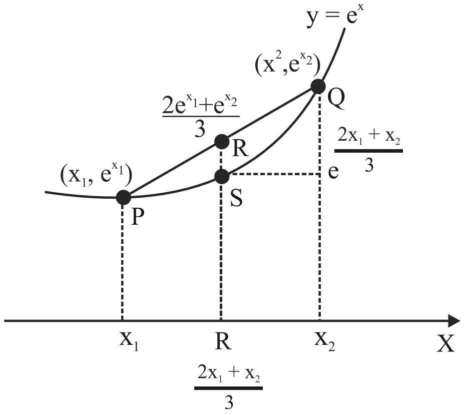
$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$


$\left(\frac{x_1+x_2}{2}, f\left(\frac{x_1+x_2}{2}\right)\right)$
Y
B
$(x_2)\ f(x_2)$
D
$\left(\frac{x_1+x_2}{2}, \frac{f(x_1)+f(x_2)}{2}\right)$
$(x_1)\ f(x_1)$
C
A
0
x_1
$\frac{x_1+x_2}{2}$
x_2
X

Example 13: Prove that for any two numbers x_1 and x_2,

$$\frac{2e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$$

Solution: Assume $f(x) = e^x$ and let x_1 and x_2 be two points on the curve $y = e^x$.



Let R be another point which divides P and Q in ratio 1 : 2.

The y coordinate of point R is $\frac{2e^{x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$.

Since f(x) = e^x is always concave up, hence point R will always be above point S.

$$\Rightarrow \frac{2e^{x_1}+e^{x_2}}{3} < e^{\frac{2x_1+x_2}{3}}$$

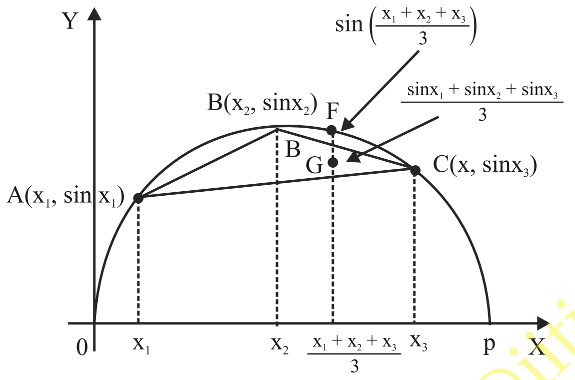
Example 14: If $0 < x_1 < x_2 < x_3 < \pi$ then prove that

$$\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}.$$

Hence or otherwise prove that if A, B, C are angles of triangle then maximum value of

$$\sin A + \sin B + \sin C \text{ is } \frac{3\sqrt{3}}{2}.$$

Solution:



Let point A, B, C form a triangle y coordinate of centroid G is $\frac{\sin x_1+\sin x_2+\sin x_3}{3}$ and y coordinate of point F is $\sin\left(\frac{x_1+x_2+x_3}{3}\right)$.

Hence $\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}$.

If A + B + C = π, then

$$\sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \sin\frac{\pi}{3} > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C$$

$\Rightarrow$ maximum value of (sin A + sin B + sin C)

$= \frac{3\sqrt{3}}{2}$. It is attained if A = B = C

Note:

Let g(x) = $f^{-1}(x)$

Since g is the inverse of f , fog (x) = gof (x) = x

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

$\Rightarrow g'(x) > 0$ {as f(x) is increasing} ...(1)

$\Rightarrow$ g(x) is increasing for all x ∈ R.

$\Rightarrow f^{-1}(x)$ is increasing for all x ∈ R.

again $g'(x) = \frac{1}{f'(g(x))}$

$$\Rightarrow g''(x) = -\frac{1}{(f'(g(x)))^2} f''(g)\, g'(x), \text{ for all } x \in R$$

$$\Rightarrow g''(x) < 0, \left\{\begin{array}{l}\text{as } g'(x) > 0, \text{ from(1)}\\ f''(x) > 0, \text{ given}\end{array}\right.$$

$\Rightarrow g'(x)$ is decreasing for all x ∈ R.

$\Rightarrow f^{-1}(x)$ is increasing and $\frac{d}{dx}\{f^{-1}(x)\}$ is decreasing.

This means that if a function f is strictly increasing and concave up, then its inverse f^{-1} is strictly increasing and concave down.

Concept Problems

E

1. Let f be a function such that $f''(x) = (x-1)(x-2)$.
   (i) For which x is f concave upward?
   (ii) For which x is f concave downward?
   (iii) List its inflection points.
2. Show that the graph of the function $y = x \tan^{-1} x$ is concave up everywhere.
3. Is it true that the concavity of the graph of a twice differentiable function y = f(x) changes every time $f''(x) = 0$? Give reasons for your answer.
4. Show that the graph of the quadratic function $y = Ax^2 + Bx + C$ is concave up if A > 0 and concave down if A < 0.
5. Let $f(x) = ax^2 + bx + c$, where a, b and c are constants, $a \neq 0$. Show that f has no inflection points.
6. Explain why a polynomial of odd degree (atleast 3) always has atleast one inflection point.
7. Let $f(x) = x^{2/3}|x|$. Is (0, 0) a point of inflection of this graph? Show that $f''(0)$ does not exist.
8. Graph the function $y = x^{1/3}$ and show that it has a point of inflection where neither the first nor the second derivative exists.
9. Sketch the graph of $f(x) = \sqrt[5]{x^3}$ and identify the inflection point. Does $f''(x)$ exist at the inflection point?

10. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

11. Prove (without referring to a picture) that if the graph of f lies above its tangent lines for all x in $[a,b]$, then $f''(x) \geq 0$ for all x in $[a,b]$.

12. If f is a function such that $f''(x) > 0$ for all x, then prove that the graph of $y = f(x)$ lies below its chords; i.e., $f(ax_1 + (1 - a)x_2) < af(x_1) + (1 - a) f(x_2)$ for any a in $(0,1)$, and for any x_1 and x_2, $x_1 \neq x_2$.

13. Assume that all of the functions are twice differentiable and the second derivatives are never 0 on an interval I.

(a) If f and g are concave up on I, show that f + g is concave up on I.

(b) If f is positive and concave up on I, show that the function $g(x) = [f(x)]^2$ is concave up on I.

(c) If f and g are positive, increasing, concave up functions on I, show that the product function fg is concave up on I

(d) Show that part (iii) remains true if f and g are both decreasing.

(e) Suppose f is increasing and g is decreasing. show, by giving examples, that fg may be concave up, concave down, or linear. Why doesn't the argument in parts (iii) and (iv) work in this case?

14. For any twice derivable function f on (a, b), prove that the function $g(x) = f(x) + cx + d$, where c and d are any real numbers, has the same concavity characteristics as f.

15. If $f(x) > 0$ and $g(x) > 0$ for all x on I and if f and g are concave up on I, then is fg also concave up on I?

16. Suppose $f(x) > 0$ on (a, b) and ln f is concave up. Prove that f is concave up.

Practice Problems E

17. Find out whether the curve $y = x^4 - 5x^3 - 15x^2 + 30$ is concave up or concave down in the vicinity of the points (1, 11) and (3, 3).

18. Prove that the curve $y = x.\ln x$ is everywhere concave up.

19. Find the points of inflection of the curve

$$y = \frac{x^2}{a^2 + x^2}.$$

20. Determine the values of a, b and c if the graph of $f(x) = ax^3 + bx^2 + c$ is to have $(-1, 1)$ as a point of inflection of f at which the slope is 2.

21. An inflection point of a graph is called a horizontal inflection point if the slope there is zero. Find the horizontal inflection points of

(i) x^3, (ii) $(x - a)^3$,

(iii) $(x - a)^3 + b$.

22. Find the ranges of values of x in which the curves

(i) $y = 3x^5 - 40x^3 + 3x - 20$

(ii) $y = (x^2 + 4x + 5)e^{-x}$

are concave up or concave down. Also find their points of inflection.

23. Find the intervals in which the curve $y = (\cos x + \sin x) e^x$ is concave up or down for $x \in (0, 2\pi)$.

24. Use the given graph of f to find the following.

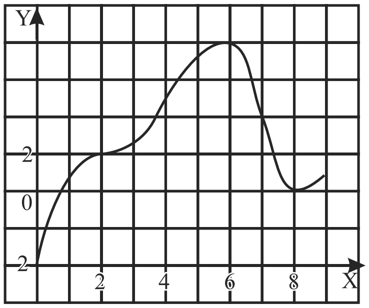
(i) The intervals on which f is increasing.

(ii) The intervals on which f is decreasing.

(iii) The intervals on which f is concave up.

(iv) The intervals on which f is concave down.

(v) The coordinates of the points of inflection.



25. Find out whether the curve $y = x^2 \ln x$ is concave up or concave down in the neighbourhoods of the points (1, 0) and $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$.

26. Find the equations of the tangent lines at all inflection points on the graph of

$f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$

27. Find the point of inflection of the graph of $y = x^2 - \frac{1}{6x^3}$

Find the equation of the tangent line to the graph at this point.

28. Show that the curve $y = \frac{1 + x}{1 + x^2}$ has three points of inflection, and that they lie in a straight line.

29. Consider the function $f(x) = \frac{\cos x}{x}$.

If $0 < x_1 < x_2 < \frac{\pi}{2}$ consider two expressions

$$\frac{\cos x_1}{x_1} + \frac{\cos x_2}{x_2} \text{ and } \frac{\cos\left(\frac{x_1 + x_2}{2}\right)}{x_1 + x_2}.$$

Prove that $\dfrac{\cos x_1}{x_1} + \dfrac{\cos x_2}{x_2} > \dfrac{\cos\left(\dfrac{x_1 + x_2}{2}\right)}{x_1 + x_2}$.

30. Show that the curve $y = e^{-x^2}$ has inflection at the points for which $x = \pm \dfrac{1}{\sqrt{2}}$.

Target Problems for JEE Advanced

Problem 1: Prove that the function $f(x) = x^2 \sin(1/x) + ax$, where $0 < a < 1$, when $x \neq 0$, and $f(0) = 0$ is not monotonic in any interval containing the origin.

Solution: We have $f'(0) = a > 0$. (using first principles) Thus the conditions of the theorem for increasing functions are satisfied.

Hence, f is strictly increasing at $x = 0$.

But for $x \neq 0$, $f'(x) = 2x\sin(1/x) - \cos(1/x) + a$

which oscillates between the limits $a - 1$ and $a + 1$ as $x \to 0$. Since $a - 1 < 0$, we can find values of x, as near to 0 as we like, for which $f'(x) < 0$; and it is therefore impossible to find any interval, including $x = 0$, througout which f(x) is a strictly increasing function.

Problem 2: Let f be the function defined by

$f(x) = (ax^2 + 2hx + b) / (px^2 + 2qx + r)$,

for all x for which the denominator does not vanish. Prove that the stationary values of f are the roots of

$(pr - q^2)\lambda^2 - (pb + ra - 2qh)\lambda + (ab - h^2) = 0$

Solution: Since $f(x) = (ax^2 + 2hx + b)/(px^2 + 2qx + r)$,

$$f'(x) = \frac{2\{(ax+h)(px^2+2qx+r)-(px+q)(ax^2+2hx+b)\}}{(px^2+2qx+r)^2}$$

If f has a stationary point at $x = x_0$, then $f'(x_0) = 0$,

i.e., $(ax_0 + h)(px_0^2 + qx_0 + r)–(px_0 + q)(ax_0^2 + 2hx_0 + b) = 0$ ...(1)

Also, the value λ of f at $x = x_0$ is given by

$\lambda = (ax_0^2 + 2hx_0 + b)/(px_0^2 + 2qx_0 + r)$, ...(2)

From (1) and (2), we have

$(ax_0 + h) - \lambda(px_0 + q) = 0$ ...(3)

Also, we may re-write (1) as

$(px_0^2 + 2qx_0 + r)/(px_0 + q) = (ax_0^2 + 2hx_0 + b)/(ax_0 + h)$,

i.e. $(qx_0 + r)/(px_0 + q) = (hx_0 + b)/(ax_0 + h)$,

i.e. $(hx_0 + b) - \lambda(qx_0 + r) = 0$, ...(4)

by using (3),

Eliminating x_0 from (3) and (4), we have

$$\begin{vmatrix} a-\lambda p & h-\lambda q \\ h-\lambda q & b-\lambda r \end{vmatrix} = 0$$

i.e. $(pr - q^2)\lambda^2 - (pb + ra - 2qh)\lambda + ab - h^2 = 0$.

Problem 3: Find the intervals of monotonicity of the function $y = 2x^2 - \ln|x|$, $x \neq 0$.

Solution: Let $f(x) = 2x^2 - \ln|x|$ ...(1)

Case I: $x < 0$.

$f(x) = 2x^2 - \ln|x| = 2x^2 - \ln(-x)$

$\therefore\ f'(x) = 4x - \dfrac{1}{(-x)}(-1) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x}$ ...(2)

Case II: $x > 0$.

$f(x) = 2x^2 - \ln|x| = 2x^2 - \ln x$

$\therefore\ f'(x) = 4x - \dfrac{1}{x}$ ...(3)

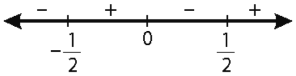
Thus when $x < 0$ or $x > 0$,

$f'(x) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x}$ ...(4)

Sign scheme for $f'(x)$ i.e. $\dfrac{4x^2 - 1}{x}$:

when $4x^2 - 1 = 0$, $x = \pm\dfrac{1}{2}$

$\therefore$ Sign scheme for $\dfrac{4x^2 - 1}{x}$ is as follows



Thus, f(x) will be a decreasing function in the interval $\left(-\infty, -\dfrac{1}{2}\right)$ and $\left(0, \dfrac{1}{2}\right)$

and increasing function in $\left(-\dfrac{1}{2}, 0\right)$ and $\left(0, \dfrac{1}{2}\right)$.

Problem 4: If $f(x) = \{-c^2 + (b - 1)c - 2\}x + \displaystyle\int_0^x (\sin^2 x + \cos^4 x)dx$ is an increasing function ∀ x ∈ R then find all possible values of b, for any c ∈ R.

Solution: Given

$$f(x) = \{-c^2 + (b-1)c - \}x + \int_0^x (\sin^2 x + \cos^4 x)dx$$

$\therefore\ f'(x) = -c^2 + (b - 1)c - 2 + \sin^2 x + \cos^4 x$

$\Rightarrow\ f'(x) = (1 - \sin^2 x)^2 + \sin^2 x - c^2 + (b - 1)c - 2$

$= \sin^4 x - \sin^2 x - c^2 + (b - 1)c - 1$

$$= \left(\sin^2 x - \frac{1}{2}\right)^2 - c^2 + (b-1)c - \frac{5}{4}$$

We should have $f'(x) \geq 0 \; \forall \; x \in R$

$$\therefore \quad \left(\sin^2 x - \frac{1}{2}\right)^2 - c^2 + (b-1)c - \frac{5}{4} \geq 0$$

$$0 - c^2 + (b-1)c - \frac{5}{4} \geq 0$$

(since the least value of $(\sin^2 x - 1/2)^2 = 0$)

or $\quad c^2 - (b-1)c + \frac{5}{4} \leq 0$ for all c.

$\therefore \quad D \leq 0 \Rightarrow (b-1)^2 \leq 5$

$\Rightarrow \quad -\sqrt{5} \leq b - 1 \leq -\sqrt{5}$

$\Rightarrow \quad 1 - \sqrt{5} \leq b \leq 1 + \sqrt{5}$

Hence $b \in [1 - \sqrt{5}, 1 + \sqrt{5}]$.

Problem 5: Find the intervals of monotonicity of the function $f(x) = x\sqrt{ax - x^2}$, $a \neq 0$

Solution: $y = x\sqrt{ax - x^2}$

$$\frac{dy}{dx} = \sqrt{ax - x^2} + \frac{x}{2}\frac{(a-2x)}{\sqrt{ax - x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}\frac{(3a-4x)}{2\sqrt{ax-x^2}}$$

Domain : $ax - x^2 \geq 0 \Rightarrow x(x-a) \leq 0$

Case I $\quad$ If $a > 0$, then $x \in [0, a]$

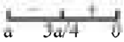
Case II $\quad$ If $a < 0$, then $x \in [a, 0]$

Critical point : $\quad x = \frac{3a}{4}$

Case I: $a > 0$, $\quad$ Sign of $f'(x)$

f(x) is increasing in $(0, \frac{3a}{4})$ and decreasing in $(\frac{3a}{4}, a)$

Case II: $a < 0$, Sign of $f'(x)$



f(x) is decreasing in $(a, \frac{3a}{4})$ and increasing in $(\frac{3a}{4}, 0)$.

Problem 6: Find the intervals of monotonicity of the function $f(x) = \frac{a}{x}\ln\frac{x}{a}$.

Solution: $y = \frac{a}{x}\ln\frac{x}{a}$

$$\frac{dy}{dx} = \frac{a}{x^2}\ln\frac{x}{a} + \frac{x}{a}\frac{1}{x/a}\cdot\frac{1}{a}$$

$$\frac{dy}{dx} = -\frac{a}{x^2}\ln\frac{x}{a} + \frac{a}{x^2} = \frac{a}{x^2}[1 - \ln x/a]$$

$$\text{Domain} : \begin{cases} x > 0 & \text{if} \quad a > 0 \\ x < 0 & \text{if} \quad a < 0 \end{cases}$$

Critical point : $\ln\left(\frac{x}{a}\right) = 1 \Rightarrow x = ae$

Sign of $f'(x)$

$a > 0$

$a < 0$

If $a > 0$, f(x) is increasing in $(0, ae)$ and decreasing in (ae, ∞).
If $a < 0$, f(x) is increasing in $(-\infty, ae)$ and decreasing in $(ae, 0)$.

Problem 7: Find the number of real roots of the equation $\sum_{i=1}^{n}\frac{a_i^2}{x - b_i} = c$, where $b_1 < b_2 < \ldots\ldots < b_n$.

Solution: Consider the function

$$f(x) = \sum_{i=1}^{n}\frac{a_i^2}{x - b_i} - c = \frac{a_1^2}{x - b_1} + \frac{a_2^2}{x - b_2} + \ldots. + \frac{a_n^2}{x - b_n} - c$$

$$\text{and } f'(x) = \left[\frac{a_1^2}{(x-b_1)^2} + \frac{a_2^2}{(x-b_2)^2} + \ldots. + \frac{a_n^2}{(x-b_n)^2}\right]$$

$< 0 \; \forall \; x \in R - \{b_1, b_2, \ldots\ldots, b_n\}$

$\Rightarrow$ f(x) strictly decreases in $(-\infty, b_1)$, (b_1, b_2), ... , (b_{n-1}, b_n)

Now, we have

$f(-\infty) = -c = f(\infty)$

$f(b_1^-) = -\infty$ and $f(b_1^+) = \infty$

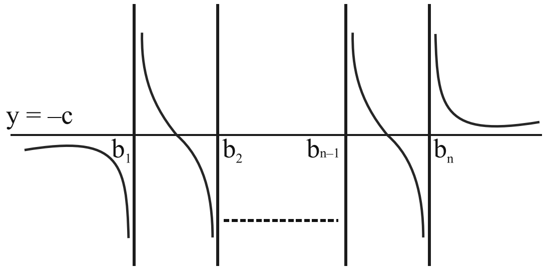
$f(b_2^-) = -\infty$ and $f(b_2^+) = \infty$

....

....

$f(b_n^-) = -\infty$ and $f(b_n^+) = \infty$

The plot of the curve $y = f(x)$ is shown alongside.



Hence, the number of real roots of the equation is $n - 1$.

Problem 8: If $f: R \rightarrow R$ and f is a polynomial with $f(x) = 0$ has real and distinct roots, show that the equation, $[f'(x)]^2 - f(x) \cdot f''(x) = 0$ cannot have real roots.

Solution: Let $f(x) = c(x - x_1)(x - x_2) \ldots\ldots (x - x_n)$

Again Let $h(x) = \frac{f'(x)}{f(x)}$

$$= \left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \ldots\ldots + \frac{1}{x - x_n}\right)$$

$$h'(x) = \frac{f(x) \cdot f''(x) - [f'(x)]^2}{f^2(x)}$$

$$= -\left(\frac{1}{(x-x_1)^2}+\frac{1}{(x-x_2)^2}+\ldots\ldots+\frac{1}{(x-x_n)^2}\right)$$

$\Rightarrow\ h'(x) < 0 \ \Rightarrow\ f(x) \,.\, f''(x) - [f'(x)]^2 < 0$

Alternatively, a function f(x) satisfying the equation

$[f'(x)]^2 - f(x)\,.f''(x) = 0$ is

$f(x) = c\,.\, e^{c_1 x}$ which cannot have any root.

Problem 9: Consider the function,

$f(x) = x^3 - 9x^2 + 15x + 6$ for $1 \le x \le 6$ and

$$g(x) = \begin{cases} \min.\, f(t) & \text{for } 1 \le t \le x,\ 1 \le x \le 6 \\ x - 18 & \text{for } x > 6 \end{cases}$$

then prove that

(i) g(x) is differentiable at x = 1 (ii) g(x) is discontinuous at x = 6 (iii) g(x) is continuous and derivable at x = 5 (iv) g(x) is monotonic in (1, 5)

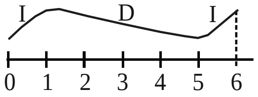
Solution: $f(x) = x^3 - 9x^2 + 15x + 6$

$f(t) = t^3 - 9t^2 + 15t + 6$

$f'(t) = 3t^2 - 18t + 15 = 3[t^2 - 6t + 5]$

$= 3(t-5)(t-1)$

Hence f is increasing in (5, 6) and f is decreasing in (1, 5)



$$\text{Now}\quad g(x) = \begin{cases} f(x) = x^3 - 9x^2 + 15x + 6 & 1 \le x < 5 \\ f(5) = -19 & 5 \le x \le 6 \\ x - 18 & x > 6 \end{cases}$$

$$\therefore\quad g(x) = \begin{cases} x^3 - 9x^2 + 15x + 6 & 1 \le x < 5 \\ -19 & 5 \le x \le 6 \\ x - 18 & x > 6 \end{cases}$$

Hence, g is continuous and differentiable at x = 1

g is continuous and differentiable at x = 5

g is neither continuous nor derivable at x = 6

g(x) is monotonic in (1, 5).

Problem 10: Given $f : [0, \infty) \to R$ be a strictly increasing function such that the functionsg (x) = f(x) – 3x and $h(x) = f(x) - x^3$ are both strictly increasing function. Then prove that the function $F(x) = f(x) - x^2 - x$ is increasing throughout (0, ∞).

Solution: $3F(x) = 3[f(x) - x^2 - x]$

$$= \underbrace{2[f(x) - 3x]}_{g(x)} + \underbrace{(f(x) - x^3)}_{h(x)} + x^3 - 3x^2 + 3x - 1 + 1$$

$3F(x) = 2g(x) + h(x) + (x-1)^3 + 1$

$\Rightarrow$ F(x) is increasing $\forall\ x \in [0, \infty]$.

***Alternative*:**

Given f(x) is increasing

$\Rightarrow\ f'(x) > 0$

$g(x) = f(x) - 3x$

$g'(x) = f'(x) - 3 > 0 \ \Rightarrow f'(x) > 3$

also $h(x) = f(x) - x^3$

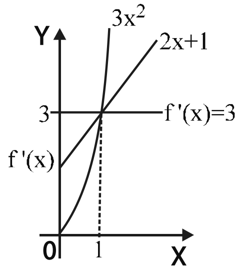
$h'(x) = f'(x) - 3x^2 > 0 \ \Rightarrow\ f'(x) > 3x^2$

To prove that $F(x) = f(x) - x^2 - x$ is increasing

i.e $\quad F'(x) = f'(x) - 2x - 1 > 0$

$F'(x) > 0$

$f'(x) > 2x + 1$



For in [0, 1), obviously $f'(x) > 3 > 2x + 1$ and in (1, 2), $3x^2 > 2x + 1$. Hence proved.

Problem 11: Let $f : (0, \infty) \to (0, \infty)$ be a derivable function and F(x) is the antiderivative of f(x) such that $2(F(x) - f(x)) = f^2(x)$ for any real positive x. Then prove that f is strictly increasing and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 1$.

Solution: Given $2(F(x) - f(x)) = f^2(x)$

and $\frac{dF}{dx} = f(x)$

$\therefore\quad F(x) = \frac{f^2(x)}{2} + f(x)$

$F'(x) = f(x) \cdot f'(x) + f'(x)$

$\therefore\quad f(x) = f'(x)(1 + f(x))$

$\therefore\quad f'(x) = \frac{f(x)}{1+f(x)} = 1 - \frac{1}{1+f(x)} > 0$ (as $f(x) > 0$)

Hence f is strictly increasing.

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty} f'(x) \quad \text{using L'Hospital's Rule}$$

$$= \lim_{x\to\infty}\left(1 - \frac{1}{1+f(x)}\right) \qquad \text{as } x \to \infty,\ f(x) \to \infty$$

$$\therefore\quad \lim_{x\to\infty}\frac{f(x)}{x} = 1.$$

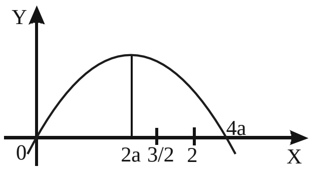
Problem 12: Find all possible values of 'a' for which $f(x) = \log_a(4ax - x^2)$ is strictly increasing for every $x \in \left[\frac{3}{2}, 2\right]$.

Solution:

Case I: If $0 < a < 1$ (obviously 'a' cannot be < 0)

then for f(x) to be increasing

$4ax - x^2$ should be decreasing in $\left(\frac{3}{2}, 2\right)$



$\Rightarrow \frac{3}{2} \geq 2a$ and $2 < 4a \quad \Rightarrow a \leq \frac{3}{4}$ and $a > \frac{1}{2}$

$\Rightarrow a \in \left(\frac{1}{2}, \frac{3}{4}\right]$

Case II: If $a > 1$ then for f (x) to be increasing

$4ax - x^2$ increasing in $\left(\frac{3}{2}, 2\right)$.

$\Rightarrow 2a \geq 2 \quad \Rightarrow a \geq 1$ but $a \neq 1$

$\Rightarrow a > 1$

Hence the final answer is $\left(\frac{1}{2}, \frac{3}{4}\right] \cup (1, \infty)$.

Problem 13: For $x \in (0, 1)$, prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$. Hence or otherwise find $\lim_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right]$.

Solution: Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$$f'(x) = 1 - x^2 - \frac{1}{1+x^2}$$

$$f'(x) = -\frac{x^4}{1+x^2}$$

$f'(x) < 0$ for $x \in (0, 1)$

$\Rightarrow$ f(x) is strictly decreasing

$\Rightarrow f(x) < f(0)$

$\Rightarrow x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow x - \frac{x^3}{3} < \tan^{-1}x$ ...(1)

Similarly, $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$$

$$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$$

$g'(x) > 0$ for $x \in (0, 1)$

$\Rightarrow$ g(x) is strictly increasing.

$\Rightarrow g(x) > g(0)$

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ...(2)

From (1) and (2), we get

$$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}.$$

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1} x}{x} < 1 - \frac{x^2}{6}$

Hence by Sandwich theorem we can prove that $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$ but it must also be noted that as $x \to 0$, the value of $\frac{\tan^{-1} x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1} x}{x} < 1$

$\Rightarrow \lim_{x \to 0} \left[\frac{\tan^{1} x}{x}\right] = 0.$

Problem 14: If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx}H(x) > 2cx\,H(x)$ for all $x \geq x_0$, where $c > 0$, then prove that H(x) cannot be zero for any $x > x_0$.

Solution: Given that $\frac{d}{dx}H(x) > 2\,cx\,H(x)$

$\Rightarrow \frac{d}{dx}H(x) - 2\,cx\,H(x) > 0$

$\Rightarrow \left\{\frac{d}{dx}H(x)\right\}e^{-cx^2} - 2cx\,e^{-cx^2}a.H(x) > 0$

$\Rightarrow \left\{\frac{d}{dx}H(x)\right\}e^{-cx^2} - H(x)\left\{\frac{d}{dx}e^{-cx^2}\right\} > 0$

$\Rightarrow \left\{\frac{d}{dx}H(x)\cdot e^{-cx^2}\right\} > 0$

$\therefore$ $H(x)\ e^{-cx^2}$ is an increasing function.

But, $H(x_0) = 0$ and e^{-cx^2} is always positive.

$\Rightarrow H(x_0) > 0$ for all $x > x_0$

$\Rightarrow$ H(x) cannot be zero for any $x > x_0$.

Problem 15: Prove that $\ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x^2}, x > 0$.

Hence, show that the function $f(x) = \left(1+\frac{1}{x}\right)^2$ strictly increases in $(0, \infty)$.

Solution: Consider the function

$g(x) = \ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x} \quad \forall\ x > 0.$

$$g'(x) = \frac{-1/x^2}{1+\frac{1}{x}} + \frac{1}{(1+x)^2} = \frac{-1}{x(1+x)} + \frac{1}{(1+x)^2}$$

$$= \frac{-1}{x(1+x)} < 0 \ \forall\ x > 0$$

$\Rightarrow$ g(x) strictly decreases in $(0, \infty)$

$\Rightarrow$ $g(x) > \lim\limits_{x \to \infty} g(x) = 0$

i.e. $\lambda n\left(1+\frac{1}{x}\right) > \frac{1}{x+1}$ ...(1)

which gives the desired result.

Now, we have

$$f(x) = \left(1+\frac{1}{x}\right)^x, \ x > 0 \text{ and}$$

$$f'(x) = \left(1+\frac{1}{x}\right)^x \ell n\left(1+\frac{1}{x}\right) + x\left(1+\frac{1}{x}\right)^{x-1}\left(\frac{-1}{x^2}\right)$$

$$= \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right] > 0 \ \forall\ x > 0$$

[using result (1)]

$\Rightarrow$ f(x) strictly increases in $(0, \infty)$.

Problem 16: Using calculus establish the inequality, $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$, where $x > 0$, $y > 0$ and $b > a > 0$.

Solution: $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$

$$\Rightarrow \left(\left(\frac{x}{y}\right)^b + 1\right)^{1/b} < \left(\left(\frac{x}{y}\right)^a + 1\right)^{1/a} \quad [\because y > 0\ ]$$

To prove that $(t^b + 1)^{a/b} < t^a + 1$ $\left[\ \frac{x}{y} = t > 0\ \right]$

Let $f(t) = (t^b + 1)^{a/b} - t^a - 1$

$$\Rightarrow f'(t) = \frac{a}{b}\left(t^b + 1\right)^{\frac{a}{b}-1} .\ b\,t^{b-1} - a\,t^{a-1}$$

$$\Rightarrow f'(t) = a\,t^{a-1}\left[t^{b-a}\left(t^b + 1\right)^{\frac{a}{b}-1} - 1\right]$$

$$= a\,t^{a-1}\left[\left(1 + \frac{1}{t^b}\right)^{\frac{a}{b}-1} - 1\right]$$

Now since $1 + \frac{1}{t^b} > 1$ and $\frac{a}{b} - 1 < 0$

therefore $\left(1 + \frac{1}{t^b}\right)^{\frac{a}{b}-1} < 1$.

Hence $f'(t) < 0$ i.e. f(t) is decreasing function

So $f(t) < f(0)$ but $f(0) = 0$

$\Rightarrow$ $(t^b + 1)^{a/b} < t^a + 1$.

Hence proved

Problem 17: Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is strictly increasing in its domain. Hence, draw the graph of f(x) and find its range.

Solution: $f(x) = \left(1+\frac{1}{x}\right)^x$

Domain of f(x) : $1 + \frac{1}{x} > 0$

$$\Rightarrow \frac{x+1}{x} > 0 \quad \Rightarrow (-\infty, -1) \cup (0, \infty)$$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}}\,\frac{-1}{x^2}\right]$

$$\Rightarrow f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of f′(x) depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So let us assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

$$g'(x) = \frac{1}{1+\frac{1}{x}}\ \frac{-1}{x^2} + \frac{1}{(x+1)^2}$$

$$\Rightarrow g'(x) = \frac{-1}{x(x+1)^2}$$

For $x \in (0, \infty)$, $g'(x) < 0$ ...(1)

$\Rightarrow$ g(x) is strictly decreasing for $x \in (0, \infty)$

$g(x) > \lim\limits_{x \to \infty} g(x)$

$g(x) > 0$.

and since $g(x) > 0 \Rightarrow f'(x) > 0$

For $x \in (-\infty, -1)$, $g'(x) > 0$ ...(3)

$\Rightarrow$ g(x) is strictly increasing for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim\limits_{x \to -\infty} g(x)$

$\Rightarrow$ $g(x) > 0 \Rightarrow f'(x) > 0$

Hence from (1) and (2) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

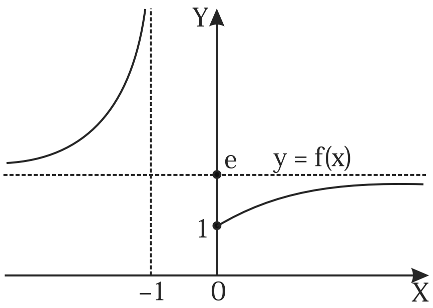
$\Rightarrow$ f(x) is strictly increasing in its domain

For drawing the graph of f(x), its important to find the value of f(x) at endpoints $\pm\infty$, 0, –1.

$$\lim_{x \to \pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0^+}\left(1+\frac{1}{x}\right)^x = 1 \quad \text{and} \quad \lim_{x \to -1}\left(1+\frac{1}{x}\right)^x = \infty$$

So the graph of f(x) is



From the graph, the range is $y \in (1, \infty) - \{e\}$.

Problem 18: Let R^+ be the set of positive real numbers. Find all functions $f : R^+ \to R^+$ such that for all $x, y \in R^+$, $f(x)\, f(yf(x)) = f(x + y)$.

Solution: First, if we assume that $f(x) > 1$ for some $x \in R^+$, setting $y = \dfrac{x}{f(x) - 1}$ gives the contradiction $f(x) = 1$.

Hence $f(x) \le 1$ for each $x \in R^+$, which implies that f is a decreasing function.

If $f(x) = 1$ for some $x \in R^+$, then $f(x + y) = f(y)$ for each $y \in R^+$, and by the monotonicity of f it follows that $f \equiv 1$.

Let now $f(x) < 1$ for each $x \in R^+$. Then f is strictly decreasing function, in particular injective. By the equalities

$$f(x)\, f(yf(x)) = f(x + y) = f(yf(x) + x + y(1 - f)))$$
$$= f(yf(x)))\, f((x + y(1 - f(x)))\, f(yf(x))))$$

We obtain that $x = (x + y(1 - f(x)))\, f(yf(x))$. Setting $x = 1$, $z = xf(1)$ and $a = \dfrac{1 - f(1)}{f(1)}$, we get $f(z) = \dfrac{1}{1 + az}$.

Combining the two cases, we conclude that $f(x) = \dfrac{1}{1 + ax}$ for each $x \in R^+$, where $a \ge 0$. Conversely, a direct verification shows that the function of this form satisfy the initial equality.

Alternative : As in the first solution we get that f is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x + y) - f(x)}{y} = f^2(x)\frac{f(yf(x)) - 1}{yf(x)}.$$

It follows that if f is differentiable at the point $x \in R^+$, then there exists the limit $\lim\limits_{z \to 0^+} \dfrac{f(z) - 1}{z} = -a$. Therefore $f'(x) = -af^2(x)$ for each $x \in R^+$, i.e. $\left(\dfrac{1}{f(x)}\right)' = a$, which means that $f(x) = \dfrac{1}{ax + b}$.

Substituting in the initial relation, we find that $b = 1$ and $a \ge 0$.

Problem 19: Does there exist a continuously differentiable function $f : R \to R$ such that for every $x \in R$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution: Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly increasing. By the monotonicity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus $f(0)$ is a lower bound for $f'(x)$, and for all $x < 0$ we have $f(x) < f(0) + x.\, f(0) = (1 + x)\, f(0)$. Hence, if $x \le -1$ then $f(x) \le 0$, contradicting the property $f(x) > 0$.
So, such a function does not exist.

Problem 20: Let $f : R \to R$ be a real function. Prove or disprove each of the following statements.

(a) If f is continuous and range (f) = R then f is monotonic.
(b) If f is monotonic and range (f) = R then f is continuous.
(c) If f is monotonic and f is continuous then range (f) = R.

Solution:

(a) False. Consider function $f(x) = x^3 - x$. It is continuous, range (f) = R but, for example, $f(0) = 0$, $f\left(\dfrac{1}{2}\right) = -\dfrac{3}{8}$ and $f(1) = 0$, therefore $f(0) > f\left(\dfrac{1}{2}\right)$, $f\left(\dfrac{1}{2}\right) < f(1)$ and f is not monotonic.

(b) True, Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim\limits_{x \to a^-} f$ and $\lim\limits_{x \to a^+} f$ exist and $\lim\limits_{x \to a^-} f \le \lim\limits_{x \to a^+} f$.
If the two limits are equal, the function is continuous at a.
Otherwise, if $\lim\limits_{x \to a^-} f = b < \lim\limits_{x \to a^+} f = c$, we have $f(x) \le b$ for all $x < a$ and $f(x) \ge c$ for all $x > a$;
therefore range $(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$ cannot be the complete R.
For non-increasing f the same can be applied writing reverse relations or $g(x) = -f(x)$.

(c) False. The function $g(x) = \tan^{-1}x$ is monotonic and continuous, but range $(g) = (-\pi/2, \pi/2) \ne R$.

Problem 21: Let f: $(a, b) \to R$, $\lim\limits_{x \to a^+} f(x) = \infty$, $\lim\limits_{x \to a^-} f(x) = -\infty$ and $f'(x) + f^2(x) \ge -1$ for $x \in (a, b)$. Prove that $b - a \ge \pi$ and give an example where $b - a = \pi$.

Solution: From the inequality we get

$$\frac{d}{dx}(\tan^{-1}x\, f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \ge 0$$

for $x \in (a, b)$. Thus $\tan^{-1}f(x) + x$ is non-decreasing in the interval and using the limits we get

$$\frac{\pi}{2} + a \le -\frac{\pi}{2} + b.$$

Hence $b - a \ge \pi$. One has equality for

$f(x) = \cot x$, $a = 0$, $b = \pi$.

Problem 22: Let $f \in c'[a, b]$, $f(a) = 0$ and suppose that $\lambda \in R$, $\lambda > 0$, is such that $|f'(x)| \le \lambda|f(x)|$ for all $x \in [a, b]$. Is it true that $f(x) = 0$ for all $x \in [a, b]$?

Solution: Assume that there is $y \in (a, b]$ such that $f(y) \ne 0$. Without loss of generality we have $f(y) > 0$.

In view of the continuity of f there exists $c \in [a, y)$ such that $f(c) = 0$ and $f(x) > 0$ for $x \in (c, y]$.

For $x \in (c, y]$ we have $|f'(x)| \le \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c, y]$ because of $g'(x) = \dfrac{f'(x)}{f(x)} - \lambda \le 0$.

Thus, $\ln f(x) - \lambda x \ge \ln f(y) - \lambda y$ and $f(x) \ge e^{\lambda x - \lambda y} f(y)$ for $x \in (c, y]$.

Thus, $0 = f(c) = f(c + 0) \ge e^{\lambda c - \lambda y} f(y) > 0$

a contradiction. Hence one has $f(x) = 0$ for $x \in [a, b]$.

Problem 23: Suppose that f(x) is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers or on at least the positive reals?

Solution: Let $u \ge v$. Suppose that $u + v \le 2$. Then, since $f(x) - 3x$ is increasing,

$$f(u) - 3u \ge f(v) - 3v,$$

$$\Rightarrow \quad f(u) - f(v) \ge 3(u - v) \ge (u + v + 1)(u - v)$$

$$= u^2 - v^2 + u - v$$

$$\Rightarrow \quad f(u) - u - u^2 \ge f(v) - v - v^2$$

Suppose that $u + v \ge 2$. Then, since $f(x) - x^3$ is increasing,

$$f(u) - u^3 \ge f(v) - v^3 \Rightarrow f(u) - f(v) \ge u^3 - v^3$$

$$= (u - v)(u^2 + uv + v^2).$$

Now $\qquad 2[(u^2 + uv + v^2) - (u + v + 1)]$

$$= (u + v)^2 + (u - 1)^2 + (v - 1)^2 - 4 \ge 0,$$

so that $u^2 + uv + v^2 \ge u + v + 1$ and

$$f(u) - f(v) \ge (u - v)(u + v + 1)$$

$$= u^2 - v^2 + u - v \Rightarrow f(u) - u - u^2 \ge f(v) - v - v^2.$$

Hence $f(u) - u - u^2 \ge f(v) - v - v^2$ whenever $u \ge v$, so that $f(x) - x - x^2$ is increasing.

Things to Remember

1. A function f(x) is said to be strictly increasing about the point $x = a$ if $f(a - h) < f(a) < f(a + h)$, where h is a small positive arbitrary number.
2. A function f(x) is said to be strictly decreasing about the point $x = a$ if $f(a - h) > f(a) > f(a + h)$, where h is a small positive arbitrary number.
3. A function f(x) is said to be non-decreasing about the point $x = a$ if $f(a - h) \le f(a) \le f(a + h)$, where h is a small positive arbitrary number.
4. A function f(x) is said to be non-decreasing about the point $x = a$ if $f(a - h) \ge f(a) \ge f(a + h)$, where h is a small positive arbitrary number.
5. Let a function f be differentiable at $x = a$.
   (i) If $f'(a) > 0$ then f(x) is strictly increasing at $x = a$.
   (ii) If $f'(a) < 0$ then f(x) is strictly decreasing at $x = a$.
   (iii) If $f'(a) = 0$ then we need to examine the signs of $f'(a - h)$ and $f'(a + h)$.
      (a) If $f'(a - h) > 0$ and $f'(a + h) > 0$ then f(x) is strictly increasing at $x = a$.
      (b) If $f'(a - h) < 0$ and $f'(a + h) < 0$ then f(x) is strictly decreasing at $x = a$.
      (c) If $f'(a - h)$ and $f'(a + h)$ have opposite signs then f(x) is neither increasing nor decreasing (non-monotonous) at $x = a$.
6. Assume that the function f is differentiable at $x = a$.
   (a) If $x = a$ is the left endpoint, we check as follows:
      (i) If $f'(a^+) > 0$, then f(x) is strictly increasing at $x = a$.
      (ii) If $f'(a^+) < 0$, then f(x) is strictly decreasing at $x = a$.
      (iii) If $f'(a^+) = 0$, but $f'(a + h) > 0$, then f(x) is strictly increasing at $x = a$.
      (iv) If $f'(a^+) = 0$, but $f'(a + h) < 0$, then f(x) is strictly decreasing at $x = a$.
   (b) If $x = a$ is the right endpoint, we check as follows:
      (i) If $f'(a^-) > 0$, then f(x) is strictly increasing at $x = a$.
      (ii) If $f'(a^-) < 0$, then f(x) is strictly decreasing at $x = a$.
      (iii) If $f'(a^-) = 0$, but $f'(a - h) > 0$, then f(x) is strictly increasing at $x = a$.
      (iv) If $f'(a^-) = 0$, but $f'(a - h) < 0$, then f(x) is strictly decreasing at $x = a$.
7. Necessary Conditions for Monotonicity
   (i) If a differentiable function f(x) increases in an interval its derivative $f'(x)$ is non-negative: $f'(x) \ge 0$.
   (ii) If a differentiable function f(x) decreases in an interval its derivative $f'(x)$ is non-positive: $f'(x) \le 0$.
   (iii) If a differentiable function f(x) does not vary in an interval (i.e. is equal to a constant) its derivative is identically equal to zero: $f'(x) = 0$.
8. Sufficient Conditions for Monotonicity
   Let f(x) be a differentiable function on the interval (a, b). Then:
   (i) If the derivative $f'(x)$ is everywhere positive (i.e. $f'(x) > 0$) in the interval (a, b), then the function f(x) is strictly increasing in the interval (a, b).

(ii) If the derivative $f'(x)$ is everywhere negative i.e. $f'(x) < 0$) in the interval (a, b), then the function f(x) is strictly decreasing in the interval (a, b).

(iii) If the derivative $f'(x)$ is everywhere equal to zero in the interval (a, b), then the function f(x) does not vary in the interval (a, b) (i.e. it is constant).

9. Monotonicity at points where $f'(x)$ does not exist Consider a continuous function f(x) whose derivative $f'(x)$ does not exist at x = c but exists in the neighbourhood of c.

(i) If $f'(c^-) > 0$ and $f'(c^+) > 0$, then f(x) is strictly increasing at x = c.

(ii) If $f'(c - h) > 0$, $f'(c^-) \geqq 0$, $f'(c^+) \geqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at x = c.

(iii) If $f'(c^-) < 0$ and $f'(c^+) < 0$, then f(x) is strictly decreasing at x = c.

(iv) If $f'(c - h) > 0$, $f'(c^-) \leqq 0$, $f'(c^+) \leqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at x = c.

10. A critical point of a function f is a number c in the domain of f such the either $f'(c) = 0$ or $f'(c)$ does not exist.

11. Steps for finding intervals of monotonicity

(i) Compute the derivative $f'(x)$ of a given function f(x), and then find the points at which $f'(x)$ equals zero or does not exist at all. These points are the critical points for the function f(x).

(ii) Using the critical points, separate the domain of definition of the function f(x) into several intervals on each of which the derivative $f'(x)$ retains its sign. These intervals will be the intervals of monotonicity.

(iii) Investigate the sign of $f'(x)$ on each of the found intervals. If on a certain interval $f'(x) > 0$, then the function f(x) increases on this interval, and if $f'(x) < 0$, then f(x) decreases on this interval.

12. Application of monotonicity in isolation of roots Suppose that

(i) f is continuous on [a, b] and differentiable on (a, b).

(ii) f(a) and f(b) have opposite signs,

(iii) $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b).

Then f has exactly one root between a and b.

13. (i) If f(x) is a strictly increasing function then its negative g(x) = – f(x) is a strictly decreasing function and vice-versa.

(ii) The reciprocal of a nonzero strictly increasing function is a strictly decreasing function and vice-versa.

(iii) If f and g are strictly increasing functions then h(x) = f(x) + g(x) is also a strictly increasing function.

(iv) If f and g are positive and both are strictly increasing then h(x) = f(x) × g(x) is also strictly increasing.

(v) If f is strictly increasing in [a, b] and g is strictly increasing in [f(a), f(b)], then gof is strictly increasing in [a, b].

(vi) If f is strictly decreasing in [a, b] and g is strictly decreasing in [f(b), f(a)], then gof is strictly increasing in [a, b].

(vii) If f is strictly increasing in [a, b] and g is strictly decreasing in [f(a), f(b)], then gof is strictly decreasing in [a, b].

(viii) If f is strictly decreasing in [a, b] and g is strictly increasing in [f(b), f(a)], then gof is strictly decreasing in [a, b].

14. If the second derivative $f''(x)$ is everywhere positive within an interval the arc of the curve y = f(x) corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve y = f(x) is concave down.

15. A hyper-critical point of a function f is a number c in the domain of f such the either $f''(c) = 0$ or $f''(c)$ does not exist.

16. Suppose the graph of a function f has a tangent line (possibly vertical) at the point P(c, f(c)) and that the graph is concave up on one side of P and concave down on the other side. Then P is called an inflection point of the graph.

17. If x = c is a hyper-critical point and the inequalities $f''(c - h) < 0$, $f''(c + h) > 0$ (or inequalities $f''(c - h) > 0$, $f''(c + h) < 0$) hold for an arbitrary sufficiently small h > 0, then the point of the curve y = f(x) with the abscissa x = c is a point of inflection.

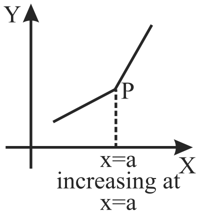
If $f''(c - h)$ and $f''(c + h)$ are of the same sign, then the point x = c is not a point of inflection.

Objective Exercises

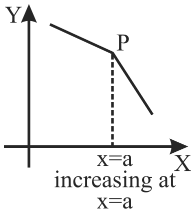
SINGLE CORRECT ANSWER TYPE

1. Which of the following conclusions does not hold true?

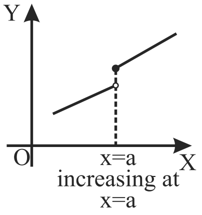
(A)



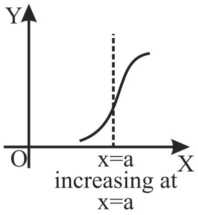
(B)



(C)



(D)



2. If $f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4$ is increasing for all $x \geq 0$, then a is an element of

(A) $[-2, 0)$
(B) $(-\infty, -2]$
(C) $[2, \infty)$
(D) $(-\infty, 2]$

3. Let $f(x) = \sin^{-1}\left(\frac{2\varphi(x)}{1+\varphi^2(x)}\right)$, where $\phi(x)$ is a decreasing function of x, then

(A) f(x) increasing when $|\phi(x)| < 1$
(B) f(x) is decreasing when $|\phi(x)| < 1$
(C) f(x) is decreasing always
(D) f(x) is increasing always

4. Let $f(x) = \sin^2 x - (2a + 1)\sin x + (a - 3)$. If $f(x) \leq 0$ for all $x \in \left[0, \frac{\pi}{2}\right]$, then range of values of a is

(A) $[-3, 0]$
(B) $[3, \infty)$
(C) $[-3, 3]$
(D) $(-\infty, 3]$

5. Let f be a function such that f(x) and f′(x) have opposite signs for all $x \in R$. Then

(A) f(x) is an increasing function
(B) f(x) is a decreasing function
(C) | f(x)| is an decreasing function
(D) |f(x)| is an increasing function

6. If $f(x) = \sin\left(\sin^{-1}\left(\log_{\frac{1}{2}} x\right)\right)$, then

(A) Domain of f(x) is $\left(0, \frac{1}{2}\right]$
(B) Range of f(x) is $[-1,1]$
(C) f(x) is decreasing in its domain
(D) f(x) is increasing in its domain

7. The function $f : R \to R$ is such that
$f(x) = a_1 x + a_3 x^3 + a_5 x^5 + \ldots\ldots$
$\ldots\ldots + a_{2n+1} x^{2n+1} - \cot^{-1} x$
where $0 < a_1 < a_2 < \ldots\ldots a_{2n+1}$ then the function f(x) is

(A) one-one into
(B) many one into
(C) one-one onto
(D) many one onto

8. If f(x) is a differentiable real valued function satisfying $f''(x) - 3f'(x) > 3 \ \forall \ x \geq 0$ and $f'(0) = -1$, then $f(x) + x \ \forall \ x > 0$ is

(A) decreasing function of x
(B) increasing function of x
(C) constant function
(D) none of these

9. Let $f(x) = ax + \sin 2x + b$, then $f(x) = 0$ has

(A) exactly one positive real root, if $a > 2, b < 0$
(B) exactly one positive real root, if $a > 2, b > 0$
(C) infinite number of positive real root, if $a < -2$
(D) None of these

10. $f : R \to R$ be a differentiable function $\forall \ x \in R$. If a tangent drawn to the curve at any point $x \in (a, b)$ always lies below the curve, then

(A) $f'(x) < 0$ and $f''(x) > 0 \ \forall \ x \in (a, b)$
(B) $f'(x) > 0$ and $f''(x) < 0 \ \forall \ x \in (a, b)$
(C) $f'(x)$ can have any value and $f''(x) > 0 \ \forall \ x \in (a, b)$
(D) none of these

11. The set of value of a, for which the function $f(x) = (4a - 3)(x + 5) + 2(a - 7)\cot(x/2)\sin^2(x/2)$ does not possess any critical point is given by

(A) $(-\infty, -4/3)$
(B) $(-\infty, -1)$
(C) $(-4/3, 2)$
(D) $(-\infty, -4/3) \cup (2, \infty)$

12. If $\tan(\pi \cos\theta) = \cot(\pi \sin\theta)$ $0 < \theta < \frac{\pi}{2}$ and $f(x) = (\cos\theta + \sin\theta)^x$, then f is

(A) increasing for all $x \in R$
(B) decreasing for all $x \in R$
(C) increasing in $(0, \infty)$
(D) decreasing in $(0, \infty)$

13. If $f(x) = x^n \sin 1/x + x^m \cos\frac{1}{2x}$, then

(A) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 2/\pi]$
(B) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 1/3\pi]$
(C) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 1/2\pi]$
(D) None of these

14. Which one is correct ?

(A) $(1999)^{2000} > (2000)^{1999}$
(B) $(1998)^{1999} < (1999)^{1998}$
(C) $(100)^{101} < (101)^{100}$
(D) $26^{25} > 25^{26}$

15. Let $f''(x) > 0 \ \forall \ x \in R$ and $g(x) = f(2 - x) + f(4 + x)$. Then g(x) is increasing in

(A) $(-\infty, -1)$
(B) $(-\infty, 0)$
(C) $(-1, \infty)$
(D) None of these

16. Let f be differentiable on [a, b] when f(A) = f(B) = 0 and f′(C) = 0, a < c < b. Then the incorrect statement is

(A) If $g(x) = k\, f(x)$ then $g'(C) = 0$
(B) If $g(x) = f(x - k)$ then $g'(c + k) = 0$
(C) If $g(x) = f(kx)$ then $g'(c/k) = 0$
(D) none of these

17. Let f (x) and g (x) are two function which are defined and differentiable for all $x \geq x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$ then

(A) $f(x) < g(x)$ for some $x > x_0$

(B) f (x) = g (x) for some $x > x_0$
(C) f (x) > g (x) only for some $x > x_0$
(D) f (x) > g (x) for all $x > x_0$

18. The number of zeros of the cubic
$f(x) = x^3 + 2x + k \;\; \forall\, k \in R$, is
(A) 0 (B) 1
(C) 2 (D) 3

19. Let f, g and h are differentiable function such that $g(x) = f(x) - x$ and $h(x) = f(x) - x^3$ are both strictly increasing functions, then the function

$$F(x) = f(x) - \frac{\sqrt{3}\,x^2}{2} \text{ is}$$

(A) strictly increasing $\forall\, x \in R$
(B) strictly decreasing $\forall\, x \in R$
(C) strictly decreasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$
(D) strictly increasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly decreasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

20. Let $f : R \to R$ be a real function. Then
(A) If f is continuous and range (f) = R then f is monotonic
(B) If f is monotonic and range (f) = R then f is continuous
(C) If f is monotonic and continuous then range f(x) = R
(D) None of these

21. The number of inflection points on the curve represented parametrically by the equations $x = t^2$, $y = 3t + t^3$ is
(A) 0 (B) 1
(C) 2 (D) 3

22. If $f : R \to R$ is the function defined by

$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}, \text{ then}$$

(A) f(x) is an increasing function
(B) f(x) is a decreasing function
(C) f(x) is onto
(D) None of these

23. Let f(x) be an increasing function and

$$f(x) \le -1 \;\; \forall\;\; x \in R \text{ then } g(x) = \frac{f(x)}{1+x^2} \text{ is}$$

(A) increasing for all x
(B) decreasing for x > 0
(C) increasing for x > 0
(D) None of these

24. If $f'(x^2 - 4x + 3) > 0 \;\; \forall \;\; x \in (2, 3)$, then f(sin x) is increasing on
(A) $(n\pi, n\pi/2), n \in I$
(B) $((2n + 1)\pi, (4n + 3)\,\pi/2), n \in I$
(C) $((4n - 1)\,\pi/2, 2n\pi), n \in I$
(D) None of these

25. A function f(x) is given by $x^2 f'(x) + 2x\,f(x) - x + 1 = 0$ $(x \ne 0)$. If f(1) = 0 then f(x) is
(A) increasing in $(-\infty, 0), (1, \infty)$ and decreasing in (0, 1)
(B) increasing in (0, 1) and decreasing in $(-\infty, 0),(1, \infty)$
(C) increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$
(D) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$

26. Let $f(x) = x\sqrt{4ax - x^2}$, (a > 0). Then f(x) is
(A) increasing in (0, 3a), decreasing in $(-\infty, 0)$ and (3a, 4a)
(B) increasing in (a, 4a), decreasing in $(4a, \infty)$
(C) increasing in (0, 4a), decreasing in $(-\infty, 0)$
(D) none of these

27. If $f'(x) = |x| - \{x\}$, where {.} denotes the fractional part of x, then f(x) is decreasing in
(A) $\left(-\frac{1}{2}, 0\right)$ (B) $\left(-\frac{1}{2}, 2\right)$
(C) $\left(-\frac{1}{2}, 2\right)$ (D) $\left(\frac{1}{2}, \infty\right)$

28. Which of the following statements is true for the function

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 1 \\ x^3, & 0 \le x \le 1 \\ \dfrac{x^3}{3} - 4x, & x < 0 \end{cases}$$

(A) f is strictly increasing $\forall\, x \in R$.
(B) $f'(x)$ fails to exist at 3 distinct values of x
(C) $f'(x)$ changes its sign twice as x varies from $-\infty$ to ∞.
(D) f attains its extreme values at x_1 and x_2, where $x_1 x_2 > 0$

29. If $f''(x) > 0$, $\forall\, x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \frac{\pi}{2}$, then g(x) is increasing in
(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$
(C) $\left(0, \frac{\pi}{3}\right)$ (D) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

30. Let f be a function such that $f'(x) = \log_{1/3}(\log_3(\sin x + a))$. If f is decreasing for all real values of x, then

(A) $a \in (1,4)$ (B) $a \in (4, \infty)$

(C) $a \in (2,3)$ (D) $a \in (2,\infty)$

31. If $f(x) = x + \sin x$, $g(x) = e^{-x}$, $u = \sqrt{c+1}-\sqrt{c}$, $v = \sqrt{c}-\sqrt{c-1}$, $(c > 1)$ then

(A) $fog(u) \geq fog(v)$ (B) $gof(u) \leq gof(v)$

(C) $gof(u) > gof(v)$ (D) $fog(u) < fog(v)$

32. The length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is

(A) $\pi/2$ (B) $\pi/4$

(C) $\pi/8$ (D) $\pi/16$

33. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is

(A) one (B) two

(C) three (D) zero

34. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a strictly decreasing function of x in the largest possible interval $[-2, -2/3]$ then

(A) $\lambda = 4$ (B) $\lambda = 2$

(C) $\lambda = -1$ (D) λ has no real value

35. Let $g(x) = \frac{1}{4}f(2x^2 - 1) + \frac{1}{2}f(1 - x^2)$ where $f'(x)$ is an increasing function, then g(x) is increasing in the interval

(A) $(-1, 1)$

(B) $\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$

(C) $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$

(D) None of these

36. If $f(x) = (ab - b^2 - 2)x + \int_0^x(\cos^4\theta+\sin^4\theta)d\theta$ is a decreasing function of x for all $x \in R$ and $b \in R$, then

(A) $a \in (0, \sqrt{6})$ (B) $a \in (-\sqrt{6}, \sqrt{6})$

(C) $a \in (-\sqrt{6}, \sqrt{6})$ (D) None of these

37. Let f(x) be an increasing function and

$$g(\theta)=\int_0^{\sin^2\theta} f(x)dx+\int_0^{\cos^2\theta} f(x)dx,$$

$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

then $g(\theta)$ is increasing in the interval

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$

(C) $\left(0, \frac{\pi}{4}\right)$ (D) $\left(-\frac{\pi}{4}, 0\right)$

38. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is

(A) 3 (B) 2

(C) 1 (D) 0

39. If $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers and $3b^2 < c^2$, is an increasing cubic function and $g(x) = af'(x) + bf''(x) + c^2$, then

(A) $\int_a^x g(t)\,dt$ is a decreasing function

(B) $\int_a^x g(t)\,dt$ is an increasing function

(C) $\int_a^x g(t)\,dt$ is neither increasing nor decreasing function

(D) None of the above

40. If the function $y = \sin(f(x))$ is monotonic for all values of x (where f(x) is continuous), then the maximum value of the difference between the maximum and the minimum values of f(x), is

(A) π (B) 2π

(C) $\pi/2$ (D) None

41. The interval in which $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, $x \in (0, \pi)$, decreases or increases are

(A) Decreases on $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ and increases on $\left(0,\frac{\pi}{2}\right),\left(\frac{2\pi}{3},\pi\right)$

(B) Decreases on $\left(\frac{\pi}{2},\pi\right)$ and increases on $\left(0,\frac{\pi}{2}\right)$

(C) Decreases on $\left(0,\frac{\pi}{2}\right),\left(\frac{2\pi}{3},\pi\right)$ and increases on $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$

(D) Decreases on $\left(0,\frac{\pi}{2}\right)$ and increases on $\left(\frac{\pi}{2},\pi\right)$

42. Let f(x) be a differentiable function such that, $f'(x) = \frac{1}{\log_3(\log_{1/4}(\cos x+a))}$. If f(x) is increasing for all values of x then

(A) $a \in (5, \infty)$ (B) $a \in \left(1,\frac{5}{4}\right)$

(C) $a \in \left(\frac{5}{4}, 5\right)$ (D) None of these

43. The intervals of monotonicity of the function of the function $f(x) = x^2 - \ln|x|$, when $(x \neq 0)$ is/are

(A) Increasing for all $x > 0$ and decreasing for all $x < 0$

(B) Increasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}\right)$ and decreasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$

(C) Increasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}\right)$

(D) Increasing when $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \{0\}$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$

44. If a function f(x) is such that $f(2) = 3$, $f'(2) = 4$, then $\lim\limits_{x \to 2}[f(x)]$, (where [.] is G.I.F.), is

(A) 2
(B) 3
(C) 4
(D) doesn't exist

45. Let $f : R \to R$ be a differentiable function such that $f(f(x)) = a(x^5 + x)$, $(a \neq 0)$, then

(A) f(x) is strictly increasing
(B) f(x) is strictly decreasing
(C) f(x) is either strictly increasing or decreasing
(D) f(x) is non-monotonic.

46. Let $f^1(x)$ denotes g(x), $f^2(x)$ denotes g(g(x)), $f^3(x)$ denotes g(g(g(x))) and so on. If g(x) is an increasing function and lies between $(0, \infty)$, $\forall$ $x \in R$, then $\lim\limits_{n \to \infty}\left(\frac{1}{f^n(x)}\right)$ is equal to

(A) 0
(B) 1
(C) does not exist
(D) none of these

47. f(x) is an increasing, concave up function for all $x \in [a, b]$, then for any $\lambda > 0$, (f is invertible)

(A) $\left(\frac{b + a\lambda}{1+\lambda}\right) > f^{-1}\left(\frac{f(b) + \lambda f(a)}{1+\lambda}\right)$

(B) $\left(\frac{b + a\lambda}{1+\lambda}\right) \leq f^{-1}\left(\frac{f(b) + \lambda f(a)}{1+\lambda}\right)$

(C) $\left(\frac{b + a\lambda}{1+\lambda}\right) < f^{-1}\left(\frac{f(b) + \lambda f(a)}{1+\lambda}\right)$

(D) $\left(\frac{b + a\lambda}{1+\lambda}\right) \leq f^{-1}\left(\frac{f(b) + \lambda f(a)}{1+\lambda}\right)$

48. If $g(x) = 2f(2x^3 - 3x^2) + f(6x^2 - 4x^3 - 3)$, $\forall$ $x \in R$, and $f''(x) > 0$, $\forall$ $x \in R$ then g(x) is increasing in the interval

(A) $\left(-\infty, -\frac{1}{2}\right), (0, 1)$
(B) $\left(-\frac{1}{2}, 0\right), (1, \infty)$
(C) $(0, \infty)$
(D) none of these

Multiple Correct Answer Type for JEE Advanced

49. Which of the following is/are true

(A) $(\ell n\,2.1)^{\ell n\,2.2} > (\ell n\,2.2)^{\ell n\,2.1}$

(B) $(\ell n\,4)^{\ell n\,5} < (\ell n\,5)^{\ell n\,4}$

(C) $(\ell n\,30)^{\ell n\,31} > (\ell n\,31)^{\ell n\,30}$

(D) $(\ell n\,28)^{\ell n\,30} < (\ell n\,30)^{\ell n\,28}$

50. If x = c is a critical point of y = f(x) and y = g(x) then x = c is also a critical point of

(A) f . g
(B) f . g,
(C) f + g
(D) none

51. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall$ $x \in R$, then

(A) $f(x + 1) > g(f(x - 1)$
(B) $f(g(x - 1)) > f(g(x + 1))$
(C) $g(f(x + 1)) < g(f(x - 1))$
(D) $g(g(x + 1)) < g(g(x - 1))$

52. Consider the function $f : R \to R$ defined as $f(x) = x + \sin x$. Which of the following is/are the correct statement(s)?

(A) The function is strictly increasing at every point on R except at 'x' equal to an odd integral multiple of π where the derivative of f (x) is zero and where the function f is not strictly increasing.

(B) The function is bounded in every bounded interval but unbounded on whole real line.

(C) The graph of the function $y = f(x)$ lies in the first and third quadrants only.

(D) The graph of the function $y = f(x)$ cuts the line $y = x$ at infinitely many points.

53. For the function $f(x) = (x^2 + bx + c)e^x$, which of the following holds?

(A) if $f(x) > 0$ for all real x $\not\Rightarrow f'(x) > 0$
(B) if $f(x) > 0$ for all real x $\Rightarrow f'(x) > 0$
(C) if $f'(x) > 0$ for all real x $\Rightarrow f(x) > 0$
(D) if $f'(x) > 0$ for all real x $\not\Rightarrow f(x) > 0$

54. The function $f(x) = (x + 2)^{1/3}$ at $x = -2$

(A) is monotonic
(B) is differentiable
(C) is such that no tangent can be drawn at this point.
(D) changes its concavity.

55. Let a function $f : R \to R$ be such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$. Then

(A) f is a continuous function

(B) f is monotonous function

(C) there are only two functions $f(x) = \pm x + c$, c is a constant

(D) None of these

56. Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are real and $3b^2 < c^2$ is an increasing function and $g(x) = af'(x) + bf''(x) + c^2$. If $G(x) = \int_{\alpha}^{x} g(t)\,dt$, $a \in R$, then

(A) G(x) is a decreasing function

(B) G(x) is an increasing function

(C) g(x) is neither increasing nor decreasing

(D) G(x) is one-one function

57. Consider a real valued continuous function $f(x)$ defined on the interval [a, b]. Which of the following statements does not hold(s) good?

(A) If $f(x) \geq 0$ on [a, b] then $\int_a^b f(x)\,dx \leq \int_a^b f^2(x)\,dx$.

(B) If f(x) is increasing on [a, b], then $f^2(x)$ is increasing on [a, b].

(C) If $f(x)$ is increasing on [a, b], then $f(x) \geq 0$ on (a, b).

(D) If $f(x)$ attains a minimum at $x = c$ where $a < c < b$, then $f'(C) = 0$.

58. If $p'(x) > p(x)$ for all $x \geq 1$ and $p(1) = 0$ then

(A) $e^{-x}\,p(x)$ is an increasing function

(B) $p(x) \,.\, e^x$ is a decreasing function

(C) $p(x) > 0$ for all x in $[1, \infty)$

(D) $p(x) < 0$ for all x in $[1, \infty)$

59. Identify the correct statements:

(A) If $y = x + c$, then $dy = dx$.

(B) If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

(C) If y is differentiable, then $\lim\limits_{\Delta x \to 0} (\Delta y - dy) = 0$.

(D) If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.

60. If $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)$ $\forall$ $x \in (-3, 4)$, where $f''(x) > 0$ $\forall$ $x \in (-3, 4)$, then h(x) is

(A) increasing in $\left(\frac{3}{2}, 4\right)$

(B) increasing in $\left(-\frac{3}{2}, 0\right)$

(C) decreasing in $\left(-3, -\frac{3}{2}\right)$

(D) decreasing in $\left(0, \frac{3}{2}\right)$

61. If $f(x) = x^3 - x^2 + 100x + 2002$, then

(A) $f(1000) > f(1001)$

(B) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

(C) $f(x - 1) > f(x - 2)$

(D) $f(2x - 3) > f(2x)$

62. If $f'(x) = g(x)(x - a)^2$ where $g(a) \neq 0$ and g is continuous at $x = a$, then

(A) f is increasing in the neighbourhood of a if $g(a) > 0$

(B) f is increasing in the neighbourhood of a if $g(a) < 0$

(C) f is decreasing in the neighbourhood of a if $g(a) > 0$

(D) f is decreasing in the neighbourhood of a if $g(a) < 0$

63. If composite function $f_1(f_2(f_3(...(f_n(x)))))$ is an increasing function and if r of f_i s are decreasing function while rest are increasing, then the maximum value of $r(n - r)$ is

(A) $\frac{n^2 - 1}{4}$, when n is an even number

(B) $\frac{n^2}{4}$, when n is an odd number

(C) $\frac{n^2 - 1}{4}$, when n is an odd number

(D) $\frac{n^2}{4}$, when n is even number

64. Which of the functions have exactly one zero in the given interval

(A) $f(x) = x^3 + \frac{4}{x^2} + 7$, $(-\infty, 0)$

(B) $g(t) = \sqrt{t} + \sqrt{1+t} - 4$, $(0, \infty)$

(C) $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8$, $(-\infty, \infty)$

(D) $r(\theta) = \tan\theta - \cot\theta - \theta$, $(0, \pi/2)$

65. If f(x) and g(x) are two positive and increasing functions, then

(A) $(f(x))^{g(x)}$ is always increasing

(B) If $(f(x))^{g(x)}$ is decreasing when $f(x) < 1$,

(C) If $(f(x))^{g(x)}$ is increasing when $f(x) > 1$

(D) If $f(x) > 1$, then $(f(x))^{g(x)}$ is increasing

66. If $\phi(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)$ $\forall$ $x \in (-3, 4)$ where $f''(x) > 0$ $\forall$ $x \in (-3, 4)$, the $\phi(x)$ is

(A) increasing in $\left(\frac{3}{2}, 4\right)$

(B) decreasing in $\left(-3, -\frac{3}{2}\right)$

(C) increasing in $\left(-\frac{3}{2}, 0\right)$

(D) decreasing in $\left(0, \frac{3}{2}\right)$

67. Let f(x) be an increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$, then the possible integral values of a is/are

(A) 1 (B) 2
(C) 3 (D) 4

68. Let f and g be functions from the interval $[0, \infty)$ to the interval $[0, \infty)$, f being an increasing function and g being a decreasing function, then

(A) $f\{g(x)\} \geq f\{g(0)\}$ (B) $g\{f(x)\} \leq g\{f(0)\}$
(C) $f\{g(2)\} \leq f(g(0))$ (D) None of these

69. If f(x) and g(x) are positive continuous function such that f(x) is an increasing function, g(x) is a monotonic function and it is given that

$$A = \int_0^{1/2} f(x).g(x)dx > \int_0^{1/2} f(x).g(1-x)dx, \text{ then}$$

(A) $f(x) < f(1-x), \forall x \in \left(0, \frac{1}{2}\right)$

(B) $g(x) < g(1-x), \forall x \in \left(0, \frac{1}{2}\right)$

(C) $f(x) > f(1-x), \forall x \in \left(\frac{1}{2}, 1\right)$

(D) $g(x) > g(1-x), \forall x \in \left(0, \frac{1}{2}\right)$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

70. Assertion (A): Both f(x) = 2cos x + 3sin x and $g(x) = \sin^{-1}\frac{x}{\sqrt{13}} - \tan^{-1}\frac{3}{2}$ are increasing, for $x \in (0, \pi/2)$.
Reason (R) : If f(x) is increasing then its inverse is also increasing.

71. Assertion (A): The function f(x) is $x^4 - 8x^3 + 22x^2 - 24x + 21$ is decreasing for $x \in (2, 3)$ and $(-\infty, 1)$.
Reason (R): f(x) is increasing for $x \in (1, 2)$ and $(3, \infty)$ and has no point of inflection.

72. Assertion (A): If f(0) = 0, $f'(x) = \ln(x + \sqrt{1+x^2})$, then f(x) is positive for all $x \in R^+$.
Reason (R): f(x) is increasing for x > 0 and decreasing for x < 0.

73. Assertion (A): The function $f(x) = \frac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is increasing function of x, then bc > ad.
Reason (R): $f'(x) > 0$ for all x.

74. Assertion (A): Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then, f is one-one.
Reason (R): f(x) is one-one if and only if f(x) is strictly monotonous.

75. Assertion (A): For $0 < x < \frac{\pi}{2}$,

cos x sin (tan x) < sin (sin x)

Reason (R): $\frac{\tan x}{x}$ is increasing function in $\left(0, \frac{\pi}{2}\right)$.

76. Assertion (A): If $0 < x < \tan^{-1}\frac{\pi}{2}$, then

$$\sqrt[3]{\cos(\tan x).\cos^2 \sin x} < \frac{1}{3}[\cos(\tan x) + 2\cos\sin x] \leq \cos\left[\frac{\tan x + 2\sin x}{3}\right] < \cos x.$$

Reason (R): We know that AM ≥ GM and if $x \in \left(0, \frac{\pi}{2}\right)$, tan x + 2 sin x > 3x and cos x is a decreasing function.

77. Assertion (A): If $x \in \left[\tan^{-1}\frac{\pi}{2}, \frac{\pi}{2}\right)$ then tan (sin x) > sin (tan x).

Reason (R): $\tan\left(\sin\left(\tan^{-1}\frac{\pi}{2}\right)\right)$

$= \tan\frac{\frac{\pi}{2}}{\sqrt{1+\frac{\pi^2}{4}}} > \tan\frac{\pi}{4}$ while sin (tan x) ≤ 1 and tan (sin x) is increasing in the given interval.

78. Assertion (A): Let a strictly decreasing function $f : R^+ \to R^+$ satisfy f(f(x)) = x for all positive x. Then $\lim_{x\to\infty} f(x) = 0$.
Reason (R): Since f is strictly decreasing and bounded below by 0, we see that $\lim_{x\to\infty} f(x)$ exists and is some non-negative number.

79. Assertion (A): $(3.14)^\pi > \pi^{3.14}$

Reason (R): Let $f(x) = \frac{\ell nx}{x}$, f(x) is decreasing for x > e.
Since $e < 3.14 < \pi$, f(3.14) > f(π).

Comprehension – 1

Consider f, g and h be three real valued function defined on R. Let $f(x) = \sin 3x + \cos x$, $g(x) = \cos 3x + \sin x$ and $h(x) = f^2(x) + g^2(x)$

80. The length of a longest interval in which the function $y = h(x)$ is increasing, is

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

81. The general solution of the equation $h(x) = 4$, is

(A) $(4n + 1)\frac{\pi}{8}$ (B) $(8n + 1)\frac{\pi}{8}$

(C) $(2n + 1)\frac{\pi}{4}$ (D) $(7n + 1)\frac{\pi}{4}$

where $n \in I$

82. The number of point(s) where the graphs of the two function, $y = f(x)$ and $y = g(x)$ intersects in $[0, \pi]$, is

(A) 2 (B) 3

(C) 4 (D) 5

Comprehension – 2

A cylinder of base radius 1 and height x is cut into two equal parts along a plane passing through the centre of the cylinder and tangent to the two base circles. Let f(x) be the ratio of surface area of each piece to the volume of the piece.

83. The value of f(2) is

(A) $2 + 3\sqrt{2}$ (B) $3 + \sqrt{2}$

(C) $3 + 2\sqrt{2}$ (D) None of these

84. The complete interval on which the function f(x) is strictly decreasing, is

(A) $(0, \infty)$ (B) $(2, 4)$

(C) $(1, \infty)$ (D) None

85. The value of $\lim\limits_{x\to\infty} f(x)$ is

(A) 2 (B) 3

(C) $\frac{2\pi}{3}$ (D) None

Comprehension – 3

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where $a, b \in R$.

86. For a = 1 if y = f(x) is strictly increasing $\forall\, x \in R$ then the largest range of values of b is

(A) $\left(-\infty, \frac{1}{3}\right]$ (B) $\left(\frac{1}{3}, \infty\right)$

(C) $\left[\frac{1}{3}, \infty\right)$ (D) $(-\infty, \infty)$

87. For b = 1, if y = f(x) is non-monotonic then the sum of all the integral values of $a \in [1, 100]$, is

(A) 4950 (B) 5049

(C) 5050 (D) 5047

88. If the sum of the base 2 logarithms of the roots of the cubic f(x) = 0 is 5 then the value of 'a' is

(A) –64 (B) –8

(C) –128 (D) –256

Comprehension – 4

Let A = {1, 2, 3, 4, 5} and B = {–2, – 1, 0, 1, 2, 3, 4, 5}. The number of

89. Increasing function from A to B is

(A) 120 (B) 72

(C) 60 (D) 56

90. Non-decreasing functions from A to B is

(A) 216 (B) 540

(C) 792 (D) 840

91. Onto functions from A to A such that $f(i) \neq i$ for all i, is

(A) 44 (B) 120

(C) 56 (D) 76

Comprehension – 5

If $\phi(x)$ is a differentiable function satisfying $\phi'(x) + 2\phi(x) \le 1$, then it can be adjusted as $e^{2x}\phi'(x) + 2e^{2x}\phi(x) \le e^{2x}$ or $\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{e^{2x}}{2}\right) \le 0$.

Here e^{2x} is called a integrating factor which helps in creating a function whose differential coefficient is given.

92. If $\phi'(x) + 2\phi(x) \le 1$ for all x then the function $f(x) = e^{2x}(2\phi(x) - 1)$

(A) is a decreasing function

(B) is a increasing function

(C) is a positive function

(D) is a negative function

93. If P(1) = 0 and $\frac{dP(x)}{dx} > P(x)$ for all $x \ge 1$ then

(A) $P(x) > 0 \;\; \forall\, x > 1$

(B) P(x) is a constant function

(C) $P(x) < 0 \;\forall\, x > 1$

(D) None of these

94. If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx}H(x) > 2cxH(x)$ for all $x \ge x_0$, where $c > 0$, then

(A) $H(x) = 0$ has root for $x > x_0$

(B) $H(x) = 0$ has no roots for $x > x_0$

(C) H(x) is a constant function

(D) None of these

MATCH THE COLUMNS FOR JEE ADVANCED

95.

	Column-I	**Column-II**
(A)	Let y = f(x) be given by $x = \frac{1}{1+t^2}$, $y = \frac{1}{(1+t)^2}$, t > 0. If f is increasing in (0, a), then the greatest value of a is	(P) 3
(B)	Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, if $A - \lambda I$ is a singular matrix, then $\lambda^2 - 3\lambda - 2$ is equal to	(Q) $\frac{2}{3}$
(C)	The number of solutions of the equation $\lvert x-1 \rvert^{\log_3 x^2 - \log_x 9} = (x-1)^7$, is	(R) 1
(D)	A ladder of length 5m leaning against a wall is being pulled along the ground at 2cm/s. When the foot of the ladder is 4m away from the wall, if the top of the ladder slides down on the wall at $\frac{8}{\lambda}$ cm/s, then λ is equal to	(S) 2

96.

	Column-I	**Column-II**
(A)	When 5^{200} is divided by 8 then remainder is	(P) –1
(B)	The value of $\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin(\tan^{-1}(\tan x))}{\tan x + \cos^2(\tan x)}}$ is	(Q) 2
(C)	If $f'(1) = -2\sqrt{2}$ and $g'(\sqrt{2}) = 4$, then the derivative of f(tan x) with respect to g(sec x) at $x = \frac{\pi}{4}$ is,	(R) 1
(D)	The length of the longest interval, in which the function $f(x) = 3\sin x - 4\sin^3 x$ is increasing is $\frac{a\pi}{6}$, then value of a is	(S) does not exist

97.

	Column-I	**Column-II**
(A)	Let $f''(x) > 0 \ \forall \ x \in R$ and g(x) = f(4 – x) + f(2 + x) then g(x) increases if x belongs to the interval	(P) $(-\infty, -2)$
(B)	The equation $x^3 - 3x + a = 0$ will have exactly one real root if a belongs to the interval	Q) $(-\infty, -1]$
(C)	If $f(x) = \cos x + a^2x + b$ is an increasing function for all values of x, then a belongs to the interval	(R) $[0, \infty)$
(D)	If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ is increasing for all values of x, then a belongs to the interval	(S) $[1, \infty)$
		(T) $(2, \infty)$

98.

	Column-I	**Column-II**
(A)	The function $f(x) = \frac{x}{(1+x^2)}$ decreases in the interval	(P) $(-\infty, -1)$
(B)	The function $f(x) = \tan^{-1} x - x$ decreases in the interval	(Q) $(-\infty, 0)$
(C)	The function $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right)$ increases in the interval	(R) $(0, \infty)$
(D)	The largest interval in which $f(x) = x^3 - \ln(1 + x^3)$ is non negative is	(S) $(1, \infty)$
		(T) $(-1, \infty)$

99.

	Column-I	**Column-II**
(A)	The value of $\lim_{n \to \infty} \sum_{r=1}^{r=4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r} + 4\sqrt{n}\right)^2} = \frac{p}{q}$ in its lowest form where p + q =	(P) 2
(B)	No. of integral values of a for which the cubic $f(x) = x^3 + ax + 2$ is non monotonic and has exactly one real root.	(Q) 0

(C) The radical centre of three circles is at the origin. The equations of two of the circles are $x^2 + y^2 = 1$ and $x^2 + y^2 + 4x + 4y - 1 = 0$. If the equation of third circle passes through the point (1, 1) and (–2, 1) is $x^2 + y^2 + 2gx + 2fy + c = 0$ then $f - c =$ (R) 1

(D) If $x^2 + y^2 + z^2 - 2xyz = 1$, then the value of $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}$ (S) 11

Review Exercises for JEE Advanced

1. Find the critical points of the function $f(x) = \frac{1}{3}\sin a \tan^3 x + (\sin a - 1)\tan x + \frac{\sqrt{a-2}}{\sqrt{8-a}}$

2. Determine the values of the number a for which the function f has no critical point.
$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$

3. Let $f(x) = a.\,9^x + 8(a-1)3^x + 2(a-1)\,x\ln 3$. Find all values of a so that f(x) is an increasing function for $x \in R$.

4. Find the intervals in which $f(x) = |x + 1|\,|x + 2|$ increases and the intervals in which it decreases.

5. A function f(x) is given by the equation, $x^2 f'(x) + 2x\,f(x) - x + 1 = 0\ (x \neq 0)$. If $f(1) = 0$, then find the intervals of monotonocity of f.

6. Prove that

$$e^x + \sqrt{1+e^{2x}} \geq (1+x) + \sqrt{2+2x+x^2} \quad \forall\ x \in R.$$

7. Find the interval to which b may belong so that the function $f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$ is increasing at every point of its domain.

8. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$.

9. If $f(x) = 2e^x - ae^{-x} + (2a+1)\,x - 3$ monotonically increases for every $x \in R$ then find the range of values of 'a'.

10. Find a polynomial f(x) of degree 4 which increases in the intervals $(-\infty, 1)$ and $(2, 3)$ and decreases in the intervals $(1, 2)$ and $(3, \infty)$ and satisfies the condition $f(0) = 1$.

11. Determine whether the function $g(x) = \tan x - 4x$ is increasing or decreasing in the interval $-\pi\ 3 < x < 0$.

12. Show that, $x^3 - 3x^2 - 9x + 20$ is positive for all values of $x > 4$.

13. If $0 < x < 1$ prove that $y = x\ln x - (x^2/2) + (1/2)$ is a function such that $d^2y/dx^2 > 0$. Deduce that $x\ln x > (x^2/2) - (1/2)$.

14. Find all numbers p for each of which the least value of the quadratic trinomial $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \leq x \leq 2$ is equal to 3.

15. Find the greatest & least values of f(x) $= \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$

16. Use the function $y = (\sin x)^{\sin x}$, $0 < x < \pi$, to determine the bigger of the two $\left(\frac{1}{2}\right)^e$ and $\left(\frac{1}{e}\right)^2$

17. Prove $e^{\cos x - \sin x} < \frac{1-\sin x}{1-\cos x}$ if $0 < x < \frac{\pi}{4}$.

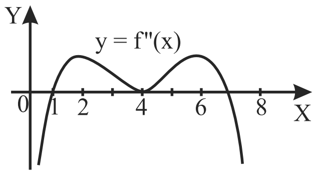
18. Show that the equation $\cos x = x\sin x$ has exactly one solution in the interval $\left(0, \frac{\pi}{2}\right)$.

19. Find all the values of the parameter b for each of which the function $f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$ decreases throughout the number line and has no critical points.

20. Identify which is smaller $\frac{1+e^2}{e}$ or $\frac{1+\pi^2}{\pi}$?

21. Find the points of inflection and the intervals of concavity of the graphs of the given functions.
(i) $y = x^3 - 5x^2 + 3x - 5$
(ii) $y = (x+2)^6 + 2x + 2$
(iii) $y = \ln(1 + x^2)$
(iv) $y = e^{\tan^{-1} x}$

22. The graph of the second derivative f″ of a function f is shown. State the x–coordinates of the inflection points of f. Give reasons for your answers.



23. Let f be a function whose second derivative is of the form $f''(x) = (x-a)^k g(x)$, where k is a positive integer, a is a fixed number, and g is a continuous function such that $g(a) \neq 0$. (a) Show that if k is odd, then a is an inflection point. (b) Show that if k is even, then a is not an inflection number.

24. Choose α and β such that the point A (2, 5/2) becomes a point of inflection of the curve $x^2y + \alpha x + \beta y = 0$. Will it have some more points of inflection? What are they?

25. Prove that $e^{\lambda\pi + (1-\lambda)e} < \lambda e^{\pi} + (1-\lambda)e^{e}$, $0 < \lambda < 1$

26. If λ & u are positive numbers whose sum is 1, using graph of $y = x^2$ prove that $(\lambda + \mu)(\lambda x_1^2 + \mu x_2^2) - (\lambda x_1 + \mu x_2)^2 \geq 0$

27. If $0 < x_1 < x_2 < x_3 < \pi$, then prove that

$$\sin\left(\frac{2x_1 + x_2 + x_3}{4}\right) > \frac{2\sin x_1 + \sin x_2 + \sin x_3}{4}$$

28. If f(x) is a monotonically increasing function $\forall$ x $\in$ R, $f''(x) > 0$ and $f^{-1}(x)$ exists, then prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right).$$

29. Show that $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for $x > 0$.

30. (i) Show that $e^x \geq 1 + x$ for $x \geq 0$.

(ii) Deduce that $e^x \geq 1 + x + \frac{1}{2}x^2$ for $x \geq 0$

(iii) Use mathematical induction to prove that for $x \geq 0$ and any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!}.$$

31. Prove that the inequality

$$5e^{1/3} < (3x^2 - 7x + 7)\,e^x < \frac{11}{3}\sqrt[3]{e^2}.$$

is valid for $x \in [0, 2/3]$.

32. Prove that the inequality

$$\cos x\,\sqrt{\sin x} \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{3}{4}}.$$

Target Exercises for JEE Advanced

1. Find the critical points of the function :

(i) $f(x) = 2\sin a \cos x + \frac{1}{3}\cos 3x + \frac{1}{\sqrt{4a - a^2}}$

(ii) $f(x) = \left(1 - \frac{\cos a}{4}\right)\sin 2x + \frac{1}{8}\sin(\pi + 4x) + x\left(\frac{\cos a - 3}{2}\right) + \sqrt{2a - a^2 + 3}$

2. Prove that $\psi(x) = \frac{1}{2}\sin x \tan x - \ln \sec x$ is positive and increasing for $0 < x < \frac{\pi}{2}$.

3. Find the intervals of monotonicity of the following functions:

(i) $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\,dt$

(ii) $f(x) = \int_0^x \{2\sqrt{2}\sin^2 t + (2 - \sqrt{2})\sin t - 1\}\,dt, x \in [0, 2\pi]$

4. Let f be defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x + 1) - f(x)$.
Prove that $g(x) \to 0$ as $x \to \infty$

5. Let (i) f be continuous for $x \geq 0$
(ii) $f'(x)$ exist for $x > 0$
(iii) $f(0) = 0$
(iv) f' be strictly increasing.

If $g(x) = \frac{f(x)}{x}$, $x > 0$, prove that g is strictly increasing.

6. Let f be differentiable on [a, b], f(a) = 0 and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on [a, b]. Prove that $f(x) = 0\ \forall\ x \in [a,b]$.

7. If f is a function such that $f(x) > 0$, $f'(x)$ is continuous for all x and $ax.f'(x) \geq 2\sqrt{f(x)} - 2a\,f(x)$, $(a.x \neq 2)$, show that

$$\sqrt{f(x)} \geq \frac{\sqrt{f(1)}}{x}, x \geq 1.$$

8. Prove that the inequality $1 \leq \sqrt[3]{\frac{x^2}{2x - 1}} \leq \sqrt[3]{\frac{4}{3}}$ holds true for $x \in [3/4, 2]$.

9. Let f be twice differentiable on $[0, \infty)$, f'' bounded on $(0, \infty)$ and let $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

10. Prove that $\pi < \frac{\sin \pi x}{x(1-x)} \leq 4$ when $0 < x < 1$.

11. Prove that, if $x^2 < 1$, $\tan^{-1}x$ lies between

$$x - \frac{1}{3}x^2 \text{ and } x - \frac{1}{3}x^2 + \frac{1}{5}x^3$$

12. Prove that the following inequality: $\frac{x}{ax^2 + b} \leq \frac{1}{2\sqrt{ab}}$.

13. If $x > -1$ then prove that $x^2 > (1 + x)[\ln(1 + x)]^2$.

14. Show that $y = \frac{\ln(1+x)}{x}$ and $y = \frac{x}{(1+x)\ln(1+x)}$ both decrease steadily as x increases from 0 towards ∞.

15. Show that $y = \frac{1}{\ln(1+x)} - \frac{1}{x}$ decreases steadily from 1 to 0 as x increases from -1 towards ∞.

16. Show that $x/(1 + x^2) < \tan^{-1}x < x$ for $x > 0$. Use this and $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ to prove that $\frac{14}{5} < \pi < \frac{10}{3}$.

17. If f(x) is continuous for $a \le x < b$, $f''(x)$ exists and $f''(x) > 0$ for $a < x < b$, then prove that $\frac{f(x) - f(a)}{x - a}$ strictly increases for $a < x < b$.

18. Find the range of values of 'b' so that for all real x, $f(x) = \int_0^x (bt^2 + t\sin t)\,dt$ is monotonic.

19. For decreasing function f in the interval [1, 10] we define $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$. Show that $y = \max\{h(x)\}$ is a tangent to the curve $y = \frac{1}{2}[\sin x + |\sin x|]$ at infinitely many points, given that $f(1) = 1$.

20. Using the graph $x\,\ell n\,x$ for $x > 0$ prove that

$a^a b^b c^c \ge \left(\frac{a+b+c}{3}\right)^{a+b+c}$ if a, b and c are all positive real numbers.

21. Let $g(x) = \int_a^x f(t)dt$ and f(x) satisfies $f(x + y) = f(x) + f(y) + 2xy - 1\ \forall\ x, y \in R$ and $f'(0) = \sqrt{3 + a - a^2}$, then prove that g(x) is increasing.

22. The functions f(x) and g(x) are continuous for $0 \le x \le a$ and differentiable for $0 < x < a$, $f(0) = 0$, $g(0) = 0$, and $f'(x)$ and $g'(x)$ are positive. Prove that

(i) if $f'(x)$ increases with x, then $f(x)/x$ increases with x,

(ii) If $f'(x)/g'(x)$ increases with x, then $f(x)/g(x)$ increases with x. Prove that the functions

$\frac{x}{\sin x}, \frac{\frac{1}{2}x^2}{1 - \cos x}, \frac{\frac{1}{6}x^3}{x - \sin x}, \ldots\ldots$ are strictly increasing in the interval $0 < x < \frac{1}{2}\pi$.

23. Let $p_n(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots. + (-1)^n\frac{x^n}{n!}$ for each positive integer n.

(i) Show that $e^{-x} > p_1(x) = -1 - x$ for all $x > 0$.

(ii) Use the result of part (i) to show that $e^{-x} < p_2(x) = 1 - x + \frac{1}{2}x^2$ for all $x > 0$.

(iii) Use the result of part (ii) to show that $e^{-x} > p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ for all $x > 0$.

(iv) Continue one step at a time in like manner until you have shown that $p_7(x) < e^{-x} < p_8(s)$ for all $x < 0$. Finally, substitute $x = 1$ in this inequality to show that $e \approx 2.718$ accurate to three decimal places.

24. (i) Let $f(x) = e^x - 1 - x$ for all x. Prove that $f'(x) \ge 0$ if $x \ge 0$ and $f'(x) \le 0$ if $x \le 0$. Use this fact to deduce the inequalities $e^x > 1 + x$, $e^{-x} > 1 - x$, valid for all $x > 0$. (When $x = 0$, these become equalities.) Integrate these inequalities to derive the following further inequalities, all valid for $x > 0$:

(ii) $e^x > 1 + x + \frac{x^2}{2!}$, $\quad e^{-x} < 1 - x + \frac{x^2}{2!}$.

(iii) $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $e^{-x} > 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(iv) Guess the generalization suggested and prove your result.

25. Prove that $\frac{1}{1 + \frac{1}{x}} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$ if $x > 0$.

26. Prove that the derivative of the function f, defined by $f(x) = e^x(x^2 - 6x + 12) - (x^2 + 6x + 12)$, is never negative for any real value of x.

Deduce that $\{e^x(x - 2) + (x + 2)\}/\,x^2(e^x - 1) < 1/6$ for all real values of x other than $x = 0$.

27. Prove that the inequality

$$\frac{1}{\sin(\pi/3 + x)} + \frac{1}{\sin(\pi/3 - x)} \ge \frac{4\sqrt{3}}{3}$$

is valid for $x \in [0, \pi/3)$

28. Show that the graph of the general cubic $y = ax^3 + 3bx^2 + 3cx + d$ is centrosymmetric about its point of inflection. (A graph is said to be centrosymmetric about a point O if for every point P of the graph there is a corresponding point P' such that the line segment PP' is bisected by O.)

29. Find the values of h, k and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called osculating circles.

30. Obtain the equation of the inflectional tangents to the curve $y = (x^3 - x)/(3x^2 + 1)$.

31. Show that the curve $y = \frac{x+1}{x^2+1}$ has three points of inflection which lie on one straight line.

32. Show that the points of inflection of the curve $y = x \sin x$ lie on the curve $y^2(4 + x^2) = 4x^2$.

33. Show that the curves $y = x^3 + x^2 - x - 1$, and $y = 2(x^3 - x^2 + x - 1)$ touch, and cross one another at the point of contact.

34. Find all strictly monotonic functions $f : (0, \infty) \to (0, \infty)$ such that $f\left(\frac{x^2}{f(x)}\right) \equiv x$.

Previous Year's Questions (JEE Advanced)

A. FILL IN THE BLANKS:

1. The larger of cos (ln θ) and ln (cos θ) if $e^{-\pi/2} < \theta < \pi/2$ is.................... **[IIT - 1983]**

2. The function $y = 2x^2 - \ln|x|$ is monotonically increasing for values of $x \neq 0$ satisfying the inequalities ......and monotonically decreasing for values of x satisfying the inequalities.......... **[IIT - 1983]**

3. The set of all x for which $ln(1 + x) \leq x$ is equal to.................... **[IIT - 1987]**

B. MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER

Questions on Increasing and Decreasing Functions

4. Let f and g be increasing and decreasing functions, respectively from [0, ∞) to [0, ∞). Let h(x) = f(g(x)). If h(0) = 0, then h(x) – h(1) is **[IIT - 1987]**

(A) always zero
(B) always negative
(C) always positive
(D) strictly increasing

5. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$ then f(x) is **[IIT - 1993]**

(A) increasing in [–1, 2]
(B) continuous in [–1, 3]
(C) greatest at x = 2
(D) all above correct

6. The function f defined by $f(x) = (x + 2)e^{-x}$ is **[IIT - 1994]**

(A) decreasing for all x
(B) decreasing in (–∞, – 1) and increasing (–1, ∞)
(C) increasing for all x
(D) decreasing in (–1, ∞) and increasing (–∞, – 1)

7. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is **[IIT - 1995]**

(A) increasing on (0, ∞)
(B) decreasing on (0, ∞)
(C) increasing on (0, π/e), decreasing on (π/e, ∞)
(D) decreasing on (0, π/e), increasing on (π/e, ∞)

8. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval **[IIT - 1997]**

(A) both f(x) and g(x) are increasing functions
(B) both f(x) and g(x) are decreasing functions
(C) f(x) is an increasing function
(D) g(x) is an increasing function

9. The function $f(x) = \sin^4 x + \cos^4 x$ increases if **[IIT - 1999]**

(A) $0 < x < \frac{\pi}{8}$
(B) $\frac{\pi}{4} < x < \frac{3\pi}{8}$
(C) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$
(D) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

10. Consider the following statement S and R

S : Both sin x and cos x are decreasing function in the interval $\left(\frac{\pi}{2}, \pi\right)$

R : If a differentiable function decreases in an interval (a, b) then its derivative also decreases in (a, b). Which of the following is true **[IIT - 2000]**

(A) both S and R wrong
(B) both S and R are correct, but R is not the correct explanation for S
(C) S is correct and R is the correct explanation for S
(D) S is correct and R is wrong

11. Let $f(x) = \int e^x (x - 1)(x - 2)\,dx$. Then f decreases in the interval **[IIT - 2000]**

(A) (–∞, –2)
(B) (–2, – 1)
(C) (1, 2)
(D) (2, ∞)

12. If $f(x) = x e^{x(1-x)}$, then f(x) is **[IIT - 2001]**

(A) increasing on [–1/2, 1]
(B) decreasing on R
(C) increasing on R
(D) decreasing on [–1/2, 1]

13. The length of the longest interval in which the function $3 \sin x - 4 \sin^3 x$ is increasing, is **[IIT - 2002]**

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{2}$
(C) $\frac{3\pi}{2}$
(D) π

14. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}.\frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is **[IIT - 2008]**

(A) even and is strictly increasing in $(0, \infty)$
(B) odd and is strictly decreasing in $(-\infty, \infty)$
(C) odd and is strictly increasing in $(-\infty, \infty)$
(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

15. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$ is
(A) 6 (B) 4
(C) 2 (D) 0 **[2013]**

C. COMPREHENSION

Consider the function $f(x) = 1 + 2x + 3x^2 + 4x^3$.
Let s be the sum of all distinct real roots of f(x) and let $t = |s|$. **[IIT - 2010]**

16. The real number s lies in the interval.
(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-1, -\frac{3}{4}\right)$
(C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

17. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval:
(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$
(C) $(9, 10)$ (D) $\left(0, \frac{21}{64}\right)$

18. The function $f'(x)$ is
(A) increasing in $\left(-t, \frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$
(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$
(C) increasing in $(-t, t)$
(D) decreasing in $(-t, t)$

D. MULTIPLE CHOICE QUESTIONS WITH ONE OR MORE CORRECT ANSWERS

19. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then **[IIT - 1993]**
(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general.

E. SUBJECTIVE PROBLEMS

20. Use the function $f(x)= x^{1/x}$, $x > 0$. to determine the bigger of the two numbers e^{π} and π^{e}. **[IIT - 1981]**

21. If $ax^2 + \frac{b}{x} \geq c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \geq 4c^3$. **[IIT - 1982]**

22. Show that $1 + x \ln(x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for all $x \geq 0$ **[IIT - 1983]**

23. Show that $2\sin x + \tan x \geq 3x$ where $0 \leq x < \frac{\pi}{2}$. **[IIT - 1990]**

24. Let $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$ Where a is a positive constant. Find the interval in which $f'(x)$ is increasing. **[IIT - 1996]**

25. Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) > x$, for $\forall\, x \in (0, \pi/4)$ **[IIT - 2003]**

26. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. **[IIT - 2003]**

27. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$. Explain the identity if any used in the proof. **[IIT - 2003]**

Previous Year's Questions (JEE Main Papers)

1. **Statement-1:** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.
Statement 2: The function $x^2 e^x$ and $x^2 e^{-x}$ are increasing for all $x > 0$ and sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).
[2013, online]

2. Let $f(x) = \sin^4 x + \cos^4 x$. Then f is an increasing function in the interval
(a) $\left]\frac{5\pi}{8}, \frac{3\pi}{4}\right[$ (b) $\left]\frac{\pi}{2}, \frac{5\pi}{8}\right[$
(c) $\left]\frac{\pi}{4}, \frac{\pi}{2}\right[$ (d) $\left]0, \frac{\pi}{4}\right[$ **[2016, online]**

3. The function defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is
(a) increasing on R
(b) decreasing on R
(c) decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$
(d) increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$
[2017, online]

4. Let $f(x) = \dfrac{x}{\sqrt{a^2 + x^2}} - \dfrac{d - x}{\sqrt{b^2 + (d - r)^2}}$, $x \in R$, where a, b and d are non-zero real constant. Then
(a) f is a decreasing function of x
(b) f is neither increasing nor decreasing function of x
(c) f′ is not a continuous function of x
(d) f is an increasing function of x **[2019]**

5. If the function f given by $f(x) = x^3 - 3(a - 2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation $\dfrac{f(x) - 14}{(x - 1)^2} = 0 \ (x \neq 1)$ is
(a) 6 (b) 5
(c) 7 (d) –7 **[2019]**

6. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then his nearest distance is
(a) $\frac{1}{2}$ (b) $\frac{1}{3}\sqrt{\frac{7}{3}}$
(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{\sqrt{5}}{6}$ **[2019]**

7. The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}$, $(x > 0)$ is
(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{5}{4}$
(c) $\frac{3}{2}$ (d) $\frac{\sqrt{3}}{2}$ **[2019]**

ANSWERS

CONCEPT PROBLEMS—A

1. (i) neither increasing nor decreasing
(ii) strictly decreasing
(iii) strictly decreasing
(iv) strictly increasing
2. Both strictly increasing at x = a
3. strictly increasing both at x = 0 and x = 1.
4. Decreases at $x_1 = \frac{1}{2}$; increases at $x_2 = 2$ and $x_3 = e$; non-monotonous at $x_4 = 1$.
5. (i) decreases (ii) non-monotonous (decreases then increases) (iii) non-monotonous (increases then decreases)
7. strictly increasing at x = 0; non-decreasing at x = 1, constant at x = 2.
8. Strictly increasing
9. (i) non-monotonous (ii) strictly increasing
(iii) strictly decreasing

CONCEPT PROBLEMS—B

1. (a) False (b) False
2. False
3. (i) True (ii) False
4. No **8.** increasing
9. g is discontinuous at x = 0.
10. (i) strictly increasing (ii) non-monotonous.
11. $\pi/4$ **12.** non-monotonous
13. strictly decreasing
14. Increasing **15.** Decreasing
16. $a \geq 1$ **17.** No
19. (i) $(1, \infty)$ (ii) $(-\infty, 1)$

PRACTICE PROBLEMS—A

24. decreasing
27. $-\frac{7}{2} \leq k \leq \frac{7}{2}$ **28.** $a \leq 0$
31. $(-\infty, -3)$
34. (i) $\left[\frac{1}{3}, \infty\right)$ (ii) $\left[-\infty, \frac{-4}{3}\right)$
39. $(-\infty, 3] \cup [1, \infty]$

PRACTICE PROBLEMS—B

2. $x = 1, 9$

3. (i) $0, (-1 \pm \sqrt{5})/2$ (ii) $0, \frac{8}{7}, 4$ (iii) $n\pi, n \in I$

4. (i) $\{\pm 1, 0\}$ (ii) $\{0\}$

(iii) $\{0, \ln 2\}$ **5.** $n/2, n \in I$

6. (i) $\{3\}$; (ii) $\{1\}$

7. (i) $\{\frac{3}{2}, \pi/2 + 2\pi n, n \in -N, -\pi/2 + 2\pi m, m \in N\}$

(ii) $\{2, (\pi/2 + 2\pi m)/6, m = 1, 0, -1, -2, \ldots;$

$\frac{1}{6}(-\pi/2 + 2\pi n), n = 3, 4, 5, \ldots\}$

8. (i) $2n\pi \pm \left(\pi - \cos^{-1}\frac{2}{3}\right), n \in I$

(ii) $\frac{n\pi}{2} + (-1)^n \frac{1}{2}\sin^{-1}\frac{a}{2} \quad -2 \le a \le 2$

9. $-6\pi, \frac{9\pi}{2}, 0$ **10.** $\left(-\infty, \frac{-4}{3}\right) \cup (2, \infty)$

CONCEPT PROBLEMS—C

1. (i) increases in $(-b/2a, \infty)$, decreases in $(-\infty, -b/2a)$,

(ii) increases in $(-\infty, \infty)$,

(iii) increases in $(-1, 1)$, decreases in $(-\infty, \infty)$

(iv) increases in $(-\infty, \infty)$

(v) increases in $(2\pi n - 2\pi/3, 2\pi n + 2\pi/3)$, decreases in $(2\pi n + 2\pi/3, 2\pi n + 4\pi/3)$, $n \in I$,

(vi) increases in $\left(\frac{1}{2n+0.5}, \frac{1}{2n-0.5}\right)$, $n \in I$, decreases in $(-\infty, -2)$, $\left(\frac{1}{2n+0.5}, \frac{1}{2n-0.5}\right)$, $n \in I, n \ne 0$ and $(2, \infty)$

(vii) increases in $(0, 2/\ln 2)$, decreases in $(-\infty, 0)$ and $(2/\ln 2, \infty)$, (h) increases in $(0, n)$, decreases in (n, ∞).

(viii) increases in $(0, n)$, decreases in (n, ∞).

2. (i) decrease in $(-\infty, -1)$ and $(0, 1)$, increases in $(-1, 0)$ and $(1, \infty)$;

(ii) decreases in $(-\infty, -1)$ and $(1, \infty)$ increases in $(-1, 1)$;

(iii) increases in $(-\infty, 1)$ and $\left(1, \frac{3}{2}\right)$, decreases in $\left(\frac{3}{2}, \infty\right)$.

3. $(-2-\sqrt{3}, -1)$ and $(-1, \sqrt{3} - 2)$.

4. $(-\infty, 1/4)$ **7.** (i) One

PRACTICE PROBLEMS—C

9. (i) increases in $(-\infty, -1/2)$, $(11/18, \infty)$; decreases in $(-1/2, 11/18)$

(ii) increases in $(0, 2)$; decreases in $(-\infty, 0)$, $(2, \infty)$

(iii) increases in (e, ∞); decreases in $(0, 1)$, $(1, e)$

(iv) increases in $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$; increases in $\left(0, \frac{\pi}{3}\right)$, $\left(\frac{5\pi}{3}, 2\pi\right)$.

10. (i) increases in $[1, 3]$; decreases in $(-\infty, 1]$, $(3, \infty)$

(ii) increases in $(-\infty, -2]$, $[0, \infty)$; decreases in $[-2, -1)$, $(-1, 0]$

(iii) increases in $\left[-\frac{1}{2}, 1\right]$; decreases in $\left(-\infty, -\frac{1}{2}\right]$, $[1, \infty)$

(iv) increases for $x \in R$

11. (i) $\left[0, \frac{7\pi}{12}\right], \left[\frac{(2k+1)\pi}{4} - \frac{\pi}{6}, \frac{(2k+1)\pi}{4} + \frac{\pi}{6}\right], k < 0,$

$\left[\frac{(2m+1)\pi}{4} + \frac{\pi}{6}, \frac{(2m+3)\pi}{4} - \frac{\pi}{6}\right], m > 0.$

(ii) $\left[\frac{2}{(4k+3)\pi}, \frac{2}{(4k+1)\pi}\right]$

12. decreases in $(-\infty, 2)$ and increases in $(2, \infty)$.

13. $(-\infty, 3 - \sqrt{1 + 1/(3e)}$ and

$\left(4, 3 + \sqrt{1 + \frac{1}{3e}}\right)$

14. $\{\cos 1 \cos 3, \sin 1 \sin 3\}$, it increases.

CONCEPT PROBLEMS—D

2. strictly decreasing

5. Let $a < x < b$. $f(a) \le f(x)$, $f(b) = f(a)$, hence $f(x) = f(a)$.

7. Yes. **8.** nothing can be said

PRACTICE PROBLEMS—D

21. Consider the inequality relating the expressions which are reciprocal of the left hand and right hand sides of the initial inequality.

22. $(-1, 0) \cup (0, \infty)$.

CONCEPT PROBLEMS—E

1. (i) $(-\infty, 1)$ and $(2, \infty)$, (ii) $(1, 2)$

(iii) 1, 2

3. No **7.** No

9. $x = 0$; No **15.** No

16. $f'' \ge f'^2 \ge 0$, so $f'' \ge 0$.

PRACTICE PROBLEMS—E

17. Concave down in the neighbourhood of the point (1, 11), concave up in the neighbourhood of the point (3, 3).

19. $x = 0, \pm a\sqrt{3}$

20. $a = -2/3$, $b = -2$, $c = 7/3$

21. (i) (0, 0), (ii) (a, 0), (c) (a, b)

22. (i) Concave up in $(-2, 0)$ and $(2, \infty)$; concave down in $(-\infty, -2)$ and $(0, 2)$. Points of inflection at $(-2, 198)$, $(0, -20)$, $(2, -238)$.

(ii) Concave up in $(-\infty, -1)$ and $(1, \infty)$; concave down in $(-1, 1)$. Points of inflection at $(-1, 2e)$ and $(1, 10/e)$.

23. Concave up in $(0, \pi/4)$ and $(5\pi/4, 2\pi)$ and concave down in $(\pi/4, 5\pi/4)$.

24. (i) (0, 6), (8, 9) (ii) (6, 8)
(iii) (2, 4), (7, 9) (iv) (0, 2), (4, 7)
(v) (2, 3), (4, 9/2), (7, 4).

25. Concave down in the vicinity of the point $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$, concave up in the neighbourhood of the point (1, 0).

26. $y = 2x$, $y = 3$

27. (1, 5/6) ; $15x - 6y = 10$.

OBJECTIVE EXERCISES

1. B	**2.** C	**3.** B
4. C	**5.** C	**6.** BC
7. C	**8.** B	**9.** A
10. C	**11.** D	**12.** B
13. B	**14.** A	**15.** C
16. D	**17.** D	
18. B	**19.** A	**20.** B
21. C	**22.** D	**23.** C
24. C	**25.** A	**26.** A
27. A	**28.** C	**29.** D
30. B	**31.** C	**32.** B
33. D	**34.** A	**35.** B
36. B	**37.** D	**38.** D
39. B	**40.** A	**41.** C
42. D	**43.** C	**44.** D
45. C	**46.** A	**47.** C
48. B	**49.** BC	**50.** C
51. BC	**52.** BCD	**53.** AC
54. AD	**55.** ABC	**56.** BD
57. ABCD	**58.** AC	**59.** AB
60. ABCD	**61.** BC	**62.** AD
63. CD	**64.** ABCD	**65.** BD
66. ABCD	**67.** BCD	**68.** BC
69. ACD	**70.** A	**71.** A
72. A	**73.** D	**74.** C
75. B	**76.** B	**77.** A
78. B	**79.** A	**80.** B
81. A	**82.** C	**83.** B
84. A	**85.** B	**86.** C
87. B	**88.** D	**89.** D
90. C	**91.** A	**92.** A
93. A	**94.** B	

95. (A)–(R) ; (B)–(S) ; (C)–(S) ; (D)–(P)

96. (A)–(R) ; (B)–(R) ; (C)–(P) ; (D)–(Q)

97. (A)–(ST) ; (B)–(PT) ; (C)–(PQST) ; (D)–(RST)

98. (A)–(PS) ; (B)–(PQRST) ; (C)–(Q); (D)–(T)

99. (A)–(S) ; (B)–(P) ; (C)–(Q) ; (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

1. $\{\pi n \pm \tan^{-1}\sqrt{\operatorname{cosec} a - 1}, n \in I\}$ for $a \in [2, \pi] \cup (2\pi, 8)$.

For a $[\pi, 2\pi]$ the function has no critical points.

2. $-3.5 < a < -2.5$

3. If n is even, limit does not exist. If n is odd limit is 0

4. Increases in $\left(-2, -\frac{3}{2}\right)$ and $(-1, \infty)$; Decreases in $(-\infty, -2)$ and $\left(-\frac{3}{2}, -1\right)$

5. Increasing in $(-\infty, 0)$, $(1, \infty)$ decreasing in $(0, 1)$

7. $[-7, -1] \cup [2, 3]$

8. $(-2, 0) \cup (2, \infty)$

9. $a \geq 0$

10. $f(x) = a(x^4 - 8x^3 + 22x^2 - 24x) + 1$, $a \in (-\infty, 0)$

11. Decreasing

14. $a = 1 - \sqrt{2}$ or $5 + \sqrt{10}$

15. $(\pi/6) + (1/2) \ln 3$, $(\pi/3) - (1/2) \ln 3$.

16. $\left(\frac{1}{2}\right)^e$

19. $b < -3 - \sqrt{3}$, $b > -1 + \sqrt{3}$.

20. $\frac{1 + e^2}{e}$

21. (i) point of inflection $\left(\frac{5}{3}, -\frac{250}{27}\right)$; concave down in $\left(-\infty, \frac{5}{3}\right)$, concave up in $\left(\frac{5}{3}, \infty\right)$.

(ii) There is no point of inflection. The graph is concave up

(iii) points of inflection $(\pm 1, \ln 2)$; concave down in $(-\infty, -1)$ and $(1, \infty)$, concave up in $(-1, 1)$

(iv) point of inflection $\left(\frac{1}{2}, e^{\tan^{-1}\frac{1}{2}}\right)$; concave up in $(-\infty, 1/2)$, concave down in $(1/2 , \infty)$.

22. x = 1, 7

24. α = – 20/3, β = 4/3. The points (– 2, – 5/2) and (0, 0) are also points of inflection.

TARGET EXERCISES for JEE ADVANCED

1. (i) $\left\{\pi n,\ \pi n \pm \frac{1}{2}\cos^{-1}\left(-\frac{1+2\sin a}{2}\right), n \in I\right\}$ for a ∈ (0, π/6] ∪ [5π/6, 4); {πn, n ∈ I} for a ∈ (π/6, 5π/6)

(ii) $\left\{\pi n,\ \pi n \pm \frac{1}{2}\cos^{-1}\frac{2-\cos a}{2}, n \in I\right\}$ for $a \in \left[-1, \frac{\pi}{2}\right]$; {πn, n ∈ I} for $a \in \left(\frac{\pi}{2}, 3\right]$.

3. (i) f decreases in (–2, –1) and (0, 1) and increases in (–∞, –2), (–1,0) and (1, ∞).

(ii) f decreases in $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{5\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and increases in $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

18. $b \geq 1$

29. $h = -4,\ k = \frac{9}{2},\ a = \frac{5\sqrt{5}}{2}$

30. x + y = 0, x – 2y ± 1 = 0.

34. f(x) = Cx, C > 0.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. cos (ln θ)

2. increases in $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \infty\right)$ and decreases in $\left(-\infty, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)$

3. x > – 1

4. A

5 D

6. D

7. B

8. C

9. B

10. D

11. C

12. A

13. A

14. C

15. C

16. C

17. A

18. B

19. AC

20. e^{π}

24. $\left(\frac{-2}{a}, \frac{a}{3}\right)$

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. (c)

2. (c)

3. (a)

4. (b)

5. (c)

6. (c)

7. (a)

HINTS & SOLUTIONS

Objective Exercises Solution

SINGLE CORRECT ANSWER TYPE

1. By Observation

Hence, B is correct

2. $f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4.$

$f'(x) = 4(\sin 2x) + 2a \geq 0$

$\Rightarrow a \in [2, \infty)$

Hence, C is correct

3. $f(x) = \sin^{-1}\left(\dfrac{2\phi(x)}{1+\phi^2(x)}\right) = 2\tan^{-1}\phi(x) \ \text{ if } |\phi(x)| < 1$

$f'(x) = \dfrac{2}{1+\phi^2(x)} \times \phi'(x) < 0$

So, f(x) is decreasing when $|\phi(x)| < 1$

Hence, B is correct

4. $f(x) = \sin^2 x - (2a+1)\sin x + (a-3)$

$f'(x) = \cos x(2\sin x - 1 - 2a)$

at $x = 0$, $f(x) = a - 3 < 0 \Rightarrow a < 3$

at $x = \dfrac{\pi}{2}$, $f(x) = -a - 3 < 0 \Rightarrow -3 < a$

Hence, C is correct

5. If $f'(x) > 0$, $f(x)$ is increasing & always $-$ve

If $f'(x) < 0$, $f(x)$ is decreasing & always +ve

As $f(x)$ is $\uparrow$ & $-$ve $\Rightarrow |f(x)|$ is $\downarrow$ & +ve

Hence, C is correct

6. $f(x) = \sin \sin^{-1} \log_{1/2} x$

$= \log_{1/2} x$ only if $-1 \leq \log_{½} x \leq 1$

Domain of $f(x)$ $x \in \left[\dfrac{1}{2}, 1\right]$

Range of $f(x) = [-1, 1]$.

$f'(x) = \dfrac{-1}{\log 2} \times \dfrac{1}{x} < 0$

Hence, B, C is correct

7. $f(x) = a_1 x + a_3 x^3 + \cdots + a_{2n+1} x^{2n+1} - \cot^{-1} x$

$f'(x) = a_1 + 3a_3 x^2 + \cdots + (2n+1)a_{2n+1} x^{2n} + \dfrac{1}{1+x^2}$

$f'(x) > 0$

It is an increasing function

Hence, C is correct

8. $f''(x) - 3f'(x) > 3, f'(0) = -1$

$g(x) = f(x) + x$

$g'(x) = f'(x) + 1$

$g''(x) = f''(x)$

Now $f''(x) > 3\left(f'(x) + 1\right)$

$\dfrac{f''(x)}{f'(x)+1} > 3$

Integrating = $f'(x) + 1 > e^{3x+c} > 0$

$\Rightarrow g'(x) > 0$

Hence, B is correct

9. $f(x) = ax + \sin 2x + b$

$f'(x) = a + 2\cos 2x$

If $a > 2$, $f'(x) > 0$, $f(x)$ is increasing

$f(0) = b$

Hence, A is correct

10. It should be concave up.

Hence, C is correct

11. $f(x) = (4a-3)(x+5) + 2(a-7)\cot\dfrac{x}{2}\sin^2\dfrac{x}{2}$

$f(x) = (4a-3)(x+5) + (a-7)\sin x$

$f'(x) = 4a - 3 + (a-7)\cos x$

$\Rightarrow 4a - 3 > -(a-7)$ or $4a - 3 < a - 7$

$5a > 10$ or $3a < -4$

$a > 2$ or $a < -4/3$

$a \in \left(-\infty, -\dfrac{4}{3}\right) \cup (2, \infty)$

Hence, D is correct

12. $\tan(\pi\cos\theta) = \cot(\pi\sin\theta), \quad 0 < \theta \leq \dfrac{\pi}{2}$

$\Rightarrow \pi\cos\theta = \dfrac{\pi}{2} - \pi\sin\theta$

$$\Rightarrow \quad \cos\theta+\sin\theta=\frac{1}{2}$$

$$\Rightarrow f(x)=2^{-x}$$

$$f'(x)=-2^{-x}<0$$

Hence, B is correct

13. $f(x)=x^n\sin\left(\frac{1}{x}\right)+x^m\cos\left(\frac{1}{2x}\right)$

$$f\left(\frac{1}{\pi}\right)=0,\quad f\left(\frac{1}{3\pi}\right)=0$$

Using Rolle's Theorem, $f'(x)=0$ will have atleast 1 root.
Hence, B is correct

14. Let $f(x)=x^{1/x}$.

$$f'(x)=x^{1/x}\frac{(1-\ln x)}{x^2}$$

$$f'(x)<0,\ \forall x\in(e,\infty)$$

$\Rightarrow f(x)$ is decreasing

As $1999<2000$

$$f(1999)>f(2000)$$

$$(1999)^{1/1999}>(2000)^{1/2000}$$

$$(1999)^{2000}>(2000)^{1999}.$$

Hence, A is correct

15. If $f''(x)>0\Rightarrow\ f'(x)$ is increasing function

$$g(x)=f(2-x)+f(4+x)$$

$$g'(x)=f'(4+x)-f'(2-x)>0$$

$$\Rightarrow \quad 4+x>2-x$$
$$2x>-2$$
$$x>-1$$

Hence, C is correct.

16. Differentiating the options
Hence, D is correct.

17. $f(x_0)=g(x_0)$ & $f'(x)>g'(x)$

As rate of change of f(x) is larger than g(x)
$f(x)>g(x)\ \ \forall x>x_0$

Hence, D is correct

18. $f(x)=x^3+2x+k$

$$f'(x)=3x^2+2>0$$

f(x) is increasing function, so, f(x) has exactly one root.

Hence, B is correct.

19. $F(x)=f(x)-\frac{\sqrt{3}x^2}{2}$

$$F'(x)=f'(x)-\sqrt{3}x \qquad \text{...(I)}$$

Now, $g(x)=f(x)-x$

$$g'(x)=f'(x)-1>0 \qquad \text{...(II)}$$

As $h(x)=f(x)-x^3$

$$h'(x)=f'(x)-3x^2$$

$$=f'(x)-3x^2>0 \qquad \text{...(III)}$$

Now, for $x\in R$, $\sqrt{3}x<1$ or $3x^2$

Hence $f'(x)-\sqrt{3}x>0$
Hence, A is correct.

20. Conceptual
Hence, B is correct

21. $x=t^2$ & $y=3t+t^3$

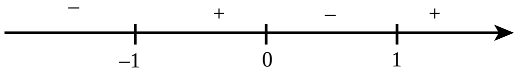
$$\frac{dx}{dt}=2t \quad \& \quad \frac{dy}{dt}=3+3t^2$$

$$\frac{dy}{dx}=\frac{3}{2}\frac{\left(1+t^2\right)}{t}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}\frac{t(2t)-\left(1+t^2\right)}{t^2}\times\frac{1}{2t}$$

$$=\frac{3}{4}\frac{(t^2-1)}{t^3}$$

$\frac{d^2y}{dx^2}\rightarrow$



Hence, C is correct

22. $f(x)=\frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}=1-\frac{2}{e^{2x^2}+1}$

$$f'(x) = \frac{8x \cdot e^{2x^2}}{\left(e^{2x^2}+1\right)^2}$$

Hence, D is correct

23. $h(x) = f \circ g(x), \quad h(0) = 0$

If x increase, h(x) decreases

As $p(x) = h\left(x^3 - 2x^2 + 2x\right) - h(4) \quad \forall \quad x \in (0, 2]$

as $x \to 0$

$$p(0) = h\left(0^3 - 2 \cdot 0^2 + 2 \cdot 0\right) - h(4)$$

$$= 0 - h(4)$$

$$= -h(4)$$

As p(x) is decreasing function.

$$\Rightarrow p(2) = h\left((2)^3 - 2.2^2 + 2.2\right) - h(4)$$

$$= h(4) - h(4)$$

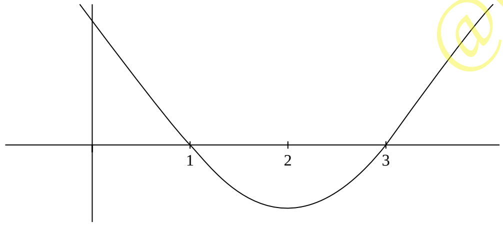
$$= 0$$

$$\Rightarrow p(x) \in (0, -h(4))$$

Hence, A is correct

24. $f'\left(x^2 - 4x + 3\right) > 0 \quad \forall \quad x \in (2, 3)$

Now $x^2 - 4x + 3 = (x-3)(x-1)$



For $x \in (2, 3)$, $x^2 - 4x + 3$ is increasing function from $(-1, 0)$

Now, let $g(x) = f(\sin x)$

$$g'(x) = f'(\sin x)\cos x$$

Hence, C is correct

25. $x^2 f'(x) + 2x f(x) - x + 1 = 0$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{x-1}{x^2}$$

Integrating

$$x^2 y = \int (x-1)\, dx \Rightarrow x^2 y = \frac{x^2}{2} - x + \frac{1}{2}$$

$$y = \frac{x^2 - 2x + 1}{2x^2} = \frac{(x-1)^2}{2x^2}$$

$$\frac{dy}{dx} = \frac{x^2\left(2(x-1) - (x-1)^2 2x\right)}{2\left(x^2\right)^2}$$

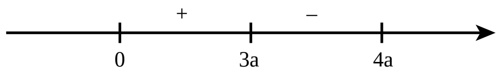
$$= \frac{x(x-1)}{x^4} = \frac{x-1}{x^3}$$

Hence, A is correct

26. $f(x) = x\sqrt{4ax - x^2}$

$$f'(x) = \sqrt{4ax - x^2} + \frac{x(2a-x)}{\sqrt{4ax - x^2}}$$

$$= \frac{6ax - 2x^2}{\sqrt{4ax - x^2}} = \frac{2x(3a-x)}{\sqrt{x(4a-x)}}$$



So, increasing in (0, 3a) & decreasing in (3a, 4a)

27. $f'(x) = \begin{cases} x - \{x\}, & x \geq 0 \\ -x - \{x\} & x < 0 \end{cases}$

$$= \begin{cases} [x], & x \geq 0 \\ -x - \{x\}, & x < 0 \end{cases}$$

For $x \in \left(-\frac{1}{2}, 0\right)$, $\{x\} = 1 + x$

so, $f'(x) = -x - 1 - x = -2x - 1$

Hence, A is correct

28. $f'(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x \geq 1 \\ 3x^2, & 0 \leq x \leq 1 \\ x^2 - 4, & x < 0 \end{cases}$

$f'(x)$ changes its sign at $x = 0$ & -2

Hence, C is correct

29. $f''(x) > 0 \Rightarrow f'(x)$ is increasing

$$g(x) = f\left(\tan^2 x - 2\tan x + 4\right)$$

$$g'(x) = f'\left(\tan^2 x - 2\tan x + 4\right)\left(2\tan x \sec^2 x - 2\sec^2 x\right)$$

$$= f'\left((\tan x - 1)^2 + 3\right) 2\sec^2 x (\tan x - 1)$$

As $f'(\tan x - 1)^2 + 3)$ is $+$ve

$\Rightarrow g'(x) > 0$ if $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Hence, D is correct

30. $f'(x) = \log_{1/3}\log_3(\sin(x)+a)) < 0$

$\log_3(\sin(x)+a)) > 1$

$\sin(x)+a > 3$

$a > 3-\sin x$

$a > 4$

Hence B is correct

31. $f(x) = x+\sin x,\ g(x) = e^{-x},\ u = \sqrt{c+1}-\sqrt{c}$

$v = \sqrt{c}-\sqrt{c-1}$

$g\circ f(x) = e^{-(x+\sin x)}$

$g\circ f'(x) == -(1+\cos x)e^{-(x+\sin x)} < 0$

$\Rightarrow g\circ f(x)$ is decreasing function

Now let $h(x) = \sqrt{x}-\sqrt{x-1}$

$h'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}}$

$= \frac{\sqrt{x-1}-\sqrt{x}}{2\sqrt{x}\sqrt{x-1}} = \frac{-1}{2\sqrt{x}\sqrt{x-1}}(\sqrt{x-1}+\sqrt{x}) < 0$

$\Rightarrow h(x)$ is decreasing

$\Rightarrow v > r$

$g\circ f(u) > g\circ f(v)$ Hence C is correct

32. $f(x) = 4x-\tan 2x$

$f'(x) = 4-2\sec^2 2x$

$= -4\tan^2 x$

$\Rightarrow x \in (0, \pi/2)$

But for continuity of $f(x) \to \pi/4$ is largest interval

Hence, B is correct

33. $f(x) = x^3+2x^2+5x+2\cos x$

$f'(x) = 3x^2+4x+5-2\sin x$

Min value of $3x^2+4x+5$ is $\frac{22}{3}$

$\Rightarrow f'(x) > 0$

$f(0) = 2\ \&\ f(\infty) = \infty,\ \ f(-\infty)-\infty$

There is no root in $[0, 2\pi]$

Hence, D is correct

34. $f(x) = x^3+4x^2+\lambda x+1$

$f'(x) = 3x^2+8x+\lambda < 0$

roots are $-2\ \&\ \frac{-2}{3}$

Product $= +\frac{4}{3}$

$\Rightarrow \lambda = +4$

Hence, A is correct

35. $g(x) = \frac{1}{4}f\left(2x^2-1\right)+\frac{1}{2}f\left(1-x^2\right)$

$g'(x) = xf'\left(2x^2-1\right)-xf'\left(1-x^2\right)$

$= x\left(f'\left(2x^2-1\right)-f'\left(1-x^2\right)\right)$

For $g(x)$ to be increasing function,

For $x > 0,\ 2x^2-1 > 1-x^2$

$3x^2-2 > 0$

$\Rightarrow \quad x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

For $x < 0,\ \ 1-x^2 > 2x^2-1$

$3x^2-2 < 0$

$x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$

Hence, B is correct

36. $f(x) = \left(ab-b^2-2\right)x+\int_0^x\left(\cos^4\theta+\sin^4\theta\right)d\theta$

$f'(x) = ab-b^2-2\ \ +\cos^4 x+\sin^4 x$

$= ab-b^2-2\ \ +1-2\cos^2 x\sin^2 x$

$= ab-b^2-1\ \ -\ 2\cos^2 x\sin^2 x$

$= ab-b^2-1-\frac{1}{2}(\sin^2 2x)$

$\Rightarrow b^2-ab+\frac{3}{2} < 0\ \ \forall\ \ b\ \in\ R$

$\Rightarrow a^2-\ \ 6 < 0$

$\Rightarrow a \in (-\sqrt{6}, \sqrt{6})$

Hence, B is correct

37. $g'(\theta)=\left(f(\sin^2\theta)-f(\cos^2\theta)\right)\sin 2\theta$

In $\left(-\frac{\pi}{4},\theta\right)$, $\sin 2\theta<0$ & $\sin^2\theta<\cos^2\theta$

& $f(\sin^2\theta)<f(\cos^2\theta)$

So, $g'(\theta)>0$

Hence, D is correct

38. Same as Q.34

Hence, D is correct

39. $f(x)=ax^3+bx^2+cx+d$

$f'(x)=3ax^2+2bx+c>0$

$\Rightarrow 4b^2-12ac<0$ & $a>0$

$\Rightarrow b^2-3ac<0$

$b^2<3ac$ & $a>0,\ 3b^2<c^2$

Now

$h(x)=\int_a^x g(t)dt$

$h'(x)=g(x)$

$=a\left(3ax^2+2bx+c\right)+b(6ax+2b)+c^2$

$=3a^2x^2+2abx+\frac{b^2}{3}+6abx+2b^2+3b^2$

$>\left(\sqrt{3}ax+\frac{4b}{\sqrt{3}}\right)^2>0.$

Hence, B is correct

40. $y=\sin(f(x))$

$\frac{dy}{dx}=\cos(f(x))\times f'(x)$

For *y* to be monotonic, Max value of difference of max & min value of f(x) can be π.

Hence, A is correct

41. $f(x)=3\cos^4x+10\cos^3x+6\cos^2x-3$

$f'(x)=-6\sin x\left(2\cos^3x+5\cos^2x+2\cos x\right)$

$=-3\sin 2x\left(2\cos^2x+5\cos x+2\right)$

$=-3\sin 2x(\cos x+2)(2\cos x+1)$

Hence, C is correct

42. $\log_3\log_{1/4}(\cos x+a)>0$

$\cos x+a<\frac{1}{4}$

$a<\frac{1}{4}-\cos x$

$a<-\frac{3}{4}.$

Hence, D is correct

43. $f(x)=x^2-\ln|x|=\begin{cases}x^2-\ln x, & x\geq 0\\ x^2-\ln(-x), & x<0\end{cases}$

$f'(x)=\begin{cases}2x-\frac{1}{x}=\frac{2x^2-1}{x}, & x\geq 0\\ 2x-\frac{1}{x}=\frac{2x^2-1}{+x}, & x<0\end{cases}$

Hence, C is correct

44. $\lim_{x\to 2^+}[f(x)]=3$

$\lim_{x\to 2^-}[f(x)]=2$

Hence, D is correct

45. $f(f(x))=a\left(x^5+x\right)$

$f'f(x)\times f'(x)=a\left(5x^4+1\right)$

As f(x) is differentiable function & $f'(x)$ can never be equal to zero.

Hence, C is correct.

46. $\lim_{n\to\infty}\frac{1}{f^n(x)}=\lim_{n\to\infty}\frac{1}{g\circ g\circ\ldots g(x)}$

As g(x) is increasing function from $(0,\infty)$

$\lim_{x\to\infty}g(x)\to\infty$

Hence, A is correct

47. $f'(x)>0$ & $f''(x)<0$

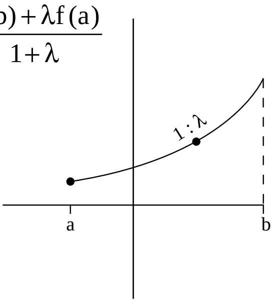
Let pt p that divides a, b in ratio $1:\lambda$

pt $p\to\frac{b+a\lambda}{1+\lambda}$

Their corresponding $f(x)=\frac{f(b)+\lambda f(a)}{1+\lambda}$

$\Rightarrow$ value at pt $p=f\left(\frac{b+a\lambda}{1+\lambda}\right)$

$\Rightarrow f\left(\frac{b+a\lambda}{1+\lambda}\right)<\frac{f(b)+\lambda f(a)}{1+\lambda}$



Hence, C is correct

48. $g'(x) = 2\left(6x^2 - 6x\right)$

$f'\left(2x^3 - 3x^2\right) + \left(12x - 12x^2\right) + f'\left(6x^2 - 4x^3 - 3\right)$

$g'(x) = \left(12x^2 - 12x\right)\left[f'\left(2x^3 - 3x^2\right) - f'\left(6x^2 - 4x^3 - 3\right)\right]$

$= 12x(x-1)\left[f'\left(2x^3 - 3x^2\right) - f'\left(6x^2 - 4x^3 - 3\right)\right]$

For increasing,

If $x \in (1, \infty)$

$2x^3 - 3x^2 > 6x^2 - 4x^3 - 3$

$6x^3 - 9x^2 + 3 > 0$

$2x^3 - 3x^2 + 1 > 0$

$x \in (1, \infty)$

If $x \in (0, 1)$

$6x^2 - 4x^3 - 3 < 2x^3 - 3x^2$

$2x^3 - 3x^2 + 1 > 0$

no soln

If $x < 0$

$2x^3 - 3x^2 + 1 > 0$

$x \in \left(-\frac{1}{2}, 0\right)$

Hence, B is correct

Multiple Correct Answer Type for JEE Advanced

49. $f(x) = x^{1/x}$

$f'(x) < 0 \quad \forall\ x \in (e, \infty) \Rightarrow f(x)$ is decreasing

& $f'(x) > 0 \quad \forall\ x \in (0, e) \Rightarrow f(x)$ is increasing

Hence, B, C is correct

50. Conceptual

C is correct

51. $g'(x) > 0 \Rightarrow g(x)$ is increasing

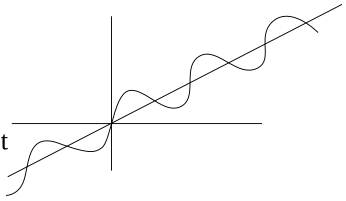
$f'(x) < 0 \Rightarrow f(x)$ is decreasing

Hence, B, C is correct

52. $f(x) = x + \sin x$

$f'(x) = 1 + \cos x$

Hence, B, C, D is correct



53. $f(x) = \left(x^2 + bx + c\right)e^x$

$f'(x) = e^x\left(x^2 + bx + c + 2x + b\right)$

Hence, A, C is correct

54. $f(x) = (x+2)^{1/3}$

$f'(x) = \frac{1}{3}(x+2)^{-2/3}$

$f'(x)$ at $\Rightarrow 2^+/-2^- > 0$

$f''(x) = -\frac{2}{9}(x+2)^{-5/3}$

$f''(x)$ changes its sign at $x = -2$

Hence, A, D is correct

55. $f(x) = \pm x + c$ (Linear function)

Hence, A, B, C is correct

56. Similar to ques 40.

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c > 0$

$\Rightarrow 4b^2 - 12ac < 0\ \&\ a > 0$

$\Rightarrow b^2 - 3ac < 0$

$b^2 < 3ac\ \&\ a > 0,\ 3b^2 < c^2$

Now

$h(x) = \int_a^x g(t)dt$

$h'(x) = g(x)$

$= a\left(3ax^2 + 2bx + c\right) + b(6ax + 2b) + c^2$

$< 3a^2x^2 + 2abx + \frac{b^2}{3} + 6abx + 2b^2 + 3b^2$

$> \left(\sqrt{3}ax + \frac{4b}{\sqrt{3}}\right)^2 > 0$

Hence, B is correct

57. A) $f^2(x)$ may or may not be larger than f(x) & increasing

B) If f(x) is < 0, $f^2(x)$ can be decreasing

C) It is not necessary

D) It is not a mandatory conclusion.

Hence, A, B, C, D is correct.

58. $\frac{p'(x)}{p(x)} > 1$

Integrating $\Rightarrow p(x) > e^{x+c}$

$\Rightarrow p(x) > 0 \quad \forall x \in [1, \infty)$

Now $\Rightarrow \quad y = e^{-x}p(x)$

$y'(x) = e^{-x}\left(p'(x) - p(x)\right) > 0$

$e^{-x}p(x)$ is increasing

Hence, A, C is correct

59. Conceptual

Hence, A, B is correct

60. $h(x) = 3f\left(\frac{x^2}{3}\right) + f\left(3 - x^2\right)$

$h'(x) = 2x\left(f'\left(\frac{x^2}{3}\right) - f'\left(3 - x^2\right)\right)$

For h(x) to be increasing

if $x < 0, \quad 3 - x^2 > \frac{x^2}{3}$

$\Rightarrow 4x^2 - 9 < 0$

$x \in (-\frac{3}{2}, 0)$

For $x > 0, \frac{x^2}{3} > 3 - x^2$

$4x^2 - 9 < 0$

$x \in (\frac{3}{2}, 4)$

Hence, A, B, C, D is correct

61. $f(x) = x^3 - x^2 + 100x + 2002$

$f'(x) = 3x^2 - 2x + 100 > 0$

$f(x)$ is increasing function

Hence, B, C is correct

62. $f'(x) = g(x)(x - a)^2$

f(x) is increasing if g(a) > 0

& $f(x)$ is decreasing if $g(a) < 0$

Hence, A, D is correct

63. When n is even,

$\frac{n}{2}$ has to be decreasing

$\Rightarrow$ Maximum value of $r(n-r) = \frac{n^2}{4}$

When n is odd,

$\frac{n-1}{2}$ has to be decreasing

$\Rightarrow$ Maximum value of $r(n-r) = \frac{(n+1)(n-1)}{4} = \frac{n^2 - 1}{4}$

Hence, C, D is correct

64. A) $f(x) = x^3 + \frac{4}{x^2} + 7$

$f'(x) = 3x^2 - \frac{8}{x^3} = \frac{3x^5 - 8}{x^3}$

$f(x)$ is ↑ in $(-\infty, \infty)$

B) $g(x) = \sqrt{t} + \sqrt{1+t} - 4$

$g'(x) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{1+t}} > 0$

g(x) is increasing in $(0, \infty)$

C) $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8$

$r'(\theta) = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right) > 0$

r(θ) is an increasing function

D) $r(\theta) = \tan\theta - \cot\theta - \theta$

$r'(\theta) = \sec^2\theta + \operatorname{cosec}^2\theta - 1$

$= \tan^2\theta + \operatorname{cosec}^2\theta$

r(θ) is an increasing function

Hence, A, B, C, D is correct

65. Conceptual

Hence, B, D is correct

66. Similar to Q.61

Hence, A, B, C, D is correct

67. As $f(x)$ is increasing function

$\Rightarrow \quad 2a^2 + a + 1 > 3a^2 - 4a + 1$

$a^2 - 5a < 0$

$a(a-5) < 0$

as $x \in (0, \infty), \quad a \in \{2, 3, 4\}$

Hence, B, C, D is correct

68. Conceptual

Hence, B, C, is correct

69. As $f(x)$ is increasing function

$f(x) < f(1-x) \quad \forall \quad x \in (0, 1/2)$

$\& f(x) > f(1-x) \quad \forall \quad x \in (1/2, 1)$

As $\int_0^{1/2} f(x)g(x)dx > \int_8^{1/2} f(x)g(1-x)dx$

$\Rightarrow g(x)$ is increasing function

$\Rightarrow g(x) < g(1-x) \ \forall x \in (0, 1/2)$

Hence, A, B, C is correct

70. $A: f(x) = 2\cos x + 3\sin x$

$f'(x) = 3\cos x - 2\sin x$

$= \sqrt{13}\cos(x+\alpha)$ where $\cos\alpha = \frac{3}{\sqrt{13}}$

$f^{-1}(x) = \sin^{-1}\frac{x}{\sqrt{13}} - \tan^{-1}\frac{3}{2}$

Hence, A is correct

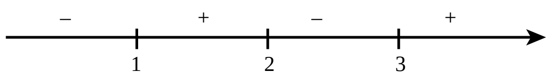
71. A: $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4\left(x^3 - 6x^2 + 11x - 6\right)$

$= 4(x-1)\left(x^2 - 5x + 6\right)$

$= 4(x-1)(x-2)(x-3)$



Hence, A is correct

72. $f'(x) = \ln\left(x + \sqrt{1+x^2}\right) > 0$

As f(x) is increasing function

$f(0) = 0$, so, $f(x)$ is +ve for $x \in R^+$

$\& f'(x) < 0$ if $x < 0$

Hence, A is correct.

73. $f(x) = \frac{ae^x + be^{-x}}{ce^x + de^{-x}}$

$f'(x) = \frac{ae^x - be^{-x}}{ce^x - de^{-x}} = \frac{ae^{2x} - b}{ce^{2x} - d} > 0$

$= \frac{e^{2x} - b/a}{e^{2x} - d/c} > 0$

Hence, D is correct

74. $f(x) = x^3 + x^2 + 3x + \sin x$

$f'(x) = 3x^2 + 2x + 3 + \cos x$

Min value of $3x^2 + 2x + 3 = \frac{32}{4\times 3} = \frac{8}{3}$

So, $f'(x) > 0$

$\Rightarrow f(x)$ is one-one function

Hence, C is correct

75. As $f(x) = \frac{\sin x}{x}$

$f'(x) = \frac{x\cos x - \sin x}{x^2} < 0$

So, f(x) is decreasing function

$\Rightarrow f(\tan x) < f(\sin x)$

$\Rightarrow \cos x \sin(\tan x) < \sin(\sin x)$

Hence, B is correct

76. A: Use A.M, G.M inequality for costan x, cossin x, cossin x

$\Rightarrow \frac{\cos\tan x + 2\cos\sin x}{3} > \left(\cos\tan x \cos^2 \sin x\right)^{1/3}$

$\& \frac{\cos\tan x + 2\cos\sin x}{3} \le \cos\left(\frac{\tan x + 2\sin x}{3}\right)$

As $\tan x > x > \sin x$

$\Rightarrow \cos\left(\frac{\tan + 2\sin x}{3}\right) < \cos x$

Hence, B is correct

77. A: $f(x) = \tan\sin x$

$f'(x) = (\sec^2 \sin x)\cos x > 0$

$\Rightarrow$ Min value of $f(x) = \tan\sin\tan^{-1}\frac{\pi}{2} > \tan\frac{\pi}{4} = 1$

Let $g(x) = \sin(\tan x) \le 1$

So, $\tan\sin x > \sin\tan x$

Hence, A is correct

78. A: As $f\,|\,f(x) = x$ & $f(x)$ is decreasing function

$\Rightarrow \lim_{x\to\infty} f(x) \to 0$

Hence, B is correct

79. Let $f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{1 - \ln x}{x^2} < 0 \quad \forall x \in (e, \infty)$

So, f(x) is decreasing $\forall x \in (e, \infty)$

Hence, A is correct.

Comprehension 1:

$f(x) = \sin 3x + \cos x, \qquad g(x) = \cos 3x + \sin x$

$h(x) = f^2(x) + g^2(x)$

$= 2 + 2(\sin 3x \cos x + \cos 3x \sin x)$

$= 2(1 + \sin 4x)$

80. $h'(x) = 8\cos 4x$

For cos x, longest interval can be 'π'

For cos x, $\frac{\pi}{4}$ is longest interval where it is +ve.

Hence, B is correct.

81. $h(x) = 2(1+\sin 4x) = 4.$

$\Rightarrow \sin 4x = 1$

$\Rightarrow 4x = (4n+1)\dfrac{\pi}{2}$

$x = \dfrac{(4n+1)\pi}{8}$

Hence, A is correct

82. $f(x) = g(x)$

$\sin 3x + \cos x = \cos 3x + \sin x$

$\sin 3x - \sin x = \cos 3x - \cos x$

$\cos 2x \ \sin x = -\sin 2x \sin x$

$\Rightarrow \sin x(\cos 2x + \sin 2x) = 0$

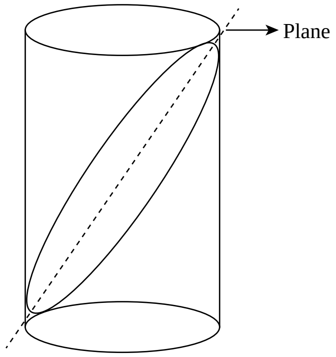
$\Rightarrow \sin x \sin\left(\dfrac{\pi}{4} + 2x\right) = 0$

$\Rightarrow$ either $\sin x = 0$ or $\sin\left(\dfrac{\pi}{4} + 2x\right) = 0$

$x = 0,\ \pi,\ \dfrac{3\pi}{8},\ \dfrac{7\pi}{8}$

Hence, C is correct

Comprehension-2:



$\text{Volume of piece} = \dfrac{\pi x}{2}$

The new surface generated by cutting will be an ellipse.

whose major axis $= \sqrt{x^2+4}$

Surface area $= \pi(1+x) + \dfrac{\pi\sqrt{x^2+4}}{2}$

$$f(x) = \frac{(1+x) + \dfrac{\sqrt{x^2+4}}{2}}{x/2}$$

83. $f(2) = 3 + \sqrt{2}$

Hence, B is correct

84. $f'(x) = \dfrac{-2}{x^2}\left(\dfrac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}} + 2\right) < 0$

Hence, A is correct

85. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\dfrac{2}{x} + 2 + \dfrac{\sqrt{x^2+4}}{x}\right)$

$= 3$

Hence, B is correct

Comprehension-3:

$f(x) = 8x^3 + 4ax^2 + 2bx + a,\ \ a, b \in R$

86. $f(x) = 8x^3 + 4x^2 + 2bx + 1$

$f'(x) = 24x^2 + 8x + 2b$

$= 8\left(3x^2 + x + \dfrac{1}{12}\right) + 2b - \dfrac{2}{3}$

$= 8\left(\sqrt{3}x - \dfrac{1}{2\sqrt{3}}\right)^2 + 2b - \dfrac{2}{3}$

$\Rightarrow \quad 2b - \dfrac{2}{3} > 0$

$\Rightarrow b > \dfrac{1}{3}$

Hence, C is correct

87. $f(x) = 8x^3 + 4ax^2 + 2x + a$

$f'(x) = 24x^2 + 8ax + 2 = 0$

$\Rightarrow 64a^2 - 8(24) \geq 0$

$\Rightarrow a^2 - 3 \geq 0$

$\Rightarrow a \in (\sqrt{3}, \infty) \cup (-\infty, -\sqrt{3})$

a can take $2,\ 3,\ 4, \cdots 100$

Hence, B is correct

88. Let roots be $\alpha,\ \beta,\ \gamma$

$\Rightarrow \alpha\beta\gamma = -\dfrac{a}{8}$

Taking log on both sides

$\log_2 \alpha + \log_2 \beta + \log_2 \gamma = \log_2(-a) - 3 = 5$

$\Rightarrow -a = 2^8 = -256$

Hence, D is correct

Comprehension-4:

$A=\{1,2,3,4,5\},\ B=\{-2,-1,0,1,2,3,4,5\}$

89. Increasing function form A to B

$^{8}C_{5}={}^{8}C_{3}=56$

Hence, D is correct

90. Non decreasing functions from A to B

$^{12}C_{5}=792$

Hence, C is correct

91. No of onto function A to A, such that $f(i)\neq i$ is equivalent to dearrangement of 5 distinct things = 44

Hence, A is correct

Comprehension-5:

92. $f(x)=2\left[e^{2x}\phi(x)-\dfrac{e^{2x}}{2}\right]$

Hence A is correct

93. Let $p(x)$ be y

$$\frac{dy}{dx}-y\geq 0$$

Integrating it using integrating factor.

$$e^{-x}y\geq 0$$

$$\Rightarrow\quad y\geq 0$$

Hence, A is correct

94. Let $H(x)=y$

$$\frac{dy}{dx}-2cxy>0$$

Integrating it

$$e^{-cx^{2}}y>0$$

$$H(x)>0$$

Hence, B is correct

Match the Columns for JEE Advanced

95.

A) $x=\dfrac{1}{1+t^{2}},\ y=\dfrac{1}{(1+t)^{2}}$

$\dfrac{dy}{dx}=0$ for all $t>0$

The possible value of x are (0, 1).

B) $A-\lambda I=\begin{bmatrix}1-\lambda & 3\\ 2 & 2-\lambda\end{bmatrix}$ is singular

$$\Rightarrow(1-\lambda)(2-\lambda)-6=0$$

$$\Rightarrow\lambda^{2}-3\lambda-4=0$$

$$\lambda=4\text{ or }-1$$

$$\Rightarrow\lambda^{2}-3\lambda-2=2$$

$$\log_{3}x^{2}-\log_{x}g$$

C) $|x-1|$

For $x>1$

$\Rightarrow x=2$ is one solution &

$$\log_{3}x^{2}-\log_{x}g-7=0$$

$$\Rightarrow 2\log_{3}x-2\log_{3}x-7=0$$

$$\Rightarrow 2\left(\log_{3}x\right)^{2}-7\left(\log_{3}x\right)-2=0$$

$$\Rightarrow\quad\log_{3}x=\frac{7\pm\sqrt{65}}{4}$$

$$\Rightarrow x=\frac{7+\sqrt{65}}{4}$$

So, $x=2\ \&\ \dfrac{7+\sqrt{65}}{4}$ are two values of x.

D) $x^{2}+y^{2}=l^{2}$. where $y=\sqrt{l^{2}-x^{2}}$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

$$4\times 2+\sqrt{25-16}\,\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=\frac{8}{3}\Rightarrow\lambda=3$$

96. A) $5^{200}=(25)^{100}=(8\times 3+1)^{100}$

It gives remainder 1 when ÷ by 8

B) $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin x}{\tan x+\cos^{2}(\tan x)}}=1$

C) $h(x)=f(\tan x)$

$r(x)=g(\sec x)$

$$\frac{dh(x)}{d(r(x))}=\frac{d(h(x))}{dx}\times\frac{dx}{d(r(x))}$$

$$=\frac{f'(\tan x)\sec^{2}x}{g'(\sec x)\sec x\tan x}$$

$$=\frac{-2\sqrt{2}\times\sqrt{2}}{4}=-1$$

D) $f(x) \le \sin 3x$ is increasing

When $3x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Longest interval $\rightarrow \frac{\pi}{3}$

$A \rightarrow R,\ B \rightarrow R,\ C \rightarrow P,\ D \rightarrow Q$

97.

A) $g(x) = f(4-x) + f(2+x)$

$g'(x) = f'(2+x) - f'(y-x) > 0$

$\Rightarrow 2 + x > 4 - x$

$x > 1$

B) $f(x) = x^3 - 3x + a$

$f'(x) = 3x^2 - 3$

$= 3\left(x^2 - 1\right)$

$a \in R$ for only one root of $f(x) = 0$

C) $f(x) = \cos x + a^2x + b$

$f'(x) = a^2 - \sin x > 0$

$\Rightarrow a^2 \ge 1$

$\Rightarrow a \in (-\infty, -1] \cup [1, \infty)$

D) $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$

$f'(x) = 2e^x + ae^{-x} + 2a + 1 > 0$

$\forall$ all positive values of 'a'

$A \rightarrow ST,\quad B \rightarrow PT,\quad C \rightarrow PQST,\quad D \rightarrow RST$

98. A) $f(x) = \dfrac{x}{1+x^2}$

$$f'(x) = \frac{\left(1+x^2\right) - 2x^2}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2} < 0$$

$\Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

B) $f(x) = \tan^{-1} x - x$

$$f'(x) = -\frac{x^2}{1+x^2} < 0$$

$\Rightarrow x \in R$

C) $f(x) = x - e^x + \tan\dfrac{2\pi}{7}$

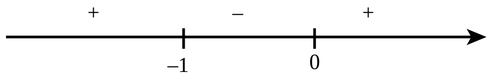
$f'(x) = 1 - e^x \ge 0$

$x \in (-\infty, 10]$

D) $f(x) = x^3 - \ln\left(1+x^3\right)$

$$f'(x) = 3x^2\left(1 - \frac{1}{1+x^3}\right)$$

$$= \frac{3x^5}{1+x^3}$$



$\Rightarrow f(x)$ is non negative for $x \in (-1, \infty)$

$A \rightarrow PS,\ \ B \rightarrow (PQRST),\ \ c \rightarrow (Q),\ \ D \rightarrow (T)$

99. A) $$\lim_{n\to\infty} \frac{1}{n} \sum \frac{1}{\sqrt{r/n}\,(3\sqrt{r/n}+4)^2} = \int_0^4 \frac{1}{\sqrt{x}\,(3\sqrt{x}+4)^2} dx$$

Put $x = t^2$

$$= 2\int_0^2 \frac{dt}{(3t+4)^2} = -\frac{2}{3}\left(\frac{1}{3t+4}\right)_0^2 = -\frac{2}{3}\left[\frac{1}{10} - \frac{1}{4}\right]$$

$$= \frac{1}{10} = \frac{p}{q}$$

$p + q = 11$

B) $f(x) = x^3 + ax + 2$

$f'(x) = 3x^2 + a = 0 \Rightarrow x = \pm\alpha$ where $\int = \sqrt{\dfrac{-a}{f}}$

$f\ |\alpha| \cdot \le f\ |-\alpha| > 0 \Rightarrow a > -3$

Hence $a = -2,\ -1$

C) eqn of circle $\Rightarrow x^2 + y^2 + 2gx + 2fy - 1 = 0$

As it passes through (1, 1) & (–2, 1)

$\Rightarrow 2g + 2f + 1 = 0$ &

$-4g + 2f + 4 = 0$

Solving, $g = \dfrac{1}{2},\quad f = -1$

$f - c = 0$

D) As $x^2 + y^2 + z^2 + 2xyz = -1$

$\Rightarrow \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi/2$

Differentiate

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$$

Review Exercises for JEE ADVANCED Solution

1. $f(x)=\frac{1}{3}\sin a\tan^3 x+(\sin a-1)\tan x+\frac{\sqrt{a-2}}{\sqrt{8-a}}$

$f'(x)=\sin a\tan^2 x\sec^2 x+(\sin a-1)\sec^2 x$

$=\sec^2 x\left(\sin a\sec^2 x-1\right)$

For critical pts,

$\sec^2 x=\operatorname{cosec} x$

$\tan^2 x=\operatorname{cosec} x-1$

$x=n\pi\pm\tan^{-1}(\operatorname{cosec} x-1),\ a\in[2,\pi)\cup(2\pi,8)$

For $[\pi,2\pi]\to f'(x)\neq 0$ or not defined

2. $f(x)=\left(a^2+a-6\right)\cos 2x+(a-2)x+\cos 1$

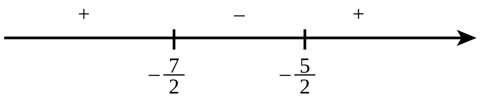
$f'(x)=-2\left(a^2+a-6\right)\sin 2x+a-2$

$=(a-2)(-2(a+3)\sin 2x+1)$

$\Rightarrow a\neq 2\ \&\ \sin 2x\neq\frac{1}{2(a+3)}$

$\Rightarrow -1<\frac{1}{2(a+3)}<1$

$\Rightarrow \frac{2a+7}{2(a+3)}>0$ or $\frac{2a+5}{2(a+3)}>0$



Hence, $a\in\left(-\frac{7}{2},-\frac{5}{2}\right)$

3. $f(x)=a\cdot 9^x+8(a-1)3^x+2(a-1)x\ln 3$

$f'(x)=\left(a\cdot 9^x\right)(2\ln 3)+8(a-1)3^x\ln 3+2(a-1)\ln 3$

$=2\ln 3\left[a\cdot 9^x+4(a-1)3^x+(a-1)\right]$

If $a\geq 1\Rightarrow f'(x)>0$

If $a\in(0,1)$

$2\ln 3\left[a\cdot\left(3^x\right)^2+2\cdot 2(a-1)3^x+a-1\right]>0$

$\Rightarrow at^2+4(a-1)t+(a-1)>0$ for $t\in[0,\infty)$

$\Rightarrow$ case-1 $D\leq 0$

$16(a-1)^2-4a(a-1)\leqslant 0$

$\Rightarrow 4(a-1)[3a-4]\leq 0$

$\Rightarrow a\in\phi$

case-2 both roots are negative

(i) $D\geq 0$

$\Rightarrow a\in(0,1)$

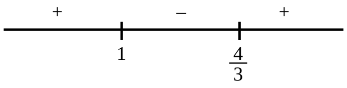
(ii) $\frac{-4(a-1)}{2a}<0$

$a\in\phi$

If $a<0$

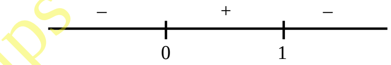
$a\cdot t^2+4(a-1)t+(a-1)>0$ will hold only

(i) $D\geq 0\Rightarrow 4(a-1)(3a-4)\geq 0$



(ii) $-\frac{b}{2a}>0$

$\Rightarrow\frac{-4(a-1)}{2a}>0$



$\Rightarrow a\in\phi$

Hence, $a\in[1,\infty)$ Ans

4. $f(x)=|x+1||x+2|$

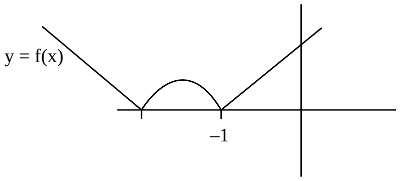
$$=\begin{cases}(x+1)(x+2), & x\leq -2\\ -(x+2)(x+1) & -2<x<-1\\ (x+1)(x+2) & x\geq -1\end{cases}$$

$f(x)$ is increasing for

$x\in\left(-2,-\frac{3}{2}\right)\cup(-1,\infty)$

& decreasing for

$x\in(-\infty,-2)\cup\left(-\frac{3}{2},-1\right)$



5. $x^2f'(x)+2xf(x)-x+1=0$

$\Rightarrow\frac{dy}{dx}+\frac{2y}{x}=\frac{x-1}{x^2}$

Integrating,

$x^2y=\int(x-1)\,dx=\frac{x^2}{2}-x+c$

At $x=1,\ y=0$

$x^2y=\left(\frac{x^2}{2}-x+\frac{1}{2}\right)$

$$y = \frac{x^2 - 2x + 1}{x^2} = \left(\frac{x-1}{x}\right)^2$$

$$y' = 2\left(\frac{x-1}{x}\right) \times \frac{1}{x^2}$$

$\Rightarrow f(x)$ is increasing for $x \in (-\infty, 0) \cup (1, \infty)$

6. Take $f(x) = x + \sqrt{1+x^2}$

$$f'(x) = \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}} > 0 \text{ for } x > 0$$

$\Rightarrow f(x)$ is increasing function $\forall x > 0$

Now $e^x \geq 1 + x$

$$\Rightarrow f\left(e^x\right) \geq f(1+x)$$

$$\Rightarrow e^x + \sqrt{1+e^{2x}} \geq 1 + x + \sqrt{2 + 2x + x^2}$$

Hence proved.

7. $f(x) = \left(1 - \dfrac{\sqrt{21 - 4b - b^2}}{b+1}\right) x^3 + 5x + \sqrt{6}$

$$f'(x) = 3\left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) x^2 + 5 > 0$$

$$\Rightarrow \quad 1 - \frac{\sqrt{21 - 4b - b^2}}{b+1} > 0$$

$$\Rightarrow \frac{(b+1) - \sqrt{21 - 4b - b^2}}{(b+1)} > 0$$

$$\Rightarrow \frac{(b+1)^2 - \left(21 - 4b - b^2\right)}{(b+1)\left(b + 1 + \sqrt{21 - 4b - b^2}\right)} > 0$$

$$= \frac{(b+5)(b-2)}{(b+1)\left(b + 1 + \sqrt{21 - 4b - b^2}\right)} > 0 \qquad \text{...(I)}$$

$$\& \; b^2 + 4b - 21 < 0 \Rightarrow (b+7)(b-3) < 0 \qquad \text{...(II)}$$

Solving (I) & (II)

$b \in (-7, -1) \cup (2, 3)$

8. Let $f(x) = 1 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$

$\Rightarrow f(x)$ is decreasing function

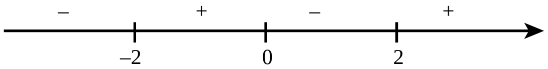
So, for $1 - f(x) - f^3(x) > f(1 - 5x)$

$\Rightarrow 1 - 5x > 1 - x - x^3$

$\Rightarrow x^3 - 4x > 0$

$\Rightarrow x(x-2)(x+2) > 0$

$\Rightarrow x \in (-2, 0) \cup (2, \infty)$



9. $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$

$f'(x) = 2e^x + ae^{-x} + (2a+1) > 0$

$\Rightarrow a \geq 0$

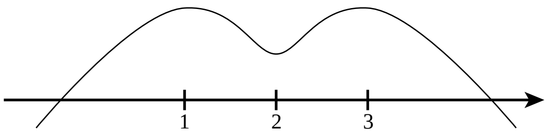
10. $\Rightarrow \dfrac{dy}{dx} = a(x-1)(x-2)(x-3)$

$$\frac{dy}{dx} = a\left(x^3 - 6x^2 + 11x - 6\right)$$

$$y = a\left(\frac{x^4}{4} - \frac{6}{3}x^3 + 11\frac{x^2}{2} - 6x\right) + c$$

$$y = a\left(x^4 - 8x^3 + 22x^2 - 24x\right) + 1$$

$\& \; a \in (-\infty, 0)$



11. $g(x) = \tan x - 4x$

$g'(x) = \sec^2 x - 4$

$$\Rightarrow g'(x) \quad < 0 \quad \forall \; x \in \left(-\frac{\pi}{3}, 0\right)$$

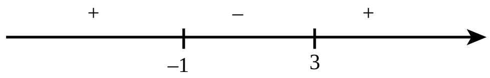
$\Rightarrow g(x)$ is decreasing

12. $f(x) = x^3 - 3x^2 - 9x + 20$

$f'(x) = 3x^2 - 6x - 9$

$= 3\left(x^2 - 2x - 3\right)$

= üüüü +

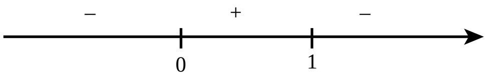


For $x > 4$, $f'(x) > 0 \Rightarrow f(x)$ is increasing

At $x = 4$, $f(4) = 0$

$\Rightarrow f(x)$ is always positive for all $x > 4$

13. $y = x \ln x - \dfrac{x^2}{2} + \dfrac{1}{2}$



$$\frac{dy}{dx} = 1 + \ln x - x$$

$$\frac{d^2y}{dx^2} = \frac{1-x}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} > 0 \quad \forall x \in (0,1)$$

$\Rightarrow \frac{dy}{dx}$ is increasing function

Value of $\frac{dy}{dx}$ at $x = 1 = 0$

$\Rightarrow \frac{dy}{dx} < 0$ & y is decreasing function

$$\Rightarrow x\ln x - \frac{x^2}{2} + \frac{1}{2} > 0$$

$$\Rightarrow x\ln x > \frac{x^2}{2} - \frac{1}{2}$$

Hence proved.

14. $f(x) = 4x^2 - 4px + p^2 - 2p + 2$

$f'(x) = 8x - 4p$

$f(x) = (2x-p)^2 + 2(1-p)$

Case 1: If $2 > p > 0$

$2(1-p) = 3$

$2 - 2p = 3$

$P = -1/2$

This is not valid.

Case 2: If $p > 2$

$(4-p)^2 + 2(1-p) = 3$

$p^2 - 10p + 15 = 0$

$$p = \frac{10 \pm \sqrt{40}}{2} = 5 + \sqrt{10}$$

Case 3: If $p < 0$

$p^2 - 2p + 2 = 3$

$p^2 - 2p - 1 = 0$

$$p = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$p = 1 - \sqrt{2}$

$\Rightarrow p = 1 - \sqrt{2}$ or $5 + \sqrt{10}$

15. $y = \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ln x$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$\Rightarrow y = \theta - \ln\tan\theta$

$$\frac{dy}{d\theta} = 1 - \frac{1}{\sin\theta\cos\theta}$$

$$= 1 - \frac{2}{\sin 2\theta}$$

$$\frac{dy}{dx} = \frac{\sin 2\theta - 2}{\sin 2\theta} \times \frac{1}{\sec^2\theta}$$

$$= \frac{\sin 2\theta - 2}{2\tan\theta} < 0$$

$\Rightarrow$ y is decreasing function

$$y_{max} = \sin^{-1}\frac{1}{2} - \ln\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6} + \ln\sqrt{3}$$

$$y_{min} = \frac{\pi}{3} - \ln\sqrt{3}$$

16. $y = (\sin x)^{\sin x}$

$$\frac{dy}{dx} = (\sin x)^{\sin x}[\cos x \ln\sin x + \cos x]$$

$$= (\sin x)^{\sin x}(\cos x)(\ln\sin x + 1)$$

y is decreasing function from $\left(0,\ \sin^{-1}\frac{1}{e}\right)$

& is increasing function from $\left(\sin^{-1}\frac{1}{e}, \frac{\pi}{2}\right)$

So, bigger value is $\left(\frac{1}{2}\right)^{1/2}$ as compared to $\left(\frac{1}{e}\right)^{1/e}$.

17. Let $f(x) = e^x(1-x)$

$f'(x) = e^x(1-x) - e^x$

$= -xe^x < 0 \quad \forall\ x \in (0,\infty)$

f(x) is decreasing

$\Rightarrow f(\cos x) < f(\sin x)$

$$\Rightarrow e^{\cos x}(1-\cos x) < e^{\sin x}(1-\sin x)$$

$$\Rightarrow e^{\cos x-\sin x} < \frac{1-\sin x}{1-\cos x}$$

18. $f(x) = \cos x - x\sin x$

$$f'(x) = -\sin x - \sin x - x\cos x$$

$$= -(2\sin x + x\cos x) < 0$$

f(x) is decreasing $\forall x \in (0, \pi/2)$

$$f(0) = 1, \quad f\left(\frac{\pi}{2}\right) = \frac{-\pi}{2}$$

So, f(x) has exactly one solution in $(0, \pi/2)$

19. $f(x) = \sin 2x - 8(b+2)\cos x - \left(4b^2+16b+6\right)x$

$$f'(x) = 2\cos 2x + 8(b+2)\sin x - \left(4b^2+16b+6\right) < 0$$

$$= 2 - 4\sin^2 x + 8(b+2)\sin x - \left(4b^2+16b+6\right) < 0$$

$$= 4\sin^2 x - 8(b+2)\sin x + 4b^2 + 16b + 4 > 0$$

$$= (2\sin x - 2(b+2))^2 - 4(b+2)^2 + 4\left(b^2+4b+1\right) > 0$$

$$= (2\sin x - 2(b+2))^2 - 12 > 0$$

$$\Rightarrow 4\left((\sin x - (b+2))^2 - 3\right) > 0$$

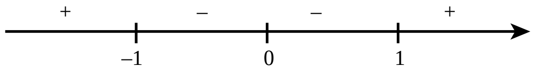
$$\Rightarrow \sin x - (b+2) > \sqrt{3} \text{ or } \sin x - (b+2) < -\sqrt{3}$$

$$\Rightarrow b < \sin x - 2 - \sqrt{3} \quad \text{or } b > \sin x - 2 + \sqrt{3}$$

$$\Rightarrow b < -3 - \sqrt{3} \text{ or } b > -1 + \sqrt{3}$$

20. $f(x) = \dfrac{1+x^2}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$



f(x) is increasing for $x \in (1, \infty)$

$$\Rightarrow f(e) < f(\pi)$$

$$\Rightarrow \frac{1+\pi^2}{\pi} \text{ is greater than } \frac{1+e^2}{e}$$

21. i) $y = x^3 - 5x^2 + 3x - 5$

$$\frac{dy}{dx} = 3x^2 - 10x + 3$$

$$= (x-3)(3x-1) = 0$$

$$\Rightarrow x = 3, \quad x = 1/3$$

$$\frac{d^2y}{dx^2} = 6x - 10 = 0$$

$$x = \frac{5}{3}$$

Point of inflection $\rightarrow \left(\dfrac{5}{3}, -\dfrac{250}{27}\right)$

Concave up $\rightarrow \left(\dfrac{5}{3}, \infty\right)$

Concave down $\rightarrow (-\infty, 5/3)$

ii) $y = (x+2)^6 + 2x + 2$

$$\frac{dy}{dx} = 6(x+2)^5 + 2$$

$$\frac{d^2y}{dx^2} = 30(x+2)^4$$

As $\dfrac{d^2y}{dx^2}$ will not change its sign, hence, there is no point of inflection.

iii) $y = \ln\left(1+x^2\right)$

$$y' = \frac{2x}{1+x^2}$$

$$y'' = \frac{2\left(1+x^2\right) - 4x^2}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

Point of inflection $\Rightarrow (1, \ln 2), (-1, \ln 2)$

Concave up $\rightarrow (-1, 1)$

Concave down $\rightarrow (-\infty, -1) \cup (1, \infty)$

iv) $y = e^{\tan^{-1} x}$

$$y' = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$y'' = \frac{1}{\left(1+x^2\right)^2}\left[e^{\tan^{-1} x}(1-2x)\right]$$

Point of inflection $\rightarrow \left(\dfrac{1}{2}, e^{\tan^{-1/2}}\right)$

Concave up $\rightarrow (-\infty, 1/2)$

Concave down $\rightarrow \left(\dfrac{1}{2}, \infty\right)$

22. Point of inflections are 1 & 7 as

$\frac{d^2y}{dx^2}$ is changing its sign.

23. $f''(x) = (x-a)^k g(x)$

a) If *k* is odd, $f''(x)$ will change its sign at $x = a$. Hence, x = a is point of inflection.

b) If k is even, there is no point of inflection

24. $x^2y + \alpha x + \beta y = 0$ (a)

$x^2y_1 + 2xy + \alpha + \beta y_1 = 0$ (I)

$x^2y_{11} + 2xy_1 + 2xy_1 + 2y + \beta y_{11} = 0$

$$y_{11} = -\frac{(2y + 4xy_1)}{x^2 + \beta}$$

For point of inflection $\Rightarrow y + 2xy_1 = 0$ (II)

Put $\left(2, \frac{5}{2}\right)$ in eq (I), eq (a)

$4y_1 + 10 + \alpha + \beta y_1 = 0$

$y_1 = -\frac{(10+\alpha)}{4+\beta}$ (III)

$10 + 2\alpha + \frac{5\beta}{2} = 0$

$20 + 4\alpha + 5\beta = 0$ (IV)

From (II) $y_1 = -\frac{5}{8}$

$\Rightarrow 80 + 8\alpha = 20 + 5\beta$

$60 + 8\alpha = 5\beta$ (V)

From (IV) & (V)

$\alpha = -\frac{20}{3}, \beta = \frac{4}{3}$

Put α, β, from $y_{11} = 0$

we get $\left(-2, -\frac{5}{2}\right)$ & $(0, 0)$ are other points of intersection

25. Prove $\rightarrow e^{\lambda(\pi - e)} > \lambda e^{\pi - e} + (1 - \lambda)$

Now, $f(x) = e^{\lambda x} - \lambda e^x + \lambda - 1$

$f'(x) = \lambda e^{\lambda x} - \lambda e^x$

$= \lambda e^x \left(e^{\lambda x - 1} - 1\right) < 0$

for $x \in (0,1)$

$f(x)$ is decreasing

$f(\pi - e) < f(0)$

$e^{\lambda(\pi - e)} - \lambda e^{(\pi - e)} + \lambda - 1 < 0$

Hence proved

26. $\lambda x_1^2 + \mu x_2^2 - (\lambda x_1 + \mu x_2)^2$

$\Rightarrow \lambda x_1^2 + \mu x_2^2 - \lambda^2 x_1^2 - \mu^2 x_2^2 - 2\lambda\mu x_1 x_2$

$\Rightarrow \lambda(1-\lambda)x_1^2 + \mu(1-\mu_1)x_2^2 - 2\lambda\mu x_1 x_2$

$\Rightarrow \lambda\mu\left(x_1^2 + x_2^2 - 2x_1x_2\right)$

$\Rightarrow \lambda\mu(x_1 - x_2)^2 \geq 0$

Hence Proved

27. Let 4 points P, Q, R, S on $y = \sin x$ as

$P \rightarrow (x_1, \sin x_1)$, $Q \rightarrow (x_2, \sin x_2)$, $R \rightarrow (x_3, \sin x_3)$,

$S \rightarrow (x_4, \sin x_4)$ where $0 < x_1, x_2, x_3, x_4 < \pi$

Now, centre of Quad P Q R S be

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{\sin x_1 + \sin x_2 + \sin x_3 + \sin x_4}{4}\right)$$

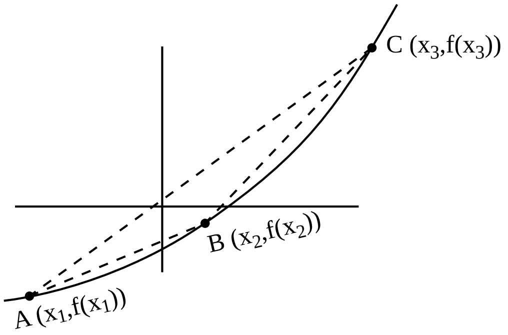
$$\Rightarrow \sin\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) > \frac{\sin(x_1) + \sin x_2 + \sin x_3 + \sin x_4}{4}$$

Now, put $x_4 = x_1$

$$\Rightarrow \sin\left(\frac{2x_1 + x_2 + x_3}{4}\right) > \frac{2\sin x_1 + \sin x_2 + \sin x_3}{4}$$

Hence proved.

28. y = f(x) ⇒



Now, In $f^{-1}(x)$, $P(f(x_1),x_1)$, $\theta(f(x_2),x_2)$, $R(f(x_3),x_3)$

be three points where $P\left(f^{-1}(x_1),x_1\right)$,

$Q\left(f^{-1}(x_2),x_2\right)$ & $R\left(f^{-1}(x_3),x_3\right)$

Now centroid of $\Delta PQR \to$

$$\left(\frac{f^{-1}(x_1)+f^{-1}(x_2)+f^{-1}(x_3)}{3}, \frac{x_1+x_2+x_3}{3}\right)$$

$$\Rightarrow \frac{f^{-1}(x_1)+f^{-1}(x_2)+f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1+x_2+x_3}{3}\right)$$

Hence proved.

29. For $y = \ln x + \frac{1}{x} - 1$

$y' = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$

Minimum value of y (x = 1) = 0

$\Rightarrow \ln x > 1 - \frac{1}{x}$

Now, $y = \ln x - x + 1$

$y' = \frac{1}{x} - 1 = \frac{1-x}{x}$

Max value at $x = 1 \Rightarrow 0$

$\ln x < x - 1$

Hence proved

30. i) $f(x) = e^x - 1 - x$

$f'(x) = e^x - 1 > 0$

f(x) is increasing function,

$e^x - 1 - x > 0$

ii) $y = e^x - x - \frac{x^2}{2} - 1$

$y' = e^x - 1 - x > 0$

$\Rightarrow y$ is increasing function $\forall\ x \geq 0$

$\Rightarrow e^x > x + \frac{x^2}{2} + 1$

iii) As $e^x \geq 1 + x$

Integrating $\int_0^x e^x dx \geq \int_0^x 1 + x dx$

$\Rightarrow e^x \geq 1 + x + \frac{x^2}{2}$

Integrating $\int_0^x e^x dx \geq \int_0^x \left(1 + x + \frac{x^2}{2}\right) dx$

$\Rightarrow e^x \geq 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}}$

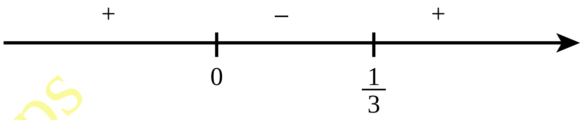
Recurring this process for n times,

$e^x \geq 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^n}{\underline{|n}}$

31. $f(x) = \left(3x^2 - 7x + 7\right)e^x$

$f'(x) = \left(3x^2 - 7x + 7 + 6x - 7\right)e^x$

$= (x(3x-1))e^x$



$\Rightarrow f(x)$ has minimum value at $x = \frac{1}{3}$

$f(1/3) = 5e^{1/3}$

$\Rightarrow f(x)$ has greater value at $x = \frac{2}{3}$

$f(2/3) = \frac{11}{3}e^{2/3}$

$\Rightarrow 5e^{1/3} < f(x) < \frac{11}{3}e^{2/3}$

32. Prove that $\cos x\sqrt{\sin x} \leq 2^{1/2}3^{-3/4}$

Let $f(x) = \cos x\sqrt{\sin x}$

$f'(x) = -\sin x\sqrt{\sin x} + \frac{\cos^2 x}{2\sqrt{\sin x}}$

$= \frac{\cos^2 x - 2\sin^2 x}{2\sqrt{\sin x}}$

$= \frac{3\cos^2 x - 2}{2\sqrt{\sin x}}$

$f\left(\cos^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{\frac{\sqrt{1-2}}{3}}$

$= \frac{\sqrt{2}}{3^{3/4}} = 2^{1/2}3^{-3/4}$

Target Exercises for JEE Advanced

1. i) $f(x)=2\sin a\cos x+\frac{1}{3}\cos 3x+\frac{1}{\sqrt{4a-a^2}}$

As $4a-a^2>0$

$a(4-a)>0$

$\Rightarrow a\in(0,4)$

$f'(x)=-2\sin a\sin x-\sin 3x$

$=-2\sin a\sin x-3\sin x+4\sin^3 x$

$=(+\sin x)\left(4\sin^2 x-3-2\sin a\right)$

As $a\cdot\varepsilon\left(\frac{\pi}{6},\frac{5\pi}{6}\right)\Rightarrow 4\sin^2 x-3-2\sin a\neq 0$

$\Rightarrow \sin x=0$

$\Rightarrow x=n\pi$

when $a\in\left(0,\frac{\pi}{6}\right]\cup\left[\frac{5\pi}{6},4\right)$

either $\sin x=0$ or $4\sin^2 x-3-2\sin a=0$

$\Rightarrow x=n\pi$ or $\sin^2 x=\frac{3+2\sin a}{4}$

$\Rightarrow x=n\pi$ or $\cos^2 x=\frac{1-2\sin a}{4}$

$\Rightarrow x=n\pi$ or $x=n\pi\pm\frac{1}{2}\cos^{-1}\left(-\frac{(1+2\sin a)}{2}\right)$

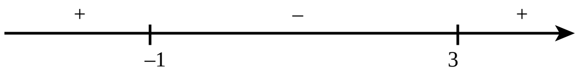
ii) $f(x)=\left(1-\frac{\cos a}{4}\right)\sin 2x+\frac{1}{8}\sin(\pi+4x)$

$+x\left(\frac{\cos a-3}{2}\right)+\sqrt{2a-a^2+3}$

As $2a-a^2+3\geq 0$

$\Rightarrow a^2-2a-3\leq 0$

$\Rightarrow (a-3)(a+1)\leq 0$



$\Rightarrow a\in[-1,3]$

$f'(x)=2\left(1-\frac{\cos a}{4}\right)\cos 2x+\frac{1}{2}\cos(\pi+4x)+\frac{\cos a-3}{2}=0$

Put $\cos 2x=t$

$t^2-2\left(1-\frac{\cos a}{4}\right)t-\left(\frac{\cos a+2}{2}\right)=0$

$t=\frac{1}{4}\left(4-\cos a\pm\sqrt{\cos^2 a}\right)$

$\cos 2x=\frac{1}{4}(4-\cos a\pm\cos a)$

$\cos 2x=1$ or $\cos 2x=1-\frac{\cos a}{2}$

$\Rightarrow x=n\pi$ or $n\pi\pm\frac{1}{2}\cos^{-1}\left(1-\frac{\cos a}{2}\right)\forall\ a\in\left(-1,\frac{\pi}{2}\right]$

For $a\in\left(\frac{\pi}{2},3\right)\Rightarrow\cos 2x=1$

$\Rightarrow x=n\pi$

2. $\psi(x)=\frac{1}{2}\sin x\tan x-\ln\sec x$

$\psi'(x)=\frac{1}{2}\sin x\sec^2 x+\frac{1}{2}\tan x\cos x-\tan x$

$=\frac{\sin x}{2}\left(\sec^2 x+1-\sec x\right)$

Now for $\sec^2 x-\sec x+1$

$\Rightarrow D<0\Rightarrow\ \sec^2 x-\sec x+1>0$

$\Rightarrow \psi'(x)>0\quad\forall\ x\in\left(0,\frac{\pi}{2}\right)$

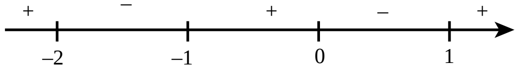
$\Rightarrow f(x)$ is increasing & $f(0)=0$

$\Rightarrow f(x)$ is always positive for $x\in(0,\infty)$

3. i) $f(x)=\int_{-1}^{x}\left(t^2+2t\right)\left(t^2-1\right)dt$

$f'(x)=\left(x^2+2x\right)\left(x^2-1\right)$

$=x(x+2)(x-1)(x+1)$



$f(x)$ is increasing in $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$

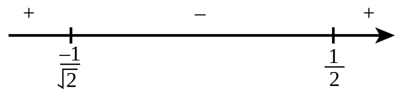
& decreasing in $(-2, -1) \cup (0, 1)$

ii) $f(x) = \int_0^x \left(2\sqrt{2}\sin^2 t + (2-\sqrt{2})\sin t - 1\right)dt$

$f'(x) = 2\sqrt{2}\sin^2 x + (2-\sqrt{2})\sin x - 1$

$= (2\sin x - 1)(\sqrt{2}\sin x + 1)$

Put $\sin x = t$



Now, $f(x)$ decreases in $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$

& increases in $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

4. As $g(x) = f(x+1) - f(x)$

$g'(x) = f'(x+1) - f'(x)$

As $x \to \infty$, $f'(x) \to 0$

As $x \to \infty$, $g'(x) \to 0$

As $x \to \infty$, $f(x+1) \to f(x)$

$\Rightarrow x \to \infty$, $f(x+1) - f(x) \to 0$

$\Rightarrow x \to \infty$, $g(x) \to 0$

5. $g(x) = \frac{f(x)}{x}$

$g'(x) = \frac{xf'(x) - f(x)}{x^2}$

Let $h(x) = xf'(x) - f(x)$

$h'(x) = f'(x) + xf''(x) - f'(x)$

$h'(x) = xf''(x) > 0 \quad \forall \ x \in [0, \infty)$

$\Rightarrow h(x)$ is increasing

$\Rightarrow$ Min value of $h(x) = h(0) = 0$

$\Rightarrow xf'(x) - f(x) > 0$

$\Rightarrow g'(x) > 0$

$\Rightarrow$ $g(x)$ is strictly increasing

6. Apply LMVT in $(0, x)$

$\Rightarrow f'(x_1) = \frac{f(x)}{x}$

$\Rightarrow |f(x)| = |xf'(x_1)| \le |Axf(x_1)|$

Apply LMVT in $(0, x_1)$

$\Rightarrow |f(x)| \le |A^2x^2f(x_2)|$

Recurring this,

$|f(x)| \le |A^n x^n|(x_n)|$

As $f(x)$ is bounded & $|x^n f(x_n)| \to 0$

$\Rightarrow |f(x)| \le 0$

$\Rightarrow f(x) = 0$

7. Take $g(x) = ax\sqrt{f(x)}$

$g'(x) = \frac{axf'(x)}{2\sqrt{f(x)}} + a\sqrt{f(x)}$

$= \frac{axf'(x) + 2af(x)}{2\sqrt{f(x)}} \ge 1$

$\Rightarrow g(x)$ is increasing function

$\Rightarrow g(x) \ge g(1)$

$\Rightarrow ax\sqrt{f(x)} \ge a\sqrt{f(1)}$

$\Rightarrow \sqrt{f(x)} \ge \frac{\sqrt{f(1)}}{x}$

8. Let $f(x) = \frac{x^2}{2x-1}$

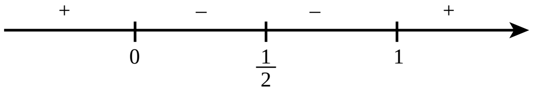
$f'(x) = \frac{2x(2x-1) - 2x^2}{(2x-1)^2}$

$= \frac{2x(x-1)}{(2x-1)^2}$

for $\left[\frac{3}{4}, 2\right] \Rightarrow$

$f\left(\frac{3}{4}\right) = \frac{9}{8}$, $f(1) = 1$, $f(2) = \frac{4}{3}$

$$\Rightarrow 1 \le \left(\frac{x^2}{2x-1}\right)^{1/3} \le \left(\frac{4}{3}\right)^{1/3}$$



9. Using LMVT

$f'(x_1)=\frac{f(x)}{x}$ where $x_1 \in (0,x)$

As $x_1 \to \infty$, $\quad x \to \infty$

as $x \to \infty$, $\quad f(x) \to 0$

$\Rightarrow \quad f'(x) \to 0$ as $x \to \infty$

10. Let $f(x)=\frac{\sin \pi x}{x(1-x)}$

$$f'(x)=\frac{\pi x(1-x)\cos \pi x-\sin \pi x(1-2x)}{(x(1-x))^2}$$

For $\quad f'(x)=0$

$$\pi x(1-x)\cos \pi x-\sin \pi x(1-2x)=0$$

$$\Rightarrow x=0 \text{ or } \frac{1}{2}$$

$$\Rightarrow \lim_{n\to 0} f(x) < f(x) \le f(1/2)$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin \pi x}{x(-x)} < f(x) \le 4$$

$$\Rightarrow \quad \pi < f(x) \le 4$$

11. $f(x)=\tan^{-1}x-x+\frac{1}{3}x^2$

$$f'(x)=\frac{1}{1+x^2}-1+\frac{2}{3}x$$

$$=\frac{3-3-3x^2+2x+2x^3}{\left(1+x^2\right)3}$$

$$=\frac{2x^3-3x^2+2x}{3\left(1+x^2\right)}$$

$$=\frac{x\left(2x^2-3x+2\right)}{3\left(1+x^2\right)}$$

$f(x) \ge f(0) \quad \forall \quad x \in [-1,1]$

$$\Rightarrow \tan^{-1}x-x+\frac{1}{3}x^2 \ge 0$$

$$\Rightarrow \tan^{-1}x > x-\frac{1}{3}x^2$$

Let $\quad g(x)=\tan^{-1}x-x+\frac{x^2}{3}-\frac{x^3}{5}$

$$g'(x)=\frac{1}{1+x^2}-1+\frac{2x}{3}-\frac{3x^2}{5}$$

$$=\frac{15-15-15x^2+10x+10x^3-9x^2-9x^4}{15\left(1+x^2\right)}$$

$$=\frac{-9x^4+10x^3-24x^2+10x}{15\left(1+x^2\right)}$$

$$=\frac{-x\left(9x^3-10x^2+24x-10\right)}{15\left(1+x^2\right)} \qquad \text{(I)}$$

Let $h(x)=9x^3-10x^2+24x-10$

$h'(x)=27x^2-20x+24>0$

$\Rightarrow \forall \ x \in (-1,1)$

$g(x)<g(0) \qquad$ as $\ g(0)$ is maximum value

$$\tan^{-1}x-x+\frac{x^2}{3}-\frac{x^3}{5}<0$$

$$\Rightarrow \tan^{-1}x < x-\frac{x^2}{3}+\frac{x^3}{5}$$

12. $f(x)=\frac{x}{ax^2+b} \quad = \frac{1}{ax+b/x}$

Apply AM ≥ GM $\quad$ for $x \in (0,\infty)$

$$f(x)=\frac{1}{ax+b/x} \quad \le \quad \frac{1}{2\sqrt{ab}}$$

13. Let $f(x)=\frac{x}{(1+x)\ln(1+x)}$

$$f'(x) \ =\frac{(1+x)\ln(1+x)-x(1+\ln(1+x))}{((1+x)\ln(1+x))^2}$$

$$=\frac{\ln(1+x)-x}{((1+x)\ln(1+x))^2} \qquad \text{(I)}$$

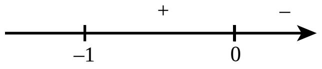
Let $g(x)=\frac{\ln(1+x)}{x}$

$$g'(x) = \frac{x/1+x-\ln(1+x)}{x^2}$$

$$g'(x) = \frac{x-(1+x)\ln(1+x)}{x^2+(1+x)} \qquad \text{(II)}$$

Now $h(x) = \ln(1+x) - x$

$$h'(x) = \frac{-x}{x+1}$$



$h(x)$ has maximum value at $x = 0$

$h(x) < h(0)$

$\Rightarrow h(x) < 0$

$\Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is decreasing

$\Rightarrow \quad f(x) < f(0) = 1$

$r(x) = x-(1+x)\ln(1+x)$

$r'(x) = 1-1-\ln(1+x) = -\ln(1+x) < 0$

$\Rightarrow r(x) < r(0) = 0$

$g'(x) < 0$

$g(x) < g(0) = 1$

$$\Rightarrow \frac{\ln(1+x)}{x} - \frac{x}{(1+x)\ln(1+x)} < 0$$

$\Rightarrow x^2 > (1+x)(\ln(1+x))^2$

14. Same as Q.13

15. $y = \dfrac{1}{\ln(1+x)} - \dfrac{1}{x}$

$$\frac{dy}{dx} = \frac{-1}{(\ln(1+x))^2} \times \frac{1}{1+x} + \frac{1}{x^2}$$

$$= \frac{(1+x)(\ln(1+x))^2 - x^2}{x^2(1+x)(\ln(1+x))^2}$$

As $(1+x)\ln(1+x)^2 - x^2 < 0$ (Proved in Q.13)

$\dfrac{dy}{dx} < 0$

$\Rightarrow y$ is a decreasing function for $x \in (-1, \infty)$ from $(1, 0)$

16. $f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$

$$f'(x) = \frac{1}{1+x^2} - \frac{(1+x^2)-2x^2}{(1+x^2)^2}$$

$$= \frac{2x^2}{(1+x^2)^2} \geq 0$$

$\Rightarrow f(x)$ is increasing function

$f(x) > f(0) \qquad \forall\ x \in (0, \infty)$

$$\tan^{-1} x > \frac{x}{1+x^2}$$

$$\Rightarrow \frac{x}{1+x^2} < \tan^{-1} x < x, \qquad x > 0$$

Now Put $x = \dfrac{1}{2}$ & $\dfrac{1}{3}$

$$\frac{2}{5} < \tan^{-1}\left(\frac{1}{2}\right) < \frac{1}{2} \qquad \text{(I)}$$

$$\frac{3}{10} < \tan^{-1}\left(\frac{1}{3}\right) < \frac{1}{3} \qquad \text{(II)}$$

Adding both eqn (I) & (II)

$$\frac{7}{10} < \tan^{-1}\frac{1}{2} + \tan^{-1}\left(\frac{1}{3}\right) < \frac{5}{6}$$

$$\frac{14}{5} < \pi < \frac{10}{3}$$

17. As $f''(x) > 0 \Rightarrow f'(x)$ is increasing function

Using LMVT,

$f'(x) = \dfrac{f(x)-f(a)}{x-a}$ is also an increasing function

18. $f(x) = \int_0^x \left(bt^2 + t\sin t\right) dt$

$f'(x) = bx^2 + x\sin x$

$= x(b+\sin x)$

For Monotonic $\rightarrow b \geq 1$

19. $h(x) = f(x) - (f(x))^2 + (f(x))^3$

$h(1) = 1$

$h'(x) = f'(x)\left(1-2f(x)+3f(x)^2\right) < 0$

$h(x)$ is a decreasing function

$y = \max\{h(x)\} = h(1) = 1$

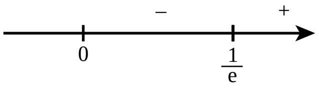
Now as $y=\frac{1}{2}[\sin x+|\sin x|]$

$$=\begin{cases} 0 & x\in\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},\pi\right]\Big] \\ 1 & x=\pi/2 \\ 1 & \left(\frac{\pi}{6},\frac{5\pi}{6}\right) \\ 1/2 & x=\pi/6 \end{cases}$$

$\Rightarrow y=1$ is tangent at infinitely many points

20. $y=x\ln x$

$$\frac{dy}{dx}=1+\ln x$$



$\lim\limits_{x\to0^+} x\ln x\to0,\ \lim\limits_{x\to\infty} x\ln x\to\infty$

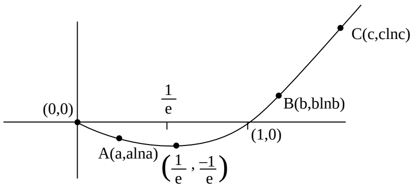
At $x=\frac{1}{e}, y=\frac{-1}{e}$

Centroid of $\Delta ABC\to\left(\frac{a+b+c}{3},\frac{a\ln a+b\ln b+c\ln c}{3}\right)$

$$\Rightarrow\quad f\left(\frac{a+b+c}{3}\right)<\frac{a\ln a+b\ln b+c\ln c}{3}$$

$$\Rightarrow\frac{(a+b+c)}{3}\ln\left(\frac{a+b+c}{3}\right)<\frac{\ln a^a+\ln b^b+\ln c^c}{3}$$

$$\Rightarrow\left(\frac{a+b+c}{3}\right)^{a+b+c}<a^a\,b^b\,c^c$$



21. $g'(x)=f(x)$ ...(I)

As

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{f(h)-1}{h}+2x$$

$f'(x)=f'(0)+2x$

Integrating wrt x.

$f(x)=x^2+f'(0)x+1$

$D=(f'(0))^2-4$

$=\ a-a^2-1$

$=-\left(a^2-a+1\right)<0$

$\Rightarrow f(x)>0$

$\Rightarrow g'(x)>0$

$\Rightarrow g(x)$ is increasing function.

22. i) $g(x)=\frac{f(x)}{x}$

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}\qquad\text{...(I)}$$

$h(x)=xf'(x)-f(x)$

$h'(x)=xf''(x)>0$

$\Rightarrow h(x)>h(0)=0$

$\Rightarrow g'(x)>0$

$\Rightarrow g(x)$ increases with x.

ii) $h(x)=\frac{f(x)}{g(x)}$

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$$

Let $r(x)=g(x)f'(x)-f(x)g'(x)$

$r'(x)=\ g(x)f''(x)-f(x)g''(x)>0$

$\Rightarrow r(x)$ is increasing

$\Rightarrow g(x)f'(x)-f(x)g'(x)>0$

$\Rightarrow h'(x)>0$

$\Rightarrow\frac{f(x)}{g(x)}$ is increasing

As $\frac{x}{\sin x}$ is always increasing.

Using above relation, $\frac{x^2/2}{1-\cos x}$ is also increasing & so on.

23. i) Let $f(x)=e^{-x}-1+x$

$f'(x)=-e^{-x}+1>0$

$f(x) > f(0)$

$e^{-x} > 1 - x = p_1(x)$

ii) Integrate $\int_0^{+x} e^{-x}dx > \int_0^{+x} 1 - x\,dx$

$\Rightarrow -e^{-x} + 1 > x - \frac{x^2}{2}$

$\Rightarrow e^{-x} < 1 - x + \frac{x^2}{2}$

III) Integrate $\int_0^{x} e^{-x} < \int_0^{x}\left(1 - x + \frac{x^2}{2}\right)dx$

$= -e^{-x} + 1 < x - \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}$

$\Rightarrow e^{-x} > 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3}$

IV) Continue integrating for $p_1(x)$ & $p_8(x)$ & put $x = 1$ to get $e = 2.718$.

24. i) $f(x) = e^x - 1 - x$

$f'(x) = e^x - 1$

$f'(x) \geq 0 \quad \forall x \in (0, \infty)$

& $f'(x) \leq 0 \quad \forall x \in (-\infty, 0]$

Now, $f(x) > f(0) \quad \forall \; x \geq 0$

$e^x > 1 + x$

Now, put $x = -x$

$e^{-x} > 1 - x, \quad \forall x \leq 0$

ii) $\int_0^x e^x dx > \int_0^x (1 + x)\, dx$

$e^x - 1 > x + \frac{x^2}{2}$

$e^x > 1 + x + \frac{x^2}{\lfloor 2}$

As $\int_0^x e^{-x} > \int_0^x 1 - x$

$-e^{-x} + 1 > x - \frac{x^2}{2}$

$e^{-x} < 1 - x + \frac{x^2}{2}$

III) $\int_0^x e^x dx > \int_0^x \left(1 + x + \frac{x^2}{\lfloor 2}\right)dx$

$e^x - 1 > x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}$

$e^x > 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}$

As $\int\limits_0^x e^{-x}dx < \int_0^x \left(1 - x + \frac{x^2}{\lfloor 2}\right)dx$

$\Rightarrow -e^{-x} + 1 < x - \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}$

$e^{-x} > 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3}$

IV) Generalising, we get

$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\infty$

& $e^{-x} = 1 + x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \cdots\infty$

25. $\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$ if $x > 0$

Put $\frac{1}{x} = t$

Let $f(t) = \ln(1+t) - \frac{t}{1+t}$

$f'(t) = \frac{1}{1+t} - \frac{1}{(1+t)^2} > 0$ for all values of $t > 0$

$\Rightarrow f(t)$ is increasing function.

so min value at $t = 0$

$\Rightarrow f(0) = 0$

$\Rightarrow \quad \ln(1+t) > \frac{t}{1+t}$ .... (1)

Now, let $f(t) = \ln(1+t) - t$

$f'(t) = \frac{1}{1+t} - 1$

$= \frac{-t}{1+t} < 0$

$\Rightarrow$ f(t) is decreasing function, f(0) = 0

$\Rightarrow \ln(1+t) - t < 0$

$\Rightarrow \ln(1+t) < t$ .... (2)

So from (1) & (2)

$\frac{t}{1+t} < \ln(1+t) < t$

$\Rightarrow \frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$

Hence proved.

26. $f(x) = e^x\left(x^2 - 6x + 12\right) - \left(x^2 + 6x + 12\right)$

$$f'(x)=e^{x}\left(x^{2}-6x+12\right)-e^{x}\left(2x-6\right)-2x-6$$

$$=e^{x}\left(x^{2}-4x+6\right)-2(x+3)$$

$$f'(x)=e^{x}\left((x-2)^{2}+2\right)-2(x+3)$$

$$f''(x)=e^{x}(2x-4)+e^{x}(x^{2}-4x+6)-2$$

$$=e^{x}\left(x^{2}-2x+2\right)-2$$

$$=e^{x}\left((x-1)^{2}+1\right)-2$$

$$f'''(x)=e^{x}\left(x^{2}-2x+2\right)+e^{x}(2x-2)$$

$$=e^{x}x^{2}>0$$

$\Rightarrow f''(x)$ is an increasing function from $(-z,\infty)$

$\Rightarrow f''(x)=0$ at $x=0$

$f'(x)$ will have minimum at $x=0$

$\Rightarrow f'(0)=0$

$\Rightarrow \quad f'(x)>0$

Now, $\dfrac{e^{x}(x-2)+(x+2)}{x^{2}\left(e^{x}-1\right)}=g(x)$

$$g'(x)=\frac{x^{2}\left(e^{x}-1\right)\left[e^{x}(x-1)+1\right]-\left(x^{2}\left(e^{x}\right)+2x\left(e^{x}-1\right)\right)\left[e^{x}(x-2)+(x+2)\right]}{\left(x^{2}\left(e^{x}-1\right)\right)^{2}}$$

At $g'(0)=0$ at $x=0$ &

$g(x)<g(0)$

$$g(0)=\lim_{x\to 0}\frac{e^{x}+e^{x}(x-2)+1}{2x\left(e^{x}-1\right)+x^{2}\left(e^{x}\right)}\ \text{(Apply L-Hospital Rule)}$$

$$\lim_{x\to 0}\frac{e^{x}x}{2\left(e^{x}-1\right)+2xe^{x}+x^{2}e^{x}+2xe^{x}}$$

$$\Rightarrow \frac{1}{2+2+2}=\frac{1}{6}\Rightarrow g(x)<\frac{1}{6}$$

27. $f(x)=\dfrac{1}{\sin\left(\dfrac{\pi}{3}+x\right)}+\dfrac{1}{\sin\left(\dfrac{\pi}{3}-x\right)}$

$$=\frac{\sin(\pi/3-x)+\sin(\pi/3+x)}{\sin^{2}\dfrac{\pi}{3}-\sin^{2}x}$$

$$=\frac{4\sqrt{3}\cos x}{3-4\sin^{2}x}$$

At x = 0, f(x) is maximum.

$$f(0)=\frac{4\sqrt{3}}{3}$$

28. $y=ax^{3}+3bx^{2}+3cx+d$ ...(I)

$y'=3ax^{2}+6bx+3c$

$y''=6ax+6b$

$\Rightarrow x=\dfrac{-b}{a}$ is point of inflection

If a function is centro symmetric

$\Rightarrow f(x)=-f(-x)$

$\Rightarrow$ we need to do transformation of axis so that point of inflection becomes zero

$$x_{1}=x+\frac{b}{3a},\quad y_{1}=y-\frac{(2b^{3}-9abc+27a^{2}d)}{27a^{2}}$$

Putting x, y in eq (I)

$$y_{1}+\frac{\left(2b^{3}-9abc+27a^{2}d\right)}{27a^{2}}=a\left(x_{1}-\frac{b}{3a}\right)^{3}+b\left(x_{1}-\frac{b}{3a}\right)^{2}+c\left(x_{1}-\frac{b}{3a}\right)+d$$

upon solving

$$y_{1}=ax_{1}^{3}+\left(\frac{3ac-b^{2}}{3a}\right)x_{1}$$

which satisfies $g\left(-x_{1}\right)=-g\left(x_{1}\right)$

Hence, y is centrosymmetric about its point of inflection

29. As circle is tangent to parabola at (1,2)

$\Rightarrow (1-h)^{2}+(2-k)^{2}=a^{2}$ (I)

As $(x-h)^{2}+(y-k)^{2}=a^{2}$

$$2(x-h)+2(y-k)\frac{dy}{dx}=0$$

$\dfrac{dy}{dx}=\dfrac{h-1}{2-k}$ (II)

As $y=x^{2}+1$

$\dfrac{dy}{dx}=2x=2$ (III)

from (II) & (III)

$h-1=2(2-k)$

Putting this into eq (I)

$$4(2-k)^2+(2-k)^2=a^2$$

$$2-k=\pm\frac{a}{\sqrt{5}}$$

$$k=2\pm\frac{a}{\sqrt{5}}$$

$$h=1\mp\frac{2a}{\sqrt{5}}$$

As $\frac{dy}{dx}=\frac{h-x}{y-k}$ or $(y-k)\frac{dy}{dx}+(x-h)=0$

Again differentiating it

$$\left(\frac{dy}{dx}\right)^2+(y-k)\frac{d^2y}{dx^2}+1=0$$

$$\left(\frac{dy}{dx}\right)^2+2(y-k)+1=0$$

$$\left(\frac{h-x}{y-k}\right)^2+2(y-k)+1=0$$

$$(h-1)^2+2(2-k)^3+(2-k)^2=0$$

$$\Rightarrow \frac{4a^2}{5}\pm\frac{2a^2}{5}\times\frac{a}{\sqrt{5}}+\frac{a^2}{5}=0$$

$$\Rightarrow a^2\pm\frac{2a^3}{5\sqrt{5}}=0$$

$$\Rightarrow a^2\left(1\pm\frac{2a}{5\sqrt{5}}\right)=0$$

$$\Rightarrow a=\pm\frac{5\sqrt{5}}{2}$$

$$\Rightarrow a=\frac{5\sqrt{5}}{2},\quad h=-4,\quad k=\frac{9}{2}$$

30. $y=\frac{x^3-x}{3x^2+1}$

$$\frac{dy}{dx}=\frac{\left(3x^2+1\right)\left(3x^2-1\right)-\left(x^3-x\right)(6x)}{\left(3x^2+1\right)^2}$$

$$=\frac{9x^4-1-6x^4+6x^2}{\left(3x^2+1\right)^2}$$

$$=\frac{3x^4+6x^2-1}{\left(3x^2+1\right)^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(3x^2+1\right)^2\left(12x^3+12x\right)-2\left(3x^2+1\right)(6x)\left(3x^4+6x^2-1\right)}{\left(3x^2+1\right)^4}$$

For Point of inflection, $\frac{d^2y}{dx^2}=0$

$$\Rightarrow 12\left(3x^2+1\right)x\left[\left(3x^2+1\right)\left(x^2+1\right)-3x^4-6x^2+1\right]=0$$

$$=12\left(3x^2+1\right)x\left(-2x^2+2\right)=0$$

$$\Rightarrow x=0,\ \pm1\ \ \&\ y=0$$

At $(0,0)$, $m=-1\Rightarrow \boxed{y+x=0}\rightarrow$ Eqn of tangent

At $(\pm1,0)$, $m=\frac{1}{2}\Rightarrow y=\frac{x}{2}\pm\frac{1}{2}$

$\Rightarrow \boxed{2y=x\pm1}\rightarrow$ Eqn of tangent

31. $y=\frac{x+1}{x^2+1}$

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)-(x+1)(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{-x^2-2x+1}{\left(x^2+1\right)^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(x^2+1\right)^2(-2x-2)+2\left(x^2+1\right)(2x)\left(x^2+2x-1\right)}{\left(x^2+1\right)^4}$$

For point of inflection,

$$\Rightarrow 2\left(x^2+1\right)\left[-\left(x^2+1\right)(x+1)+2x^3+4x^2-2x\right]=0$$

$$\Rightarrow 2(x^2+1)(x^3+3x^2-3x-1]=0$$

$$\Rightarrow x^3+3x^2-3x-1=0$$

$$\Rightarrow (x-1)\left(x^2+4x+1\right)=0$$

$$\Rightarrow\quad x=1\ \&\ x=-2\pm\sqrt{3}$$

A: For $x=1\Rightarrow y=1$

B: For $x=-2+\sqrt{3} \Rightarrow y=\frac{\sqrt{3}-1}{8-4\sqrt{3}}$

C: For $x=-2-\sqrt{3} \Rightarrow y=\frac{-\sqrt{3}-1}{8+4\sqrt{3}}$

Slope of AB $=\frac{\sqrt{3}-1-8+4\sqrt{3}}{(8-4\sqrt{3})(\sqrt{3}-3)}$

$=\frac{5\sqrt{3}-9}{4(2-\sqrt{3})\sqrt{3}(1-\sqrt{3})}=\frac{1}{4}$

Slope of AC $=\frac{-\sqrt{3}-1-8-4\sqrt{3}}{4(2+\sqrt{3})(-3-\sqrt{3})}$

$=\frac{\sqrt{3}(5+3\sqrt{3})}{4\sqrt{3}(2+\sqrt{3})(1+\sqrt{3})}=\frac{1}{4}$

Hence, A, B, C lie on the line.

32. $y=x\sin x$

$\frac{dy}{dx}=\sin x+x\cos x$

$\frac{d^2y}{dx^2}=\cos x+\cos x-x\sin x=2\cos x-x\sin x$

Now, point of inflection lies on $y = x\sin x$

& $2\cos x-x\sin x=0$

$\Rightarrow \quad x\sin x=y \;\&\; y=2\cos x$

$\Rightarrow 4y^2+(xy)^2=4x^2$

$\Rightarrow y^2\left(x^2+y\right)=4x^2$

33. $y=x^3+x^2-x-1,\ y=2\left(x^2-x^3+x-1\right)$

$y=\left(x^2-1\right)(x+1),\ \ y=-2(x-1)\left(x^2-1\right)$

Point of Contact $\Rightarrow$

$\Rightarrow\left(x^2-1\right)(x+1)=-2(x-1)\left(x^2-1\right)$

$\Rightarrow\left(x^2-1\right)(3x-1)=0$

$\Rightarrow x=\pm 1,\frac{1}{3}$

$y=x^3+x^2-x-1$

$\frac{dy}{dx}=3x^2+2x-1$

$\frac{dy}{dx}$ at $x=1,-1,\frac{1}{3}$ are $4,0,0$ respectively.

$y=2x^2-2x^3+2x-2$

$\frac{dy}{dx}=4x-6x^2+2$

$\frac{dy}{dx}$ at $x=1,-1,\frac{1}{3}$ are $0,\,-8,\,\frac{8}{3}$ respectively.

Both curves cross one another.

34. As $f\left(\frac{x^2}{f(x)}\right)=x$

$f'\left(\frac{x^2}{f(x)}\right)\left(\frac{2xf(x)-x^2f'(x)}{f^2(x)}\right)=1$

Previous Year's Question Solution (JEE Advanced)

1. $f(x)=\cos\ln x-\ln\cos x$

Now, $e^{-\pi/2}<\theta<\frac{\pi}{2}$

$-\frac{\pi}{2}<\ln\theta<\ln\frac{\pi}{2}$

$0<\cos\ln\theta<\cos\ln\pi/2$ ...(1)

As, $\quad -1\le\cos\theta\le 1$

$\quad -\infty<\ln\cos\theta\le 0$

$\Rightarrow \ln\cos\theta\le 0$ ...(2)

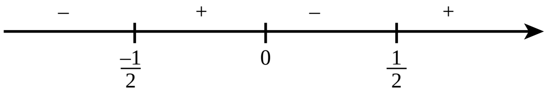
From (1) & (2)

$\cos\ln\theta>\ln\cos\theta$

2. $y=2x^2-\ln|x|$

$\frac{dy}{dx}=4x-\frac{1}{x}=\frac{(2x+1)(2x-1)}{x}$

Critical points are $0,\frac{1}{2},\frac{-1}{2}$



$f(x)$ increases in $\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$ and decreases in $\left(-\infty,\frac{-1}{2}\right)\cup\left(0,\frac{1}{2}\right)$.

3. $f(x)=\log(1+x)-x,\quad x>-1$

$f'(x)=\frac{-x}{1+x}$

$f'(x)>0\quad \forall\ x\in(-1,\,0)$

$f'(x)<0$ for $x>0$

Also, $f(0)=0$

$\Rightarrow f(x)\le f(0)$

$\log(1+x)-x\le 0$

$\Rightarrow \log(1+x)\le x\quad \forall x\ge 0$

4. Since g is decreasing in $[0,\infty)$

For $x\ge y\ge 0,\ g(x)\le g(y)$

Also, $g(x),\ g(y)\in[0,\infty)$ & f is increasing from $[0,\infty)$

For $g(x)\in[0,\infty)$

$g(x)\le g(y)$

or $f(g(x)\le f(g(y))$ where $x\ge y$

or $h(x)\le h(y)$

Thus $h(x)$ is decreasing in $[0,\infty)$

$\Rightarrow h(x)\le h(0)$ for $x\ge 0$

$h(x)\le 0\quad \forall\, x\ge 0$

Also $h(x)\ge 0\quad \forall\, x\ge 0$

$\Rightarrow h(x)=0\quad \forall\, x\ge 0$

$\Rightarrow h(x)-h(1)=0\quad \forall\, x\ge 0$

Hence, A is correct

5. $f(x)=\begin{cases}3x^2+12x-1, & -1\le x\le 2\\ 37-x, & 2<x\le 3\end{cases}$

In $x\in[-1,2],\quad f'(x)=6x+12=0$

$x=-2$

$f\left(2^-\right)=35,\ \ f\left(2^+\right)=35$

$\Rightarrow f(x)$ is continuous.

$f'(x)=\begin{cases}6x+12 & -1\le x\le 2\\ -1 & 2<x\le 3\end{cases}$

$f'\left(2^-\right)=24,\quad f'\left(2^+\right)=-1$

$\Rightarrow f(x)$ is not derivable at $x=2$.

Also $f\left(2^+\right)<f(2)$ & $f(2^-)<f(2)\Rightarrow x=2$ is maxima.

Hence, D is correct

6. $f(x)=2x^3-15x^2+36x+1$

$f'(x)=6(x-2)(x-3)$

f(x) is increasing in [0, 2] & decreasing in [2, 3]
Therefore, f(x) is many one.
$f(0)=1,\ f(2)=29,\ f(3)=28$

$\Rightarrow$ Range $=[1,\,29]$

$\Rightarrow f(x)$ is many one onto function.
Hence, D is correct

7. $f'(x)=\dfrac{\dfrac{\log(e+x)}{\pi+x}-\dfrac{\log(\pi+x)}{e+x}}{\left(\log(e+x)\right)^2}$

$=\dfrac{(\log(e+x))(e+x)-(\pi+x)\log(\pi+x)}{(\pi+x)(e+x)(\log(e+x))^2}$

Since, log is an increasing function & $e<\pi$.

$\log(e+x)<\log(\pi+x)$.

$\Rightarrow (e+x)\log(e+x)<(e+x)\log(\pi+x)$
$\qquad<(\pi+x)\log(\pi+x)\ \forall\, x>0$

$\Rightarrow f'(x)<0$

$\Rightarrow f(x)$ is decreasing for $x\in(0,\infty)$

Hence, B is correct

8. $f(x)=\frac{x}{\sin x},0<x\le 1$

$f'(x)=\frac{\sin x-x\cos x}{x^2}=\frac{\cos x(\tan x-x)}{x^2}$

$\Rightarrow f'(x)>0\quad \forall\, x\in(0,1]$

$\Rightarrow f(x)$ is increasing function.

$g(x)=\frac{x}{\tan x}$

$g'(x)=\frac{\tan x-x\sec^2 x}{\tan^2 x}=\frac{\sin 2x-2x}{2\sin^2 x}<0$

$\Rightarrow g\left(x\right)$ is decreasing function

Hence, C is correct

9. $f(x) = \sin^4 x + \cos^4 x = \frac{3}{4} + \frac{1}{4}\cos 4x$

$f'(x) = -\sin 4x$

Now $f'(x) > 0$,

$\Rightarrow -\sin 4x > 0$

$\Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

Hence, B is correct

10. As sin x & cos x in $\left(\frac{\pi}{2}, \pi\right)$ are decreasing function

Therefore, S is correct

Let us $f(x) = \sin x$

$f'(x) = \cos x$

Now $f(x)$ is increasing in $(0, \frac{\pi}{2})$

but $f'(x)$ is decreasing in $(0, \frac{\pi}{2})$

Therefore, R is wrong

Hence, D is correct

11. $f(x) = \int e^x (x-1)(x-2)dx$

For decreasing function, $f'(x) < 0$

$e^x (x-1)(x-2) < 0$

$\Rightarrow x \in (1, 2)$

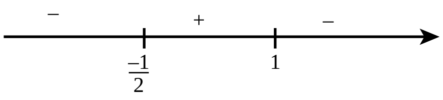
Hence, C is correct

12. $f(x) = x\ e^{x(1-x)}$

$f'(x) = e^{x(1-x)} + (1-2x)x\ e^{x(1-x)}$

$= -e^{x(1-x)}(2x+1)(x-1)$

sign scheme of $f'(x)$



$\Rightarrow$ g(x) is increasing in $\left[\frac{-1}{2}, 1\right]$

Hence, A, is correct

13. $f(x) = x^2 + bx - b$

$f'(x) = 2x + b$

$\left.\frac{dy}{dx}\right|_{x=1} = b + 2$

$y - 1 = (b+2)(x-1) \rightarrow$ Eqn of tangent

$x \text{ int } = \left(1 - \frac{1}{b+2}, 0\right)$

$y \text{ int } = (0, -b-1)$

$\text{Area} = \frac{1}{2}\left|\frac{(b+1)(b+1)}{(b+2)}\right| = 2$

$\Rightarrow \quad b^2 + 1 + 2b = -4b - 8$

$\Rightarrow b^2 + 6b + 9 = 0$

$\Rightarrow (b+3)^2 = 0$

$b = -3$

Hence, C is correct

14. $y = 3\sin x - 4\sin^3 x = \sin 3x$

$\frac{dy}{dx} = 3\cos 3x$

Largest interval for $\cos x$ to be negative is π

Hence, A is correct

15. $g(\mu) = 2\tan^{-1} e^{\mu} - \frac{\pi}{2}$

$g(-\mu) = 2\tan^{-1} e^{-\mu} - \frac{\pi}{2}$

$= 2\cot^{-1} e^{\mu} - \frac{\pi}{2}$

$= 2\left(\frac{\pi}{2} - \tan^{-1} e^{\mu}\right) - \frac{\pi}{2}$

$= \frac{\pi}{2} - 2\tan^{-1} e^{\mu}$

$= -g(\mu)$

So, $g(\mu)$ is an odd function

$g'(\mu) = \frac{2\ e^{\mu}}{1 + e^{2\mu}} > 0$

Hence, C is correct

Comprehension

$f(x) = 1 + 2x + 3x^2 + 4x^3$

16. f(x) is increasing function

$f\left(\frac{-3}{4}\right) f\left(\frac{-1}{2}\right) < 0$

$f'(x) = 12x^2 + 6x + 2 > 0 \quad x \in R$

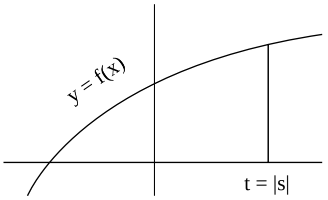
$\Rightarrow f(x)$ has only one root lying in $\left(\frac{-3}{4}, \frac{-1}{2}\right)$

Hence, C is correct

17. $A = g(t) = \int_{\sigma}^{t} f(x)dx$

$= t + t^2 + t^3 + t^4$

$\Rightarrow \frac{t\left(1-t^4\right)}{1-t}$



$g'(t) = f(x) > 0, \ x > 0$

$\Rightarrow g(t)$ is increasing $\forall \ t > 0$

$g\left(\frac{1}{2}\right) < A < g\left(\frac{3}{4}\right)$

$\frac{15}{16} < A < \frac{525}{256}$ which lies in $\left(\frac{3}{4}, 3\right)$

Hence, A is correct

18. $f'(x) = 12x^2 + 6x + 2$

$f''(x) = 24x + 6$

For $f''(x) > 0$

$x > -\frac{1}{4}$

For $f''(x) < 0$

$x < -\frac{1}{4}$

Hence, B is correct

19. $h(x) = f(x) - f^2(x) + f^3(x)$

$h'(x) = f'(x)\left[1 - 2f(x) + 3f^2(x)\right]$

Let $f(x) = t$

$\Rightarrow$ $3t^2 - 2t + 1$ will always be positive

So, sign of $h'(x)$ will depend only on $f'(x)$.

Hence, A, C is correct

20. Let $f(x) = x^{1/x}$

$\Rightarrow \log f(x) = \frac{1}{x}\log x$

Differentiating wrt x .

$\Rightarrow \quad f'(x) = \frac{x^{1/x}}{x^2}(1 - \log x)$

Obviously, for $x > e, \quad f'(x) < 0$

$\Rightarrow \ f(\pi) < f(e)$

$\Rightarrow \ \pi^{\frac{1}{\pi}} < e^{\frac{1}{e}}$

$\Rightarrow \pi^e < e^{\pi}$

21. $ax^2 + \frac{b}{x} \geq c \quad \forall \ \ x > 0, \ a > 0, \ b > 0$

Let $f(x) = ax^2 + \frac{b}{x} - c$

$\Rightarrow f'(x) = 2ax - \frac{b}{x^2} = 0$

$\Rightarrow x = \left(\frac{b}{2a}\right)^{1/3}$

$f''(x) = 2a + \frac{2b}{x^3}$

$f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 6a > 0$

Therefore, at $x = \left(\frac{b}{2a}\right)^{1/3}$, f is minimum

As $a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{\left(b/2a\right)^{1/3}} \geq c$

$\Rightarrow \frac{a\left(b/2a\right) + b}{\left(b/2a\right)^{1/3}} \geq C$

Taking cube on both sides

$\frac{27b^3}{8} \times \frac{2a}{b} \geq C^3$

$27ab^3 \geq 4c^3$

22. $$f(x)=1+x\ln\left(x+\sqrt{x^2+1}\right)-\sqrt{x^2+1}$$

$$f'(x)=\ln\left(x+\sqrt{1+x^2}\right)$$

as $x+\sqrt{x^2+1}\ge 1$ for $x\ge 1$

$\Rightarrow f'(x)\ge 0$

$\Rightarrow$ For $x\ge 0, f(x)\ge f(0)$

$$1+x\ln\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2}\ge 0$$

$$\Rightarrow 1+x\ln\left(x+\sqrt{x^2+1}\right)\ge\sqrt{1+x^2}$$

23. $f(x)=2\sin x+\tan x-3x$

$$f'(x)=2\cos x+\sec^2 x-3$$

$$=\sec^2 x\left(2\cos^3 x-3\cos^2 x+1\right)$$

$$=\left(\sec^2 x\right)(1-\cos x)^2(1+2\cos x)$$

$$\Rightarrow f'(x)\ge 0\quad\forall\quad 0\le x<\frac{\pi}{2}$$

Thus $f(x)$ is increasing function

$\Rightarrow f(x)\ge f(0)$

or $f(x)\ge 0$

$$\Rightarrow 2\sin x+\tan x\ge 3x\,\forall\, x\in\left[0,\frac{\pi}{2}\right]$$

24. $$f(x)=\begin{cases}xe^{ax}, & x\le 0\\ x+ax^2-x^3, & x>0\end{cases}$$

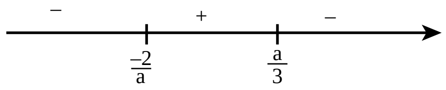
$$f'(x)=\begin{cases}axe^{ax}+e^{ax}, & x<0\\ 1+2ax-3x^2, & x>0\end{cases}$$

$$f''(x)=\begin{cases}2ae^{ax}+a^2xe^{ax}, & x<0\\ 2a-6x, & x>0\end{cases}$$

For $\quad f''(x)=0$

$$\Rightarrow x=-\frac{2}{a}\text{ if }x<0\text{ or }x=\frac{a}{3}\text{ if }x>0$$

$$f'(x)\text{ increases in }\left(\frac{-2}{a},\frac{a}{3}\right)$$



25. Given that $2(1-\cos x)<x^2,\quad x\ne 0$ ...(I)

Let $f(x)=\sin(\tan x)-x$

$$f'(x)=\cos(\tan x)\sec^2 x-1$$

$$=\frac{\cos(\tan x)-\cos^2 x}{\cos^2 x}\qquad\text{...(II)}$$

Using (I), $\cos x>1-\dfrac{x^2}{2}$

Put $x=\tan x$

$$\Rightarrow\cos\tan x>1-\frac{\tan^2 x}{2}\qquad\text{...(III)}$$

Use (III) in eq (II)

$$f'(x)>\frac{1-\frac{1}{2}\tan^2 x-\cos^2 x}{\cos^2 x}$$

$$=\frac{\sin^2 x\cos 2x}{2\cos^4 x}>0\ \forall\ x\in\left[0,\frac{\pi}{4}\right]$$

Therefore, $f'(x)$ is increasing function.

$\Rightarrow f(x)\ge f(0)$

$\Rightarrow\sin\tan x-x\ge 0$

$\Rightarrow\sin\tan x\ge x$

26. Given $\dfrac{dP(x)}{dx}>P(x)$

$$\Rightarrow\frac{dP(x)}{dx}-P(x)>0$$

Multiplying by e^{-x}, we get

$$\frac{d}{dx}\left(e^{-x}p(x)\right)>0$$

$\Rightarrow e^{-x}p(x)$ is an increasing function.

$\Rightarrow e^{-x}P(x)>e^{-1}P(1)$

$\Rightarrow e^{-x}\left(P(x)>0\right.$

$\Rightarrow P(x)>0\quad\forall x>1$

27. $$f(x)=\sin x+2x-\frac{3x(x+1)}{\pi}$$

$$f'(x)=\cos x+2-\frac{3}{\pi}(2x+1)$$

$$f''(x)=-\sin x-\frac{3}{\pi}(2)<0$$

$\Rightarrow f'(x)$ is a decreasing function

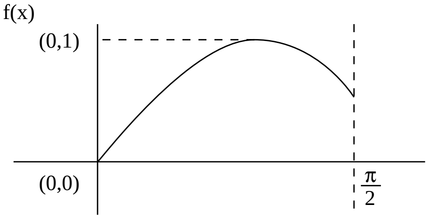
Also, For $x \in \left[0, \frac{\pi}{2}\right], f'(x) = 0$

$\Rightarrow \cos x = \frac{6x}{\pi} + \frac{3}{\pi} - 2$

As graph of cos x & $\frac{6x}{\pi} + \frac{3}{\pi} - 2$ intersect only once.

$\Rightarrow f'(x) = 0$ has one root in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x)$ changes its sign from +ve to –ve .



$\Rightarrow f(x) \geq 0$

$\Rightarrow \sin x + 2x - \frac{3x}{\pi}(x+1) \geq 0$

Also, f(x) has a point of Maxima.

Previous Year's Questions (AIEEE/JEE MAINS)

1. Statement 1 → $f(x) = x^2(e^x + e^{-x})$

$f'(x) = x^2(e^x - e^{-x}) + 2x(e^x + e^{-x})$

$e^x > e^{-x}$ for $x > 0$

$\Rightarrow f'(x) > 0$

So, f(x) is always increasing.

Statement 2 → $f(x) = x^2 e^x$ & $g(x) = x^2 e^{-x}$ is increasing for $x > 0$

$\Rightarrow g(x) = f(x) + g(x)$ is increasing for $x > 0$.

Hence, C is correct.

2. $f(x) = \sin^4 x + \cos^4 x$

$f(x) = 1 - 2\sin^2 x \cos^2 x$

$f(x) = 1 - \frac{1}{2}(\sin 2x)^2$

$f'(x) = -2\sin 2x \cos 2x = -\sin 4x > 0$

$\Rightarrow \sin 4x < 0$

$\Rightarrow \pi < 4x < 2\pi$

$\Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

Hence, C is correct.

3. $f(x) = x^3 - 3x^2 + 5x + 7$

$f'(x) = 3x^2 - 6x + 5 > 0$

$\Rightarrow$ f(x) is always increasing.

Hence, A is correct.

4. $f(x) = \frac{x}{\sqrt{a^2+x^2}} + \frac{d-x}{\sqrt{b^2+(d-x)^2}}$

$$f'(x) = \frac{\sqrt{a^2+x^2}\cdot 1 - \frac{x\cdot x}{\sqrt{a^2+x^2}}}{a^2+x^2} + \frac{\sqrt{b^2+(d-x)^2}(-1) + 1\frac{(d-x).(d-x)}{\sqrt{b^2+(d-x)^2}}}{b^2+(d-x)^2}$$

$$\Rightarrow \frac{a^2}{\left(a^2+x^2\right)^{3/2}} - \frac{b^2}{\left(b^2+(d-x)^2\right)^{3/2}}$$

$f'(x)$ will depend on value of a & b.

Hence, option B is correct.

5. $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$

$f'(x) = 3x^2 - 6(a-2)x + 3a$

x = 1 is a root for $f'(x) = 0$

$\Rightarrow 3 - 6a + 12 + 3a = 0$

$\Rightarrow a = 5$

$\Rightarrow f(x) = x^3 - 9x^2 + 15x + 7$

Now $\frac{f(x)-14}{(x-1)^2} = \frac{x^2 - 9x^2 + 15x - 7}{(x-1)^2}$

$= \frac{(x-1)\left(x^2 - 8x + 7\right)}{(x-1)^2}$

$= \frac{(x-1)(x-1)(x-7)}{(x-1)^2} = x - 7$

Hence, C is correct.

6. As $y - x^{3/2} = 7$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\Rightarrow \frac{3}{2}\sqrt{x}\left(\frac{7-y}{1/2-x}\right) = -1$$

$$\Rightarrow \frac{3}{2}\frac{\sqrt{x}\left(-x^{3/2}\right)}{\left(\frac{1}{2}-x\right)} = -1$$

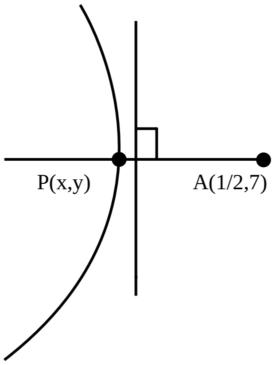
$$\Rightarrow (x+1)(3x-1) = 0$$

$$\Rightarrow x = -1 \text{ or } 1/3$$

$$\Rightarrow x = \frac{1}{3}, \quad y = 7 + \left(\frac{1}{3}\right)^{3/2}$$

$$l_{AB} = \frac{1}{6}\sqrt{\frac{7}{3}}$$

Hence, C is correct.



7. Let $P(x_1, y_1)$ on $y = \sqrt{x}$

$$D = \sqrt{\left(x_1 - \frac{3}{2}\right)^2 + y_1^2}$$

$$D^2 = x_1^2 + \frac{9}{4} - 3x_1 + x_1$$

$$D = \sqrt{x_1^2 + \frac{9}{4} - 2x_1}$$

for $\frac{dD}{dx} = 0 \Rightarrow x = 1$

$$\Rightarrow y = 1 \text{or } -1$$

$$D = \sqrt{1 + \frac{9}{4} - 2} = \frac{\sqrt{5}}{2}$$

Hence, A is correct.

7

MAXIMA AND MINIMA

7.1 Introduction

The notion of optimizing functions is one of the most important application of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home.

To control the ideas of greatest or least, highest or lowest, farthest or closest, etc., we introduce the technical terms maximum and minimum as applied to functions. However, the terms maximum and minimum without further qualification turn out to be insufficient to cope with all types of optimizing problems.

For example, when a stone is thrown upward, it attains a maximum height and a minimum height, and the story ends there; whereas, when a rubber ball is tossed up, it attains a highest point, descends to a lowest point, but then bounces up again to attain another "maximum" height, then falls, bounces up again to still another "maximum" height, and continues this rise and fall process until it comes to rest. These "maxima" in quotes are "local" maxima in the sense that only when restricted to competition in the time interval from bounce to bounce are they guaranteed to be the maximum in the strictest sense of the word.

A similar situation shows up in the oscillation of a pendulum which is slowly coming to rest. In this case the bob of the pendulum reaches extreme right and left positions over intervals of time. Of course, the first extreme right position would be the undisputed maximum, while the first extreme left position would be the undisputed minimum; but the extreme positions of each subsequent interval of swing qualify as maxima and minima only relative to the corresponding time interval of swing.

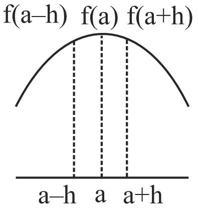
In view of the above illustrative situations we are led to introduce the adjectives local and global to distinguish the two type of extreme positions. But the only characteristic which distinguishes the local from the global will be the domain of validity. In fact, the local concept will merely be the global concept if a small part of the given domain is considered.

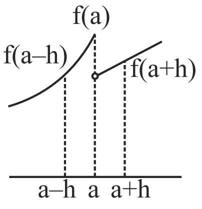
In terms of graph, the global maximum is the y-coordinate of the highest point of the graph. But the local maxima are y-coordinates of the locally highest points of the graph, namely, the points which are highest in competition with only the points of the graph contained within a suitably thin vertical strip centred at the highest point. Similar remarks hold when minimum and lowest replace maximum and highest, respectively.

7.2 Concept of Local Maxima and Local Minima

Let $y = f(x)$ be a function defined at $x = a$ and also in the neighbourhood of the point $x = a$. Then, $f(x)$ is said to have a local maximum at $x = a$, if the value of the function at $x = a$ is greater than the values of the function at all points in the neighbourhood of $x = a$.

Mathematically, $f(a) > f(a - h)$ and $f(a) > f(a + h)$ where $h > 0$ is a small positive arbitrary number.

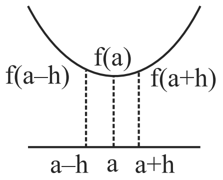


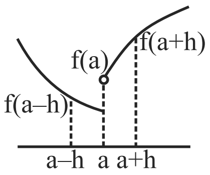


In other words, a function $f(x)$ is said to have a local maximum at $x = a$ if $f(a) > f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$, where h is a small positive arbitrary number.

Similarly, $f(x)$ is said to have a local minimum at $x = a$, if the value of the function at $x = a$ is smaller than the values of the function at all points in the neighbourhood of $x = a$.

Mathematically, $f(a) < f(a-h)$ and $f(a) < f(a+h)$ where $h > 0$ is a small positive arbitrary number.





In other words, a function $f(x)$ is said to have a local minimum at $x = a$ if $f(a) < f(x)\ \forall\ x \in (a-h, a+h) - \{a\}$, where h is a small positive arbitrary number.

Note:

(i) The above definition is applicable to all functions continuous or discontinuous, differentiable or non-differentiable at $x = a$.

(ii) If the graph of a function f attains a local maximum at the point $(a, f(a))$, then $x = a$ is called the point of local maximum and $f(a)$ is called the local maximum value. A similar terminology is used for local minimum.

(iii) The local maximum and minimum values of a function are also known as relative maxima or relative minima as these are the greatest and least values of the function relative to some neighbourhood of the point in question.

(iv) The generic terms for maxima and minima of a function are extremum (plural extrema) or extreme values of the function. The term 'extremum' or 'turning point' is used both for local maximum or minimum values.

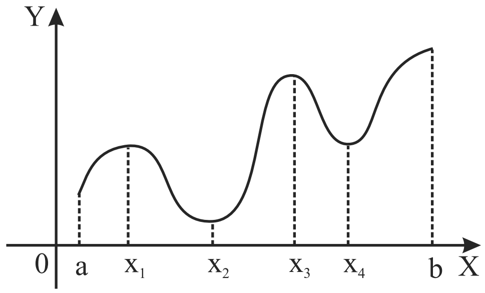
(v) A local maximum (minimum) value of a function may not be the greatest (least) value in an interval.

(vi) A function can have several local maximum and minimum values, and a local minimum value may even be greater than a local maximum value.

Here is a function, defined on the interval $[a, b]$, which has a local maximum at $x = x_1$ and $x = x_3$,

has a local minimum at $x = x_2$ and $x = x_4$,

Note that the minimum of the function at $x = x_4$ is greater than the maximum of the function at $x = x_1$. At $x = b$, the value of the function is greater than any local maximum of the function in the interior of the interval under consideration.



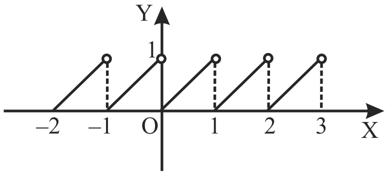
(vii) Usefully we deal with functions having only a finite number of extrema in a given finite interval.

Consider the function $f(x) = x \sin 1/x$, $x \neq 0$, $f(0) = 0$. In every interval containing the point 0, it is continuous and has an infinite number of points of extrema.

(viii) If a function is strictly increasing or strictly decreasing at an interior point $x = a$ it cannot have an extremum at $x = a$ and vice versa.

Example 1: Test $f(x) = \{x\}$ for the existence of a local maximum and minimum at $x = 1$, where $\{.\}$ represents the fractional part function.

Solution:



Clearly $x = 1$ is a point of discontinuity.

$f(1) = 0$

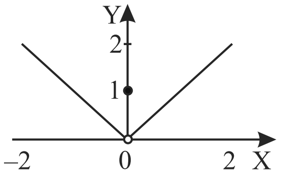
Now, $f(1-h) > 0$ and $f(1+h) > 0$, i.e., the value of the function at $x = 1$ is less than the values of the function at the neighbouring points.

Thus, $x = 1$ is the point of local minimum.

Example 2: Let $f(x) = \begin{cases} |x| & 0 < |x| \leq 2 \\ 1 & x = 0 \end{cases}$.

Examine the behaviour of $f(x)$ at $x = 0$.

Solution: The graph of $y = f(x)$ is shown below:



$f(x)$ has local maxima at $x = 0$ because $f(0) = 1$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = 0$.

Note: When speaking of a maximum or a minimum at a point $x = a$, we usually mean a strict extremum. In such a case, there exists a neighbourhood of the point of extremum $x = a$ for all whose points (except the point a itself) there holds the strict inequality

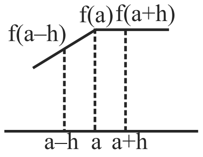
$f(x) < f(a)$ for the point of maximum

or the strict inequality

$f(x) > f(a)$ for the point of minimum

If we relax our definition, and the signs $>$ and $<$ in these inequalities are replaced, respectively, by $\geq$ and $\leq$, the point $x = a$ is called a point of **non-strict extremum.**

Consider the graph of a function in the neighbourhood of the point $x = a$ as shown in the figure:

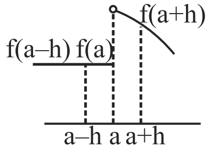


We observe that in the given figure,

$f(a-h) < f(a) = f(a+h)$.

Here, we say that the function has a point of local maximum (non-strict) at x = a.

The function shown below has a point of local minimum (non-strict) at x = a, since $f(a-h) = f(a) < f(a+h)$.



If f(x) is constant in the neighbourhood of the point x = a then it said to have both a local maximum and a local minimum at x = a.

Thus, a function f(x) is said to have a local maximum at x = a if $f(x) \le f(a)\ \forall\ x \in (a-h, a+h)$, when h is a small positive arbitary number.

And, it is said to have a local maximum at x = a if $f(x) \ge f(a)\ \forall\ x \in (a-h, a+h)$.

Extremum at End Points

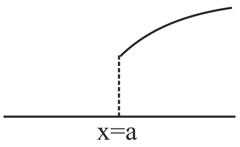
A point (a, f(a)) is called an endpoint of the graph of the function f if there exists an interval (α, β) containing a such that the domain of f contains every number of the interval (α, a) and no number of the interval (a, β), or vice versa.

If x = a happens to be an endpoint of the function, then we compare f(a) with appropriate values of the function in either the left or right neighbourhood of x = a.

Consider a function f(x) defined in the interval [a, b].

Then, the function is said to have a local maximum at the left end point x = a if $f(a) > f(a+h)$, where h is a small positive arbitrary number.

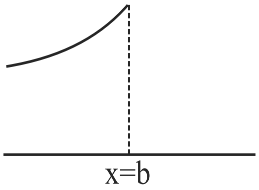
It is said to have a local minimum at the left end point x = a if $f(a) < f(a+h)$.



Local minimum
at left endpoint x = a

The function is said to have a local maximum at the right end point x = b if $f(b-h) < f(b)$.

It is said to have a local minimum at the right end point x = b if $f(b-h) > f(b)$.



Local maximum
at right endpoint x = b

However, if a function is not defined at its endpoint then there is no extremum at such a point.

We can see from these figures that endpoints (provided the function is defined) are almost always points of local maxima or minima.

Consider $f(x) = \sqrt{4-x^2}$.

Clearly, f(–2) = 0 and f(2) = 0 are local minima of f, since 0 is the least of all the values of the function in the right neighbourhood of x = –2 and the left neighbourhood of x = 2 respectively.

Example 3: Let $f(x)=\begin{cases}|x-1|+a, & x<1\\ 2x+3, & x\ge 1\end{cases}$. If f(x) has a local minima at x = 1, then find all possible values of a.

Solution: $f(x)=\begin{cases}1-x+a, & x<1\\ 2x+3, & x\ge 1\end{cases}$

Since f(1) = 5, the local minimum value of f(x) at x = 1 will be 5. Hence 5 should be smaller than every other value assumed by f(x) in the immediate neighbourhood of x = 1. The function increases above 5 in the right neighbourhood of x = 1. In the left neighbourhood of x = 1, the function is decreasing. So the left hand limit of the function should be either equal to or greater than 5 to have a local minimum at x = 1.

$f(1^-) \ge f(1)$

$\Rightarrow\ a \ge 5$.

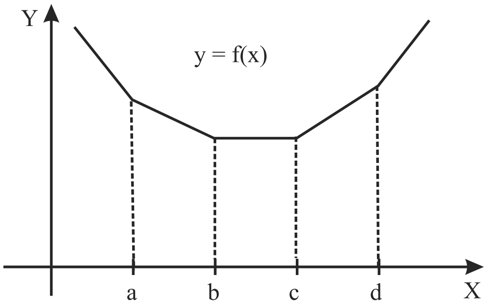
Example 4: If a < b < c < d and x ∈ R then find the points of extrema of the function,

$f(x) = |x-a| + |x-b| + |x-c| + |x-d|$

Solution: $f(x) = |x-a| + |x-b| + |x-c| + |x-d|$

$$= \begin{cases} 4x-(a+b+c+d) & x < a \\ 2x - a + (b+c+d) & a \le x < b \\ c+d-(a+b) & b \le x < c \\ 2x-(a+b+c)+d & c \le x < d \\ 4x-(a+b+c+d) & x \ge d \end{cases}$$

The graph of y = f(x) is shown below:



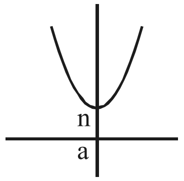
We can observe in the figure that the function has a local minimum or maximum (non-strict) at each x between b and c i.e. b < x < c. It has a local minimum at x = b and c.

x = a is not a point of extremum because the function is strictly decreasing at x = a. Similarly the function is strictly increasing at x = d.

Example 5: If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}[f(x)]$ $(a \in R)$, where $[\cdot]$ denotes greatest integer function and f(x) is a non constant continuous function, then show that f(x) has a local minimum at x = a.

Solution: Clearly $\lim\limits_{x\to a}[f(x)]$ is an integer and LHL and RHL should be same for existence of $\lim\limits_{x\to a} f(x)$. The function should approach an integer from both sides of x = a.

Let $\lim\limits_{x\to a}[f(x)] = n$ $(n \in I)$ then $\lim\limits_{x\to a} f(x) = n$.



Clearly the function should be just greater than n on both sides of x = a since f(x) is continuous at x = a.

∴ f(x) has local minimum at x = a.

Example 6: If $f(x) = (\sin^2 x - 1)^n (2 + \cos^2 x)$, then prove that x = π/2 is a point of maximum, if n is an odd positive integer.

Solution: We have f(π/2) = 0.

If x = a is a point of local maximum, then

$f(a - h) < 0$ and $f(a + h) < 0$

$\Rightarrow$ $f(\pi/2 - h) < 0$ and $f(\pi/2 + h) < 0$

$f(\pi/2 - h) = (-ve)^n$ ... (1)

$f(\pi/2 + h) = (-ve)^n$ ... (2)

$\Rightarrow$ $f(\pi/2 - h)$ and $f(\pi/2 + h)$ are negative when n is odd. So, f(x) has local maximum at x = π/2 when n is odd .

Note that both $f(\pi/2 - h)$ and $f(\pi/2 + h)$ are positive if n is even in which case the function has a local minimum at x = π/2. So , f(x) has maxima or minima at x = π/2 according as n is odd or even .

7.3 Fermat Theorem

It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. The following theorem says that this is always true for differentiable functions.

Fermat Theorem. If a belongs to an open interval in the domain of f, if f′(a) exists, and if (a, f(a)) is a point of local extremum (either a maximum or a minimum), then f′(a) = 0.

Geometrically, Fermat theorem says that if there is a tangent at a highest point or lowest point of the graph and further if the point of contact is not an end point of the graph, then the tangent line is necessarily horizontal.

Proof : Geometrically this theorem is obvious. We shall prove it, only in the case that (a, f(a)) is a local maximum point, since a similar proof (with the inequalities reversed) is valid for a local minimum point.

Since (a, f(a)) is a local maximum point, f(a – h) ≤ f(a), and f(a + h) ≤ f(a) for all positive h in some open interval containing 0. Thus f(a – h) – f(a) ≤ 0.

Since h is positive, $\dfrac{f(a-h)-f(a)}{-h} \ge 0$ and

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \ge 0$$

and, $\dfrac{f(a+h)-f(a)}{h} \le 0$ and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \le 0$$

Since f′(a) exists, the two limits above must have a common value which is f′(a). Thus f′(a) is both greater than or equal to zero and also less than or equal to zero. The only number which satisfies both of these conditions is zero, hence f′(a) = 0. This completes the proof.

This theorem is instrumental in discovering both local and global extrema.

Example 1: Suppose $x^3 + ax^2 + bx + c$ satisfies $f(-2) = -10$ and takes the extreme value $\dfrac{50}{27}$ where $x = \dfrac{2}{3}$. Find the value of a, b and c.

Solution: $f(x) = x^3 + ax^2 + bx + c$

$f(-2) = -8 + 4a - 2b + c = -10$

$4a - 2b + c + 2 = 0$ ...(1)

$f'(x) = 3x^2 + 2ax + b$

$f'(2/3) = 0$ by Fermat theorem

$$\therefore \quad 3 \cdot \frac{4}{9} + 2a\left(\frac{2}{3}\right) + b = 0 \Rightarrow \frac{4}{3} + \frac{4a}{3} + b = 0$$

$$\Rightarrow \quad 4 + 4a + 3b = 0 \qquad ...(2)$$

Also $f(2/3) = \dfrac{50}{27}$

This gives $\dfrac{8}{27} + \dfrac{4a}{9} + \dfrac{2b}{3} + c = \dfrac{50}{27}$

$$8 + 12a + 18b + 27c = 50$$

$$12a + 18b + 27c = 42$$

$$4a + 6b + 9c = 14 \qquad ...(3)$$

Solving (1), (2) and (3) we get

$a = -1$, $b = 0$ and $c = 2$.

Example 2: Given that the function $f(x) = (x - p)^2 + (x - q)^2 + (x - r)^2$ has a minimum, find the corresponding value of x.

Solution: We have,

$$f(x) = (x - p)^2 + (x - q)^2 + (x - r)^2 \qquad ...(1)$$

It is a differentiable function for $x \in (-\infty, \infty)$. Hence by Fermat theorem, a minimum can be attained when $f'(x) = 0$.

Differentiating (1) w.r.t. x, we get

$$f'(x) = 2(x - p) + 2(x - q) + 2(x - r) \qquad ...(2)$$

$f'(x) = 0$ when

$$2(x - p) + 2(x - q) + 2(x - r) = 0$$

$$\Rightarrow \quad 3x - (p + q + r) = 0 \Rightarrow x = \frac{1}{3}(p + q + r).$$

Since a single value of x is obtained, without further investigation we can say the minimum is attained at

$$x = \frac{1}{3}(p + q + r).$$

Example 3: Check whether the function $f(x) = 1 - 2x^2$, $x \in [-1, 0]$ satisfies the conditions of Fermat theorem.

Solution: The given function is strictly decreasing in $[-1, 0]$ and consequently attains maximum at $x = -1$ and minima at $x = 0$.

Since, these points are not interior points of the interval $[-1, 0]$ we can not apply Fermat theorem to assert that $f'(-1) = f'(0) = 0$

In fact $f'(-1) = 4$ and $f'(0) = 0$. We see that $f'(0) = 0$ but this inference could not have been drawn using Fermat theorem (which is not applicable in this problem).

If $f'(x) = 0$ at $x = a$, then we say that $x = a$ is a stationary point of the function f. Also, f(a) is then said to be a stationary value of f. We say so because the rate of change of the function is zero at such a point.

Example 4: If $f(x) = x^3 + ax^2 + bx + c$ has stationary point at $x = -1$ and $x = 3$. Find a, b, c.

Solution: Since f(x) has stationary point at $x = -1$ and $x = 3$,

$$f'(-1) = 0 = f'(3)$$

$$f'(x) = 3x^2 + 2ax + b$$

$$f'(-1) = 3 - 2a + b = 0$$

$$f'(3) = 27 + 6a + b = 0$$

$$\Rightarrow \quad a = -3, b = -9, c \in R.$$

Example 5: Show that the function f defined by $f(x) = x^4 - 2x^3 + 3x - 1$ has a stationary point in the interval $(-1, 2)$.

Solution: We have $f'(x) = 4x^3 - 6x^2 + 3$

and $f'(-1) = -7$, $f'(2) = 11$. Since f' is continuous in $[-2, 3]$, f has a stationary point a in $(-1, 2)$.

Evidently, $f'(a) = 0$. Thus, a is root of the equation $4x^3 - 6x^2 + 3 = 0$.

We find the set of values of x for which $f'(x) = 0$ and carry out investigation to locate the points of extrema.

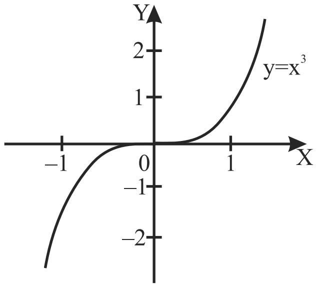
However, vanishing of $f'(c)$ is only a necessary condition and not a sufficient condition for $x = c$ to be a point of extremum.

Consider the graph of $y = x^3$. We can see that $f'(0) = (3x^2)_{x=0} = 0$ but $x = 0$ is not a point of extremum.

Indeed, no matter how close the point x is to origin, we will always have $x^3 < 0$ when $x < 0$

and $x^3 > 0$ when $x > 0$.

The graph is rising for $x < 0$ and also for $x > 0$, as shown in the figure.



Thus, Fermat theorem tells us that a local extremum of a differentiable function f can occur only at a stationary point, but it does not say that a local extremum must occur at each stationary point.

The converse of Fermat's theorem is false in general. Furthermore, there may be an extreme value at $x = a$ even when $f'(a)$ does not exist.

Theorem Suppose that f is a function defined on an interval containing the number $x = a$. If the function has a point of local extremum at $x = a$, then either $f'(a) = 0$ or $f'(a)$ does not exist or a is an endpoint of the interval.

Critical Points

Critical points are interior points in the domain of the function where either $f'(x) = 0$ or $f'(x)$ does not exist.

In Fermat theorem we have considered the critical points at which the derivative is equal to zero i.e stationary points. Let us now deal with the critical points at which the derivative does not exist.

Example 6: Find the critical points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$ if (i) $x \in [0, 3]$ (ii) $x \in [-3, 3]$ (iii) $x \in [-1, 2]$.

Solution: $f'(x) = 12(x^2 - x - 2)$

$= 12(x - 2)(x + 1)$

$f'(x) = 0$

$\Rightarrow x = -1$ or 2

(i) If $x \in [0, 3]$, $x = 2$ is the critical point.

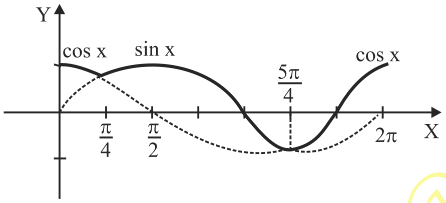
(ii) If $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.

(iii) If $x \in [-1, 2]$, then there is no critical point as both $x = 1$ and $x = 2$ become endpoints.

Note: Critical points are always interior points of an interval.

Example 7: Find the number of critical points for $f(x) = \max(\sin x, \cos x)$, $x \in (0, 2\pi)$.

Solution: The graph of $y = f(x)$ is shown below :



We can see from the figure that

$f'(x) = 0$ at $x = \dfrac{\pi}{2}$ and

$f'(x)$ does not exist at $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$.

Thus, the critical points of $f(x)$ are $x = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}$.

Hence, $f(x)$ has three critical points.

Example 8: Find the critical points of the function $f(x) = |x^2 - 2x|$ and find whether these points are points of extrema.

Solution: We have $f(x) = |x^2 - 2x|$

$$f(x) = \begin{cases} x^2 - 2x & x \le 0, x \ge 2 \\ 2x - x^2 & 0 < x < 2 \end{cases}$$

$$f'(x) = \begin{cases} 2(x-1) & x < 0, x > 2 \\ 2(1-x^2) & 0 < x < 2 \end{cases}$$

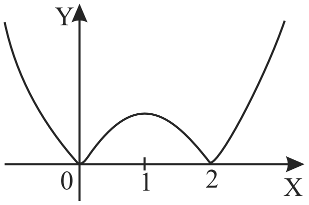
$f'(x) = 0$ at $x = 1$ and

$f'(x)$ does not exist at $x = 0, 2$.

Thus, the critical points of $f(x)$ are $x = 0, 1, 2$.

Note that $x = 1$ is also a stationary point.

The graph of $y = f(x)$ is shown below:



From the graph, we can see that $x = 1$ is a point of local maximum and $x = 0, 2$ are points of local minima.

Note: It is also quite possible for a continuous function f to have no local extremum at a point a where f'(x) does not exist.

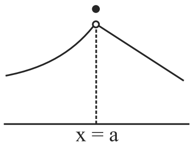
For example, the function $y = \sqrt[3]{x}$ has no derivative at $x = 0$. ($y' \to \infty$ as $x \to 0$). Thus, $x = 0$ is a critical point of the function where the derivative does not exist. At this point the function has neither a maximum nor a minimum because $f(0) = 0$, $f(x) < 0$ for $x < 0$, $f(x) > 0$ for $x > 0$.

We see that $f(x)$ is strictly increasing at $x = 0$ and hence, has no extremum at $x = 0$.

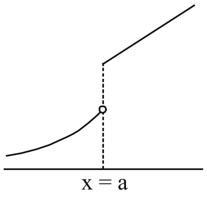
Concept Problems A

1. In each of following cases, identify if $x = a$ is point of local maxima, minima or neither of them.

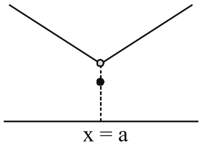
(i)



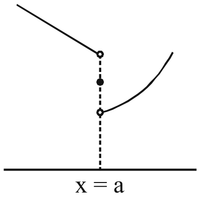
(ii)



(iii)



(iv)



2. Draw the graph of the following functions and locate the points of extrema :

(i) $y = |x^2 - 1|$ (ii) $y = 2 - \dfrac{1}{|x+1|}$

(iii) $y = |x + 1| + 2|x|$

3. Find the points of extrema of the function $f(x) = 2|x - 2| + 5|x - 3|$.

4. Let $f(x) = \begin{cases} x+1, & x < 2 \\ a, & x = 2 \\ a - x, & x > 2 \end{cases}$, Find 'a' if f(x) has local maximum at $x = 2$.

5. Find the points of extrema for the function $y = |x - 2| + |x - a|$.

6. If $f(x) = \begin{cases} (x+\lambda)^2 & x < 0 \\ \cos x & x \geq 0 \end{cases}$, find possible values of λ such that f(x) has local maxima at x = 0.

7. Show that $f(x) = 1-(x-1)^{1/3}$ has a critical point at x = 1, but the function has neither a maximum nor a minimum at x = 1.

8. The greatest integer function f(x) = [x], defined for all values of x, assumes a local maximum value of 0 at each point of [0, 1). Could any of these local maximum values also be local minimum values of f ? Give reasons for your answer.

9. If a function f is continuous in an interval I and if f′(a) > 0, f′(b) < 0 for some a, b in I, then prove that f has a critical point in I.

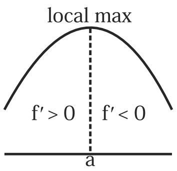
10. Suppose f is differentiable on [a, b], and f′(a) < k < f′(b) or f′(a) > k > f′(b). Prove there exists c ∈ (a, b) such that f′(c) = k.

7.4 The First Derivative Test

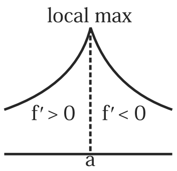
From what has been discussed in the previous section, it follows that not for every critical point does a function have a maximum or a minimum. However, if at some interior point the function attains a maximum or a minimum, this point is definitely critical. And so to find the extrema of a function, do as follows: find all the critical points, and then, investigating separately each critical point, find out whether the function will have a maximum or a minimum at that point, or whether there will be neither maximum nor minimum.

Usually the behaviour of a continuous function y = f(x) at a critical point x = a can be determined from the algebraic sign of the derivative near a. Consider the following graphs in the neighbourhood of a critical point x = a :

(i) If f ′ changes from positive to negative at a (f ′ > 0 for x < a and f ′ < 0 for x > a), then f has a local maximum value at a.

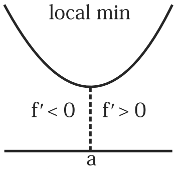


(a) f′(a) = 0

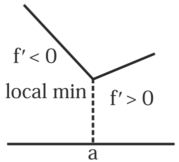


(b) f′(a) does not exist

(ii) If f ′ changes from negative to positive at a (f ′ < 0 for x < a and f ′ > 0 for x > a), then f has a local minimum value at a.

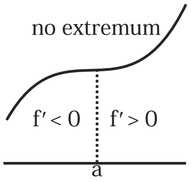


(a) f′(a) = 0

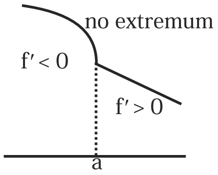


(b) f′(a) does not exist

(iii) If f ′ does not change sign at c (f ′ has the same sign on both sides of c), then f has no local extreme value at c.



(a) f′(a) = 0



(b) f′(a) does not exist

The First Derivative Test

Suppose that x = a is a critical point of a continuous function y = f(x).

(i) If f′(x) changes sign from positive to negative at x = a, then f has a local maximum at x = a.

(ii) If f′(x) changes sign from negative to positive at x = a, then f has a local minimum at x = a.

(iii) If f′(x) does not change sign at x = a (that is, if f′(x) is positive on both sides of x = a or negative on both sides), then f has no local maximum or minimum at x = a.

In part (i), when the sign of f′(x) changes from positive to negative at x = a, f is increasing on the left of x = a and decreasing on the right of x = a, it follows that f has local maximum at x = a.

It is easy to remember the first derivative test by visualizing the above diagrams. Informally, the derivative test says, "If the derivative changes sign at x = a, then the function has either a local minimum or a local maximum." To decide which it is, just make a crude sketch of the graph near (a, f(a)) to show on which side of x = a the function is increasing and on which side it is decreasing.

Mathematically, if x = a is a critical point of the function f(x) and the inequalities

$$f'(a-h) > 0,\ f'(a+h) < 0$$

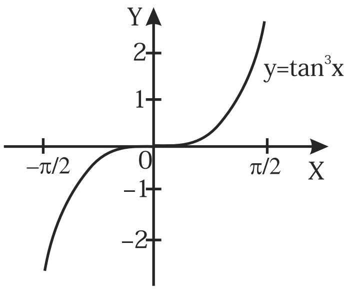
are satisfied, where h is a small positive arbitrary number, then the function f(x) possesses a maximum at the point x = a.

If $f'(a-h) < 0,\ f'(a+h) > 0$,

then the function f(x) possesses a minimum at the point x = a.

If the signs of f′(a – h) and f′(a + h) are the same, then the function f(x) does not possess an extremum at the point x = a.

Saddle Point



Let x = a be a stationary point of the function y = f(x) i.e. f′(a) = 0. If f′(x) has the same sign on either side of x = a, then f has a saddle point at x = a. The point (a, f(a)) is known as a saddle point, as it can be envisioned as a level resting spot on the side of an ascending or descending portion of a functional curve. A saddle point is a special case of point of inflection.

For example, if $f(x) = \tan^3 x$, then $f'(x) = 3\tan^2 x \sec^2 x$ and $f'(0) = 0$, $f'(0-h) > 0$, $f'(0+h) > 0$.

Hence, the graph is rising for $-\frac{\pi}{2} < x < 0$ and also for $0 < x < \frac{\pi}{2}$.

The origin is a saddle point of f. The function has a point of inflection at $x = 0$. There is no extremum at $x = 0$ on the graph of f.

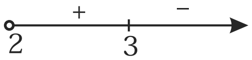
Example 1: Let $f(x) = 2\ln(x-2) - x^2 + 4x + 1$. Investigate the function for extremum at $x = 3$.

Solution: The function $f(x) = 2\ln(x-2) - x^2 + 4x + 1$ is defined for $x > 2$. Let us find its derivative :

$$f'(x) = \frac{2}{x-2} - 2x + 4 = \frac{2(x-1)(3-x)}{x-2}.$$

The point $x = 3$ is a critical point of the given function.

Sign scheme of f '(x)



On the interval (2, 3), the derivative $f'(x)$ is positive and, consequently, on the interval the function increases.

And on the infinite interval $(3, \infty)$, the derivative $f'(x)$ is negative and, therefore, the function f(x) decreases on that interval.

The point $x = 3$ is a point of local maximum.

Note: By considering the function f defined as

$$f(x) = x^4\left(2 + \sin\frac{1}{x}\right), \text{ if } x \neq 0,\ f(0) = 0,$$

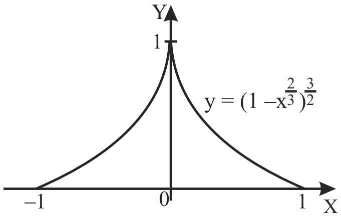
it can be seen that the conditions of the first derivative test are sufficient but not necessary. In fact, f is derivable everywhere, f '(x) does not change sign from negative to positive as x passes through 0 and yet f has a minimum at $x = 0$.

We have considered the critical points at which the derivative is equal to zero i.e. the stationary points. Let us now deal with the critical points at which the derivative does not exist.

The function $f(x) = \left(1 - x^{\frac{2}{3}}\right)^{3/2}$ has no derivative at $x = 0$,

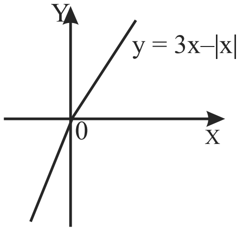
since $f'(x) = -\left(1 - x^{\frac{2}{3}}\right)^{1/2} x^{-\frac{1}{3}}$ approaches infinity at $x = 0$,

but the function has a maximum at this point.



We can see from the figure that $f(0) = 1$, and $f(x) < 1$ for all x different from zero.

Let $f(x) = 3x - |x|$ The given function has no derivative at the point $x = 0$. Since for all $x < 0$, $f(x) < f(0)$, and for all $x > 0$, $f(x) > f(0)$, the given function has no extremum at the point $x = 0$.



We cannot say that the change of sign of the derivative always helps in determining maxima/minima.

For example, let

$$f(x) = \begin{cases} x^2, & x < 0 \\ 5, & x = 0 \\ 2\sin x, & x > 0 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 2x, & x < 0 \\ 2\cos x, & x > 0 \end{cases}$$

f '(x) does not exist and f is discontinuous.

Here also the derivative is changing sign from negative to positive, but $x = 0$ is not a point of minimum. On the contrary, it is a point of maximum since $f(0) = 5$ and $f(0-h) < f(0)$ and $f(0+h) > f(0)$.

This type of difficulty appears in case of discontinuous functions.

Test for Local Maximum/Minimum When f(x) Is Not Differentiable at x = a

Case 1: When f(x) is continuous at $x = a$ and $f'(a^-)$ and $f'(a^+)$ exist and are non-zero, then f(x) has a local maximum or minimum at $x = a$ according to the following :

If $f'(a^-) > 0$ and $f'(a^+) < 0$, then $x = a$ is a point of local maximum . If $f'(a^-) < 0$ and $f'(a^+) > 0$, then $x = a$ is a point of local minimum.

Case 2: When f(x) is continuous and one or both of $f'(a^-)$ and $f'(a^+)$ are either zero or does not exist, then we should consider the signs of $f'(a-h)$ and $f'(a+h)$, where h is a small positive arbitrary number.

If $f'(a-h) > 0$ and $f'(a+h) < 0$, then $x = a$ is a point of local maximum . If $f'(a-h) < 0$ and $f'(a+h) > 0$, then $x = a$ is a point of local minimum.

Case 3: If f(x) is discontinuous at $x = a$, then we should find about the existence of local maxima/minima using the basic definition of local maxima/minima i.e. compare the values of f(x) at the neighbouring points of $x = a$.

We should find the values of $f(a^-)$, $f(a^+)$ and f(a). If f(a) is the greatest of all these values, then there is a local maximum at a. If f(a) is the least of all these values, then there is a local minimum at a.

It is also advisable to draw the graph of the function in the vicinity of the point x = a, because the graph would give us a clear picture about the existence of local maxima/minima at x = a.

If f(a) is undefined, then there is no question of extremum at x = a.

Example 2: Test the function $f(x) = (x-2)^{2/3}(2x+1)$ for extrema at x = 1 and 2.

Solution: We find $f'(x) = \frac{10}{3}\cdot\frac{x-1}{\sqrt[3]{x-2}}$.

The critical points are x = 1 (the derivative is zero) and x = 2 (the derivative does not exist). The function is continuous.

The inequalities $f'(1-h) > 0$, $f'(1+h) < 0$,

$f'(2-h) < 0$, $f'(2+h) > 0$ hold at a sufficiently small $h > 0$.

Consequently, at the point x = 1 the function possesses a local maximum and at the point x = 2 it possesses a local minimum since $f'(2-h) < 0$ and $f'(2+h) > 0$.

Example 3: Let $f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \ge 0 \end{cases}$.

Investigate the function for extremum at x = 0.

Solution: Clearly f(x) is continuous at x = 0 but non-differentiable at x = 0. We have

$$f'(0^-) = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-h^3+h^2-10h-0}{-h} = 10$$

$$\text{But } f'(0^+) = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-3\sin h}{h} = -3$$

Since $f'(0^-) > 0$ and $f'(0^+) < 0$, x = 0 is the point of local maximum.

Alternative :

f(x) is continuous at x = 0.

$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$$

$f'(0^-) = -10$ and $f'(0^-) = 3$.

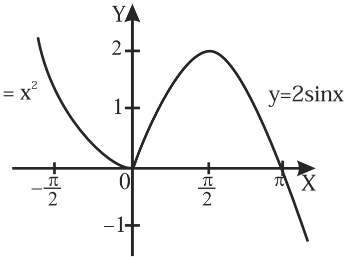
Thus, f(x) is non-differentiable at x = 0

⇒ x = 0 is a critical point.

Also the derivative changes sign from negative to positive. So x = 0 is a point of local minima.

Example 4: Let $f(x) = \begin{cases} x^2, & x \le 0 \\ 2\sin x, & x > 0 \end{cases}$. Investigate the function for extremum at x = 0.

Solution: We sketch the graph of f(x) in the vicinity of x = 0. We see that x = 0 is a point of local minimum.



Alternative:

$$f'(x) = \begin{cases} 2x, & x < 0 \\ 2\cos x, & x > 0 \end{cases}$$

f'(0) does not exist but f is continuous at x = 0.

x = 0 is a critical point where $f'(0^-) = 0$ and $f'(0^+) = 2 > 0$. On the left of x = 0, $f'(0-h) < 0$.

Thus, the derivative changes sign from negative to positive. Hence, x = 0 is a point of minimum.

Example 5: Test the function $f(x) = 1 - (x-2)^{4/5}$ for extremum at x = 2.

Solution: We find $f'(x) = -\frac{4}{5}(x-2)^{-1/5} = -\frac{4}{5\sqrt[5]{x-2}}$.

The derivative does not exist at x = 2 (critical point).

The one-sided derivatives at x = 2 do not exist.

But $f'(2-h) > 0$ and $f'(2+h) < 0$, where h is a small positive arbitrary number.

Hence, we infer that the function possesses a local maximum at x = 2.

Example 6: Investigate the following functions for extrema :

(i) $f(x) = \begin{cases} -2x, & x < 0, \\ 3x+5, & x \ge 0 \end{cases}$

(ii) $f(x) = \begin{cases} 2x^2+3, & x \ne 0, \\ 4, & x = 0 \end{cases}$

Solution: (i) Though the derivative $f'(x) = \begin{cases} -2, & x < 0 \\ 3, & x > 0 \end{cases}$ exists at all points, except the point x = 0, and changes sign from minus to plus passing through the point x = 0, there is no minimum here because $f(0-h) < f(0) = 5$.

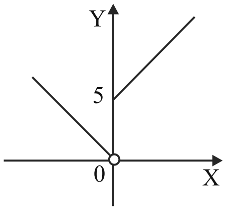


Figure (a)

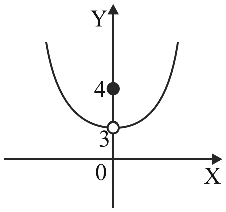


Figure (b)

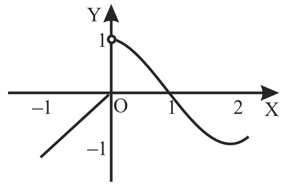
This is explained by the fact that the function is discontinuous at the point x = 0.

(b) Here the derivative $f'(x) = 4x (x \neq 0)$ also exists at all points except at $x = 0$, and it changes sign from minus to plus when passing through the point $x = 0$. Nevertheless, we have here a maximum but not a minimum, which can readily be checked.

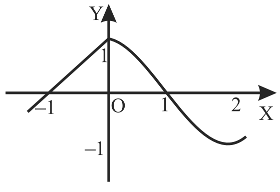
It is explained by the fact that the function is discontinuous at the point $x = 0$.

Example 7: Let $f(x) = \begin{cases} \cos\dfrac{\pi x}{2}, & x > 0 \\ x + a, & x \leq 0 \end{cases}$. Find the values of a if $x = 0$ is a point of local maximum.

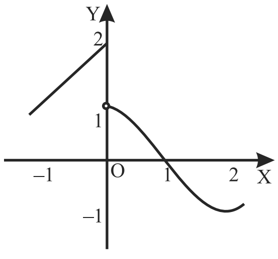
Solution: We draw the graph of with different values of a.



(a)



(b)



(c)

Clearly, f(x) increases before $x = 0$ and decreases after $x = 0$. $f(0) = a$.

Note that in figure (a) for $a = 0$, $x = 0$ is the not point of maximum. The graph of $y = x + a$ must shift atleast 1 unit upwards, for $x = 0$ to be the point of maximum.

For $x = 0$ to be the point of local maxima,

we have $f(0) \geq \lim\limits_{x \to 0^+} f(x)$

$$\Rightarrow \quad f(0) \geq \lim_{x \to 0^+} \cos\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad a \geq 1$$

Hence, $a \geq 1$.

Example 8: Let

$$f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \geq 2 \end{cases}$$

Find the value of 'a' for which f(x) has local minimum at $x = 2$.

Solution: We have,

$$f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \geq 2 \end{cases}$$

Given f(x) has local minima at $x = 2$.

Since, $f(x) = 2x - 3$ is strictly increasing for $x \geq 2$

$$\lim_{x \to 2^-} f(x) \geq f(2)$$

$$\Rightarrow \quad \lim_{h \to 0} f(2 - h) \geq \perp \quad \{\because f(2) = 2 \times 2 - 3 = 1\}$$

$$\Rightarrow \quad \lim_{h \to 0} \{|2 - h - 2| + a^2 - 9a - 9\} \geq 1$$

$$\Rightarrow \quad a^2 - 9a - 10 \geq 0$$

$$\Rightarrow \quad (a + 1)(a - 10) \geq 0$$

$$\Rightarrow \quad a \leq -1 \text{ or } a \geq 10.$$

Example 9: Given a continuous function

$$f(x) = \begin{cases} \left(2 - \sin\dfrac{1}{x}\right)|x| & \text{at } x \neq 0, \\ 0 & \text{at } x = 0. \end{cases}$$

Show that f(x) has a minimum at the point $x = 0$, but is not monotonic either on the left or on the right of $x = 0$.

Solution: For $x \neq 0$, $f(x) > 0$, and $f(0) = 0$, hence f(0) is a minimum.

For $x > 0$ the derivative $f'(x) = 2 - \sin\dfrac{1}{x} + \dfrac{1}{x}\cos\dfrac{1}{x}$ is positive at the points $x = \dfrac{1}{2\pi n}$ and negative at the points $x = \dfrac{1}{(2n+1)\pi}$. Hence, f is not monotonic on the right of $x = 0$.

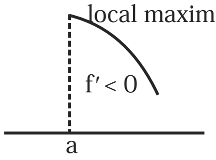
The case $x < 0$ is investigated analogously.

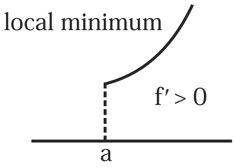
Extremum at Endpoints

Let $y = f(x)$ be continuous at the endpoint of the interval [a, b]

At a left endpoint a:

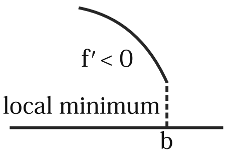
If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.

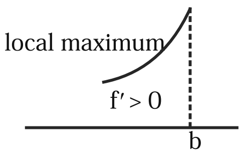




At right endpoint b :

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.





Example 10: Find the extrema of the function $f(x) = 3x - (x - 1)^{3/2}$ at $x = 1$.

Solution: Since $(x-1)^{3/2}$ is real only if $x \geq 1$, the domain of f is the set $[1, \infty)$. Thus x = 1 is the left endpoint of the graph of f. Note that f is continuous a x = 1.

We have $f'(x) = 3 - \frac{3}{2}(x-1)^{1/2}$.

Since $f'(1+h) > 0$, x = 1 is a point of local minimum.

Steps for Finding the Extrema of a Function Defined in an Interval

Let f(x) be defined and continuous on a certain interval (a, b) and have a derivative everywhere in the interval (a, b) except possibly at a finite number of points, and have a finite number of stationary points. Then, in order to find the extrema of the function, it is necessary:

(i) To find the critical points of the function f(x), i.e. the points at which either f'(x) = 0 or f'(x) does not exist;

(ii) To investigate the sign of the derivative f'(x) in a certain neighbourhood of each critical point. And, if , as x passes through such a point, f'(x) changes sign, then f(x) has an extremum at this point.

If minus is changed to plus, then the function has a local minimum at this point; if the sign is altered from plus to minus, then a local maximum is the case. But if the sign of f'(x) remains unchanged as x passes through the point under consideration, then the function f(x) has no extremum at this point.

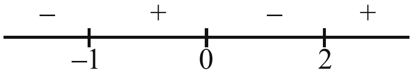
Example 11: Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution: $f'(x) = 12x(x-2)(x+1)$

$f'(x) = 0 \Rightarrow x = -1, 0, 2.$

The critical points are : $x_1 = -1$, 0 and $x_2 = 2$.

Sign scheme of f′(x)



f′(x) changes from negative to positive at –1, so f(–1) = 0 is a local minimum value by the first derivative test.

Similarly, f′ changes from negative to positive at 2, so f(2) = –27 is also a local minimum value. f(0) = 5 is a local maximum value because f′(x) changes from positive to negative at 0.

Note: In case of continuous functions, points of maxima and minima are found alternately.

Example 12: Show that $f(x) = x^3 - 6x^2 + 12x - 8$ does not have any point of local maxima or minima.

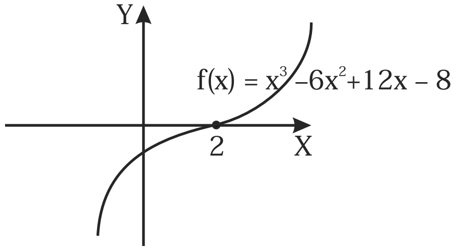
Solution: $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x-2)^2$

$f'(x) = 0 \Rightarrow x = 2$

Clearly f′(x) does not change sign about x = 2. $f'(2^+) > 0$ and $f'(2^-) > 0$. So f(x) has no point of maxima or minima. In fact, f(x) is a strictly increasing function for $x \in R$ and x = 2 is a saddle point or point of inflection of the function.



Note: It will be clear from this example that neither maxima nor minima values can arise from the vanishing of such factors of f′(x) that have even indices.

Example 13: Find the extremum of the function $f(x) = x^{\frac{2}{3}}(x-3)$, $x \in R$.

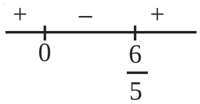
Solution: We find the derivative of the given function: $f'(x) = \frac{5x-6}{3\sqrt[3]{x}}$ and determine the critical points.

$f'(x) = 0$ if $x = \frac{6}{5}$ and f'(x) does not exist at x = 0.

Thus, the critical points are : $x_1 = 0$ and $x_2 = \frac{6}{5}$.

Note that f is continuous everywhere. We now investigate the sign of the derivative f'(x) in a certain neighbourhood of each critical point and tabulate the results :

Sign of f'(x)



x	f'(x)	f(x)
$-\infty < x < 0$	+	increases
x = 0	does not exist	local maximum
$0 < x < \frac{6}{5}$	–	decreases
$x = \frac{6}{5}$	0	local minimum
$\frac{6}{5} < x < \infty$	+	increases

Example 14: Find the local maximum and minimum values of the function $g(x) = x + 2\sin x$, $0 \leq x \leq 2\pi$.

Solution: To find the critical point of g, we differentiate:

$g'(x) = 1 + 2\cos x$

So $g'(x) = 0$ when $\cos x = -\frac{1}{2}$.

The solutions of this equation are $2\pi/3$ and $4\pi/3$.

Because g is differentiable everywhere, the only critical points are $2\pi/3$ and $4\pi/3$ and so we analyze g in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g (x)
$0 < x < 2\pi/3$	+	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	–	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	+	increasing on $(4\pi/3, 2\pi)$

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the first derivative test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}$$

$$= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46.$$

The specification of the change of sign of the derivative as x passes through the boundaries of the intervals of monotonicity, that is through the points x_i, indicates which of these points are points of maxima and which points of minima. It may also turn out that some of the points x_i are not points of extremum. This is the case when the function has the same character of monotonicity in two adjoining subintervals (x_{i-1}, x_i) and (x_i, x_{i+1}) separated by the point x_i, i.e. when the derivative has the same sign in them. In this case x_i is not a point of extremum of the function (for instance, for the function $y = x^3$, the point $x = 0$ belongs to this type).

The substitution of the critical numbers $x = x_i$ into f(x) yields the corresponding values of the function :

$$f(x_1), f(x_2), \ldots\ldots, f(x_n)$$

As has been already mentioned, not all the them are necessarily extreme. If the values $f(x_1), f(x_2), \ldots.f(x_n)$ have been computed and the endpoint values f(a) and f(b) have also been found, the character of variation of the function becomes clear without investigating the sign of the derivative. For, since we know that the function has no extrema within each of the subintervals $(a, x_1), (x_1, x_2), \ldots, (x_n, b)$ and thus is monotonous on them, the comparison of the values of the function at the end points of each subinterval indicates directly whether the function increases or decreases on that subinterval.

It depends on the circumstances which of the techniques should be used. It is sometimes more convenient to consider the sign of the derivative in the subintervals, but sensually we are interested in the values of the function at the critical points the second approach may prove preferable.

Example 15: Find the extrema of the function

$$f(x) = 3\sqrt[3]{x^2} - x^2.$$

Solution: The function is defined and continuous for all x. Now,

$$f'(x) = 2\left(\frac{1}{\sqrt[3]{x}} - x\right).$$

From the equation $f'(x) = 0$ we find the roots of the derivative: $x = \pm1$.

Furthermore, the derivative goes to infinity at the point $x = 0$. Thus, the critical points are $x_1 = -1, x_2 = 0, x_3 = 1$.

The substitution of the critical points into f(x) yields the corresponding values of the function :

$$f(-1) = 2,\ f(0) = 0,\ f(1) = 2.$$

The endpoint values are $f(-\infty) = -\infty$, $f(\infty) = -\infty$.

Thus, f increases from $f(-\infty) = -\infty$ to $f(-1) = 2$ and then decreases from $f(-1) = 2$ to $f(0) = 0$. Hence, the function has a maximum at $x = -1$.

Similarly, f decreases from $f(-1) = 2$ to $f(0) = 0$ and then increases from $f(0) = 0$ to $f(1) = 1$. Hence, the function has a local minimum at $x = 0$.

Further investigation shows that the function has a local maximum at $x = 1$.

Example 16: If the function $f(x) = ax\,e^{-bx}$ has a local maximum at the point (2, 10) then find a and b.

Solution: $f(2) = 10$, hence $2ae^{-2b} = 10$.

$\Rightarrow \quad ae^{-2b} = 5 \qquad$ ...(1)

$f'(x) = a[e^{-bx} - bx\,e^{-bx}] = 0$

$f'(2) = 0$

$\Rightarrow \quad a(e^{-2b} - 2be^{-2b}) = 0$

$\Rightarrow \quad ae^{-2b}(1 - 2b) = 0 \Rightarrow \quad b = 1/2$

From (1) if $b = 1/2$, $a = 5e$ or $a = 0$ (rejected)

$\therefore \quad a = 5e$ and $b = 1/2$.

Example 17: Find the extremum of the function

$$f(x) = \sqrt{2x^2 - x + 2}$$

Solution: We find the derivative

$$f'(x) = \frac{1}{2}\,\frac{4x-1}{\sqrt{2x^2 - x + 2}} \qquad \text{...(1)}$$

Then we equate the derivative $f'(x)$ to zero :

$$\frac{1}{2}\,\frac{4x-1}{\sqrt{2x^2 - x + 2}} = 0$$

From this we find a critical point: $x = 1/4$.

The denominator of expression (1) is positive for all $x \in R$. Consequently, the function has no other critical point except for $x = 1/4$.

It can be seen from expression (1) that

$f'(x) > 0$ for $x > 1/4$ and $f'(x) < 0$ for $x < 1/4$

i.e., when the derivative passes through the point $x = 1/4$ it changes sign from minus to plus. Consequently, $x = 1/4$ is a point of local minimum, with $f(1/4) = \sqrt{15/8}$.

Example 18: Find the points of extrema of the function

$$\begin{aligned} f(x) &= 2 + x^4, \ -1 \le x \le 1 \\ &= 4 - x^2, \ 1 < x \le 2 \\ &= 2x - 5, \ 2 < x < 3 \\ &= 0, \ x = 3. \end{aligned}$$

Also sketch the curve $y = f(x)$.

Solution: The function is continuous at all points except $x = 2$.

$$\begin{aligned} f'(x) &= 4x^3, \ -1 < x < 1 \\ &= -2x, \ 1 < x < 2 \\ &= 2, \ 2 < x < 3. \end{aligned}$$

Thus, the critical points of f(x) are

$x = 0$ (points where $f'(x)$ vanishes)

and $x = 1, 2, 3$ (points where $f'(x)$ does not exist)

Now, we have

$f'(0^-) < 0$ and $f'(0^+) > 0$

$\Rightarrow$ local minima at $x = 0$

$f'(1^-) > 0$ and $f'(1^+) < 0$

and $f(1^-) = f(1) = 3$, $f(1^+) = 3$

$\Rightarrow$ local minima at $x = 1$.

Since f is discontinuous at $x = 2$, we use the basic definition test rather then the first derivative test.

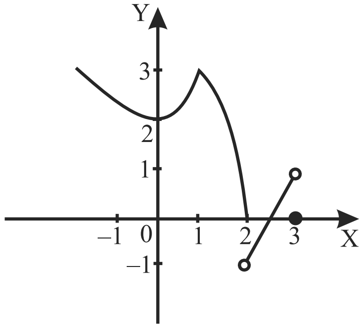
and $f(2^-) = f(2) = 0$, $f(2 - h) > 0$, $f(2^+) = -3$

$\Rightarrow$ no extremum at $x = 2$.

At the end points, we have

$f(-1) > f(-1^+) \Rightarrow$ local maxima at $x = -1$

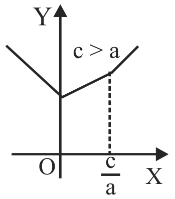
and $f(3) < f(3^-) \Rightarrow$ local maxima at $x = 3$.



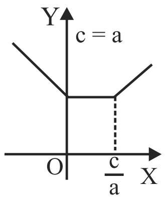
The above results can be easily understood from the graph shown above.

Example 19: Let $f(x) = |ax - b| + c|x|$, where $a > 0$, $b > 0$, $c > 0$. Find the condition if f(x) attains the minimum value only at one point.

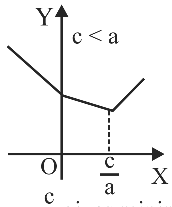
Solution: $f(x) = \begin{cases} b - (a + c)x, & x < 0 \\ b + (c - a)x, & 0 \le x < \dfrac{b}{a} \\ (a + c)x + b, & x \ge \dfrac{b}{a} \end{cases}$



x= 0 gives minima



minima at several points

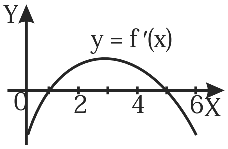


$x = \frac{c}{a}$ gives minima

The figures clearly indicate that for exactly one point of minima, we must have $a \ne c$.

Concept Problems B

1. Prove that the function $2x^3 - 3x^2 - 36x + 10$ has a maximum when $x = -2$, and a minimum when $x = 3$.
2. Find the local maximum and minimum value of $f(x) = (x - 1)(x - 2)(x - 3)$.
3. Find the local maximum and minimum value of $f(x) = 2x^3 - 21x^2 + 36x - 20$.
4. The graph of the derivative f ' of a function f is shown.
   (i) On what intervals is f increasing or decreasing?
   (ii) At what values of x does f have a local maximum or minimum ?



5. Prove that the function $4x^3 - 18x^2 + 27x - 7$ has no maxima or minima.
6. Show that 5 is a critical point of the function $g(x) = 2 + (x - 5)^3$ but g does not have a local extremum at 5.

7. Find the points of local maxima or minima of the following functions :

(i) $f(x) = (x-1)^3 (x+2)^2$

(ii) $f(x) = \sin 2x - x$

(iii) $f(x) = x^3 + x^2 + x + 1$.

8. For a certain curve

$$\frac{dy}{dx} = (x-1)(x-2)^2(x-3)^3(x-4)^4.$$

Discuss the character of the curve at the points $x = 1, x = 2, x = 3, x = 4$.

9. Prove that if at the point of a minimum there exists a right hand derivative, then it is nonnegative, and if there exists a left hand derivative, then it is non-positive.

10. Determine the points of extremum of the function $y = |3x-1| - x^3$.

11. A differentiable function f has only one critical point $x = 5$. Identify the relative extrema of f at the critical point if $f'(4) = -2.5$ and $f'(6) = 3$.

12. Show that the function $y = \begin{cases} 1/x^2 & (x > 0), \\ 3x^2 & (x \le 0) \end{cases}$ has a minimum at the point $x = 0$, though its first derivative does not change sign when passing through this point.

Practice Problems A

13. Find the extrema of the given functions :

(i) $y = 2x^3 - 3x^2$ (ii) $y = -x^2\sqrt{x^2+2}$

(iii) $y = x - \ln(1+x^2)$

14. Find the points of extrema of the followings functions:

(i) $f(x) = |3-x| + |2+x| + |5-x|$

(ii) $f(x) = \begin{cases} 2x^2+3 & x < 0 \\ 4 & x = 0 \\ x+3 & 0 < x \le 1 \\ 4-x & x > 1 \end{cases}$

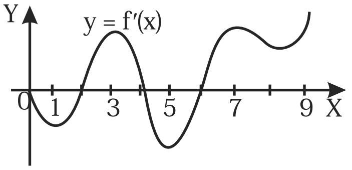
15. The graph of the first derivative f' of a function f is shown.

(i) On what intervals is f increasing ?

(ii) At what values of x does f have a local maximum or minimum ?

(iii) On what intervals is f concave up ?

(iv) What are the x-coordinates of the inflection points of f ?



16. Prove that the function $f(x) = x^{101} + x^{51} + x + 1$ does not have an extremum.

17. Prove that the function $\frac{(x-1)^2}{(x+1)^3}$ has a maximum value $\frac{2}{27}$, and a minimum value 0.

18. Show that the function f, defined by $f(x) = x^5 - 5x^4 + 5x^3 - 1$, has a maximum value when $x = 1$, a minimum value when $x = 3$ and neither when $x = 0$.

19. Find the points of maximum and minimum of the curve for which

$$\frac{dy}{dx} = (x-2)^3(2x-3)^4(3x-4)^5(4x-5)^6.$$

20. Let $f(x) = \sin x(1+\cos x)$, $x \in (0, 2\pi)$. Find the number of critical points of f(x). Also identify which of these critical points are points of maxima/minima.

21. Discuss the extremum of

$f(x) = \begin{cases} 1+\sin x, & x < 0 \\ x^2 - x + 1, & x \ge 0 \end{cases}$ at $x = 0$.

22. Prove that the function

$$f(x) = \begin{cases} x^2\sin^2(1/x) & \text{for } x \ne 0, \\ 0 & \text{for } x = 0. \end{cases}$$

has a minimum at the point $x = 0$ but not a strict minimum.

23. A certain function $y = f(x)$ has the property that $y' = \sin y + 2y + x$. Show that at a critical point the function has a local minimum.

24. If $\frac{dy}{dx} = (x-a)^{2n}(x-b)^{2p+1}$, where n and p are positive integers, show that $x = a$ gives neither maximum nor minimum values of y, but that $x = b$ gives a minimum.

25. Suppose f is a differentiable function with

$$f'(x) = \frac{(2x-1)\ell n(2x^2-3x+2)}{(x-2)^2}.$$

Find all critical numbers of f and determine whether each corresponds to a local maximum, a local minimum or neither.

26. Find constants A, B, C and D that guarantee that the graph of $f(x) = 3x^4 + Ax^3 + Bx^2 + Cx + D$ will have horizontal tangents at (2, –3) and (0, 7). There is a third point that has a horizontal tangent. Find this point. Then, for all three points, determine whether each corresponding to a relative maximum, a relative minimum or neither.

27. Let $f(x) = x^{1/3}\sqrt{Ax+B}$,where A and B are positive constants. Find all critical numbers of f and classify the corresponding value of f as a relative maximum, a relative minimum, or neither.

7.5 The First Derivative Procedure for Sketching the Graph of a Continuous Function

Step 1. Compute the derivative f'(x) and determine the critical points of f, that is, where f'(x) = 0 or f '(x) does not exist.

Step 2. Substitute each critical point into f(x) to find the y-coordinate. Plot these critical points on the coordinate plane.

Step 3. Determine where the function is increasing or decreasing by checking the sign of the derivative on the intervals whose endpoints are the critical points found in step 2.

Step 4. Sketch the graph so that it rises on the intervals where f'(x) > 0, passes through the critical points, and has a horizontal tangent where f'(x) = 0

Example 1: Sketch the graph of $f(x) = 2x^3 + 3x^2 - 12x - 5$.

Solution: We begin by computing and factoring the derivative:

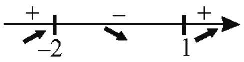
$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$$

We see that f '(x) exists for all x, and from the factored form of the derivative we see that f '(x) = 0 when x = –2 and when x = 1. The corresponding y-coordinates are found as follows :

f(–2) = 15; critical point (–2, 15)

f(1) = –12; critical point (1, –12)

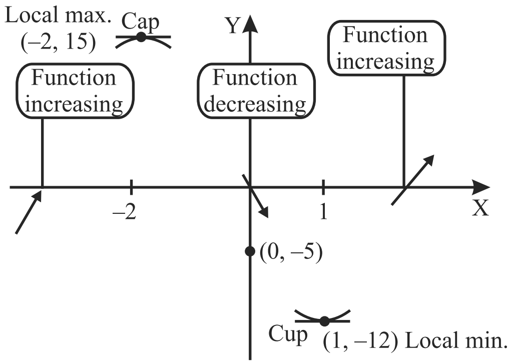
Next, to find the intervals of increase and decrease of the function, we plot the critical points on a number line and check the sign of the derivative at values to the left and right of –2 and 1. We find that f is increasing and decreasing, as indicated in the figure.



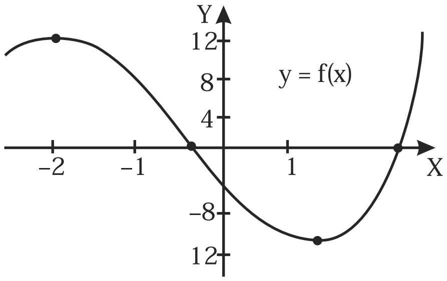
The arrow pattern in figure suggests that the graph of f has a relative maximum at (–2, 15) and a relative minimum at (1, –12).

We begin the sketch by plotting these points on a coordinate plane. We put a "cap" at the relative maximum (–2, 15) and a "cup" at the relative minimum (1, –12), as shown in figure (a). Finally, we note that because f(0) = –5, the graph has its y-intercept at (0, –5).

We complete the sketch by drawing the curves so that it increases for x < –2 to the relative maximum at (–2, 15), decreases for –2 < x < 1, passing through the y-intercept (0, –5) on its way to the relative minimum at (1, –12), and then increases again for x > 1. The completed graph is shown in figure (b).



(a) Preliminary sketch



(b) Final graph

Example 2: Determine where the function f(x)= (ℓnx)/x is increasing and where it is decreasing. Sketch the graph of f.

Solution: f is defined only for x > 0.

$$f'(x) = \frac{x(1/x) - (\ell nx).1}{x^2} = \frac{1 - \ell nx}{x^2}$$

Because $x^2 > 0$ for all x > 0, it follows that f '(x) = 0 only when ℓn x= 1; thus e is the only critical point of f. We use the first derivative test and look at the sign of f '(x) to the left of e (at x = 1, for example) and to the right (at x = 3, for example). We see that

f '(x) > 0 for x < e and

f '(x) < 0 for x > e.

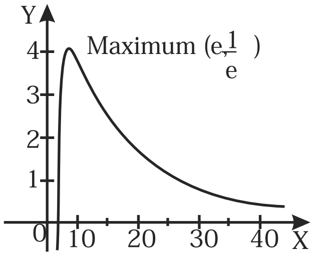
Thus, f is increasing to the left of e and decreasing to the right, as shown in the figure.

We now have most of the information we need to sketch the graph of f. In particular, we know that the graph lies entirely to the right of the y-axis (because f is defined only for x > 0) and that it rises to a relative maximum at x = e, after which it falls indefinitely.

Moreover, because we obtain $\frac{\ell nx}{x} = 0$ only for x = 1, it follows that the graph crosses the x-axis on the way up but not on the way down, and this causes us to suspect that the graph must "flatten out" in some way as $x \to \infty$.

But what happens at the two "ends" of the graph, as $x \to 0^+$ and as $x \to \infty$?

As x approaches 0 from the right, $\ell n\, x$ decreases without bound and $f(x) = (\ell n\, x)/x$ does the same, so the graph approaches the negative y-axis asymptotically. As x increases without bound, we have seen that f(x) decreases, but by how much ? The graph cannot cross the x-axis a second time because $\ell n\, x = 0$ only for x = 1, so the graph must "flatten out" in some way. Using limits we find that $f(x) \to 0$ as $x \to \infty$.



Example 3: Sketch the graph of $f(x) = \cos^2 x + \cos x$ on $[0, 2\pi]$

Solution: The derivative of f is

$$f'(x) = -2\cos x \sin x - \sin x$$

Because f'(x) exists for all x, we solve f '(x) = 0 to find the critical points :

$$-2\cos x \sin x - \sin x = 0$$

$$\sin x(2\cos x + 1) = 0$$

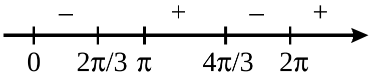
$$\sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}$$

The critical points on the interior of the interval $[0, 2\pi]$ are $x = \frac{2\pi}{3}$, π and $\frac{4\pi}{3}$.

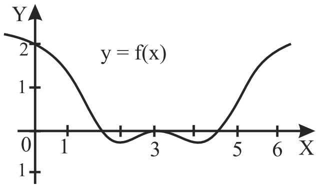
Because $f\left(\frac{2\pi}{3}\right) = -\frac{1}{4}$, $f(\pi)=0$, and $f\left(\frac{4\pi}{3}\right) = -\frac{1}{4}$, The points corresponding to these are

$$\left(\frac{2\pi}{3}, -\frac{1}{4}\right), (\pi, 0) \text{ and } \left(\frac{4\pi}{3}, -\frac{1}{4}\right)$$

Next, we plot the critical points on a number line and determine the sign of the derivative f '(x) in each interval, as indicated in the figure.



The sign scheme in the diagram indicates that we have relative minima at $\left(\frac{2\pi}{3}, -\frac{1}{4}\right)$ and $\left(\frac{4\pi}{3}, -\frac{1}{4}\right)$ and a relative maximum at $(\pi, 0)$. We plot these points on a coordinate plane and place "cups" at the relative minima and a "cap" at the relative maximum. Finally, we sketch a smooth curve through these points, obtaining the graph shown in the figure. If the curve is defined over a particular interval, then one should also plot the endpoints.



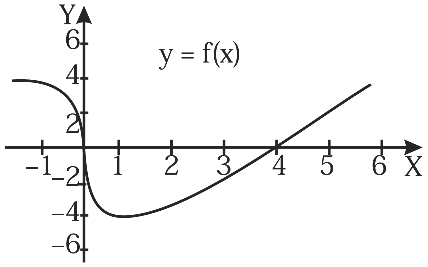
The graph of $f(x) = \cos^2 x + \cos x$

Example 4: Sketch the graph of $f(x) = x^{1/3}(x - 4)$.

Solution: $f(x) = x^{4/3} - 4x^{1/3}$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1)$$

We see that there is a relative minimum at (1, –3) and no extremum at (0, 0). We plot the critical points on a coordinate plane with a "cup" at (1, –3). Finally, we pass a smooth curve through these points, obtaining the graph shown in the figure. Note that at (0, 0) the curve has a vertical tangent.

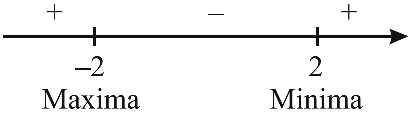


Example 5: Find the points of maxima/minima of $f(x) = x^3 - 12x$ and also draw the graph.

Solution: $f(x) = x^3 - 12x$

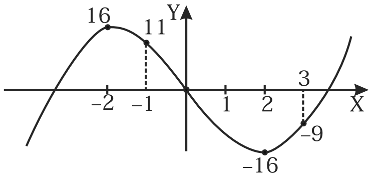
$$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2)$$

$$f'(x) = 0 \Rightarrow x = \pm 2$$



For tracing the graph let us find the maximum and minimum values of f(x).

$$f(-2) = 16, \quad f(2) = -16.$$

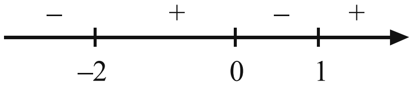


Example 6: Find the values of a for which all roots of the equation $3x^4 + 4x^3 - 12x^2 + a = 0$ are real and distinct.

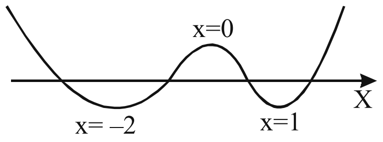
Solution: Consider the function

$f(x) = 3x^4 + 4x^3 - 12x^2 + a$.

Then $f'(x) = 12(x^3 + x^2 - 2x) = 12x(x - 1)(x + 2)$.



From the sign scheme for $f'(x)$, we can see that the shape of the curve will be as shown below.



For four real and distinct roots, the two minima must lie below the x-axis and the maxima must lie above the x-axis.

Thus, we have $f(-2) < 0$

i.e. $48 - 32 - 48 + a < 0$ ...(1)

i.e. $a < 32$

and $f(1) < 0$

i.e. $3 + 4 - 12 + a < 0$ ...(2)

i.e. $a < 5$

and $f(0) > 0$

i.e. $a > 0$ ...(3)

Taking intersection of inequalities (1), (2) and (3) we have $a \in (0, 5)$.

Example 7: Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) attains maxima at some positive value of x, then find the possible values of a.

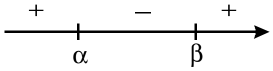
Solution: We have

$f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$

and $f'(x) = 3x^2 + 6(a - 7)x + 3(a^2 - 9)$

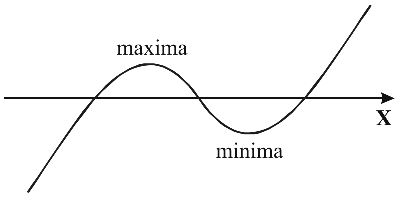
There are two critical points.

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Also, we have $f(-\infty) = -\infty$ and $f(\infty) = \infty$

From the above facts, the graph of the curve $y = f(x)$ can be drawn as shown below. Thus, if the maxima occurs at some positive value of x, then the minima must also occur at some positive value of x.



This basically implies that both of roots $f'(x) = 0$ must be positive and distinct, which it possible if discriminant $D > 0$

$\Rightarrow$ $(a - 7)^2 > a^2 - 9$

$\Rightarrow$ $-14a + 58 > 0$

$\Rightarrow$ $a < 29/7$ ...(1)

and product of the roots > 0

$$-\frac{B}{2A} > 0$$

$\Rightarrow$ $a^2 - 9 > 0$

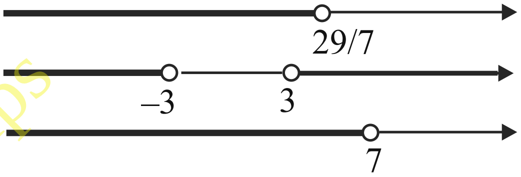
$\Rightarrow$ $a^2 < -3$ or $a > 3$ ...(2)

and sum of the roots > 0

$\Rightarrow$ $a - 7 < 0$

$\Rightarrow$ $a < 7$ ...(3)

Drawing the number line for inequalities (1), (2), (3)



and taking intersection, gives

$$a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right).$$

Example 8: For what values of 'a' the point of local minima of $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ is less than 4 and point of local maxima is greater than –2.

Solution: $f'(x) = 3(x^2 - 2ax + a^2 - 1)$

Clearly roots of the equation $f'(x) = 0$ must be distinct and lie in the interval (–2, 4)

$\therefore$ $D > 0 \Rightarrow a \in R$ ...(1)

$f'(-2) > 2 \Rightarrow a^2 + 4a + 3 > 0$

$\Rightarrow$ $a < -3$ or $a > -1$ ...(2)

$f'(4) > 0 \Rightarrow a^2 - 8a + 15 > 0$

$\Rightarrow$ $a > 5$ or $a < 3$ ...(3)

and $-2 < -\frac{B}{2A} < 4 \Rightarrow -2 < a < 4$

From (1), (2) and (3) $-1 < a < 3$.

Alternative:

$f'(x) = 3(x - (a - 1))(x - (a + 1))$

Clearly $-2 < a + 1 < 4$ and $-2 < a - 1 < 4$.

$\Rightarrow$ $-1 < a < 3$.

Practice Problems B

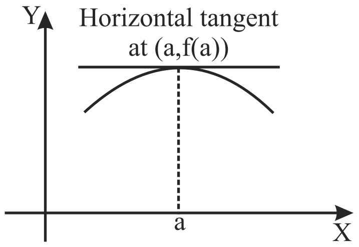
1. Find the extrema of the function $y = x\sqrt{2-x^2}$ and draw its graph.
2. Find the points of local minima of the function, $f(x) = 4x^3 - x\,|x-2|,\ x \in [0, 3]$.
3. Find the values of 'a' so that $f(x) = e^{-2x} + (6-2a)e^{-x} + 6ax$ has all its point(s) of extremum in (0, 1).
4. Determine the points of extrema of $f(x) = |x-a|^3 + |(x-b)^3|$
5. Prove that the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has only one local minimum.
6. Find the least integral value of a for which the function $f(x) = \frac{a}{3}x^3 + (a+2)x^2 + (a-1)x + 2$ possess a negative point of local minimum.
7. For what real values of a and b are all the extreme values of the function, $f(x) = a^2x^3 + ax^2 - x + b$ negative and the local maximum is at the point $x_0 = -1$?
8. For what real values of 'a' are all the extrema for the function $f(x) = \frac{5}{3}a^2x^3 + 2ax^2 - 9x + b$ are positive and the function has a local maximum at the point $x = -5/9$.
9. Let $f(x) = \frac{x}{2} + \frac{2}{x}$. Find local maximum and local minimum value of f(x). Can you explain this discrepancy of locally minimum value being greater than locally maximum value?
10. If $f(x) = 4x^3 - x^2 - 2x + 1$,
$$g(x) = \begin{cases} \text{Min}\{f(t) : 0 \le t \le x, 0 \le x \le 1 \\ 3 - x & 1 < x \le 2 \end{cases},$$
examine the differentiability of g(x) and find its points of extrema.
11. Construct a function which has a local maximum point, with local maximum point defined as in this section, but would not have a local maximum if the definition were changed to demand $f(a) > f(x),\ x \ne a$, instead of $f(a) \ge f(x)$.

7.6 Second Derivative Test

The second derivative is useful in testing whether at a stationary point there is a local maximum or local minimum.

For instance, let x = a be a stationary point for the function f and assume that $f''(a)$ happens to be negative. If f'' is continuous in some open interval that contains a, then $f''(a)$ remains negative for a suitably small open interval that contains a. This means that the graph of f is concave down near (a, f(a)), hence lies below its tangent lines.

In particular, it lies below the horizontal tangent line at the stationary point (a, f(a)), as illustrated in the figure. Thus, the function has a local maximum at the stationary point a.



Graph is concave down near (a, f(a)).

This observation suggests the following test for a local maximum or minimum.

If $f'(a) = 0$ and $f''(a) < 0$ the tangent to the graph of the function has a horizontal tangent at the point a and is concave down in the vicinity of that point; therefore x = a is a point of local maximum.

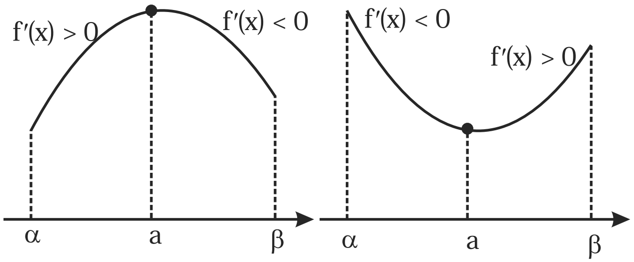
Similarly, if $f'(a) = 0$ and $f''(a) > 0$ the tangent to the graph of the function y = f(x) at the point a is horizontal, and the graph is concave up in the vicinity of that point; therefore x = a is a point of local minimum.

The Second Derivative Test

Let x = a be a stationary point of a function f (i.e. $f'(a) = 0$) in an open interval (α, β), that is, assume α < a < β. Assume also that the second derivative f'' exists in (α, β). Then we have the following:

(i) If f'' is negative in (α, β), f has a local maximum at a.
(ii) If f'' is positive in (α, β), f has a relative minimum at a.

The two cases are illustrated in the figure.



(i) Local maximum at a (ii) Local minimum at a

Proof Consider case (i), $f'' < 0$ in (α, β). The function f' is strictly decreasing in (α, β). But $f'(a) = 0$, so f' changes its sign from positive to negative at a, as shown in Figure (i). Hence, f has a local maximum at a. The proof in case (ii) is entirely analogous.

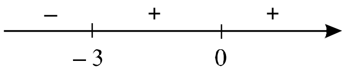
Special Case of the Second Derivative Test

If f'' is continuous at a, and if $f''(a) \neq 0$, there will be a neighbourhood of a in which f'' has the same sign as $f''(a)$. Therefore, if $f'(a) = 0$, the function f has a local maximum at a if $f''(a)$ is negative, and a relative minimum if $f''(a)$ is positive. This test suffices for many examples that occur in practice.

Geometrically, it tells us that locally the highest (lowest) points of a graph are attained when the tangent line is horizontal and the graph bulges upward (downward) at the point of contact.

Let us take the function $f(x) = x^5 + 5x^4$.

Here $f'(x) = 5x^3 (x + 4)$, and $f''(x) = 20x^2 (x + 3)$
$f'(x) = 0$ at $x = -4$ and $x = 0$. We now apply the second derivative test at these stationary points.
Consider the sign scheme of $f''(x)$



$f''(x)$ is negative in the neighbourhood of $x = -4$ and positive in the neighbourhood of $x = 0$.

When x passes through the point $x = -4$, the graph of the given function is concave down on both sides of the point, therefore $x = -4$ is a point of local maximum.
When x passes through the point $x = 0$, the graph of the given function is concave up on both sides of the origin, therefore $x = 0$ is a point of local minimum.

Note:
If we apply the special case of the second derivative test we have:
At $x = -4$, $f''(-4) = 20(-4)^2 (-4 + 3) = -320 < 0$.

Therefore $x = -4$ is a point of local maximum.

At $x = 0$, $f''(0) = 20(0)^2 (-4 + 3) = 0$. Here the special test cannot be applied. We need to consider the sign of the second derivative in the neighbourhood of $x = 0$ as done above.

Note: When it is relatively easy to find the second derivative and if the roots of this function are easy to find, then the second derivative test provides a quick means for classifying the stationary points. However, if it is difficult to compute $f''(a)$, it may be easier to first derivative test.

Now, consider the function $f(x) = x^5 + 5x^3$.
$f'(x) = 5x^2 (x + 3)$, and $f''(x) = 10x (x^2 + 3)$
$f'(x) = 0$ at $x = -3$ and $x = 0$.

We apply the special case of the second derivative test:
At $x = -3$, $f''(-3) < 0$.

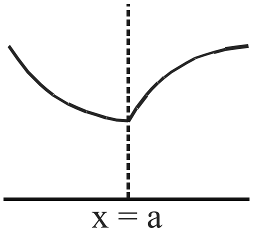
Therefore $x = -3$ is a point of local maximum.

At $x = 0$, $f''(0) = 0$. Here, again the special test cannot be applied. We need to consider the sign of the second derivative in the neighbourhood of $x = 0$.

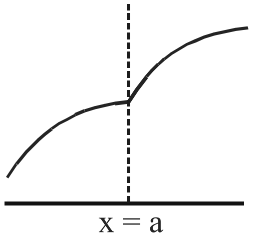
We have $y'' < 0$ for $x < 0$ and $y'' > 0$ for $x > 0$.

When x passes through the point $x = 0$, $f''(x) = 10x (x^2 + 3)$ changes sign from negative to positive. Consequently, the origin is a point of inflection of the graph of the given function, the interval of concave down lying on the left of it and the interval of concavity on the right.

Note: The second derivative test is not applicable to such critical points where the first derivative does not exist. We can conclude this from the following figures:



The function is not differentiable at $x = a$. We observe that it has a local minimum at $x = a$, even if the concavity is changing when x passes through the point $x = a$.



Here also function is not differentiable at $x = a$. We observe that it does not have a local maximum at $x = a$, even if the graph is concave down on both sides of $x = a$.

Example 1: Using the second derivative test, find the extrema of the function $f(x) = 2 \sin x + \cos 2x$.

Solution: Since the function is a periodic one we may confine ourselves to the interval $[0, 2\pi]$.
We find the first and second derivatives :

$f' = 2 \cos x - 2 \sin 2x = 2 \cos x(1 - 2 \sin x)$;
$f'' = -2 \sin x - 4 \cos 2x$.

From the equation $2 \cos x(1 - 2 \sin x) = 0$ we get the critical points on the interval $[0, 2\pi]$:

$x_1 = \pi/6$, $x_2 = \pi/2$, $x_3 = 5\pi/6$, $x_4 = 3\pi/2$

Now we find the sign of the second derivative at each critical point:
Since $f''(\pi/6) = -3 < 0$, f(x) has a maximum $f(\pi/6) = 3/2$;
$f''(\pi/2) = 2 > 0$, f(x) has a minimum $f(\pi/2) = 1$;
$f''(5\pi/6) = -3 < 0$, f(x) has a maximum $f(5\pi/6) = 3/2$;
$f''(3\pi/2) = 6 > 0$, f(x) has a minimum $f(3\pi/2) = -3$.

Example 2: Find the points of extrema of the function $f(x) = 12x^5 - 45x^4 + 40x^3 + 40$ using second derivative test.

Solution: We have

$f(x) = 12x^5 - 45x^4 + 40x^3 + 40$
$f'(x) = 60x^4 - 180x^3 + 120x^2 = 60x^2 (x - 1) (x - 2)$

and $f''(x) = 60(4x^3 - 9x^2 + 4x) = 60x(4x^2 - 9x + 4)$.
The stationary points of f(x) are $x = 0, 1, 2$
At the stationary points, we have

$f''(0) = 0 \Rightarrow$ more investigation is required.

$f''(0 - h) < 0$ and $f''(0 + h) > 0$

∴ $x = 0$ is not a point of extremum since the concavity is changing in the neighbourhood of a stationary point.

$f''(1) = 60\,(4 - 9 + 4) < 0$

$\Rightarrow$ local maxima at $x = 1$ and

$f''(2) = 60(32 - 36 + 8) > 0$

$\Rightarrow$ local minima at $x = 2$.

Example 3: Find the points of maxima and minima of the function $f(x) = \sin 2x + 2\cos x,\ x \in [0, 2\pi]$.

Solution: We have

$f(x) = \sin 2x + 2\cos x$

$f'(x) = 2(\cos 2x - \sin x)$

and $f''(x) = -2(2\sin 2x + \cos x)$

$= -2\cos x(1 + 2\sin x)$

The stationary points of f(x) are given by

$\cos 2x - \sin x = 0$

$\Rightarrow\ 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow\ \sin x = -1, \frac{1}{2}$

$$\Rightarrow\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \qquad [\because\ 0 \le x \le 2\pi]$$

Now, we have

$$f''\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right)\left[1 + 2\sin\frac{\pi}{6}\right] < 0$$

$\Rightarrow$ local maxima at $x = \frac{\pi}{6}$

$$f''\left(\frac{5\pi}{6}\right) = -2\cos\left(\frac{5\pi}{6}\right)\left[1 + 2\sin\frac{5\pi}{6}\right]$$

$$= \sqrt{3}\ (1 + 1) > 0$$

$\Rightarrow$ local minima at $x = \frac{5\pi}{6}$.

$$\text{and}\quad f''\left(\frac{3\pi}{2}\right) = -2\cos\left(\frac{3\pi}{2}\right)\left[1 + 2\sin\frac{3\pi}{2}\right] = 0$$

$\Rightarrow$ more investigation is required

To determine the nature of the stationary point $x = 3\pi/2$, let us check the sign of $f''(x)$ in the neighbourhood of $3\pi/2$.

$$f''\left(\frac{3\pi}{2} - h\right) = -2\cos\left(\frac{3\pi}{2} - h\right)\left[1 + 2\sin\left(\frac{3\pi}{2} - h\right)\right] < 0$$

$$f''\left(\frac{3\pi}{2} + h\right) = -2\cos\left(\frac{3\pi}{2} + h\right)\left[1 + 2\sin\left(\frac{3\pi}{2} + h\right)\right] > 0$$

$x = 3\pi/2$ is not a point of extremum since the concavity is changing in the neighbourhood of a stationary point.

Alternatively, we may use first derivative test as follows.

$$f'\left(\frac{3\pi}{2} - h\right) = 2\left[\cos 2\left(\frac{3\pi}{2} - h\right) - \sin\left(\frac{3\pi}{2} - h\right)\right]$$

$$= [-\cos 2h + \cos h] > 0 \quad [\because\ \cos 2h < \cos h]$$

$$\text{and}\quad f'\left(\frac{3\pi}{2} + h\right) = 2\left[\cos 2\left(\frac{3\pi}{2} + h\right) - \sin\left(\frac{3\pi}{2} + h\right)\right]$$

$$= 2[-\cos 2h + \cos h] > 0$$

Since $f'(x)$ does not change sign as x passes through $3\pi/2$, there is no extremum at $x = 3\pi/2$.

Example 4: Find the extrema and the points of inflection of the function $y = (x + 1)^2(x - 2)$.

Solution: Let us find the first derivative: $y' = 3(x^2 - 1)$.

The roots of the first derivative are $x_1 = -1$, $x_2 = 1$. Let us now find the second derivative : $y'' = 6x$. Then we find the values of the second derivative at the stationary points : $y''(-1) = -6 < 0$, i.e. y has a local maximum at $x = 1$; $y''(1) = 6 > 0$, i.e. y has a local minimum at $x = \perp$.

We shall now find the inflection point, for which purpose we shall equate the second derivative to zero: $6x = 0$, i.e. $x = 0$.

To the left of the point $x = 0$ we have $y''(0 - h) < 0$, the curve is concave down , and to the right of the point $x = 0$ we have $y''(0 + h) > 0$, the curve is concave up; consequently, the point with the abscissa $x = 0$ is an inflection point.

In conclusion we note that when determining points of inflection and intervals of concavity we use the same rules as in the case of the determination of points of extremum but apply these rule not to the given function itself, but to its first derivative.

Our investigation show that to an interval of increase of the first derivative there corresponds an interval of concavity of the graph of the function and to an interval of decrease of the first derivative an interval of convexity; accordingly, a point of extremum of the first derivative is that of a point of inflection of the graph of the function.

Example 5: Find the extrema of the function

$$y = 2x + 3\sqrt[3]{x^2}\ .$$

Solution: We find the derivative

$$y' = 2 + \frac{2}{\sqrt[3]{x}} = \frac{2}{\sqrt[3]{x}}(\sqrt[3]{x} + 1) \qquad \text{...(1)}$$

Equating the derivative y' to zero, we get the stationary point:

$\sqrt[3]{x} + 1 = 0$. $\quad x_1 = -1$.

It is possible to test the behaviour of the function at the point $x = -1$ by means of the second derivative test:

$$y'' = -\frac{2}{3x\sqrt[3]{x}}.$$

Here, $y'' < 0$ for $x = -1$ and, hence, $x = -1$ is a point of local maximum of the function.

Observing the denominator of the expression of y' in (1), we find the point where there is no derivative. This gives the second critical point (but not stationary point) of the function : $x = 0$.

Here it is not possible to apply the second derivative test. We apply the first derivative test.

For $x = -h$, we obviously have $y' < 0$; for $x = h$ we have $y' > 0$. Consequently, $x = 0$ is a point of local minimum of the function y, and $y_{min} = 0$.

We could have as well applied the first derivative test at $x = -1$.

From (1) we have : if $x = -1 - h$, where h is a sufficiently small positive number, then $y' > 0$;
but if $x = -1 + h$, then $y' < 0$.

Hence, $x = -1$ is a point of local maximum of the function y, and $y_{max} = 1$.

If it is difficult to determine the sign of the derivative y', one can calculate arithmetically by taking for h a sufficiently small positive number.

Example 6: If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find a.

Solution: We have, $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore$ $f'(x) = 6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow$ $6[x^2 - 3ax + 2a^2] = 0$

$\Rightarrow$ $x^2 - 3ax + 2a^2 = 0$ $\Rightarrow$ $x^2 - 2ax - ax + 2a^2 = 0$

$\Rightarrow$ $x(x - 2a) - a(x - 2a) = 0$

$\Rightarrow$ $(x - a)(x - 2a) = 0$ $\Rightarrow$ $x = a, x = 2a$

Now, $f''(x) = 12x - 18a$

$\therefore$ $f''(a) = 12a - 18a = -6a < 0$

$\therefore$ f(x) will be maximum at $x = a$

i.e. $p = a$

Also, $f''(2a) = 24a - 18a = 6a$

$\therefore$ f(x) will be minimum at $x = 2a$

i.e. $q = 2a$

Given, $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a = 2$.

Example 7: Suppose f(x) is real valued polynomial function of degree 6 satisfying the following conditions:

(a) f has minimum value at x = 0 and 2

(b) f has maximum value at x = 1

(c) for all x, $\lim_{x\to 0} \frac{1}{x} \ln \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$.

Determine f(x).

Solution: The determinant $= 1 + \frac{f(x)}{x^3}$.

$$\therefore \quad \lim_{x\to 0} \ln\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = 2$$

$\Rightarrow$ The coefficients of x^3, x^2, x and the constant term in f(x) should be zero in order that the limit may exist. Let us assume $f(x) = ax^6 + bx^5 + cx^4$

$$\Rightarrow \quad \ln e^{\lim_{x\to 0} \frac{1}{x} \cdot \frac{f(x)}{x^3}} = \lim_{x\to 0} \frac{f(x)}{x^4} = 2$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{ax^6 + bx^5 + cx^4}{x^4} = 2 \Rightarrow c = 2.$$

$\therefore$ $f(x) = ax^6 + bx^5 + 2x^4$

$f'(x) = x^3(6ax^2 + 5bx + 8)$

$f'(1) = 0$ and $f'(2) = 0$ give

$6a + 5b + 8 = 0$ and $24a + 10b + 8 = 0$

$$\Rightarrow \quad a = \frac{2}{3}\ ;\ b = -\frac{12}{5}$$

$$\Rightarrow \quad f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4.$$

Concept Problems C

1. Find the extrema of the function and also find the extreme values:
   (i) $y = x^2(a - x)^2$ (ii) $y = x + \sqrt{1 - x}$
   (iii) $y = x^{1/x}$
2. Find the points of extrema of the following functions:
   (i) $y = \frac{x^3}{3} - 2x^2 + 3x + 1$
   (ii) $y = (x - 1)\sqrt[3]{x^2}$
3. Find the points of extrema of the following functions:
   (i) $y = 2\sin x + \cos 2x$, $x \in [0, 2\pi]$
   (ii) $y = (x - 2)^3(2x + 1)$
4. Suppose the derivative of the function $y = f(x)$ is $y' = (x - 1)^2(x - 2)$. At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

5. (i) Show that both of the functions $f(x) = (x-1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$.

(ii) What does the second derivative test tell you about the nature of these stationary points ?

(iii) What does the first derivative test tell you about the nature of these stationary points ?

6. (i) Show that $f(x) = 1 - x^5$ and $g(x) = 3x^4 - 8x^3$ both have stationary points at $x = 0$.

(ii) Can we determine the nature of these stationary points using the second derivative test?

7. Prove that if the graph of a function is everywhere concave up or everywhere concave down, then this function cannot have more than one extremum.

8. Let x_0 be the abscissa of the point of inflection on the curve $y = f(x)$. Will the point x_0 be a point of extremum for the function $y = f'(x)$?

9. Prove that in any twice differentiable function between two points of extremum there lies atleast one point of inflection, provided it is not a constant function.

10. Taking the function $y = x^4 + 8x^3 + 18x^2 + 8$ as an example, verify that between the abscissas of the inflection points of the graph of a function can be no points of extremum.

Practice Problems C

11. Find the extrema of the function and also find the extreme values :

(i) $f(x) = xe^{x-x^2}$ (ii) $f(x) = \dfrac{2x}{x^2+9}$

(iii) $f(x) = \dfrac{x}{\ln x}$ (iv) $f(x) = \dfrac{x^2-2x+2}{x-1}$

12. Find the points of extrema of the following functions:

(i) $f(x) = x(x+1)^3(x-3)^2$

(ii) $f(x) = \cos x + \dfrac{1}{2}\cos 2x - \dfrac{1}{3}\cos 3x$

(iii) $f(x) = x^2 \ell n\, x$

(iv) $f(x) = \sin 3x - 3\sin x$

13. Find the points of extrema of the following functions:

(i) $f(x) = \dfrac{x^2-3x+2}{x^2+2x+1}$

(ii) $f(x) = a^2\sec^2 x + b^2\operatorname{cosec}^2 x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2},\ a, b > 0$

(iii) $f(x) = 3\sqrt[3]{x^2} - x^2$

(iv) $f(x) = \sqrt[3]{(x-1)^2} + \sqrt[3]{(x+1)^2}$

14. Prove that the function $10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$ has a minimum value when $x = 1$, and no other maxima or minima.

15. For what a does the function $f(x) = a\sin x + \dfrac{1}{3}\sin 3x$ have an extremum at $x = \dfrac{\pi}{3}$? Will it be a maximum or a minimum?

16. Let $f(x) = 2x - 3x^{2/3}$. Prove that the function has extrema at $x = 0, 1$.

17. Consider the function $f(x) = (\ell n x)/\sqrt{x}$.

(i) Find the intervals on which f is increasing or decreasing.

(ii) Find the local maximum and minimum values of f.

(iii) Find the intervals of concavity and the inflection points.

18. Suppose f'' is continuous on $(-\infty, \infty)$.

(i) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f ?

(ii) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

19. Find the value of x at which $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ increases most rapidly.

20. Use implicit differentiation to show that a function defined implicitly by $\sin x + \cos y = 2y$ has a critical point whenever $\cos x = 0$. Then use either the first or second derivative test to classify these critical points as relative maxima or minima.

21. Let f be a function that is differentiable at 0 and satisfies $f(1) = -1$. If $y = f(x)$, suppose

$$\frac{dy}{dx} = \frac{3y^2+x}{y^2+2}$$

(i) Find an equation for the tangent line to the graph of f at the point where $x = 1$.

(ii) Note that the origin is a critical point. What kind of relative extremum (if any) occurs at this point ?

7.7 Higher Order Derivative Test

It was noted that if at a certain point $x = a$ we have $f'(a) = 0$ and $f''(a) = 0$, then at this point there may be either a maximum or a minimum or neither.

One way of solving the problem is to test the sign of the second derivative on the left and on the right of the point $x = a$.

Another way is to use the first derivative test.

Now we will show that it is possible in this case to investigate by means of Taylor's Theorem.

First we observe that the second derivative test can also be obtained when considering the Taylor series :

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{1}{2!}f''(a)\cdot(x-a)^2 + \ldots \quad \ldots(1)$$

Let $f'(a) \neq 0$. For example, let $f'(a) > 0$. For x close to a, the quantities $(x-a)^2, (x-a)^3, \ldots$ may be neglected when compared with $(x-a)$. We obtain

$$f(x) = f(a) + f'(a)\cdot(x-a)$$

$$\text{or}\quad f(x) - f(a) = f'(a)\cdot(x-a) \quad \ldots(2)$$

From this equation we see for $x > a$, $f(x) - f(a) > 0$, that is $f(x) > f(a)$. But if $x < a$, then $f(x) < f(a)$. Therefore, when $x = a$ there is neither a maximum nor a minimum. Similarly when $f'(a) < 0$. But if $f'(a) = 0$, then we cannot neglect the term as compared with $(x-a)^2$, we get, from (1)

$$f(x) = f(a) + \frac{1}{2!}f''(a)(x-a)^2$$

From this we see that for $f''(a) > 0$, $f(x) > f(a)$ irrespective of whether $x < a$ or $x > a$. Hence, $f(a)$ is less than any adjacent value of $f(x)$ and therefore $f(a)$ is a minimum value of the function. If $f''(a) < 0$, then $f(x) < f(a)$ and $f(a)$ is a maximum value of the function.

It may however happen that $f''(a) = 0$. How do we investigate the values of the function near $x = a$ in that case? We then have to take the next derivatives of the function $f(x)$. If $f'''(a) \neq 0$, then, neglecting $(x-a)^4$, etc., as compared with $(x-a)^3$, we get from (1)

$$f(x) = f(a) + \frac{1}{6}f'''(a)(x-a)^3$$

The difference $f(x) - f(a)$ changes sign depending on whether $x > a$ or $x < a$. For $x = a$ we have neither a maximum nor a minimum.

But if $f'''(a) = 0$ and $f^{(4)}(a) \neq 0$, then

$$f(x) = f(a) + \frac{1}{24}f^{(4)}(a)(x-a)^4$$

The sign of the expression $f(x) - f(a)$ is the same for $x < a$ and for $x > a$; it is determined by the sign of $f^{(4)}(a)$.

If $f^{(4)}(a) > 0$, then we have a minimum; if $f^{(4)}(a) < 0$, then a maximum. Now we can guess that if for $x = a$ the first nonzero derivative is of odd order (first, third, fifth, etc.), then there is neither a maximum nor a minimum. But if the first nonzero derivative is of even order (second, fourth, etc.), then we either have a maximum or a minimum depending on the sign of the derivative.

Example 1: Test the functions (i) $y = (x-1)^4$ (ii) $y = x^4 + 2x^5$ possible extrema.

Solution:

(i) We find the derivative : $y' = 4(x-1)^3$; $(x-1)^3 = 0$; $x = 1$ is a stationary point. The second derivative $y'' = 12(x-1)^2$ is equal to zero at $x = 1$. The third derivative $y''' = 24(x-1)$ also vanishes at $x = 1$. The fourth derivative $y^{(4)} = 24 > 0$. Therefore, in accordance with the above dicussion, we infer that at the point $x = 1$ the function possesses a minimum $y_{min} = 0$.

(ii) For the function $y = x^4 + 2x^5$ the critical points are represented by the solutions of the equation $y' = 4x^3 + 10x^4 = 0$ that is , $x_1 = 0$ and $x_2 = -0.4$.

At $x_1 = 0$ we have

$y''(0) = 12x^2 + 40x^3 |_{x=0} = 0,$

$y'''(0) = 24x + 120x^2 |_{x=0} = 0,$

$y^{(4)} = 24 + 240x |_{x=0} = 24.$

Thus, at $x_1 = 0$ the first nonzero derivative has an even order (four and is positive). Hence, $x_1 = 0$ is a point of local minimum of the function.

At $x_2 = -0.4$ we have

$y''(-0.4) = 12x^2 + 40x^3 |_{x=-0.4} = -0.64 < 0.$

Consequently, at $x_2 = -0.4$ the first nonzero derivative has an even order (two) and is negative. Therefore $x_2 = -0.4$ is a point of local maximum.

Example 2: Prove that $x^5 - 5x^4 + 5x^3 - 10$ has a maximum for $x = 1$, a minimum for $x = 3$, and for $x = 0$, it has neither a maximum nor a minimum.

Solution: Let $f(x) = x^5 - 5x^4 + 5x^3 - 10$.

For a maximum or minimum of $f(x)$ it is necessary that $f'(x) = 0$ i.e., $5x^4 - 20x^3 + 15x^2 = 0$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0 \Rightarrow 5x^2(x-3)(x-1) = 0$$

$$\Rightarrow x = 0, 1 \text{ or } 3.$$

Thus f(x) is stationary at $x = 0$, 1 and 3.
Now $f''(x) = 20x^3 - 60x^2 + 30x$.
We have $f''(1) = 20 - 60 + 30 = -10 < 0$.
Hence f(x) has a maximum value at $x = 1$.
Again $f''(3) = 540 - 540 + 90 = 90 > 0$
Hence f(x) has a minimum value at $x = 3$.

Further we have $f''(0) = 0$. So we find $f'''(x)$. We have $f'''(x) = 60x^2 - 12x + 30$ which is nonzero when $x = 0$.

Hence at $x = 0$, f(x) has neither a maximum nor a minimum.

For greater generality, we assume that not only f"(x), but also all derivatives of the function f(x) up to the nth order inclusive vanish at $x = a$;

$$f'(a) = f''(a) = \ldots f^{(n)}.(a) = 0 \qquad \ldots(1)$$

and $f^{(n+1)}(a) \neq 0$

Further, we assume that f(x) has continuous derivatives up to the $(n+1)^{th}$ order inclusive in the neighbourhood of the point $x = a$.

Write the Taylor's expansion series for f(x), taking account of equalities (1);

$$f(x) = f(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \ldots(2)$$

where ξ is a number that lies between a and x.

Since $f^{(n+1)}(x)$ is continuous in the neighbourhood of the point a and $f^{(n+1)}(a) \neq 0$, there will be a small positive number h such that for any x that satisfies the inequality $|x - a| < h$, it will be true that $f^{(n+1)}(x) \neq 0$. And if $f^{(n+1)}(a) > 0$, then at all points of the interval $(a - h, a + h)$ we will have $f^{(n+1)}(x) > 0$; if $f^{(n+1)}(a) < 0$, then at all points of this interval we will have $f^{(n+1)}(x) < 0$;

$$f(x) - f(a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \ldots(3)$$

and consider various special cases.

Case 1: n is odd.

(a) Let $f^{(n+1)}(a) < 0$. Then there will be an interval $(a - h, a + h)$ at all points of which the $(n + 1)$th derivative is negative. If x is a point of this interval, then ξ likewise lies between $a - h$ and $a + h$ and consequently, $f^{(n+1)}(\xi) < 0$. Since $n + 1$ is an even number, $(x - a)^{n+1} > 0$ for $x \neq a$, and therefore the right side of formula (3) is negative.

Thus, for $x \neq a$ at all points of the interval $(a - h, a + h)$ we have $f(x) - f(a) < 0$ and this means that at $x = a$ the function has a maximum.

(b) Let $f^{(n+1)}(a) > 0$. Then we have $f^{(n+1)}(\xi) > 0$ for a sufficiently small value of h at all points x of the interval $(a - h, a + h)$. Hence, the right side of formula (3) will be positive; in other words, for $x \neq a$ we will have the following at all points in the given interval :
$f(x) - f(a) > 0$
and this means that at $x = a$ the function has a minimum.

Case 2: n is even.
Then $n + 1$ is odd and the quantity $(x - a)^{n+1}$ has different signs for $x < a$ and $x > a$.
If h is sufficiently small in absolute value, then the $(n + 1)$ th derivative retains the same sign at all points of the interval $(a - h, a + h)$ as at the point a. Thus, $f(x) - f(a)$ has different signs for $x < a$ and $x > a$. But this means that there is neither maximum nor minimum at $x = a$.
It will be noted that if $f^{(n+1)}(a) > 0$ when n is even, then $f(x) < f(a)$ for $x < a$ and $f(x) > f(a)$ for $x > a$.
But if $f^{(n+1)}(a) < 0$ when n is even, then $f(x) > f(a)$ for $x < a$ and $f(x) < f(a)$ for $x > a$.
The results obtained may be formulated as follows :

The Higher Order Derivative Test

If at $x = a$ we have $f'(a) = f''(a) = \ldots = f^{(n)}(a) = 0$
and the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of even order, then at the point a
f(x) has a maximum if $f^{(n+1)}(a) < 0$
f(x) has a minimum if $f^{(n+1)}(a) > 0$

But if the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of odd order, then the function has neither maximum nor minimum at the point a. Here,
f(x) increases if $f^{(n+1)}(a) > 0$
f(x) decreases if $f^{(n+1)}(a) < 0$.

Example 3: Test the function
$f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$
for maximum and minimum.

Solution: We find the critical values of the function
$f'(x) = 4x^3 - 12x^2 + 12x - 4 = 4(x^3 - 3x^2 + 3x - 1)$
From equation $4(x^3 - 3x^2 + 3x - 1) = 0$
we obtain the only critical point $x = 1$
(since this equation has only one real root).
We investigate the character of the critical point $x = 1$:
$f''(x) = 12x^2 - 24x + 12 = 0$ for $x = 1$
$f'''(x) = 24x - 24 = 0$ for $x = 1$
$f^{(4)}(x) = 24 > 0$ for any x
Consequently, for $x = 1$ the function f(x) has a minimum.

Example 4: Find whether $x = 1$ is a point of maximum or minimum of the function
$f(x) = 8x^5 - 15x^4 + 10x^3$.

Solution: We have $f'(x) = 40x^4 - 60x^3 + 30x^2$
$= 20x(x - 1)^2(2x + 1)$.
$f'(x) = 0$ for $x = 0, 1, -1/2$.
For $x = 1$, $f''(1) = 0$.
$f'''(x) = 20(16x^2 - 18x)$
$f'''(1) \neq 0$.
The first non vancing derivative is a derivative of odd order.
$\therefore$ The function has neither a maximum nor a minimum at $x = 1$.

Example 5: Find points of local maxima or minima of $f(x) = x^5 - 5x^2 + 5x^3 - 1$

Solution: $f(x) = x^3 + 5x^3 - 5x^2 - 1$

$$f'(x) = 5x^2 (x - 1)(x - 3)$$

$$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$$

$$f''(x) = 10x(2x^2 - 6x + 3)$$

Now, $f''(1) < 0 \Rightarrow$ Local maxima at $x = 1$

$f''(3) > 0 \Rightarrow$ Local minima at $x = 3$

and, $f''(0) = 0$

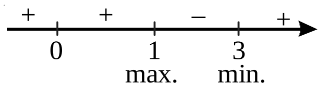
so, $f'''(x) = 30(2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Note: It was very convenient to check maxima/minima using the first derivative test by examining the sign change of $f'(x)$.

$$f'(x) = 5x^2 (x - 1)(x - 3)$$



Since, there is no sign change of $f'(x)$ at $x = 0$, it is not a point of extremum.

Example 6: Investigate the function

$$y = \cos x - 1 + \frac{x^2}{2!} - \frac{x^3}{3!}$$

for an extremum at the point $x = 0$.

Solution: $y' = -\sin x + x - \dfrac{x^2}{2}$; $y'(0) = 0$;

$y'' = -\cos x + 1 - x$; $y''(0) = 0$;

$y''' = \sin x - 1$; $y'''(0) = -1 \neq 0$.

And so, the first non-zero derivative at the point $x = 0$, is a derivative of the third order, i.e. of an odd order. This means that there is no extremum at the point $x = 0$.

7.8 Extrema of parametric functions

Consider a function defined parametrically :

$x = x(t)$, $y = y(t)$

Suppose that we are interested in finding the points of extrema of this function.

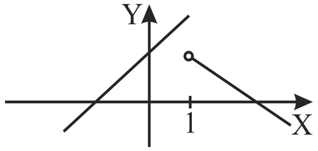
We can **eliminate** the parameter to get y as a function of x. For example, consider

$$x = t + 1,\ y = \begin{cases} t + 2, & t \le 0 \\ 1 - 2t, & t > 0 \end{cases}$$

On elimination of t, we get

$$y = \begin{cases} x + 1, & x \le 1 \\ 3 - 2x, & x > 1 \end{cases}$$

If we draw the graph of $y = f(x)$ we can apply the basic definition of local maximum to claim that the function has a local maximum at $x = 1$.



First Derivative Test for Parametric Functions

Assume that the function is continuous.

We find the critical points :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

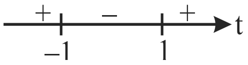
We find values of t, where dy/dx is zero or does not exist. Then we find the sign scheme of dy/dx on the number line of t. Now, we convert the sign scheme of dy/dx on the number line of x.

If $x = x(t)$ is a strictly increasing function of t, then the sign scheme in x is same as the sign scheme in t. If $x = x(t)$ is a strictly decreasing function of t, then the sign scheme in x is obtained by reversing the number line in t.

Consider $x = t^3 + 3t$, $y = t^3 - 3t + 2$.

$$\frac{dy}{dx} = \frac{t^2 - 1}{t^2 + 1} = 0 \text{ at } x = -1, 1.$$

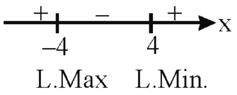
Sign scheme of dy/dx in t :



$x = t^3 + 3t$ is a strictly increasing function of t.

$x(-1) = -4$, $x(1) = 4$

Sign scheme of dy/dx in x :



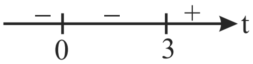
Hence, $x = -4$ is a point of local maximum and $x = 4$ is a point of local minimum

Now, consider $x = 1 - t^3$, $y = 6t - t^2$

$$\frac{dy}{dx} = \frac{2(3 - t)}{-3t^2}$$

The critical points in terms of t are 0 and 3.

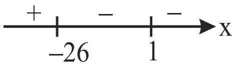
Sign scheme of dy/dx in t:



$x(0) = 1$, $x(3) = -2b$

$x = x(t)$ is a strictly decreasing function of t.

Sign scheme of dy/dx in x:



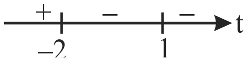
$x = 1$ is not a point of extremum. $x = -26$ is a point of local maximum.

Let $x = -2t + \cos\frac{\pi}{2}t,\ \ y = t^4 - 6t^2 + 8t$

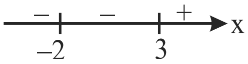
$$\frac{dy}{dx} = \frac{4(t-1)^2(t+2)}{-2-\frac{\pi}{2}\sin\frac{\pi t}{2}}$$

The critical points are $t = -2, 1$

Sign scheme of dy/dx in t :



Sign scheme of dy/dx in x :



$x = -2$ is not a point of extremum. $x = 3$ is a point of local minimum.

Second Derivative Test for Parametric Functions

Assume that the function is differentiable.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

First we get the stationary points : we find the values of $t = t_c$ where $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$

If $\frac{dx}{dt} = 0$, then this test is not applicable.

Now $$\frac{d^2y}{dx^2} = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{(\dot{x})^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=t_c} = \left.\frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{(\dot{x})^3}\right|_{t=t_c} = \left.\frac{\ddot{y}}{(\dot{x})^2}\right|_{t=t_c}$$

Now, $\left.\frac{d^2y}{dx^2}\right|_{t=t_c} > 0$, if $\left.\frac{d^2y}{dt^2}\right|_{t=t_c} > 0$, then $x = x(t_c)$ is a point of local minimum.

Further if $\left.\frac{d^2y}{dt^2}\right|_{t=t_c} < 0$, then $x = x(t_c)$ is a point of local maximum.

Consider $x = \tan^{-1}t + 1,\ y = \ell n(4 - t^2)$

$$\frac{dy}{dt} = 0 \ \Rightarrow\ t = 0.$$

We confirm that at $t = 0$, $\frac{dx}{dt} \neq 0$.

The sign of $\left.\frac{d^2y}{dx^2}\right|_{t=0}$ is same as sign of $\left.\frac{d^2y}{dt^2}\right|_{t=0}$

$$\therefore\ \frac{d^2y}{dt^2} = -\left.\frac{2(t^2+4)}{(4-t^2)^2}\right|_{t=0} < 0.$$

Hence, $x = x(0) = 1$ is a point of local maximum.

Example 1: Find the extrema of the function $y = f(x)$ represented parametrically as

$$\begin{cases} x = \phi(t) = t^5 - 5t^3 - 20t + 7 \\ y = \psi(t) = 4t^3 - 3t^2 - 18t + 3\ (-2 < t < 2). \end{cases}$$

Solution: We have

$\phi'(t) = 5t^4 - 15t^2 - 20.$

In the interval $(-2, 2)$, $\phi'(t) \neq 0$.

Now, we find $\psi'(t)$ and equate it to zero

$\psi'(t) = 12t^2 - 6t - 18 = 0$

$\Rightarrow\ t_1 = -1$ and $t_2 = 3/2$.

These roots are interior points of the given interval of the parameter t.

Further, $\psi''(t) = 24t - 6$;

$\psi''(-1) = -30 < 0,\quad \psi''(3/2) = 30 > 0.$

Consequently, the function $y = f(x)$ has a local maximum $y = 14$ at $t = -1$ (i.e. at $x = 31$) and a local minimum $y = -69/4$ at $t = 3/2$ (i.e. at $x = -1033/32$).

7.9 Operations on functions having points of extrema

1. If $y = f(x)$ has a local maximum at $x = a$ then $y = -f(x)$ has a local minimum at $x = a$ and vice-versa.

   For example, $y = \sin\left(x - \frac{\pi}{4}\right)$ has a local maximum at $x = \frac{3\pi}{4}$. Hence, $y = \sin\left(\frac{\pi}{4} - x\right)$ has a local minimum at $x = \frac{3\pi}{4}$.

   Similarly, $y = \log_2(x^2 - 4x)$ has a local minimum at $x = 2$. Hence, $y = \log_{1/2}(x^2 - 4x)$ has a local maximum at $x = 2$.

2. If $f(x)$ and $g(x)$ both have a local maximum (minimum) at $x = a$ then $y = f(x) + g(x)$ also has a local maximum (minimum) at $x = a$.

   If $f(a-h) < f(a) > f(a+h)$
   $g(a-h) < g(a) > g(a+h)$

   then $f(a-h) + g(a-h) < f(a) + g(a) > f(a+h) + g(a+h)$

   For example, $y = \cos x + e^{-|x|}$ has a local maximum at $x = 0$ because both $y = \cos x$ and $y = e^{-|x|}$ have a local maximum at $x = 0$.

   For similar reasons, $y = (x - 1)^{4/5} + |x^2 - 1|$ has a local minimum at $x = 1$, and $y = 2\sin^2 x - 3|\cos x|$ has a local maximum at $x = \frac{\pi}{2}$.

Note: If f and g both have a local minimum at $x = a$ only, it does not mean that $y = f(x) + g(x)$ has only one point of local minimum at $x = a$.

Consider, $f(x) = \begin{cases} x+1, & x<0 \\ x, & x \geq 0 \end{cases}$ and $g(x) = x^2$.

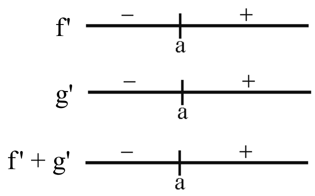
f and g both have a local minimum at x = 0.

$$(f+g)(x) = \begin{cases} x^2+x+1, & x<0 \\ x^2+x, & x \geq 0 \end{cases}$$

We can check that f + g has two points of local minima:

$$x = 0, -\frac{1}{2}.$$

However, if both f and g are differentiable, and they have a local minimum at x = a only, then y = f(x) + g(x) has only one point of local minimum at x = a.



We have a similar result in case of local maximum.

3. We have no comments for product and quotient of functions, in general.

Let f and g be twice differentiable functions each having a point of local maximum at x = a. If f(a) > 0 and g(a) > 0, then we can prove that y = f(x). g(x) has a local maximum at x = a.

We have f '(a) = 0, f "(a) < 0 and g '(a) = 0, g "(a) < 0.

y = f(x). g(x)

$$\frac{dy}{dx} = f'g + g'f = 0 \text{ at } x = a.$$

$$\frac{d^2y}{dx^2} = f''g + g''f$$

$$\frac{d^2y}{dx^2} = \underset{-}{f''}\,\underset{+}{g} + \underset{-}{g''}\,\underset{+}{f} < 0$$

Hence, y = f(x). g(x) has a local maximum at x = a.

We can apply this result to y = $(1 - x^2)$cosx at x = 0.

4. Composition

Let f and g be twice differentiable functions.

(i) If g has a local maximum (minimum) at x = f(a) then gof also has a local maximum (minimum) at x = a.

Let y = (gof)(x)

Assume that g has a local maximum at x = f(a).

y' = g'(f(x)) f '(x) = 0 at x = a, since g'(f(a)) = 0.

$y'' = g''(f(x))(f'(x))^2 + g'(f(x)).f''(x)$

At x = a, $y'' = g''(f(a))(f'(a))^2 < 0$ as g"(f(a)) < 0.

Consider $f(x) = x^3 + 1$ and $g(x) = 2x - x^2$.

According to the above result, gof has a local maximum at x = 0, because g has a local maximum at x = f(0) = 1.

If we use $f(x) = 1 - x^3$ then gof = $1 - x^6$. Again gof has a local maximum at x = 0.

If we use $f(x) = x^2 + 1$, then gof = $1 - x^4$. We have the same result. This implies that the monotonic behaviour of f does not affect the result.

(ii) If f has a local maximum (minimum) at x = a and g is increasing at x = f(a), then gof has a local maximum (minimum) at x = a.

Let y = (gof)(x)

y' = g'(f(x)) f '(x) = 0 at x = a, since f '(a) = 0.

$y'' = g''(f(x))(f'(x))^2 + g'(f(x))f''(x)$

At x = a, y"= g'(f(a))f"(a) < 0

as g'(f(a)) > 0 and f"(a) < 0.

The above result can be verified for y = ℓn(cosx) at x = 0.

(iii) If f has a local maximum (minimum) at x = a and g is decreasing at x = f(a) then gof has a local minimum (maximum) at x = a.

Thus, y = $\cot^{-1}(2x - x^2)$ has a local maximum at x = 1.

Example 1: Find the local maximum and minimum values of the function $f(x) = \dfrac{50}{3x^4+8x^3-18x^2+60}$;

Solution: Here it is simpler to find the extrema of the function $g(x) = 3x^4 + 8x^3 - 18x^2 + 60$.

$g'(x) = 12x^3 + 24x^2 - 36x = 12x(x^2 + 2x - 3)$,

$g''(x) = 12(3x^2 + 4x - 3)$,

The critical points are : $x_1 = -3$, $x_2 = 0$, $x_3 = 1$

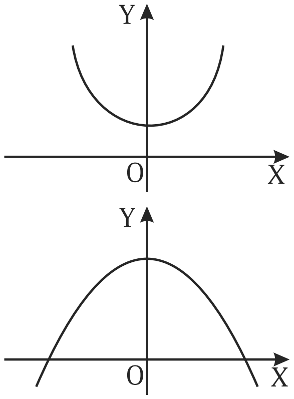
The character of the extrema is readily determined from the sign of the second derivative g" (–3) > 0; hence at the point $x_1 = -3$ the function g(x) has a minimum, and the given function f(x) obviously has a maximum f(–3) = –2/3,

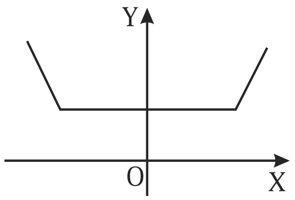
g" (0) < 0; hence, at the point $x_2 = 0$ the function g(x) has a maximum, and f(x) a minimum f(0) = 5/6;

g"(1) > 0; hence at the point $x_3 = 1$ the function g(x) has a minimum, and f(x) a maximum f(1) = 50/53.

5. Even/Odd Functions

(i) An even function has an extremum at x = 0.





Note that odd function cannot have an extremum at $x = 0$.

(ii) If an even function f has a local maximum (minimum) at $x = a$ then it also has a local maximum (minimum) at $x = -a$.
For example, $y = x^4 - 2x^2$ has local minimum at $x = \pm 1$ and a local maximum at $a = 0$.

(iii) If an odd function f has a local maximum (minimum) at $x = a$ then it has a local minimum (maximum) at $x = -a$.
For example, $y = \sin x$ has a local maximum at $x = \pi/2$ and a local minimum at $x = -\pi/2$.

Concept Problems D

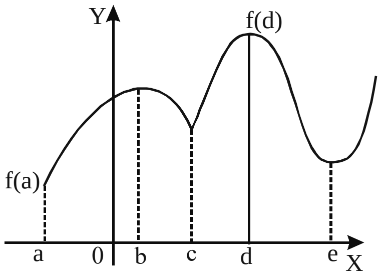
1. Suppose $f'(c) = f''(c) = 0$ but $f'''(c) \neq 0$. Prove f(c) is neither a maximum nor a minimum.
2. Suppose $f'(c) = f''(c) = f'''(c) = 0$, but $f^{(4)}(c) > 0$. Prove f(c) is a minimum.
3. Investigate the following functions for an extremum at the point $x = 0$:
   (i) $f(x) = \sin x - x + x^3/3$
   (ii) $f(x) = \sin x - x + \frac{x^3}{3!} - \frac{x^4}{4!}$
   (iii) $f(x) = \begin{cases} e^{1/x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$
4. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c ? Does f have a point of inflection at c?
5. Find the points of extremum of f(x) represented as $x = 2t + \sin t$, $y = t^3 - 3t$.
6. Show that $y = \tan^{-1}(\cos 3\sin x)$ has a local maximum at $x = 0$ and local minima at $x = \pm\frac{\pi}{2}$.
7. Find the points of extrema :
   (i) $y = \ell n(x^4 - 2x^2 + 3)$
   (ii) $y = \ell n(x^3 - 3x + 2)$ $y = x^3 - 3x + 2$
   (iii) $y = \sqrt{3x^2 - 2x^3}$
   (iv) $y = \frac{1}{\cos x + 2}, x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$
   (v) $y = \frac{1}{x^2 - 3x^{2/3}}$
8. Show that the odd function $y = \sin 3x - 3\sin x$ has a local mininimum at $x = \frac{\pi}{2}$ and a local maximum at $x = -\frac{\pi}{2}$.
9. If $f(x) = x^5 - 5x^3 + 10x$ has a local minimum at $x = 2$ then find the kind of extremum at the stationary point $x = -2$.

7.10 Global maximum and minimum

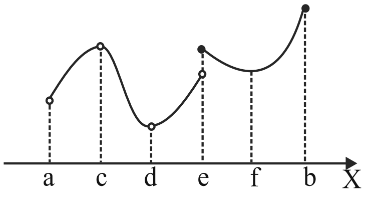
Definition. A function f has an **absolute maximum** (or **global maximum**) at $x = a$ if $f(a) \geq f(x)$ for all x in D, where D is the domain of f. The number f(a) is called the **maximum value** of f on D.

Similarly, f has an **absolute minimum** at $x = a$ if $f(a) \leq f(x)$ for all x in D and the number f(a) is called the **minimum value** of f on D. The maximum and minimum values of f are called the extreme values of f.

The figure shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that (d, f(d)) is the highest point on the graph and (a, f(a)) is the lowest point. The maximum of a function occurs at the highest point on its graph and the minimum occurs at the lowest point.



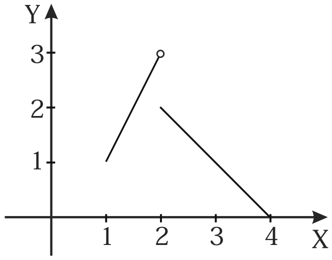
Now consider the graph of another function :



f(x) has local maxima at x = c, e, b and local minima at x = a, d, f. It can also be easily seen that f(b) is the greatest value and hence it is the absolute maximum and similarly f(d) is the absolute minimum.

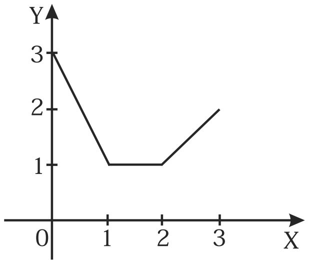
Also we have to be careful about the fact that a function has absolute maximum or minimum value when it actually achieves these values.

(i) Let us take graph of $f(x) = \begin{cases} 2x-1 & 1 \le x < 2 \\ 4-x & 2 \le x \le 4 \end{cases}$.



This function has a absolute minimum at x = 4 and f(4) = 0 is the global minimum value but the absolute maximum value is not defined, even if the value of the function can be made as close to 3 as we may please. The function is not able to attain the value of 3 at any x.

(ii) Also consider the graph of $f(x) = \begin{cases} 3-2x & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$

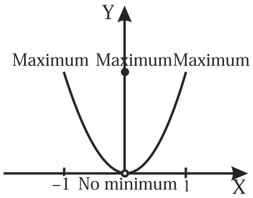


f(x) has absolute maximum at x = 0 and f(0) = 3. It has the absolute minimum value of 1 which is attained at all points in the interval [1, 3].

(iii) The discontinuous function defined by

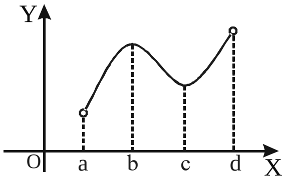
$$h(x) = \begin{cases} x^2 & \text{for } x \ne 0 \\ 1 & \text{for } x = 0 \end{cases}$$

has an absolute maximum on the closed interval [–1, 1] but no absolute minimum, as shown in the figure. Incidentally, this graph also illustrates the fact that a function may assume an absolute extre-mum at more than one point. In this case, the maximum is at (–1,1), (0, 1) and (1,1).



The discontinuous function h has a maximum, but not a minimum on the closed interval [–1,1]

Now, consider the graph of a function which does not have either absolute maximum or absolute minimum.



Function has no global maxima and minima.
Range : (f(a), f(d))

Extreme Value Theorem

If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value of f(d) at some numbers c and d in [a, b].

The Extreme Value Theorem is illustrated in Figure 1. Note that an extreme value can be attained at more than one point. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

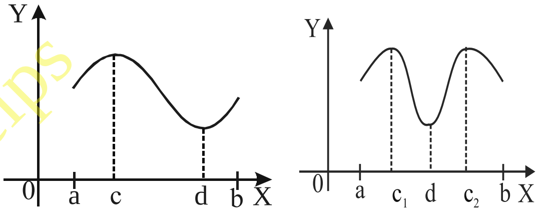


Figure 1

This theorem does not apply if the function is not continuous; neither does the theorem apply if the interval is not closed or not bounded.

Figures 2 and 3 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

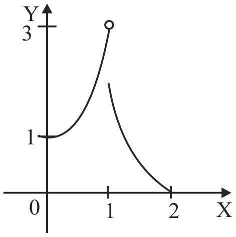


Figure 2

This function has minimum value f(2) = 0, but no maximum value.

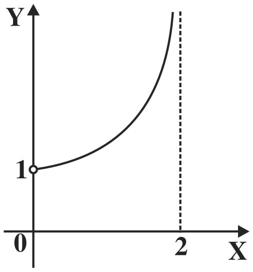


Figure 3

This continuous function has no maximum or minimum

The function f whose graph is shown in Figure 2 is defined on the closed interval [0, 2] but has no maximum value. (Notice tht the range of f is [0, 3). The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem.

The function f shown in Figure 3 is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value. The range of f is $(1, \infty)$. The function takes on arbitrarily large values. This does not contradict the Extreme Value Theorem because the interval (0, 2) is not closed.

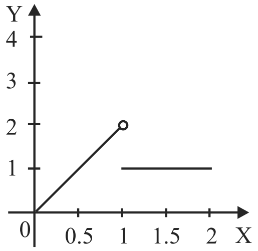
Example 1: In each case, explain why the given function does not contradict the Extreme Value Theorem.

(a) $f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \end{cases}$

(b) $g(x) = x^2$ on $0 < x \le 2$.

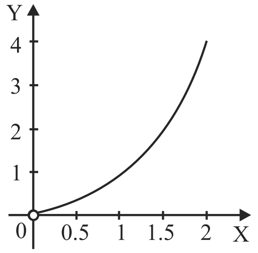
Solution:

(a) The function f has no maximum. It takes on all values arbitrarily close to 2, but it never reaches the value 2. The Extreme Value Theorem is not violated because f is not continuous on [0, 2].



f does not have a maximum value

(b) Although the functional values of g(x) become arbitrarily small as x approaches 0, it never reaches the value 0, so g has no minimum. The function g is continuous on the interval (0, 2], but the Extreme Value Theorem is not violated because the interval is not closed.



g does not have a minimum value
(but it does have a maximum value.)

The extreme value theorem guarantees that both global extrema are attained by continuous functions on a closed interval.

Note that the Extreme Value Theorem (like the Intermediate Value Theorem) is an existence theorem because it tells of the existence of minimum and maximum values, but does not show how to find these values.

We will now address methods for determining the locations of absolute extrema under the conditions of the Extreme Value Theorem.

If f is continuous on the finite closed interval [a, b], then the absolute extrema of f can occur either at the endpoints of the interval or inside the open interval (a, b). If the absolute extrema happen to fall inside, then the following theorm tells us that they must occur at critical points of f.

Theorem If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.

Proof If f has an absolute maximum on (a, b) at x_0, then $f(x_0)$ is also a local maximum for f; for if $f(x_0)$ is the largest value of f on all of (a, b), then $f(x_0)$ is certainly the largest value for f in the immediate vicinity of x_0. Thus, x_0 is a critical point of f. The proof for absolute minima is similar.

Thus, an absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighbourhood. Hence, a list of all local maxima will automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

Example 2: Find the greatest and least values of $f(x) = x^3 - 12x$, $x \in [-1, 3]$

Solution: The possible points of absolute maxima/minima are the critical points and the endpoints.

$f'(x) = 3x^2 - 12$

$x = 2$ is the only critical point in $[-1, 3]$.

The points of local maxima/minima are $x = -1, 2, 3$. Examining the values of f(x) at these points, we can find the greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

The minimum value of f(x) is –16 and the maximum value is 11.

Procedure for Finding Global Maximum/ Minimum of a Continuous Function

Global maximum/minimum in [a, b]

In order to find the global maximum and minimum of a continuous function f(x) in [a, b] :

Find out all the critical points of f(x) in (a, b).

Let $c_1, c_2, \dots, c_n$ be the different critical points.

Find the value of the function at these critical points. Let $f(c_1), f(c_2), \dots, f(c_n)$ be the values of the function at the critical points.

Let $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$

Then M_1 is the global maximum (greatest value) of f(x) in [a, b] and M_2 is the global minimum (least value) of f(x) in [a, b].

Example 3: Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1, -\frac{1}{2} \le x \le 4$.

Solution: $f(x) = x^3 - 3x^2 + 1,\ x \in \left[-\frac{1}{2}, 4\right]$. Since f is continuous on a closed interval, it must have both absolute maximum and minimum values.

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since f'(x) exists for all x, the only critical points of f occur when f'(x) = 0, that is x = 0 or x = 2. Notice that each of these critical points lies in the interval $\left[-\frac{1}{2}, 4\right]$.

the values of f at these critical numbers are

$f(0) = 1, \qquad f(2) = -3$

The values of f at the endpoints of the interval are

$f\left(-\frac{1}{2}\right) = \frac{1}{8}, \qquad f(4) = 17$

Comparing these four numbers, we see that the absolute maximum value is f(4) = 17 and the absolute minimum value is f(2) = –3.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical point.

Example 4: Find the maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 12x + 15$ on the closed interval [0, 3].

Solution: The derivative of f is

$$f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1)$$

So the critical points of f are the solutions of the equation

$$6(x - 2)(x + 1) = 0$$

and the numbers c for which f'(c) does not exist.

There are none of the latter, so the critical of f occur at x = –1 and x = 2. The first of these is not in the domain of f ; we discard it, and thus the only critical point of f in [0, 3] is x = 2.

Including the two endpoints, our list of all values of x that yield a possible maximum or minimum value of f consists of 0, 2, and 3. We evaluate the function f at each:

$f(0) = 15,$ ← absolute maximum

$f(2) = -5,$ ← absolute minimum

$f(3) = 6.$

Therefore the maximum value of f on [0, 3] is f(0) = 15 and its minimum value is f(2) = –5.

Example 5: Find the greatest and the least value of the function $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$ on the interval [1, 4].

Solution: The function f(x) is defined and has a derivative for all x ∈ R.

We find the derivative of the function :

$$f'(x) = x^2 - 5x + 6 = (x - 2)(x - 3).$$

The function f(x) has two critical points x = 2 and x = 3. Both points belong to the interval [1, 4]. Consequently, the least value of the function on the interval [1, 4]

$$= \min\{f(1), f(2), f(3), f(4)\}$$

$$= \min\left\{\frac{23}{6}, \frac{28}{6}, \frac{27}{6}, \frac{32}{6}\right\} = \frac{23}{6}.$$

The greatest value of the function on the interval $[1, 4] = \max\{f(1), f(2), f(3), f(4)\} = \frac{32}{6}$.

Example 6: Find the global maximum of

$f(x) = 1 + 12|x| - 3x^2$ on [–1, 4].

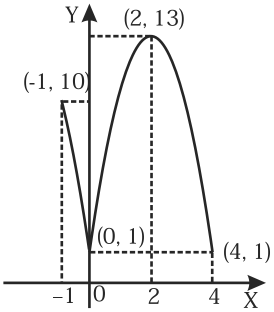
Solution: If is derivable on R – {0}, since |x| fails to be derivable precisely at 0. Thus 0 is a critical point. Moreover, –1 and 4 are the end points.

Since $f'(x) = (1 + 12x - 3x^2)' = 12 - 6x$ if x > 0 and $f'(x) = (1 - 12x - 3x^2)' = -12 - 6x$ if x < 0, we see that $f'(x) = 0 \Rightarrow x = \pm 2$.

On excluding –2 since it falls outside the domain the set of critical points and end points is {0, –1, 4, 2}.

Since f is continuous on the closed interval [–1, 4] we are assured by the Extreme Value Theorem that both global extrema are attained.

Comparing f(0) = 1, f(–1) = 10, f(4) = 1, and f(2) = 13, we have max f = 13 and min f = 1. Notice that the maximum is attained at a stationary point while the minimum is attained at both an end point and a point of non-derivability (see figure).



Example 7: Find the absolute extrema of $f(x) = x^{2/3}(5 - 2x)$ on the interval [–1, 2].

Solution: To find the derivative, we write the given function as $f(x) = 5x^{2/3} - 2x^{5/3}$. Then

$$f'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}x^{-1/3}(1 - x)$$

f′(x) = 0 when x = 1

Even though f(0) is defined, we note that f′(x) does not exist at x = 0 (notice the division by zero when x = 0). Thus, the critical numbers are x = 0 and x = 1.

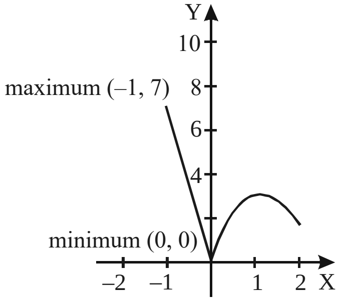
Values at endpoints : $f(-1) = 7$

$f(2) = 2^{2/3} \approx 1.58$

Values at critical points : $f(0) = 0$

$f(1) = 3$

The absolute maximum of f occurs at x = –1 and is f(–1) = 7. The absolute minimum of f occurs at x = 0 and f(0) = 0. The graph of f is shown below.



Example 8: Find the greatest value of the function $f(x) = 2.3^{3x} - 3^{2x} . 4 + 2.3^x$ in the interval [– 1, 1].

Solution: $f'(x) = 2.3^x . \ln 3 [3.3^{2x} - 4.3x + 2]$

$$= 3.2 . 3^x \ln 3\left[\left(e^x - \frac{2}{3}\right)^2 + \frac{2}{3}\right] > 0 \text{ in } [-1, 1]$$

f is strictly increasing in [–1, 1].

Hence, f(x) is greatest when x = 1 and f(1) = 24.

Example 9: Find the maximum value of the function $f(x) = \frac{1}{2}x - \sin x$ for $0 \le x \le 4\pi$.

Solution: The maximum occurs either at 0 or 4π, or at a root of the derivative,

$$f'(x) = \frac{1}{2} - \cos x.$$

Thus, f′(x) = 0 if $\cos x = \frac{1}{2}$.

In the range $0 \le x \le 4\pi$, the roots of the derivative are

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

Hence, there are six possible places where the maximum can occur. To eliminate some of them, consider the second derivative, f″(x) = sin x :

$$f''\left(\frac{\pi}{3}\right) = f''\left(\frac{7\pi}{3}\right) = \sin\frac{\pi}{3} > 0,$$

$$f''\left(\frac{5\pi}{3}\right) = f''\left(\frac{11\pi}{3}\right) = \sin\frac{5\pi}{3} < 0.$$

The value 5π/3 and 11π/3, for which f″(x) is negative, are candidates for the maximum. The others, π/3 and 7π/3, are candidates for the minimum; eliminate them. Hence the maximum of f(x) occurs at one of the points $0, \frac{5\pi}{3}, \frac{11\pi}{3}, 4\pi$.

We evaluate $f(x) = \frac{1}{2}x - \sin x$ at each of these points :

$$f(0) = 0,$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} - \sin\frac{5\pi}{3} = \frac{5\pi}{6} + \frac{\sqrt{3}}{2},$$

$$f\left(\frac{11\pi}{3}\right) = \frac{11\pi}{6} - \sin\frac{11\pi}{3} = \frac{11\pi}{6} + \frac{\sqrt{3}}{2},$$

$$f(4\pi) = 2\pi = \frac{11\pi}{6} + \frac{\pi}{6}.$$

The largest of these is $f(11\pi/3) = \frac{11\pi}{6} + \frac{\sqrt{3}}{2}$.

Darboux Theorem

If f(x) is differentiable for $a \le x \le b$, $f'(a) = \alpha$, $f'(b) = \beta$, and γ lies between α and β, then there is a ξ between a and b for which $f'(\xi) = \gamma$.

Proof: Suppose, for example, that $\alpha < \gamma < \beta$, and let

$\psi(x) = \phi(x) - \gamma(x - a)$

Then ψ(x) is continuous, and therefore attains its lower bound in (a, b) at some point ξ of (a, b). this point ξ cannot be a or b, because

$$\psi'(a) = \alpha - \gamma < 0, \ \psi'(b) = \beta - \gamma > 0$$

Hence ψ(x) has a minimum at a ξ between a and b, and $\psi'(\xi) = 0$, i.e. $\phi'(\xi) = \gamma$.

Global Maximum/Minimum in (a, b)

We know that a continuous function may or may not have absolute extrema on an open interval. However, certain conclusions about the existence of absolute extrema of a continuous function f on a finite open interval (a, b) can be drawn from the behaviour of f(x) as $x \to a^+$ and as $x \to b^-$.

The method for obtaining the greatest and least values of f(x) in (a, b) is almost same as the above method, however with a caution.

Let y = f(x) be a continuous function and $c_1, c_2 \ldots c_3$ be the critical points of the function in (a, b).

Let $M_1 = \max. \{f(c_1), f(c_2), f(c_3) \ldots. f(c_n)\}$

and $M_2 = \min \{f(c_1), f(c_2), f(c_3) \ldots f(c_n)\}$

Now if $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ are less than or equal to M_1 then f(x) would have a global maximum M_1 in (a, b).

If $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ are greater than or equal to M_2, then f(x) would have a global minimum M_2 in (a, b).

This means that if the limiting values at the endpoints are greater than M_1 or less than M_2, then f(x) would not have global maximum/minimum in (a, b).

Example 10: Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Find the global maximum and minimum of f(x) in [0, 2] and (1, 3).

Solution: $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow$ $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

$= 6(x - 1)(x - 2)$

Clearly the critical point of f(x) in [0, 2] is x = 1.

Now, f(0) = 6, f(1) = 11, f(2) = 10.

Thus, the global maximum of f(x) in [0, 2]

= f(1) = 11

and the global minimum of f(x) in [0, 2]

= f(0) = 6.

Now consider the interval $x \in (1, 3)$

x = 2 is the only critical point in (1, 3).

$f(2) = 10,\ \lim\limits_{x\to1^+} f(x) = 11$ and $\lim\limits_{x\to3^-} f(x) = 15$

Thus, the global minimum of f(x) in (1, 3)

= f(2) = 10.

Since the greatest value 15 is a limiting value, the global maximum does not exist in (1, 3).

Example 11: Find the extrema of the function $f(x) = x^4 - 2x^2 + 5$ in the intervals (i) $-2 \le x \le 1$, and (ii) $-2 < x < 1$

Solution:

(i) Since f(x) is a polynomial function, f′(x) exists for all x in the domain. We must find all x in $-2 < x < 1$ at which $f'(x) = 0$.

$f'(x) = 4x^3 - 4x = 4(x^3 - x) = 4x(x + 1)(x - 1)$, and so $f'(x) = 0$ in $-2 < x < 1$ only at $x = -1$ and $x = 0$.

That is, –1 and 0 are the critical points of f.

We evaluate f(–1) = 4 and f(0) = 5.

Now we evaluate f at the endpoints.

f(1) = 4 and f(–2) = 13.

The greatest and least of the values computed above are 13 and 4 respectively.

Thus, f(–2) = 13 is the maximum value of f(x) for $-2 \le x \le 1$ and f(–1) = f(1) = 4 is the minimum value of f for $-2 \le x \le 1$. See figure 1.

Notice that there is a local maximum at x = 0 that fails to be a maximum relative to the entire interval $-2 \le x \le 1$.

(ii) The graph of f is similar, except that f is not defined when x = –2 and x = 1. (See Figure 2 and compare it to Figure 1.) Using the above information, we see that f does not have a maximum at x = –2 since it is not defined there. However, f has a minimum at x = –1.

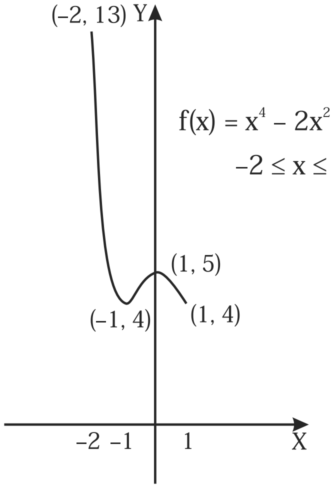


Figure 1

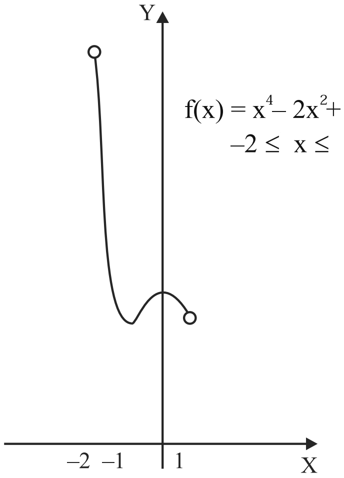


Figure 2

Example 12: Determine whether the function $f(x) = \dfrac{1}{x^2 - x}$ has any absolute extrema on the interval (0, 1). If so, find them and state where they occur.

Solution: Since f is continuous on the interval (0, 1) and

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x^2 - x} = \lim_{x\to0^+} \frac{1}{x(x-1)} = -\infty$$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{1}{x^2 - x} = \lim_{x\to0^-} \frac{1}{x(x-1)} = -\infty$$

the function f has an absolute maximum but no absolute minimum on the interval (0, 1). The absolute maximum must occur at a critical point of f in the interval (0, 1). We have

$f'(x) = -\dfrac{2x-1}{(x^2 - x)^2}$, so the only solution of the equation $f'(x) = 0$ is $x = 1/2$. Although f is not differentiable at x = 0 or at x = 1, these values are doubly disqualified since they are neither in the domain of f nor in the interval (0, 1). Thus, the absolute maximum occurs at x = 1/2, and this absolute maximum is

$$f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -4.$$

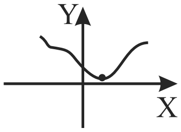
Global Maximum/Minimum on Infinite Intervals

We observe that a continuous function may or may not have absolute extrema on an infinite interval. However, certain conclusions about the existence of absolute extrema of a continuous function f on (–∞, ∞) can be drawn from the behaviour of f(x) as x → – ∞ and as x → ∞ (See Table).

Table

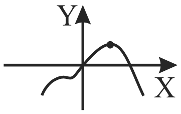
Conclusions on functions continuous everywhere

(i) $\lim\limits_{x\to-\infty} f(x) = \infty$ $\lim\limits_{x\to\infty} f(x) = \infty$



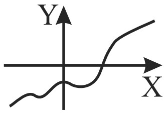
f has an absolute minimum but no absolute maximum on (–∞, ∞).

(ii) $\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to\infty} f(x) = -\infty$



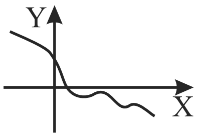
f has an absolute maximum but no absolute minimum on (–∞, ∞).

(iii) $\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to\infty} f(x) = \infty$



f has neither an absolute maximum nor an absolute minimum on (–∞, ∞)

(iv) $\lim\limits_{x\to-\infty} f(x) = \infty$ $\lim\limits_{x\to\infty} f(x) = -\infty$



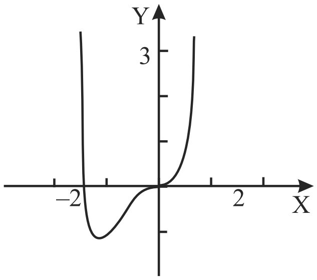
f has neither an absolute maximum nor an absolute minimum on (–∞, ∞).

Example 13: Determine by inspection whether $f(x) = 3x^4 + 4x^3$ has any absolute extrema. If so, find them and state where they occur.

Solution: Since f(x) → ∞ as x → ∞. Thus, there is an absolute minimum but no absolute maximum. The absolute minimum must occur at a critical point of f. Since f is differentiable everywhere, we can find all critical points by solving the equation

$$f'(x) = 0 \Rightarrow 12x^3 + 12x^2 = 12x^2(x + 1) = 0$$

from which we conclude that the critical points are x = 0 and x = –1. Evaluating f at these critical points yields f(0) = 0 and f(–1) = –1



$f(x) = 3x^4 + 4x^3$

Therefore, f has an absolute minimum of –1 at x = –1.

Example 14: Find all absolute extrema of the function $f(x) = x^3 - 3x^2 + 4$ on the interval

(a) (–∞, ∞) (b) (0, ∞).

Solution:

(a) Because f is a polynomial of odd degree, it follows from the discussion that there are no absolute extrema on the interval (–∞, ∞).

(b) Since $\lim\limits_{x\to\infty} (x^3 - 3x^2 + 4) = \infty$

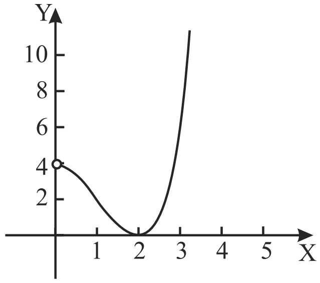
we know that f cannot have an absolute maximum on the interval (0, ∞). However, the limit

$$\lim_{x\to 0^+} (x^3 - 3x^2 + 4) = 4$$

is not infinite, so there is a possibility that f may have an absolute minimum on this interval. In this case it would have to occur at a stationary point. which suggests that we look for solutions of the equation f′(x) = 0. But,

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

so f has critical points x = 0 and x = 2. However, the only critical point inside the interval (0, ∞) is x = 2. Since $f''(x) = 6x - 6$ we have $f''(2) = 6 > 0$, so a relative minimum occurs at x = 2 by the second derivative test. Thus, f(x) has an absolute minimum at x = 2, and this absolute minimum is f(2) = 0.



$f(x) = x^3 - 3x^2 + 4,\ x > 0$

Example 15: Find the global maximum of

$$f(x) = \frac{1}{x^2+1} - 1 \text{ on R.}$$

Solution: f is derivable on R.

$$f'(x) = -\frac{2x}{(x^2+1)^2} = 0 \Rightarrow x = 0.$$

The only critical point is 0. Thus, if there is a maximum it must be f(0) = 0. To verify that it is indeed the maximum we show directly that $f(x) \le f(0)$ for all $x \in R$:

$$f(x) \le f(0) \Leftrightarrow \frac{1}{x^2+1} - 1 \le 0 \Leftrightarrow \frac{1-(x^2+1)}{x^2+1} \le 0$$

$$\Leftrightarrow -x^2 \le 0 \Leftrightarrow x \in R.$$

Observe that the only application of the Fermat theorem was to locate a prospective maximum. But this could have been accomplished directly by inspection.

Example 16: Find the global minimum of the function $f(x) = \cos x + |x|$.

Solution: f is derivable on R – {0} since both cos x and |x| are derivable on R – {0} and is not derivable at 0.

Thus x = 0 is a critical point. To obtain other critical points, we note that

for $x > 0$, $f'(x) = (\cos x + x)' = \sin x + 1$, and
for $x < 0$, $f'(x) = (\cos x - x)' = -\sin x - 1$,
so that $f'(x) = 0 \Rightarrow -\sin x + 1 = 0$ or $-\sin x - 1 = 0$
$\Rightarrow x = \pm \pi/2 + 2\pi n$, n any integer. Thus the critical set consists of 0 and $\pm\pi/2 + 2\pi n$ (n is any integer). Since f(0) = 1 and $f(\pm \pi/2 + 2\pi n) = |\pm \pi/2 + 2\pi n| > 1$ we see that f(0) = 1 is the global minimum.

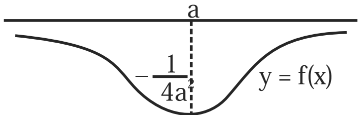
Example 17: Find the greatest value of

$$f(x) = \frac{1}{(2ax - x^2 - 5a^2)}$$ in [–3, 5] depending upon the parameter a.

Solution: Given $f(x) = \dfrac{1}{(2ax - x^2 - 5a^2)}$

$$\frac{1}{f(x)} = -x^2 + 2ax - 5a^2$$ has a maximum value at x = a.

Hence f(x) has the minimum value at x = a.



The graph is symmetrical about x = a and the interval [–3, 5] is symmetrical about x = 1.

If $-3 \le a < 1$, the greatest value of f(x) occurs at x = 5.

$$f(5) = \frac{1}{10a - 25 - 5a^2}$$

If $1 \le a < 5$, the greatest value of f(x) occurs at x = –3.

$$f(-3) = \frac{1}{-(9 + 6a + 5a^2)} .$$

Global Maximum/Minimum of Parametric Curves

Suppose that a curve C is given parametrically by the equations

$$x = f(t), \qquad y = g(t) \qquad (a \le t \le b)$$

where f and g are continuous on the finite closed interval [a, b]. It follows from the Extreme Value Theorem that f(t) and g(t) have absolute maxima and absolute minima for $a \le t \le b$. This means that a particle moving along the curve cannot move away from the origin indefinitely–there must be a smallest and largest x-coordinate and a smallest and largest y-coordinate. Geometically, the entire curve is contained within a box determined by these smallest and largest coordinates.

Example 18: Suppose that the equations of motion for a paper airplane during its first 10 seconds of flight are

$$x = t - 3 \sin t, \qquad y = 4 - 3 \cos t \quad (0 \le t \le 10)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points ?

Solution: The trajectory is as shown in the figure. We want to find the absolute maximum and minimum values of y over the time interval [0, 10] and the values of t for which these absolute extrema occur. The absolute extrema must occur either at the endpoints of the closed interval [0, 10] or at critical points in the open interval (0, 10).

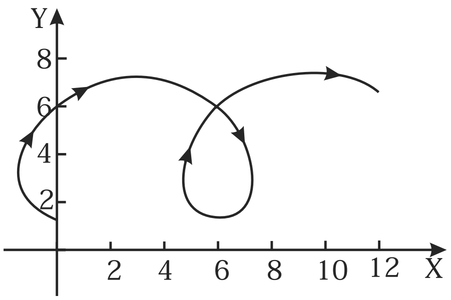
To find the critical points, we must solve the equation dy/dt = 0, which is

$$3 \sin t = 0$$

Thus, there are critical points in the interval (0, 10) at $t = \pi, 2\pi$, and 3π. Evaluating y = 4 – 3 cost at the endpoints and the critical points yields

$$y = 4 - 3 \cos 0 = 4 - 3 = 1$$
$$y = 4 - 3 \cos \pi = 4 - (-3) = 7$$
$$y = 4 - 3 \cos 2\pi = 4 - 3 = 1$$
$$y = 4 - 3 \cos 3\pi = 4 - (-3) = 7$$
$$y = 4 - 3 \cos \approx 6.517$$

Thus, a high point of y = 7 is reached at times $t = \pi$ and $t = 3\pi$, and a low point of y = 1 is reached at times t = 0 and $t = 2\pi$.



Note: If f is a periodic function, then the locations of all absolute extrema on the interval $(-\infty, \infty)$ can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest.

Global Maximum/Minimum of Discontinuous Functions

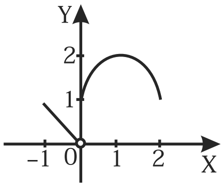
A function may attain its greatest and least values at its (i) critical points (ii) end points.

After having found all the above points, find the value of the function at those points. At points where f(x) is discontinuous, we also need to find the one-sided limits. The largest and the smallest values among these, are respectively the greatest and least values of the function, provided it is a function's value and not purely a limiting value.

Example 19: Find the greatest and least values of

$$f(x) = -x, \quad -1 \le x < 0$$
$$= 2 - (x-1)^2, \quad 0 \le x \le 2.$$

Solution: From the graph of the function shown below, we can see that the greatest value of the function is 2. However, we can see that as $x \to 0$, f(x) approaches 0 but is not equal to 0 and hence the least value does not exist.



Alternative:

We have

$$f(x) = -x, -1 \le x < 0$$
$$= 2 - (x-1)^2, 0 \le x \le 2$$

and $f'(x) = -1, -1 \le x < 0$

$$= -2(x-1), 0 < x < 2$$

The critical points x = 0, 1

and the end points x = – 1, 2

Now we have $f(-1) = 1$, $f(1) = 2$ and $f(2) = 1$.

Since f is discontinuous at x = 0, we also need to find the limiting values of f(x) as $x \to 0$.

we have $f(0^-) = 0$, $f(0^+) = 1$ and $f(0) = 1$.

The largest and the smallest among the above six values are 2 and 0 respectively.

Hence, the greatest value is 2 but the least value does not exist since the function approaches 0 but is never equal to 0.

Example 20: Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5 & , \ x \le 1 \\ -2x + \log_2(b^2 - 2) , & x > 1 \end{cases}$

Find all possible real values of b such that f(x) has greatest value at x = 1.

$$f'(x) = 3x^2 - 2x + 10 = 3\left\{x^2 - \frac{2x}{3} + \frac{10}{3}\right\}$$

$$= 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{29}{9}\right\} > 0$$

$\therefore$ f(x) is increasing for $x \le 1$.

Now for $x > 1$, $f'(x) = -2 < 0$

So, f(x) is decreasing for $x > 1$.

Now f(x) will have greatest value at x = 1 if

$$\lim_{x \to 1^+} f(x) \le f(1)$$

$$\Rightarrow \lim_{h \to 0} f(1+h) \le 5$$

$$\Rightarrow \lim_{h \to 0} \{-2(1+h) + \log_2(b^2 - 2)\} \le 5$$

$$\Rightarrow -2 + \log_2(b^2 - 2) \le 5 \Rightarrow \log_2(b^2 - 2) \le 7$$

$$\Rightarrow 0 < b^2 - 2 \le 2^7$$

$$\Rightarrow 2 < b^2 \le 130$$

$$\therefore b \in [-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130}].$$

Example 21: Let

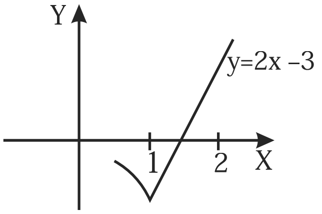
$$f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & , 0 \le x < 1 \\ 2x - 3 & , 1 \le x \le 3 \end{cases}$$

Find all possible real values of b such that f(x) has the smallest value at x = 1.

Solution: Obviously f is decreasing in $0 \le x < 1$ and increasing in [1, 3]. Also $f(1) = -1$

Note that $f(1) \le f(1-h)$ in order that f(x) has the smallest value at x = 1.

$$\lim_{h \to 0} f(1-h) \ge f(1)$$



$$\lim_{h \to 0} -(1-h)^3 + \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge -1$$

or, $\dfrac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0 \Rightarrow b \in (-2, -1) \cup [1, \infty).$

The First Derivative Test for Global Extrema

If f has only one critical point c in (a, b), then the first derivative test can tell us that f(c) is the desired extreme value and can even determine whether it is a maximum or a minimum value of f(x). Let f : [a, b] → R be continuous on [a, b] and derivable on (a, c) ∪ (c, b), then

(i) if $f' > 0$ on (a, c) and $f' < 0$ on (c, b)
$\Rightarrow$ f(c) = max f on [a, b]; and

(ii) if $f' < 0$ on (a, c) and $f' > 0$ on (c, b)
$\Rightarrow$ f(c) = min f on [a, b]

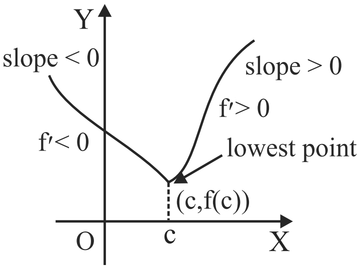
Theorem If a function is continuous on an interval and derivable on the interior except possibly at an interior point c, then the positivity [resp. negativity] of the derivative to the left of c and the negative [resp. positivity] of the derivative to the right of c implies that the functional value at c is the global maximum [resp. minimum].

Proof We apply the concept of monotonicity to [a, c] and [c, b]. For (i), if $f' > 0$ on (a, c) and $f' < 0$ on (c, b), then f is strictly increasing on [a, c] and strictly decreasing on [c, b].

Therefore, for $x \in [a, c]$ we have $f(x) < f(c)$, and for $x \in [c, b]$ we have $f(c) > f(x)$.

Thus f(c) = max f on [a, b]. Similarly for (ii).

It is easy to see that the theorem extends to the case in which the domain of f is an unbounded interval (say) I on which $f' > 0$ on $I \cap (c, \infty)$ and $f' < 0$ on $I \cap (-\infty, c)$. Then min f = f(c). Similarly, for the case of the global maximum.



Let us find the global extrema of $f(x) = (x - a)^{2/3}$ on R.

Since $f'(x) = \frac{2}{3}(x-a)^{-1/3}$ on R – {a}, clearly $f' > 0$ on (a, ∞) and $f' < 0$ on $(-\infty, a)$. Therefore by the above theorem the minimum value of f = f(c) = 0.

Example 22: Find the absolute maximum value of $f(x) = \frac{2 \ln x}{x}$.

Solution: f(x) is defined on the open interval $(0, \infty)$.

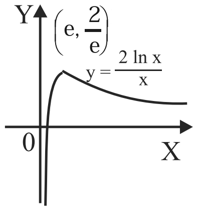
Because $f'(x) = \frac{2}{x} \cdot \frac{1}{x} - \frac{2}{x^2} \cdot \ln x = \frac{2}{x^2}(1 - \ln x)$,

there is a single critical point at x = e. Note that

If x < e, then ln x < 1, so $f'(x) > 0$ if x < e ;

If x > e, then ln x > 1, so $f'(x) < 0$ if x > e.

Therefore the first derivative test implies that f(e) = 2/e is a local maximum value of f. The figure shows the graph of the function.



Indeed, because f is increasing in 0 < x < e and decreasing for x > e, it follows that 2/e is the absolute maximum value of f.

Example 23: Let $f(x) = ax^2 - 4ax + b$ (a > 0) be defined in $1 \le x \le 5$. Suppose the average of the maximum value and the minimum value of the function is 14, and the difference between the maximum value and minimum value is 18. Find the value of $a^2 + b^2$.

Solution: $f(x) = ax^2 - 4ax + b$ (a > 0)

$f'(x) = 2ax - 4a = 0$

$x = 2$

Also, $f'(x) = 2a(x - 2)$

$\Rightarrow$ For $x \in (1, 2)$, f is strictly decreasing and for $x \in (2, 5)$, f is strictly increasing.

Hence, the minimum occurs as x = 2.

$f(2) = 4a - 8a + b$

$f(2) = b - 4a$

The maximum will occur at f (5) and

$f(5) = 25a - 20a + b = b + 5a$

$\therefore$ $M = b + 5a$

$m = b - 4a$

$M - m = 9a = 18 \Rightarrow a = 2$

Also $\frac{M+m}{2} = 14 \Rightarrow M + m = 28 = 2b + a$

$\Rightarrow$ $b = 13$

Hence, a = 2 and b = 13

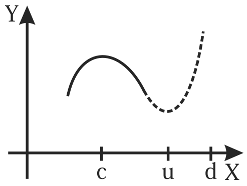
$\therefore$ $a^2 + b^2 = 4 + 169 = 173$.

Fork Extremum Theorem

Assume that f is a continuous function defined on an interval I. The interval I can be finite or infinite interval. If f has a unique local extremum in I, then that local extremum is also an absolute extremum on I.

Proof To see why this is so, look at the figure, where f is assumed to have a unique extremum, a local maximum at c.

Consider any other number d in I. The graph moves downward on both sides of c. So, if f(d) were greater than f(c), then, by the Extreme Value Theorem for the closed interval with endpoints c and d, f would have an absolute minimum at some point u between c and d. (u could not be equal to c or d.) Then f would have a local minimum at u, contradicting our hypothesis that f has a local extremum only at c. We can extend this argument to the case where f has a local minimum at c by applying the result we have just obtained to – f.



Remark: Suppose that we want to maximize or minimize the function f on the open interval I, and we find that f has only one critical point in I, a number c at which $f'(c) = 0$. If $f''(x)$ has the same sign at all points of I, then the above theorem implies that f(c) is an absolute extremum of f on I – a minimum if $f''(x) > 0$ and a maximum if $f''(x) < 0$. This absolute interpretation of the second derivative test can be useful in applied open-interval maximum-minimum problems.

Example 24: Find the maximum value of x ln 1/x in $0 < x < \infty$.

Solution: Let $f(x) = x \ln 1/x = -x \ln x$

Then $f'(x) = -1 - \ln x$

$f'(x) = 0 \Rightarrow x = 1/e$.

Now $f''(x) = -1/x$

$\therefore \quad f''(1/e) = -e < 0$

Therefore f(x) has a local maximum at x = 1/e.

Using Fork extremum theorem, since x = 1/e is the only point of extremum, the absolute maximum value of

$f(x) = f(1/e) = 1/e \ln e = 1/e$.

Example 25: If $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, where A is a positive constant . Find the smallest A such that $f(x) \geq 24$ for all $x > 0$.

Solution: $f'(x) = 10x - 5Ax^{-6}$ and

$f''(x) = 10 + 30Ax^{-6} > 0$

i.e. $f'(x) = 0$ gives a minima .

$$\Rightarrow x^7 = \frac{A}{2} \Rightarrow x = \left(\frac{A}{2}\right)^{1/7}.$$

Since A > 0, we get only one minima and no maxima.

Hence the smallest value of f(x) will be at $x = \left(\frac{A}{2}\right)^{1/7}$.

$$\text{i.e. } f(x)_{\min} = 5 .\left(\frac{A}{2}\right)^{2/7} + A\left(\frac{A}{2}\right)^{-5/7} = 24$$

$$\text{or, } 5\left(\frac{A}{2}\right)^{2/7} + 2\left(\frac{A}{2}\right)^{2/7} = 24 \quad \Rightarrow A = 2\left(\frac{24}{7}\right)^{7/2}$$

Example 26: Find the coordinates of the point on the parabola $y = x^2$ which is nearest to the point (3, 0).

Solution: Let (x, y) be any point on the parabola

$y = x^2$ ...(1)

Let s be the distance of (x, y) from (3, 0).

Then $s^2 = (x - 3)^2 + (y - 0)^2 = (x - 3)^2 + y^2$

$= (x - 3)^2 + x^4$, [from (1)]

Now s is maximum or minimum according as s^2 is maximum or minimum.

Let $u = s^2 = (x - 3)^2 + x^4$.

Then $du/dx = 2(x - 3) + 4x^3$.

$du/dx = 0 \Rightarrow 2(x - 3) + 4x^3 = 0$

$\Rightarrow 2x^3 + x - 3 = 0$ giving x = 1.

Now $d^2u/dx^2 = 2 + 12x^2 > 0$, when x = 1.

Hence x = 1 is a local minimum.

When x = 1, we have from (1), y = 1.

Using Fork extremum theorem, since x = 1 is the only point of extremum, the absolute minimum value occurs at x = 1.

Thus the required point is (1, 1).

One way of proving that $f(x) \leq g(x)$ for all x in a given interval is to show that $0 \leq g(x) - f(x)$ for all x in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of g(x) – f(x) on the interval is nonnegative.

Example 27: Prove the inequality $e^x > 1 + x$ when $x \neq 0$.

Solution: Consider the function $f(x) = e^x - (1 + x)$.

In the usual way we find that this function has absolute minimum f(0) = 0.

Hence, $f(x) > f(0)$ when $x \neq 0$,

and so $e^x > 1 + x$ when $x \neq 0$.

Example 28: Prove that $e^x \leq 1/(1 - x)$ for $-\infty < x < 1$ with equality only for x = 0.

Solution: Let $f(x) = (1 - x)e^x$.

Then $f(x) > 0$ on $(-\infty, 1)$, $f(1) = 0$,

and $f(x) \to 0$ as $x \to -\infty$.

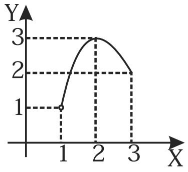
Also $f'(x) = -xe^x$,

so $f'(x) > 0$ for $x < 0$, $f'(0) = 0$, $f'(x) < 0$ for $0 < x \leq 1$. This proves that f has a maximum at x = 0, $f(x) \leq f(0) = 1$, with equality only for x = 0.

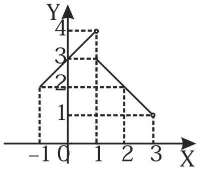
Concept Problems E

1. Examine the graph of following functions in each case identify the points of global maximum/minimum and local maximum / minimum.

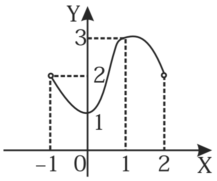
(i)



(ii)

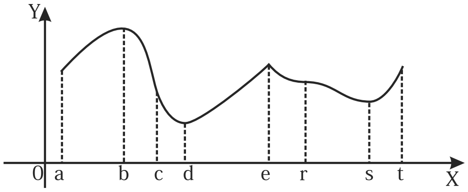


(iii)

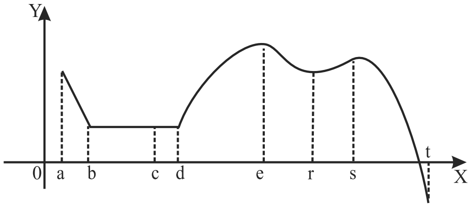


2. For each of the numbers a, b, c, d, e, r, s, and t, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

(i)

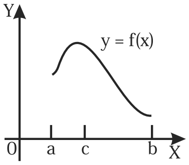


(ii)

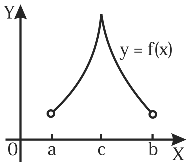


3. Determine from the graph whether the function has any absolute extreme values on [a, b].

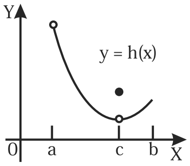
(i)



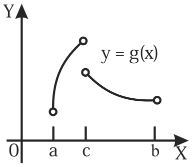
(ii)



(iii)



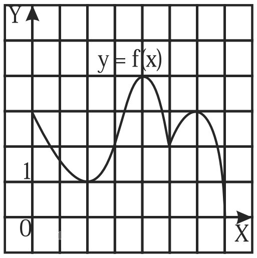
(iv)



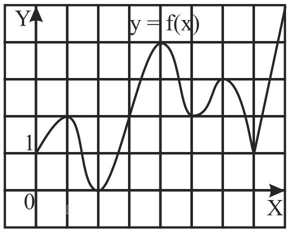
4. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

5. Use the graph to state the absolute and local maximum and minimum values of the function.

(i)



(ii)



6. Consider the function $f(x) = x^2$ only for x in [–1, 2].

(a) Graph the function f(x) for x in [–1, 2]

(b) What is the maximum value of f(x) for x in the interval [–1, 2]?

(c) Does $f'(x)$ exists at the maximum?

(d) Does $f'(x)$ equal 0 at the maximum?

(e) Does $f'(x)$ equal 0 at the minimum?

7. Let $f(x) = 2x^3 - 9x^2 + 12x + 6$

(i) Find the possible points of local extrema of f(x) for $x \in R$.

(ii) Find the number of critical points of f(x) for $x \in [0, 2]$.

(iii) Find absoluble maxima/minima value of f(x) for $x \in [0, 2]$.

(iv) Prove that for $x \in (1, 3)$, the function does not has a global maximum.

8. Find the maximum and minimum values of $f(x) = x^2 \ln x$, $x \in [1,e]$.

9. Find the maximum and minimum values of $f(x) = x + \sin 2x$, $x \in [0,2\pi]$

10. Show that the graph of $x^3 - 12x$ has a local maximum point but no absolute maximum point and that it also has a local minimum point which is strictly local.

11. Find the greatest and the least values of the functions on the given intervals.

(i) $y = x + 2\sqrt{x}$, [0, 4]

(ii) $y = x^5 - 5x^4 + 5x^3 + 1$, [–1, 2]

(iii) $y = \sin 2x - x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

12. Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur.

(i) $f(x) = \frac{3x}{\sqrt{4x^2+1}}$, [–1, 1]

(ii) $f(x) = 1 + |9 - x^2|$, [–5, 1]

(iii) $f(x) = \frac{x^2}{x+1}$, (–5, –1)

13. Draw the graph of function $f(x) = 2|x - 2| + 5|x - 3|$ $(x \in R)$. Also identify points of local extrema and also find the global maximum/minimum values.

14. Apply Fork extremum theorem to obtain the global extrema of the following functions :

(i) $(x - 1)^2 (x - 2)^2$ on R

(ii) $x(1 - 2\pi x^2)$ on $[0, \infty)$.

15. In each part, determine whether the statement is true or false, and justify your answer.

(i) If f is differentiable on the open interval (a, b), and if f has an absolute extremum on the that interval, then it must occur at a stationary point of f.

(ii) if f is continuous on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.

16. Does every polynomial of odd degree $n \ge 3$ have atleast one critical point? a global maximum or minimum

17. Suppose that f is continuous on [a, b] and differentiable on (a, b) and that $f'(x)$ is never zero at any point of (a, b). Explain why the maximum and minimum values of f must occur at the endpoints of the interval [a, b].

18. Can you show that the function f must have an absolute minimum value if $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = \infty$ and f is continuous on the open interval (a, b)?

19. Let f : R → R be a continuous function such that $\lim\limits_{x\to-\infty} f(x) = 0 = \lim\limits_{x\to\infty} f(x)$. Prove that if f is strictly negative somewhere on R then f attains a finite absolute minimum on R, and that if f is strictly positive somewhere on R then f attains a finite absolute maximum on R.

Practice Problems D

20. Find the maximum and minimum values of

$$f(x) = \sin x + \frac{1}{2}\cos 2x,\ x \in \left[0, \frac{\pi}{2}\right].$$

21. Find the maximum and minimum values of $f(x) = \{(1-x^2)(2x^2+1)\}^{1/2}$, $x \in [-1, 1]$.

22. Find the greatest and the least value of the following functions on the indicated intervals :

(i) $f(x) = \tan^{-1}x - \frac{1}{2}\ln x$ on $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$

(ii) $f(x) = 2\sin x + \sin 2x$ on $\left[0, \frac{3}{2}\pi\right]$

(iii) $f(x) = \begin{cases} 2x^2 + \dfrac{2}{x^2} & \text{for } -2 \le x \le 2,\ x \ne 0 \\ 1 & \text{for } x = 0 \end{cases}$

23. Find the absolute maximum and minimum values of $f(x) = \begin{cases} 4x-2, & x<1 \\ (x-2)(x-3), & x \ge 1 \end{cases}$ on $\left[\frac{1}{2}, \frac{7}{2}\right]$.

24. Prove that the function $f(x) = 3|x| + 4|x-1|$, has a minimum value, and that this value is 3.

25. Find the greatest and the least value of the function $f(x) = \sqrt{1-2x+x^2} - \sqrt{1+2x+x^2}$, $x \in R$.

26. Find the largest and smallest values of each continuous function on the closed, bounded interval.

(i) $f(x) = \begin{cases} 9-4x & \text{if } x<1 \\ -x^2+6x & \text{if } x \ge 1 \end{cases}$ on [1, 4]

(ii) $f(t) = \begin{cases} -t^2-t+2 & \text{if } t<1 \\ 3-t & \text{if } t \ge 1 \end{cases}$ on [–2, 3]

27. Find the set onto which the derivative of the function $f(x) = x(\ln x - 1)$ maps the ray $[1, \infty)$.

28. Find the greatest and least values of the given functions on the indicated intervals.

(i) $f(x) = x + \cos^2 x$, $x \in [0, \pi/2]$.

(ii) $f(x) = \tan x + \cot 2x$, $x \in [\pi/6, \pi/3]$.

29. Show that $f(x) = \frac{64}{\sin x} + \frac{27}{\cos x}$ has a minimum value but no maximum value on the interval $(0, \pi/2)$. Find the minimum valune.

30. Show that the absolute minimum value of

$$f(x) = x^2 + \frac{16x^2}{(8-x)^2},\ x > 8 \text{ occurs at } x = 4(2 + \sqrt[3]{2}).$$

31. An object moves along the t-axis with position $s(t) = t^4 - 2t^3 - 12t^2 + 60t - 10$. Find the largest value of its velocity on [0, 3].

32. The parametric equations of a curve are $x = 2t(t^2+3) - 3t^2$, $y = 2t(t^2+3) + 3t^2$. Find the maximum slope of the curve and the corresponding point on it.

33. Find the intersection of the sets onto which the derivatives of the function

$$y = \frac{x+3}{x-5},\ y = \sqrt{6x+5} \text{ map the interval } [0, 1].$$

34. Show that $e^x/x \ge e$ for all $x \ge 0$ and that $e^x/x = e$ only for $x = 1$.

35. Prove $\ln x \le x - 1$ for $x > 0$ with equality only for $x = 1$.

7.11 Boundedness

The number M is called the least upper bound of the function y = f(x) on a set X if

(i) ∀ x ∈ X the inequality f(x) ≤ M holds true,

(ii) ∀ M' < M there exists. x' ∈ X such that f(x') > M'.

Remark Condition (i) signifies that the number M is one of the upper bounds of the function y = f(x) on the set X.

Condition (ii) signifies that M is the least of the upper bounds of the function y = f(x) on the set X, i.e. there is no number M', smaller than M, which is an upper bound.

The least upper bound of the function y = f(x) on the set X is designed as sup f(x), the supremum of f(x). If the function y = F(x) is unbounded from above on the set X, then we write sup f(x) = ∞. The greatest lower bound, in f(x), the infimum of f(X) is defined by analogy. The difference sup f(x) – inf f(x) is the oscillation or span of the function y = f(x) on the set X.

Example 1: Prove that the function $y = \frac{1}{1+x^2}$ is bounded for $x \in (-\infty, \infty)$. Find the least upper and the greatest lower bound of the function and find out whether it possesses a maximum and a minimum value.

Solution: Since $x^2 \ge 0$, we have $1 + x^2 \ge 1$ and , consequently, $\forall x \in (-\infty, \infty)$ there hold inequalities

$$0 \le \frac{1}{1+x^2} \le 1. \qquad ...(1)$$

It follows that the function $y = \frac{1}{1+x^2}$ is bounded on $(-\infty, \infty)$.

It is clear that $\frac{1}{1+x^2} \to 0$ as $x \to \infty$, and therefore this function can assume values arbitrarily close to zero. It is natural to assume that $\inf \frac{1}{1+x^2} = 0$. Let us prove this using the definition of the greatest lower bound. We must show that the following two conditions are fulfilled:

(i) $\forall\ x \in (-\infty, \infty)$, $\frac{1}{1+x^2} \geq 0$,

(ii) $\forall\ m > 0$, there exists x' such that $\frac{1}{1+x'^2} < m$.

Condition (i) is fulfilled by virtue of the left inequality (1). We shall prove that condition (ii) is also fulfilled. We specify an arbitrary $m > 0$. If $m \geq 1$, then the inequality $\frac{1}{1+x'^2} < m$ is fulfilled by virtue of (1) for all x'. Let $m < 1$. Then $\frac{1}{m} - 1 > 0$.

We take, as x', any number such that $x' > \sqrt{\frac{1}{m} - 1}$. Then $x'^2 > \frac{1}{m} - 1$ Þ $1 + x'^2 > \frac{1}{m}$ and , consequently, $\frac{1}{1+x^2} < m$, and this is what we wished to prove.

Thus in f $\frac{1}{1+x^2} = 0$.

We can prove by analogy that $\sup \frac{1}{1+x^2} = 1$. We take note now of the fact that the function $y = \frac{1}{1+x^2}$ assumes a value of 1 (for x = 0) but does not assume a value of 0 for any x $\left(\frac{1}{1+x^2} > 0 \quad \forall\ x\right)$.

Therefore, this function has a maximum value 1 on $(-\infty, \infty)$ but does not have a minimum value.

Concept Problems F

1. Can a function unbounded on a set X be continuous on this set if : (a) X is a closed interval, (b) X is an open interval?
2. Find out whether the following statement is true : "A function bounded from above (from below) on a set X has the absolute minimum (absolute maximum) on this set".
3. Find out whether the following statement is true : "A function continuous and bounded on an open interval attains its least upper and greatest lower bounds on that interval".
4. Find out whether the following statement is true : "If a function does not have an absolute minimum (absolute maximum) on the closed interval [a, b], then it is discontinuous on this interval".
5. Find out whether the following statement is true : "A function discontinuous on the closed interval [a, b] does not attain its least upper and greatest lower bounds on this interval".
6. Find out whether each of the following statements is true:
   (a) "A function y = f(x) bounded on the closed interval [a, b] has absolute maximum and absolute minimum".
   (b) "A function continuous on the closed interval [a, b] has absolute maximum and absolute minimum".
7. Find out whether the following functions are bounded :
   (a) $y = x^2$ on $[-5, 10]$,
   (b) $y = x^2$ on $[-5, \infty]$,
   (c) $y = x \cos(1/x)$ on $(-\infty, \infty)$
   (d) $y = \begin{cases} 2^{1/(x-1)} & \text{for} \quad x \neq 1, \\ 0 & \text{for} \quad x = 1 \end{cases}$ on (0, 2),
   (e) $y = 2^{1/(x-1)}$ on (0, 1).
8. Let the function f(x) be defined on a set X and assume that $\forall\ x \in X$ there is a neighbourhood in which f(x) is bounded. Does it follow that f(x) is bounded on X if : (a) X is an open interval, (b) X is a closed interval?
9. Prove that if a function is differentiable but unbounded on an open interval, then its derivative is also unbounded on that interval. Give an example showing that the converse statement is not true.

Practice Problems E

10. Prove the boundedness of the function.
    (a) $y = \frac{1+x}{1+x^2}$ on the half-open interval $[0, \infty)$,
    (b) $y = \frac{2x}{1+x^2}$ on the number line $(-\infty, \infty)$,
    (c) $y = x \sin \frac{1}{x}$ on $(-\infty, \infty)$,
    (d) $y = \tan^{-1} 2^x$ on $(-\infty, \infty)$,
    (e) $y = xe^{-x}$ on $(0, \infty)$.
11. Find the least upper and the greatest lower bound of the following functions and mention whether they are attained:
    (a) $f(x) = \frac{2x}{1+x^2}$ on $(0, \infty)$,

(b) $f(x) = x^2$ on $[-5, 10]$,
(c) $f(x) = \tan^{-1} 2^x$ on $(-\infty, \infty)$
(d) $f(x) = \sin x + \cos x$ on $[0, \pi]$,
(e) $f(x) = 2^{1/(x-1)}$ on $(0, 1)$.

12. Find the oscillations of the following functions :
(a) $f(x) = x^2$ on $(-1, 2)$
(b) $f(x) = \sin(1/x)$ on $(0, \varepsilon)$, where ε is an arbitrary number
(c) $f(x) = x \sin(1/x)$ on $(0, 1)$,
(d) $f(x) = x\,|\sin(1/x)|$ on $(0, 1)$.

13. We designate the greatest lower and the least upper bound of the function f(x) on a set X as m [f] and M[f] respectively. Let $f_1(x)$ and $f_2(x)$ be defined and bounded on X. Prove that

$m[f_1 + f_2] \geq m[f_1] + m[f_2]$, and
$M[f_1 + f_2] \leq M[f_1] + M[f_2]$.

7.12 Algebra of Global Extrema

(i) If two functions f and g attain their greatest (least) value at x = a then y = f(x) + g(x) also attains its greatest (least) value at x = a.

For example, the greatest value of $y = \cos x + \frac{1}{x^2+1}$ is 2 when x = 0, since both cosx and $\frac{1}{x^2+1}$ attain their greatest values at x = 0.

(ii) If f and g are nonnegative functions which attain their greatest (least) values at x = a, then y = f(x). g(x) also attains its greatest (least) value at x = a.

For example, $y = (1 + \sin x)\sin^2 x$ attains the maximum value of 2 at $x = \frac{\pi}{2}$.

(iii) If f is non-negative and g is positive so that f attains its greatest (least) value at x = a and g attains its least (greatest) value at x = a then $y = \frac{f(x)}{g(x)}$ attains its greatest (least) value at x = a.

For example, $y = \frac{1+\sin x}{2+\cos 2x}$ attains the maximum value of 2 at $x = \frac{\pi}{2}$.

$y = \frac{e^x + e^{-x}}{3+\cos 2\pi x}$ attains the minimum value at x = 0, and the minimum value is $\frac{2}{3+1} = \frac{1}{2}$.

(iv) If f(x) is continuous on [a, b] and g(x) is continuous on [m, M], where m and M are the absolute minimum and the absolute maximum of f on [a, b]

then $\max\limits_{x \in [a,b]} gof = \max\limits_{x \in [m,M]} g(x)$

and $\min\limits_{x \in [a,b]} gof = \min\limits_{x \in [m,M]} g(x)$

Example 1: Find the greatest and the least value of the function $F(x) = \frac{\sin 2x}{\sin(x+\pi/4)}$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution: Using the formulae

$$\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}}(\sin x + \cos x),$$

$$\sin 2x = (\sin x + \cos x)^2 - 1,$$

we represent the given function as a composite function

$$F(x) = f(g(x)),$$

where $f(y) = \sqrt{2}\,\frac{y^2-1}{y}$, $y = g(x) = \sin x + \cos x$.

Let us find the greatest and the least value of g(x). The critical points of g(x) are the roots of the equation

$$\cos x - \sin x = 0$$

from which only $x = \pi/4$ lies in the interval $[0, \pi/2]$. Comparing g(0), g(π/4), and g(π/2), we infer that the interval $[1, \sqrt{2}]$ is the range of g(x).

It is easy to see that $f'(y) = \sqrt{2}\left(1 + \frac{1}{y^2}\right) > 0$, for $y \in [1, \sqrt{2}]$. Consequently, the function f(y) increases on the interval $[1, \sqrt{2}]$ and attains its greatest and least values at the right and left endpoints of the interval, respectively :

$$\min_{y \in [1,\sqrt{2}]} f(y) = f(1) = 0.$$

The same values are the greatest and least values of the initial function F(x) as well.

$$\max_{x \in [0,\pi/2]} F(x) = 1, \quad \min_{x \in [0,\pi/2]} F(x) = 0.$$

Example 2: Find the least and greatest values of the function f(x) = sin (cos (sin x)) on the closed interval [π/2, π].

Solution: Let $\pi/2 \leq x_1 < x_2 \leq \pi$, then $0 \leq \sin x_2 < \sin x_1 \leq 1$, and the points $\sin x_1$, $\sin x_2$ lie in the first quadrant since $1 < \pi/2$. Since the function cos x decreases on the interval [0, π/2], we have

$$0 < \cos(\sin x_1) < \cos(\sin x_2) \leq 1.$$

But the points $\cos(\sin x_1)$, $\cos(\sin x_2)$ also lie in the first quadrant and the function sin x increases on the interval [0, π/2], therefore

$$0 < \sin(\cos(\sin x_1)) < \sin(\cos(\sin x_2)) < 1.$$

that is, the function f(x) = sin (cos (sin x)) is increasing on the interval [π/2, π], consequently, the minimum value of f(x) on this interval is equal to f(π/2) = sin (cos 1), while the maximum value to f(π) = sin (cos 0) = sin 1.

Example 3: Find the minimum value of the function $f(x) = 8^x + 8^{-x} + 4(4^x + 4^{-x})$, $\forall\ x \in R$.

Solution: $f(x) = \left(8^x + \frac{1}{8^x}\right) + 4\left(4^x + \frac{1}{4^x}\right)$

Both the functions $8^x + \frac{1}{8^x}$ and $4^x + \frac{1}{4^x}$ attain then least values at x = 0.

Hence, the minimum value of f(x) = 1 + 1 + 4(1 + 1) = 10.

Example 4: Find the minimum value of the function $f(x) = x^{\frac{3}{2}} + x^{\frac{-3}{2}} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x.

Solution: $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$

$$f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 - 3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$- 4\left[\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 2\right]$$

Let $\sqrt{x} + \frac{1}{\sqrt{x}} = t \ (x > 0)$

Note: $y = x + \frac{1}{x} \geq 2$ for $x > 0 \quad \leq -2$ for $x < 0$

Let $g(t) = t^3 - 3t - 4t^2 + 8$

Now $g(t) = t^3 - 4t^2 - 3t + 8$ where $t \in [2, \infty]$

$g'(t) = 3t^2 - 8t - 3 = (t - 3)(3t + 1)$

$g'(t) = 0 \Rightarrow t = 3 \ (t \neq -1/3)$

$g''(t) = 6t - 8$

$g''(3) = 10 > 0 \Rightarrow g(3)$ is the absolute minimum.

$g(3) = 27 - 9 - 36 + 8 = -10.$

Hence $f_{\min} = g(3) = -10.$

Concept Problems G

1. Let h = fg be the product of two differentiable functions of x.
   (i) If f and g are positive, with local maximum at x = a, and if f′ and g′ change sign at a, does h have local maximum at a?
   (ii) If the graphs of f and g have inflection points at x = a, does the graph of h have an inflection point at a? In either case, if the answer is yes, give a proof. If the answer is no, give a counter example.
2. Let h and g have relative maxima at x_0. Prove or disprove:
   (a) h + g has a relative maximum at x_0.
   (b) h – g has a relative maximum at x_0.
3. Suppose f(x) and g(x) are both continuous for all x and differentiable at x = c. Show that if c is a critical number for both f and g, then it is also a critical number for the product function fg. If a relative maximum occurs at c for both f and g, is it true that a relative maximum of fg occurs at c ?
4. Find whether the following statements about continuous functions are true in general :
   (i) If f and g are both differentiable and f and g both have relative extremum at c, then so does f + g.
   (ii) If f is strictly increasing for x > c and strictly decreasing for x < c, then f '(c) = 0.
   (iii) If f '(0) = 0, $f'\left(-\frac{1}{2}\right) < 0$, and $f'\left(\frac{1}{2}\right) > 0$, then f has a relative minimum at (0, f(0)).
   (iv) If f and g are each strictly increasing on an interval I, then f – g is also strictly increasing on I.
5. Find the greatest and the least values of the continuous functions given below in the indicated intervals.
   (i) $f(x) = e^{x^2-4x+3}$ in [– 5, 5]
   (ii) $f(x) = \cos 3x - 15\cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$.
6. Find the greatest and least values of the function tan (cos x) on the interval [π/2, π].

Practice Problems F

7. Find the values of x where f attains its maximum.
   (i) $f(x) = \cos x + \cos\sqrt{2}\,x$
   (ii) $f(x) = 2\sin^2 x - 3|\cos x|$.
8. Show that for every real number x,
   $$\frac{1}{5} \leq \frac{x^2 - 4x + 9}{x^2 + 4x + 9} \leq 5.$$
9. Investigate the following functions for extrema :
   (i) $f(x) = \frac{4x}{x^2 + 4}$
   (ii) $f(x) = \frac{14}{x^4 - 8x^2 + 2}$
   (iii) $f(x) = \sqrt[3]{2x^3 + 3x^2 - 36x}$
10. Find the global maximum and minimum values of the functions :
    (i) $f(x) = \frac{4x}{x^2 + 4}$
    (ii) $f(x) = \frac{14}{x^4 - 8x^2 + 2}$

(iii) $f(x) = \sqrt[3]{2x^3 + 3x^2 - 36x}$

11. Find the greatest and the least value of the function:

$f(x) = \dfrac{1}{\sin x + 4} - \dfrac{1}{\cos x - 4}, x \in R.$

12. Find the least value of the function

$f(x) = \left(\dfrac{2 + \cos x}{\sin x}\right)^2$ on the interval $[0, \pi]$.

13. Prove that the inequality $\min\limits_{x \in [-\pi, \pi]} f(x) > -7/9$ holds true for the function $f(x) = \cos x \sin 2x$.

14. Find the greatest and the least value of the function $f(x) = (x - 1)^2 \sqrt{x^2 - 2x + 3}$, $x \in [0, 3]$

7.13 Miscellaneous Methods

There are several problems in algebra and geometry in which it is required to find the range of certain proposed functions in real variables. This investigation can often be conducted in an elementary manner. Not all problems in maxima and minima require calculus for their solution.

As an example, it is obvious that if $f(x) = \dfrac{1}{1+x^2}$, then $f(0) = 1$ is the only maximum value of f.

It is to be noticed that some important problems of maxima and minima can be solved by elementary algebraical methods, without recourse to calculus.

Example 1: Find the the minimum value of $f(x) = 2x^2 - 3x + 2$.

Solution: The function $f(x) = 2x^2 - 3x + 2$, may be put in the form

$$f(x) = 2(x^2 - 3x/2 + 1) = 2(x - 3/4)^2 + 7/8$$

Hence the function has the minimum value 7/8, corresponding to $x = \dfrac{3}{4}$.

Example 2: Find the least value of $y = ax + \dfrac{b}{x}$, a, b, x > 0.

Solution: The function $ax + \dfrac{b}{x}$ may be written in the form $\left(\sqrt{ax} - \sqrt{\dfrac{b}{x}}\right)^2 + 2\sqrt{ab}$, from which it is obvious that the expression can never be less than $2\sqrt{ab}$, the value it attains when $\sqrt{ax} = \sqrt{\dfrac{b}{x}}$, or $x = \sqrt{\dfrac{b}{a}}$.

This is because the square of a real quantity is essentially positive and therefore any value of x other than $x = \sqrt{\dfrac{b}{a}}$ will give to the expression a greater value than $2\sqrt{ab}$.

Hence, the least value is $2\sqrt{ab}$.

Example 3: Find the minimum and maximum values of the expression $\dfrac{3x^2 - 4x + 3}{3x^2 + 4x + 3}$ for real values of x.

Solution: Let $\dfrac{3x^2 - 4x + 3}{3x^2 + 4x + 3} = y$

Then $3(1 - y)x^2 - 4(1 + y)x + 3(1 - y) = 0$.

If x be real, we must have $D \geq 0$ i.e. $4(1 + y)^2 - 9(1 - y)^2 > 0$, i.e. $(5y - 1)(5 - y)$ must be positive.

Hence y must lie between the values 5 and $\dfrac{1}{5}$.

Therefore the maximum value of the expression is 5 and the minimum value is $\dfrac{1}{5}$.

Example 4: For $x \geq 0$, find the smallest value of the function $f(x) = \dfrac{4x^2 + 8x + 13}{6(1+x)}$.

Solution: We have f(x)

$$= \frac{4x^2 + 8x + 13}{6(1+x)} = \frac{4(x+1)^2 + 9}{6(1+x)}$$

$$= \frac{2}{3}(x+1) + \frac{3}{2(x+1)} \geq 2\sqrt{\frac{2}{3}\cdot\frac{3}{2}} = 2 \text{ [A.M.}\geq\text{G.M.]}$$

Equality occurs when $(x + 1)^2 = \dfrac{9}{4}$

$\Rightarrow\ x + 1 = \dfrac{3}{2}$ i.e. $x = \dfrac{1}{2}$ and the minimum value $= 2$.

Example 5: If the sum of two positive quantities be given, when is their product a maximum?

Solution: Let $x + y = a$, a constant,

then $4xy = (x + y)^2 - (x - y)^2 = a^2 - (x - y)^2$.

The right hand side has its maximum value when $(x - y)^2$ has its minimum, i.e. when $x = y$, for being a square it cannot be negative.

Thus the maximum value of xy is $\dfrac{a^2}{4}$.

This may be shown geometrically as follows : the problem is to divide a given line AB in such manner that the rectangle of the segments is as great as possible. Let C be the centre and P any other point of the line. Then

rect. AP . PB + sq. on CP = sq. on AC = rect. AC . CB,

i.e. the rect. AP . PB is less than the rect. AC . CB.

Hence the point of division is the mid-point, or the line must be bisected.

Example 6: A cylinder is inscribed in a frustum of a paraboloid of revolution about the x-axis, of the curve $y^2 = 4ax$, . For the volume of the cylinder to be the greatest, prove that the height of the cylinder is one-half that of the frustum.

Solution: $y^2 = 4ax$, ...(1)

Let h be the length of the axis, and x the abscissa of the end of the cylinder nearest the origin.

Volume of the cylinder

$$= \pi y^2 (h - x) = \pi y^2 \left(h - \frac{y^2}{4a}\right) \quad ...(2)$$

Now the sum of the quantities y^2 and $4ah - y^2$ is constant; their product is therefore greatest when they are equal, i.e. when

$y^2 = 2ah$, or $x = 1/2h$ ...(3)

The height of the cylinder is therefore one-half that of the frustum.

Example 7: If $x + y + z + w$ be constant, when will xyzw have its maximum value?

Solution: So long as any two, say x and y, are unequal we can without altering z and w (and thus keeping $x + y$ constant) increase xy, and therefore also xyzw by making x and y more nearly equal. Hence xyzw does not attain its maximum value until $x = y = z = w$. This is called virtue of symmetry.

The same argument obviously applies to the product of any number of quantities whose sum is constant.

If we are searching for the maximum value of such an expression as xy^2z^3, say, with condition $x + y + z = a$, we proceed thus $xy^2z^3 = 2^2 3^3 . x . \frac{y}{2} . \frac{y}{2} . \frac{z}{3} . \frac{z}{3} . \frac{z}{3}$ and is to be a maximum, where $x + \frac{y}{2} + \frac{y}{2} + \frac{z}{3} + \frac{z}{3} + \frac{z}{3} = a$; and by the preceding work we are to make $\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = \frac{a}{6}$, from whence the maximum value is $\frac{2^2 . 3^3 a^6}{6^6} = \frac{a^6}{2^4 . 3^3}$.

Example 8: In any triangle, prove that the maximum value of cos A cos B cos C is $\frac{1}{8}$.

Solution: Since 2cos A cos B cos C

$= \cos A \{\cos (B - C) - \cos A\}$,

and therefore as long as B and C are unequal we may increase the expression by making them more nearly equal and keeping their sum constant. Thus, cos A cos B cos C does not attain its maximum value until $A = B = C = \frac{\pi}{3}$, and then its value is $\frac{1}{8}$.

Example 9: What are the greatest and least values of a sin x + b cos x?

Solution: Let $a = c \cos \alpha$ and $b = c \sin \alpha$,

so that $c^2 = a^2 + b^2$ and $\tan \alpha = \frac{b}{a}$

Thus $a \sin x + b \cos x = c \sin x \cos \alpha + c \cos x \sin \alpha$

$= \sqrt{a^2 + b^2} \sin (x + \alpha)$,

and as the greatest and least values of a sine are 1 and – 1, the maximum and minimum values required are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively.

If a student is required to find the maximum or minimum value of a function of two or more variables, where the variables are connected by one or more relations, then change the function into a function of one variable by using the relation(s), and then work with the resulting function of one variable.

Example 10: If $x, y \in R$ and satisfy the equation $xy(x^2 - y^2) = x^2 + y^2$ where $x \neq 0$ then find the minimum possible value of $x^2 + y^2$.

Solution: If $x = r \cos \theta$ and $y = r \sin \theta$ then $x^2 + y^2 = r^2$. Hence we have to minimise r^2.

Now in the given equation substituting

$x = r \cos \theta$ and $y = r \sin \theta$,

we get $r^2 = 4 \operatorname{cosec} 4\theta$

$\Rightarrow r^2_{min} = 4$.

Example 11: Suppose that the temperature T at every point (x, y) in the plane cartesian is given by the formula $T = 1 - x^2 + 2y^2$

Show that the minimum temperature along the line $x + y = 1$, is -1.

Solution: $T = 1 - x^2 + 2y^2$ where $x + y = 1$

$T = 1 - x^2 + 2(1 - x)^2$

$= 1 - x^2 + 2(1 + x^2 - 2x)$

$T = x^2 - 4x + 3$

$= (x - 2)^2 - 1$

$T_{min} = -1$.

Example 12: The range of the function

$f(x) = \sqrt{4 - x^2} + \sqrt{x^2 - 1}$.

Solution: Let $x^2 = 4 \cos^2\theta + \sin^2\theta$

then $(4 - x^2) = 3 \sin^2\theta$ and $(x^2 - 1) = 3 \cos^2\theta$

$\therefore \quad f(x) = \sqrt{3} |\sin \theta| + \sqrt{3} |\cos \theta|$

$\Rightarrow \quad y_{min.} = \sqrt{3}$ and $y_{max.} = \sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{6}$.

Hence, the range of f(x) is $[\sqrt{3}, \sqrt{6}]$.

Example 13: A particle is moving on the curve defined parametrically by the system of equation

$x = 1 - 2 \cos^2 t$ and $y = \cos t$.

If the particle is closest to the origin when $t = \cos^{-1}(a)$ for $0 \leq t \leq \pi/2$, then find the value of 'a'.

Solution: Distance 'd' from the origin is $= \sqrt{x^2 + y^2}$

$= \sqrt{(1 - 2\cos^2 t)^2 + \cos^2 t}$

$\Rightarrow \quad d = \sqrt{4\cos^4 t - 3\cos^2 t + 1}$

$= 2\sqrt{\left(\cos^2 t - \frac{3}{8}\right)^2 + \frac{16}{14} - \frac{9}{64}}$

$= 2\sqrt{\left(\cos^2 t - \frac{3}{8}\right)^2 + \frac{7}{64}}$

This is a quadratic in $\cos^2 t$, this will be minimum when $\cos^2 t = \frac{3}{8}$

$\Rightarrow \quad \cos t = \dfrac{\sqrt{3}}{2\sqrt{2}} \qquad \Rightarrow \quad \cos t = \dfrac{\sqrt{6}}{4}$

$\Rightarrow \quad a = \dfrac{\sqrt{6}}{4}$.

Example 14: Find the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$ when x and y are subjected to the restrictions $0 \le x \le 1$ and $0 \le y \le 1$.

Solution: We have $f(x, y) = x^2 + y^2 - 4x + 6y$

Let $(x, y) = (\cos\theta, \sin\theta)$, then $\theta \in [0, \pi/2]$ and

$f(x, y) = f(\theta) = \cos^2\theta + \sin^2\theta - 4\cos\theta + 6\sin\theta$

$f'(\theta) = 6\cos\theta + 4\sin\theta > 0 \ \forall\ \theta \in [0, \pi/2]$

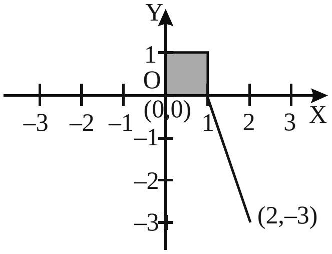
$\therefore \quad f'(\theta)$ is strictly increasing in $[0, \pi/2]$

$\therefore \quad f(\theta)_{\min} = f(0) = 1 - 4 + 0 = -3$.

Alternative:

$f(x, y) = (x - 2)^2 + (y + 3)^2 - 13$.

So, to minimize f over the unit square is the same as minimizing the distance from a point in this square to the point of coordinates $(2, -3)$.



The closest point in the square to $(2, -3)$ is then $(1, 0)$. So $f(1, 0) = -3$ must be the minimum.

Example 15: Find the least and the greatest value of $f(x, y) = x^2 + y^2 - xy$ where x and y are connected by the relation $x^2 + 4y^2 = 4$.

Solution: Here $x^2 + 4y^2 = 4$

$\Rightarrow \quad \dfrac{x^2}{4} + y^2 = 1$

Let $x = 2\cos\theta$, $y = \sin\theta$

Hence, $f(x, y) = x^2 + y^2 - xy$

$= 4\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta$

$= 2(1 + \cos 2\theta) + \dfrac{1}{2}(1 - \cos 2\theta) - \sin 2\theta$

$= \left(2 - \dfrac{1}{2}\right)\cos 2\theta - \sin 2\theta + \dfrac{5}{2}$

$= \dfrac{3}{2}\cos 2\theta - \sin 2\theta + \dfrac{5}{2}$

Since we know that $a\sin\theta + b\cos\theta$ lies between

$-\sqrt{a^2 + b^2}$ to $\sqrt{a^2 + b^2}$

the maximum and minimum values of $\left(\dfrac{3}{2}\cos 2\theta - \sin 2\theta\right)$ are $\sqrt{\left(\dfrac{3}{2}\right)^2 + 1}$ and $-\sqrt{\left(\dfrac{3}{2}\right)^2 + 1}$

i.e. $\dfrac{\sqrt{13}}{2}$ and $-\dfrac{\sqrt{13}}{2}$.

Hence, the greatest value of $f(x, y)$

$= \dfrac{\sqrt{13}}{2} + \dfrac{5}{2} = \dfrac{5 + \sqrt{13}}{2}$,

and the least value of $f(x, y) = -\dfrac{\sqrt{13}}{2} + \dfrac{5}{2} = \dfrac{5 - \sqrt{13}}{2}$.

Example 16: Find the maximum and minimum values of $x^2 + y^2$ where $ax^2 + 2hxy + by^2 = 1$.

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$.

Then $x^2 + y^2 = r^2$, $r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) = 1$

Now the problem reduces to that of finding the extreme values of r^2, where $\dfrac{1}{r^2} = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

(When (x, y) lies on $ax^2 + 2hxy + by^2 = 1$, then $r \ne 0$. hence it is meaningful to talk of $1/r^2$)

Let us now define a function f as

$f(\theta) = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$,

for $\theta \in [0, 2\pi]$. ...(1)

Since r^2 has a maximum or a minimum according as $1/r^2$ has a minimum or a maximum, therefore, the extreme values of r^2 are the same as the extreme values of f, a maximum value of r^2 being a minimum value of f, and a minimum value of r^2 being a maximum value of f.

We can write (1) as.

$f(\theta) = \dfrac{1}{2}(a + b) + \dfrac{1}{2}(a - b)\cos 2\theta + h\sin 2\theta$,

$= \dfrac{1}{2}(a + b) + \rho\cos(2\theta - \alpha)$ ...(2)

where $\rho\cos\alpha = \dfrac{1}{2}(a - b)$, $\rho\sin\alpha = h$ ...(3)

The extreme values of f occur when $2\theta - \alpha = k\pi$. Then $f(\theta) = \dfrac{1}{2}(a + b) \pm \rho$,

$= \dfrac{1}{2}(a + b) \pm \dfrac{1}{2}\{(a - b)^2 + 4h^2\}^{1/2}$.

Hence the maximum value of $x^2 + y^2$ i.e. r^2 is $\left[\dfrac{1}{2}(a + b) - \dfrac{1}{2}\{(a - b)^2 + 4h^2\}^{1/2}\right]^{-1}$ and the minimum value of $x^2 + y^2$ is $\left[\dfrac{1}{2}(a + b) + \dfrac{1}{2}[(a - b)^2 + 4h^2\}^{1/2}\right]^{-1}$.

Maxima and Minima of Functions of Several Variables

We give a few indications concerning the extension of some of the preceding results to functions of two or more independent variables.

In the first place let us seek for the maxima and minima of a function.

$u = \phi(x, y)$ ...(1)

A first condition is that we must have simultaneously

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \qquad ...(2)$$

where the differential coefficients are "partial".

For if u be greater (or less) than any other value of the function obtained by varying x, y within certain limits, u will be a maximum (or minimum) when y is kept constant and x alone is varied. This requires in general that $\partial f/\partial x = 0$. Similarly, u must be a maximum (or minimum) when x is kept constant and y alone varies; this requires that $\partial f/\partial y = 0$.

As before, these conditions, though necessary, are not sufficient. The further examination of the question, in its general form, is beyond the scope of the text; but it often happens that the existence of maxima and minima can be inferred, and the discrimination between them effected, by independent consideration. The conditions (2) then supply all that is analytically necessary.

Example 17: Find the cuboid of the least surface area for a given volume.

Solution: Let x,y,z be the edges, and a^3 the given volume. Since $xyz = a^3$, ...(1)

the function to be made a minimum is

$$u = xy + yz + zx = xy + \frac{a^3}{x} + \frac{a^3}{y} \qquad ...(2)$$

The conditions $\partial u/\partial x = 0$, $\partial u/\partial y = 0$ give

$x^2y = a^3$, $xy^2 = a^3$,

the only real solution of which is x = y = a, which implies, z = a.

It appears from (2) that, x and y being essentially positive in this problem, there is a lower limit to the surface area of the parallelepiped. And the above investigation shows that this limit is not attained unless the figure be a cube.

Maximum and Minimum for Discrete Valued Functions

Example 18: Find the largest term in the sequence

$$a_n = \frac{n}{n^2+10} \quad (n \in N).$$

Solution: Consider the function

$$f(x) = \frac{x}{x^2+10}, \ x > 0$$

Then $f'(x) = \frac{(x^2+10)-2x^2}{(x^2+10)^2}$

$$= \frac{-(x+\sqrt{10})\,(x-\sqrt{10})}{(x^2+10)^2} > 0 \text{ for } 0 < x < \sqrt{10}$$

$$< 0 \text{ for } x > \sqrt{10}$$

$\Rightarrow$ f(x) is strictly increasing in $(0, \sqrt{10}\,)$

strictly decreasing in $(\sqrt{10}, \infty)$

$\Rightarrow$ f(x) has greatest value at $x = \sqrt{10} \approx 3.16$..

Hence, the given sequence has the greatest value either at n = 3 or n = 4.

Now, we compare $a_3 = \frac{3}{19}$ and $a_4 = \frac{4}{26}$

Thus, $a_3 = \frac{3}{19}$ is the largest term of the given sequence.

Example 19: In how many parts an integer $N \geq 5$ should be dissected so that the product of the parts is maximized.

Solution: Using A.M. ≥ G.M.

$$\frac{x_1+x_2+....+x_n}{n} \geq (x_1x_2........x_n)^{1/n}$$

$$\Rightarrow \quad x_1x_2 \,.....\, x_n \leq \left(\frac{x_1+x_2+....+x_n}{n}\right)^n$$

Therefore maximum value of $x_1\, x_2\, x_3$............ x_n is obtained when $x_1 = x_3 = x_3 = ... = x_n$.

$\Rightarrow$ the parts are all equal.

Now $x_1 + x_2 + x_3 + ... + x_n = N$.

Now, function to be maximized

is $\left(\frac{x_1+x_2+....+x_n}{n}\right)^n$ which is a discrete function of n. In order to arrive at some possible neighbourhood we make it continuous first.

Thus changing the variables n to x, we write

$$f(x) = \left(\frac{N}{x}\right)^x.$$

For maxima $f'(x) = 0$ i.e. $f'(x) = f(x)\left(\ell n\left(\frac{N}{x}\right)-1\right)$.

$$f'(x) = 0 \text{ for } x = \frac{N}{e}$$

The nearest integer $\left[\frac{N}{e}\right]$ or $\left[\frac{N}{e}\right]$ + 1 where [.] denotes the greatest integer function.

We compare $f\left(\left[\frac{N}{e}\right]\right)$ and $f\left(\left[\frac{N}{e}\right]+1\right)$ to get the maximum value.

Example 20: A manufacturer determines that the demand for n lamps priced at p rupees each is given by the function $n = f(p) = 60 - \frac{1}{2}p$ where p is an even integer, $0 \le p \le 120$. The range of f(p) is a set of integers that are on the interval $0 \le n \le 60$. Suppose that the lamp manufacturer has determined that the cost of producing n lamps is (140 + 25n) rupees. How many lamps should be manufactured to maximize profit ?

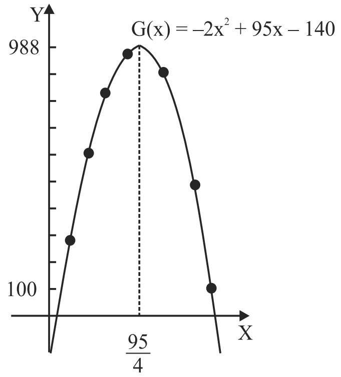
Solution: The total profit F(n) obtained by producing n lamps at p rupees is

$$F(n) = np - (140 + 25n)$$
$$= n(120 - 2n) - (140 + 25n)$$
$$= -2n^2 + 95n - 140.$$

We consider the function $G(x) = -2x^2 + 95x - 140,\ 0 \le x \le 60$, and observe that whenever x is an integer, n, G(n) = F(n), and thus all points (n, F(n)) are on the graph of G(x).

Observe also that $G\left(\frac{1}{2}\right) = -93$, but $F\left(\frac{1}{2}\right)$ is undefined. Since $G'(x) = -4x + 95 > 0$ if $x < \frac{95}{4}$, and $G'(x) < 0$ if $x > \frac{95}{4}$, we see that G(x) is increasing in $0 \le x \le \frac{95}{4}$ and is decreasing in $\frac{95}{4} \le x \le 60$.

Hence, G has its maximum at $x = \frac{95}{4}$.



The dots on the graph indicate some points on the graph of G(x) corresponding to integer values of s; that is, for integers n, they are the points (n, G(n)) = (n, F(n)). We find G(23) = F(23) = 987 and F(24) = G(24) = 988. Since G(24) is larger, a maximum profit of 988 rupees results from the production of 24 sets.

It is necessary only to consider F(23) and F(24), since the function G(x), hence the function F(n), is increasing for x < 23 so that no value of n < 23 can possibly exceed F(23); a similar remark holds for n > 24. Finally, since $n = 60 - \frac{1}{2}p$, we see that, for n = 24, the price that must be charged to yield a maxmum profit is 72 rupees.

Example 21: A bus company will charter a bus that holds 50 people to groups of 35 or more. If a group containes exactly 35 people, each person pays 60 rupees. In larger groups, everybody's fare is reduced by 1 rupee for each person in excess of 35. Determine the size of the group for which the bus company's revenue will be the greatest.

Solution: We wish to maximize the revenue :

Revenue = (Number of People in the Group) (Fare Per Person)

Let x be the number of people in excess of 35 who take the trip. Then,

Number of people in the group = 35 + x

Fare per Person = 60 – x

Let R(x) be the revenue for the bus company :

$$R(x) = (35 + x)(60 - x) = 2,100 + 25x - x^2$$

Next, find the domain : We note that there must be at least 25 people (x = 0) and at most 50 people (x = 15); thus $0 \le x \le 15$, but because x represents the number of people, it must be an integer.

The critical numbers are found by solving

$$R'(x) = 25 - 2x = 0.$$

Because the derivative exists throughout the interval, the only critical number is x = 12.5. But x must be an integer, so x = 12.5 is not in the domain. To find the optimal integer solution. observe that R is increasing on (0, 12.5) and decreasing on (12.5, 15)

It follows that the optimal integer value of x is either x = 12 or x = 13. Because

$$R(12) = 2,256 \text{ and } R(13) = 2,256$$

we conclude that the bus company's revenue will be greatest when the group contains either 12 or 13 people in excess of 35 — that is, for groups of 47 or 48. In either case, the revenue will be 2,256 rupees.

Concept Problems **H**

1. Find the least value of the following functions :
   (i) $y = 2\log_{10}x - \log_x 0.01$ for $x > 1$
   (ii) $y = 2\cos x + \sec^2 x$ in $\left[0, \frac{\pi}{2}\right)$
   (iii) $y = \frac{x^3 + x + 2}{x}$, $x > 0$
   (iv) $y = x^2 + \frac{1}{x^2 + 1}$.

2. Find the greatest value of $y = \dfrac{x}{ax^2 + b}$, $a > 0, b > 0$ for $x > 0$.

3. If $a + b + c = 4$, $a, b, c, \geq 0$ then find the maximum and minimum value of $ab + bc + ca$.

4. Find the greatest and the least values of

$$y = \sin^2\left(\frac{\pi}{3} + x\right) + \sin^2\left(\frac{\pi}{3} - x\right)$$

5. What if any, is the maximum or minimum value of y, where $y = x^2 + \dfrac{250}{x}$, $x > 0$?

6. If $x + y = 4$, find the maximum value of x^2y^3, $x > 0$, $y > 0$.

Practice Problems G

7. Under the condition that $3x + y = 80$, maximize the product xy^3 when $x \geq 0$ and $y \geq 0$.

8. (i) Show that $\dfrac{1}{2}$ is the number that exceeds its own square by the greatest amount.

(ii) Which nonnegative number exceeds its own cube by the greatest amount ?

(iii) Which nonnegative number exceeds its nth power $(n > 0)$ by the greatest amount ?

9. If $f(x) = 2x^2 + \dfrac{2}{x^2}$, $0 < |x| \leq 2$

$= 1, x = 0$

then find the least value of $f(x)$.

10. Show that if a, b, c and d are positive integers, then

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16$$

11. Prove that the function $y = \dfrac{x^2 + x + 1}{x^2 + 1}$ cannot have values greater than 3/2 and values smaller than 1/2 for any real values of x.

12. Determine the relationship among the numbers a, b and c in order that the function f defined by $f(x) = (x - a)(x - b)/(x - c)$ has $(-\infty, \infty)$ as its range.

13. Find the greatest and least values of the function

$$y = \frac{a^2}{x} + \frac{b^2}{1-x} \quad (0 < x < 1)$$

14. Show that $PA^2 \sin A + PB^2 \sin B + PC^2 \sin C$ has its minimum value when P is at the incentre.

15. Find the maximum value of xyz, having given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

16. Find the largest term in the sequence

$$t_n = \frac{n^2}{n^3 + 200}$$

17. Find the highest and lowest points on the graph of $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(i) Without using calculus.

(ii) Using calculus.

18. Show that (2, 2) is the point on the graph of the equation $y = x^3 - 3x$ that is nearest the point (11, 1).

19. Examine $2x^2 - 4xy + 3y^2 - 8x + 8y - 1 = 0$ for maximum and minimum values of y.

7.14 Optimisation Problems

Working Rule

1. When possible, draw a figure to illustrate the problem and label those parts that are important in the problem. Constants and variables should be clearly distinguished.
2. Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations. Also find the domain.
3. If $y = f(x)$ is the quantity to be maximized or minimized, find those values of x for which $f'(x) = 0$ or $f'(x)$ does not exist.
4. Test each value of x to determine whether it provides a local maximum or minimum or neither. The usual tests are the First Derivative Test or Second Derivative Test.
5. If the function $y = f(x)$ is defined for only a limited range of values $a \leq x \leq b$ then also examine $x = a$ and $x = b$ for possible extreme values.
6. If $y = f(x)$ is to be maximized (minimized), then choose the greatest (least) value out of the local maxima (minima) found above.

We can also apply the Fork Extremum Theorem discused earlier in case the function has a single critical point.

Example 1: Represent the number 26 as the sum of three positive terms the sum of whose squares is the least, if it is known that the second term is thrice as large as the first.

Solution: We designate the unknown terms as x, y, z. By the hypothesis, the terms we have introduced satisfy the following system of equations :

$$x + y + z = 26, \qquad \text{...(1)}$$

$$y = 3x.$$

Using (1), we express the unknowns y and z in terms of x : $y = 3x,\ z = 26 - 4x$. ...(2)

Let us now form a function whose minimum is to be found:

$S(x) = x^2 + 9x^2 + (26 - 4x)^2$.

In this case, domain of the argument is determined from the condition that all the terms be positive. Solving the system of inequalities $x > 0,\ 26 - 4x > 0$,

we find that the required interval is $\left(0, \frac{13}{2}\right)$. We have thus reduced the problem to seeking the minimum of the function S(x) on the interval $\left(0, \frac{13}{2}\right)$.

The only critical point of the function S(x) on the interval $\left(0, \frac{13}{2}\right)$ is the point x = 4.

When passing through that point, the derivative of the function S(x) changes sign from minus to plus, and consequently, S(x) decreases on the interval (0, 4) and increases on the interval $\left(4, \frac{13}{2}\right)$.

Thus, for x = 4 the function S(x) attains its minimum. Substituting x = 4 into equation (2), we get the values of the other unknowns.

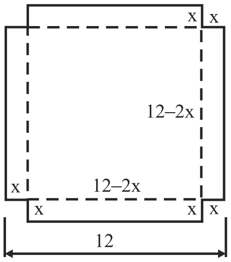
Hence, we should represent 26 = 4 + 12 + 10.

Note: If in the interval at hand the first derivative has only one root, then obvious considerations frequently permit dispensing with a formal investigation by means of the second derivative.

Example 2: If we cut four congruent squares out of the corner of a square piece of cardboard 12 inches on each side, we can fold up the four remaining flaps to obtain a tray without a top. What size squares should be cut in order to maximize the volume of the tray?

Solution: Let us remove squares of side x, as shown in the figure. Folding on the dotted lines, we obtain a tray of volume

$$V(x) = (12 - 2x)^2\,(x) = 4x^3 - 48x^2 + 144x.$$



Since each side of the carboard square has length 12 inches, the only values of x which make sense are those in the closed interval [0, 6]. Thus we wish to find the number x in [0, 6] that maximize V(x).

Notice that $V(x) = (12 - 2x)^2(x)$ is small when x is near 0 (that is, what we try to economize by making the height of the tray small) and small when x is near 6 (that is, when we try to economize by making the base small). We have a "two-influence" problem; to find the best balance between them, we use calculus.

The maximum value of V(x) for x in [0,6] occurs either at 0, 6 or a critical point [where V'(x) = 0]. Now, V(0) = 0 (the tray has height 0), and V(6) = 0 (the tray has a base of area 0). These are minimum values for the volume, certainly not the maximum volume, so the maximum must occur at some critical point in (0, 6).

Now $\quad V'(x) = 12x^2 - 96x + 144$

$\qquad\quad = 12(x - 6)\,(x - 2).$

The equation 12(x – 6) (x – 2) = 0 has two roots, namely, 2 and 6. The critical points are 2 and 6. As already remarked, the maximum does not occur at 0 or 6. Hence it occurs at x = 2. When x = 2, the volume is

$$V(2) = (12 - 2 \cdot 2)^2(2) = x^2 \cdot 2$$

$$= 128 \text{ cubic inches.}$$

This is the largest possible volume and is obtained when the length of the cut is 2 inches.

Thus, the maximum value of V(x) is the maximum of the three numbers V(0), V(2), and V(6). A quick computation shows that the maximum must occur at x = 2.

In the next example implicit differentiation is used to answer the question on optimization. Not only will the algebra be simpler than before, but the answer will provide more information, since also the general shape–the proportion between height and radius–is revealed.

An Endpoint Maximum

Example 3: Four feet of wire is to be used to form a square and a circle. How must of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area ?

Solution: Let x be the side of the square and r be the radius of the circle.

The total area is given by $A = x^2 + \pi r^2$.
Because the total amount of wire is 4 feet,

$$4 = 4x + 2\pi r.$$

Thus, $r = 2(1 - x)/\pi$, and by substituting in the primary equation we have

$$A = x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2$$

$$= x^2 + \frac{4(1-x)^2}{\pi}$$

$$= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4].$$

The feasible domain is $0 \le x \le 1$.

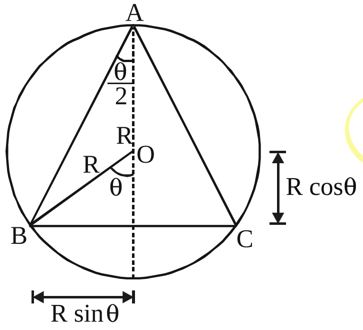
Since $dA/dx = [2(\pi + 4)x - 8]/\pi$, the only critical point is $x = 4/(\pi + 4) \approx 0.56$. This gives the minimum value of A.

Now $A(0) = 4/\pi$ and $A(1) = 1$. Thus, the maximum value of A is 1.

We can conclude that the maximum area occurs when x = 0. That is, all the wire is used for the circle.

Example 4: What is the radius of the smallest circular disk large enough to cover every isosceles triangle of a given perimeter L ?

Solution:



$$AB = (R + R\cos\theta)\sec\frac{\theta}{2}$$

Hence

$$L = 2AB + BC = 2R\left[(1 + \cos\theta)\sec\frac{\theta}{2} + \sin\theta\right]$$

$$= 2R\left[2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$L = 4R\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$$

$$R = \frac{L}{4\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)}$$

Let $f(\theta) = \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$.

For R to be minimum, f(θ) should be maximum.

$$\Rightarrow f'(\theta) = -\frac{1}{2}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right) + \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$f'(\theta) = \frac{1}{2}\cos\theta - \frac{1}{2}\sin\frac{\theta}{2} = 0 \Rightarrow \theta = \frac{\pi}{3}$$

But $0 < \theta \le \frac{\pi}{2}$

So, $f(0) = 1$ maximum

$$f\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{4} = \frac{2 + \sqrt{3}}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0.207$$

If $\theta = 0$, $R = \frac{L}{4}$ is the smallest required radius

Though θ = 0 is not possible, practically $R = \frac{L}{4}$ is the smallest area.

How to Use Implicit Differentiation in an Extremum Problem

1. Name the various quantities in the problem by letters, such as x, y, A, V.
2. Identify which quantity is to be maximized (or minimized).
3. Express the quantity to be maximized (or minimized) in terms of other quantities, such as x and y.
4. Obtain an equation relating x and y. (This equation is called a constraint.)
5. Differentiate implicitly both the constraint and the expression to be maximized (or minimized), interpreting all the various quantities to be function of x (or, perhaps, of y).
6. Set the derivative of the expression to be maximized (or minimized) equal to 0 and combine with the derivative of the constraint to obtain an equation relating x and y at a maximum (or minimum).
7. Step 6 gives only a relation or proportion between x and y at an extremum. If the explicit values of x and y are desired, find them by using the fact that x and y also satisfy the constraint.

Example 5: Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?

Solution: The height h and radius r of any can of volume 100 cubic inches are related by the equation.

$$\pi r^2 h = 100 \qquad ...(1)$$

The surface area S of the can is

$$S = 2\pi r^2 + 2\pi rh \qquad ...(2)$$

Consider h, and hence S, as functions of r. However, it is not necessary to find these functions explicitly.

Differentiation of equation (1) and (2) with respect to r yields

$$\pi\left(r^2\frac{dh}{dr}+2rh\right)=\frac{d(100)}{dr}=0 \qquad ...(3)$$

and $\dfrac{dS}{dr}=4\pi r+2\pi\left(r\dfrac{dh}{dr}+h\right).$ ...(4)

Since when S is a minimum, dS/dr = 0, we have

$$0 = 4\pi r + 2\pi\left(r\frac{dh}{dr}+h\right) \qquad ...(5)$$

Equation (3) and (5) yield, with a little algebra, a relation between h and r, as follows :

Factoring πr out of equation (3) and 2π out of equation (5) show that

$$r\frac{dh}{dr} + 2h = 0 \text{ and } 2r + r\frac{dh}{dr} + h = 0 \qquad ...(6)$$

Elimination of dh/dr from equation (6) yields

$$2r + r\left(\frac{-2h}{r}\right) + h = 0.$$

which simplifies to $2r = h$ ...(7)

Equation (7) asserts that the height of the most economical can is the same as its diameter. Moreover, this is the ideal shape, no matter what the prescribed volume happens to be.

The specific dimensions of the most economical can are found by combining the equations (7) and (1)

$2r = h$ and $\pi r^2h = 100$.

Elimination of h from these two equations shows that

$$\pi r^2(2r) = 100 \text{ or } r^3 = \frac{50}{\pi}.$$

Hence, $r = \sqrt[3]{\dfrac{50}{\pi}}$ and $h = 2r = 2\sqrt[3]{\dfrac{50}{\pi}}$

Thus, implicit differentiation finds the proportions of a general solution before finding the exact values of the variables. Often it is the proportion, rather than the (perhaps complicated) explicit values, that gives more insight into the answer. For instance, equation (7) tells that the diameter equals the height for the most economical can.

The procedure illustrated in this example is quite general. It may be of use when maximizing (or minimizing) a quantity that at first is expressed as a function of two variables which are linked by an equation. The equation that links them is called the constraint. In this example, the constraint is $\pi r^2h = 100$.

Example 6: Show that the semi-vertical angle of the right circular cone of given total surface (including area of the base) and maximum value is $\sin^{-1} 1/3$.

Solution: Let x be the radius of the base, h be the height and y be the slant height of the cone.

Then, the total surface of the cone = constant

$\Rightarrow \quad \pi x^2 + \pi xy = \text{constant}.$ ...(1)

Now, the volume of the cone

$V = 1/3\pi x^2h = 1/3\pi x^2 (y^2 - x^2)^{1/2}$,

since $h = \sqrt{(y^2 - x^2)}$.

Therefore, $V^2 = 1/9\pi^2x^4 (y^2 - x^2)$.

Now, V is maximum or minimum according as V^2 or $9V^2/\pi^2$ is maximum or minimum.

Let $z = 9V^2/\pi^2 = x^4 (y^2 - x^2)$.

We have $dz/dx = 4x^3 (y^2 - x^2) + x^4 \{2y(dy/dx) - 2x\}$. ..(2)

y is connected with x by (1).

Differentiating (1) w.r.t. x, we get

$$\pi\left(2x + y + x\frac{dy}{dx}\right) = 0 \text{ or } \frac{dy}{dx} = -\frac{2x+y}{x}$$

Substituting this value of dy/dx in (2), we get

$$dz/dx = 4x^3y^2 - 4x + x^4\left[-2y\frac{(2x+y)}{x} - 2x\right]$$

$= 2x^3y^2 - 6x^5 - 4x^4y.$

For a maximum or a minimum value of z, dz/dx = 0

$\Rightarrow \quad 2x^3y^2 - 6x^5 - 4x^4y = 0$

$\Rightarrow \quad 2x^3 (y^2 - 2xy - 3x^2) = 0$

$\Rightarrow \quad 2x^3 (y - 3x) (y + x) = 0$

$\Rightarrow \quad y = 3x$, since $x \neq 0$ and $y \neq -x$.

Now dz/dx is positive when y is slightly less than 3x and is negative when y is slightly greater than 3x. Therefore, z is maximum when y = 3x.

Hence, V is maximum when y = 3x

or x/y = 1/3

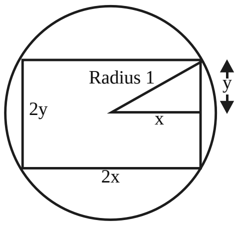
or $\sin \alpha = 1/3$ where α is the semi-vertical angle of the cone.

Geometry Problems

Example 7: Suppose that you need to cut a beam with maximal rectangular cross section from a circular log of radius 1 ft. What are the shape and cross-sectional area of such a beam ?

Solution: Let x and y denote half the base and half the height, respectively, of inscribed rectangle $x^2 + y^2 = 1$,

so $y = \sqrt{1 - x^2}$.



The area of the inscribed rectangle is A = (2x) (2y) = 4xy. We may now express A as a function of x alone:

$A(x) = 4x\sqrt{1 - x^2}$.

The practical domain of definition of A is (0, 1).

$$\frac{dA}{dx} = 4\,.\,(1-x^2)^{1/2} + 2x(1-x^2)^{-1/2} = \frac{4-8x^2}{(1-x^2)^{1/2}}$$

We observe that A'(1) does not exist, but this causes no trouble, because x = 1 is an endpoint.

A'(x) = 0 when 4 – 8x² = 0. Thus, the only critical point of A in the open interval (0, 1) is

$$x = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2} \text{ (and } 2x = 2y = \sqrt{2}\text{).}$$

We evaluate A here and at the two endpoints to find that

$A(0^+) = 0 = A(1^-)$,

$A\left(\frac{1}{2}\sqrt{2}\right) = 2$. This is the absolute maximum

Therefore, the beam with rectangular cross section of maximal area is square, with edges $\sqrt{2}$ ft long and with cross-sectional area 2 ft².

Example 8: Rectangles are inscribed inside a semi-circle of radius r. Find the rectangle with maximum area.

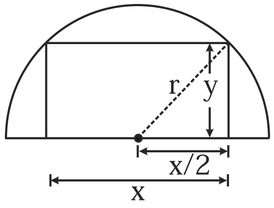
Solution: Let sides of rectangle be x and y.

$\Rightarrow$ A = xy

Here x and y are not independent variables and are related by pythogoreas theorem with r.

$$\frac{x^2}{4} + y^2 = r^2 \qquad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$$

$$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$$



$$\Rightarrow \quad A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$$

Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $\qquad x \in (0, r)$

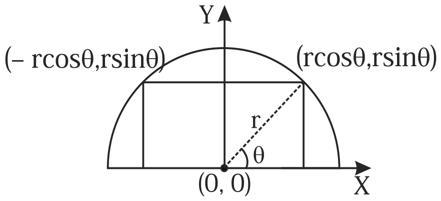
A(x) is maximum when f(x) is maximum

Hence, $f'(x) = x(2r^2 - x^2) = 0 \quad \Rightarrow x = r\sqrt{2}$

Also $\quad f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming that f(x) is maximum when $x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Alternative: Let use choose coordinate system with origin as centre of circle A = xy



$\Rightarrow \quad A = 2\,(r\cos\theta)\,(r\sin\theta)$

$\Rightarrow \quad A = r^2 \sin 2\theta \qquad \theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4}$

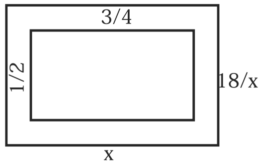
$\Rightarrow \quad x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example 9: A sheet of paper of a poster is to be 18 ft² in area. The margins at the top and bottom are to be 9 inches, and the margins at the sides 6 inches. What should be the dimensions of the sheet to maximize the printed area?

Solution: Let x be one dimension, measured in feet. Then 18/x is the other dimension. (See figure). The only restriction of x is that x > 0.

The printed area in square feet is $A = (x-1)\left(\frac{18}{x} - \frac{3}{2}\right)$,

and $\frac{dA}{dx} = \frac{18}{x^2} - \frac{3}{2}$.



Solving dA/dx = 0 yields the critical point $x = 2\sqrt{3}$. Since $d^2A/dx^2 = -36/x^3$ is negative when $x = 2\sqrt{3}$, the second derivative test tells us that A has a relative maximum at $x = 2\sqrt{3}$.

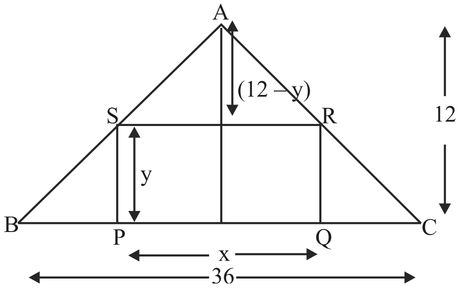
Since $2\sqrt{3}$ is the only critical number in the interval (0, ∞), the Fork extremum theorem tells us that A has an absolute maximum at $x = 2\sqrt{3}$. Thus, one side is $2\sqrt{3}$ ft and the other side is $18/(2\sqrt{3}) = 3\sqrt{3}$ ft.

Example 10: Find the greatest area of the rectangular plot which can be laid out within a triangle of base 36 ft. and altitude 12 ft. (Assume that one side of the rectangle lies on the base of the triangle)

Solution: Area of rectangle = A = xy ...(1)

Also $\quad \frac{36}{x} = \frac{12}{12-y} \quad \Rightarrow \quad 3y = (36-x)$ ...(2)

∴ $A = \frac{x}{3}(36 - x) = \frac{1}{3}(36x - x^2)$



Now $A'(x) = 0 \Rightarrow 36 - 2x = 0 \Rightarrow x = 18$

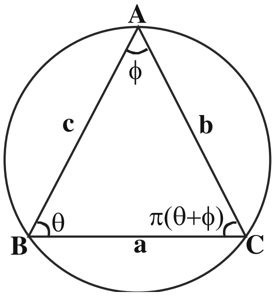
$$A''(x) = \frac{1}{3}(-2) < 0$$

Also $y = \frac{36 - x}{3} = \frac{36 - 18}{3} = 6$

∴ $A_{max} = 18 \times 6 = 108$ sq. feet

Example 11: Through a point A on the circumference of a circle of radius r, two straight lines are drawn enclosing an angle ϕ. If the straight lines meet the circle again at B & C, find the maximum area of triangle ABC.

Solution:



$$\frac{a}{\sin\phi} = \frac{b}{\sin\theta} = \frac{c}{\sin(\theta + \phi)}$$

$$A = \frac{1}{2} bc \sin\phi$$

$$A = 2r^2 \sin\phi \sin\theta \sin(\theta + \phi)$$

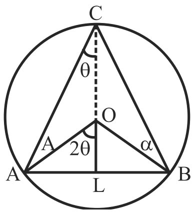
$$= r^2 \sin\phi\, [\cos\phi - \cos(2\theta + \phi)]$$

$$\frac{dA}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{2} - \frac{\phi}{2}$$

Hence, the maximum area of the triangle is $r^2 \sin\phi\,(1 + \cos\phi)$ sq. units.

Example 12: Show that the triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.

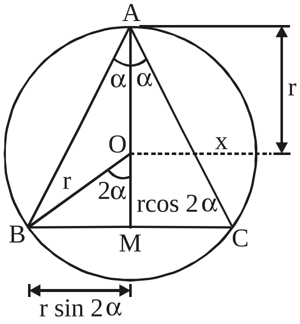
Solution: First, we observe that among all the inscribed triangles having AB as base, the area of that triangle is greatest for which the altitude of C w.r.t. AB is greatest. Evidently such a triangle is an isosceles triangle.
Let θ be the semivertical angle of such a triangle ABC inscribed in a given circle of radius a.



S = area of the triangle ABC
$= 2\,\Delta AOC + \Delta AOB$
$= 2 \cdot 1/2a^2 \sin(\pi - 2\theta) + 1/2a^2 \sin 4\theta$
$= a^2 \sin 2\theta + 1/2a^2 \sin 4\theta$
$\frac{dS}{d\theta} = 2a^2 \cos 2\theta + 2a^2 \cos 4\theta.$
$dS/d\theta = 0$
$\Rightarrow \cos 2\theta + \cos 4\theta = 0,$
$\Rightarrow 2\cos 3\theta \cos\theta = 0,$
$\Rightarrow 3\theta = \pi/2$ or $\theta = \pi/2,$
$\Rightarrow \theta = \pi/6$ or $\theta = \pi/2.$
But $\theta = \pi/2$ is not admissible. Hence $\theta = \pi/6$.
Also, $d^2S/d\theta^2 = -4a^2 \sin 2\theta - 8a^2 \sin 4\theta < 0$ for $\theta = \pi/6$.
Using Fork extremum theorem, the area is maximum when $\theta = \pi/6$ or $2\theta = \pi/3$. i.e., when the triangle is equilateral.

Example 13: An isosceles triangle is inscribed in a circle of radius r. If the angle 2α at the apex is restricted to lie between 0 and $\frac{\pi}{2}$, find the largest and the smallest value of the perimeter of the triangle.

Solution:



$$P = 2x + 2r\sin 2\alpha,\ 0 < 2\alpha < \frac{\pi}{2} \Rightarrow 0 < \alpha < \frac{\pi}{4}$$

$$= \frac{2r(1 + \cos 2\alpha)}{\cos\alpha} + 2r\sin 2\alpha$$

$$= 2r\left[\frac{1 + \cos 2\alpha + \sin 2\alpha \cos\alpha}{\cos\alpha}\right]$$

$P(\alpha) = 4\,r\,(1 + \sin\alpha)\cos\alpha$

$P'(\alpha) = 4\,r\,[\cos^2\alpha - (1 + \sin\alpha)\cos\alpha]$

$= 4\,r\,(1 - 2\sin\alpha)(1 + \sin\alpha)$

$P'(\alpha) = 0$

$\Rightarrow \quad \sin\alpha = \dfrac{1}{2}$ or $\sin\alpha = -1$ (not possible)

When through $\alpha = \dfrac{\pi}{6}$, $P'(\alpha)$ changes sign from plus to minus.

$$\text{Hence } P_{\max} = 4\,r\left(\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}\ r$$

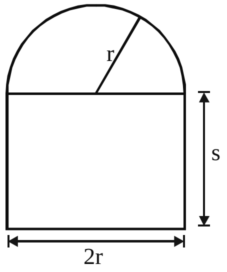
$$P(0^+) = 4\,r \text{ and } P\left(\frac{\pi}{4}\right)$$

$$= 4\,r\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right).\frac{1}{\sqrt{2}} = 2\,r\,(\sqrt{2}+1) > 4r.$$

When $\alpha \to 0$, the triangle becomes a diameter. This is not a feasible case. Hence, there is no minimum value of the perimeter.

Example 14: The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's' . If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum .

Solution:



$$A = \frac{\pi r^2}{2} + 2\,rs \qquad \text{...(1)}$$

$$P = 2s + 2r + \pi r$$

$$P = \left(\frac{\pi r}{2} + 2s\right) + 2r + \frac{\pi r}{2}$$

$$P = \frac{A}{r} + 2r + \frac{\pi r}{2}$$

$$\text{Now } \frac{dP}{dr} = 0 \Rightarrow r = \sqrt{\frac{2A}{\pi + 4}}$$

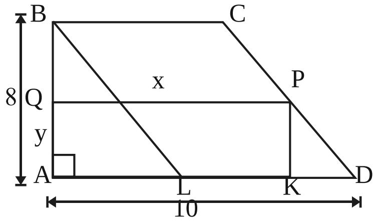
Since this is the only critical point, the minimum value of P occurs for this value of r. When this is the case, the value of s is found using (1).

$$\therefore \quad s = \sqrt{\frac{2A}{\pi + 4}}$$

Example 15: Inscribe a rectangle of the greatest area into the trapezoid ABCD, one of whose nonparallel sides AB (8 cm long) is perpendicular to the base, so that one of its sides lies on the larger base of the trapezoid. The base of the trapezium are 6 and 10 cm in length respectively. Calculate the area of the rectangle.

Solution: Let us consider two cases.

Case 1 The vertex P of the rectangle lies on one of the nonparallel sides CD of the trapezium



In the first case, let the sides of the rectangle PQ = x and AQ = y. We set up an equation relating the unknowns x and y. For that purpose we draw a line segment BL parallel to the side CD and consider two right triangles ABL and KPD which are similar.

Now, AB = 8, AL = 10 – 6 = 4, KD = 10 – x, PK = y.

$$\frac{PK}{KD} = \frac{AB}{AL} \Rightarrow \frac{y}{10 - x} = \frac{8}{4} = 2 \Rightarrow y = 20 - 2x.$$

The area of the rectangle AKPQ is

$S(x) = x\,(20 - 2x)$

In the first case the domain can be found from the condition that the point Q is the projection of the point P lying on the side CD and, consequently, $x \geq 6$.

Thus the problem has reduced to seeking the least value of the function S(x) on the interval [6, 10].

The only critical point of the function S(x), x = 5, does not belong to the interval obtained. Consequently, the derivative of the function S(x) does not change sign on this interval. Calculating the derivative of S(x) at an arbitary point of the interval [6, 10], we ascertain that it is negative.

Thus, the greatest value of S(x) is attained at the left end of the interval, i.e.,

$$S(x)_{\max} = S(6) = 48\text{cm}^2.$$

Case 2 The vertex P lies on the base BC of the trapezium. In this case, the area of the rectangle does not exceed 48 cm^2 since with same nonparallel side equal to 8 cm, the length of the base cannot be greater than 6 cm.

Hence the greatest area of the rectangle is 48 cm^2.

Solid Geometry Problems

Useful Formulae of Measuration to Remember :

1. Volume of a cuboid = ℓbh.

2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.

3. Volume of cube = a^3

4. Surface area of cube = $6a^2$

5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.

6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)

7. Curved surface of a cylinder = $2\pi rh$.

8. Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

9. Volume of a sphere = $\frac{4}{3}\pi r^3$.

10. Surface area of a sphere = $4\pi r^2$.

11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

12. Volume of a prism = (area of the base) × (height).

13. Lateral surface of a prism = (perimeter of the base) × (height).

14. Total surface of a prism = (lateral surface) + 2 (area of the base)

(Note that lateral surfaces of a prism are all rectangle).

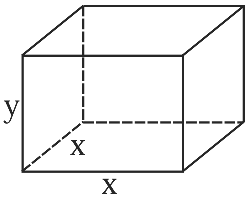
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

16. Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).

Example 16: A sheet of area 40 m^2 in used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Solution: Let length of base be xm and height be y m.

$V = x^2 y$



Also, x and y are related to surface area of this tank which is equal to 40 m^2.

$\Rightarrow \quad x^2 + 4xy = 40$

$\therefore \quad y = \frac{40 - x^2}{4x}, \ x \in (0, \sqrt{40})$

$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{(40x - x^3)}{4}$

For maximizing volume,

$V'(x) = \frac{(40 - 3x^2)}{4} = 0 \Rightarrow \quad x = \sqrt{\frac{40}{3}}$ m

and $V''(x) = -\frac{3x}{2} \Rightarrow V''\left(\frac{\sqrt{40}}{3}\right) < 0$.

Confirming that volume is maximum at $x = \frac{\sqrt{40}}{3}$ m.

Example 17: A box with a square base is constructed so that the length of one side of the base plus the height is 10 in. What is the largest possible volume of such a box ?

Solution: Let b be the length of one side of the base and h be the height of the box.

The volume $V = b^2h$.

However, we know that $b + h = 10$; therefore, $h = 10 - b$, and we can now write V as a function of b alone

$V(b) = b^2(10 - b)$

The domain is not stated, but we must have $b > 0$ and $10 - b = h > 0$, so that $0 < b < 10$. First, we find the critical points.

$V'(b) = 20b - 3b^2$.

$V'(b) = 0 \Rightarrow \ 20b - 3b^2 = 0 \Rightarrow b = 0, \frac{20}{3}$

(note that b = 0 is an endpoint)

Checking the endpoints and the critical points, we have

$V(0^+) = 0$

$V(10^-) = 10^2(10 - 10) = 0$

$$V\left(\frac{20}{3}\right) = \left(\frac{20}{3}\right)^2\left(10 - \frac{20}{3}\right) = \frac{4000}{27}$$

Thus, the largest volume V is $\frac{4000}{27} \approx 148.1$ in.^{3}.

It occurs when the square base has a side of length $\frac{20}{3}$ in. and the height is $h = 10 - \frac{20}{3} = \frac{10}{3}$ in.

Example 18: Find the height of a cone of the greatest volume if its slant height is equal to ℓ.

Solution: The volume of the cone of area of whose base radius is R and the height is H, is

$V = \frac{1}{3}\pi R^2 H$.

By Pythagoras theorem, $R^2 + H^2 = \lambda^2$.

Using this equality, we express V as a function of only one variable H:

$$V = \frac{1}{3}\pi(\lambda^2 - H^2)H.$$

Solving $V'(H) = \frac{\pi}{3}(\lambda^2 - 3H^2) = 0$,

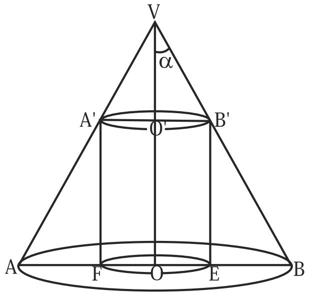
we find two critical points of the function: $H_1 = +\frac{\lambda}{\sqrt{3}}$ and $H_2 = -\frac{\lambda}{\sqrt{3}}$, from which only the point H_1 belongs to the interval $(0, \lambda)$.

When passing through the point H_1, the function $V'(H) = \frac{1}{3}(\lambda^2 - 3H^2)$ changes sign from plus to minus and, consequently, on the interval $(0, \lambda/\sqrt{3})$ the function V (H) increases and on the interval $(\lambda/\sqrt{3}, \lambda)$ it decreases.

Thus, $H = \lambda/\sqrt{3}$ is the height of the cone of the maximum volume for the given length of the slant height λ.

Example 19: Find the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α.

Solution: Let x be the radius of the cylinder A'B'EF.



In $\Delta VO'B'$.

$VO' = x\cot\alpha$

$\therefore$ Height of cylinder $= OO' = h - VO' = (h - x\cot\alpha)$

$\therefore$ Volume of cylinder $V = \pi.x^2.OO'$

$= \pi x^2(h - x\cot\alpha)$

$= \pi(x^2h - x^3\cot\alpha)$ ...(1)

$$\therefore \frac{dV}{dx} = \pi(2xh - 3x^2\cot\alpha)$$

$$\frac{dV}{dx} = 0 \Rightarrow x = \frac{2}{3}h\tan\alpha$$

Since $\frac{d^2V}{dx^2} = \pi(2h - 6x\cot\alpha) = -2\pi h < 0$ at the critical point, V has a local maximum.

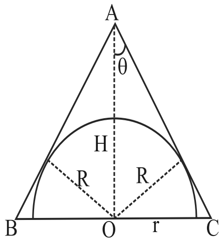
Since $x = \frac{2}{3}h\tan\alpha$ is the only critical point, V has an absolute maximum at this point.

From (1), the maximum volume of cylinder

$$= \pi.\frac{4}{9}h^2\tan^2\alpha\left(h - \frac{2h}{3}\right) = \left(\frac{4}{27}\right)\pi h^3\tan^2\alpha.$$

Example 20: Find the altitude of a cone of the least volume that can be drawn around a hemisphere of radius R (the centre of the base of the cone falls on the centre of the sphere).

Solution:



Let AO = H, BO = r = radius of the base of the cone

R = radius of the hemisphere, $\angle OAC = \theta$ ($\theta \in (0, \pi/2)$)

$$\frac{r}{H} = \tan\theta \text{ and } \frac{R}{H} = \sin\theta$$

$$\Rightarrow r = \frac{R}{\sin\theta}.\tan\theta = \frac{R}{\cos\theta}$$

$$V(\theta) = \frac{1}{3}\pi\left(\frac{R}{\cos\theta}\right)^2\left(\frac{R}{\sin\theta}\right) = \frac{1}{3}\pi R^3.\frac{1}{\cos^2\theta}.\frac{1}{\sin\theta}$$

$$= \frac{1}{3}\pi R^3.\frac{1}{\sin\theta - \sin^3\theta}$$

Now $V'(\theta) = \frac{\pi R^3\cos\theta}{(\sin\theta - \sin^3\theta)^2}\left(\sin\theta + \frac{1}{\sqrt{3}}\right)\left(\sin\theta - \frac{1}{\sqrt{3}}\right)$

Clearly V(θ) has only one critical point namely

$\theta = \sin^{-1}\frac{1}{\sqrt{3}}$. Using sign scheme for $V'(\theta)$

we get, $\theta = \sin^{-1}\frac{1}{\sqrt{3}}$ to be the point of maximum.

Hence, corresponding altitude $H = \frac{R}{\sin\theta} = R\sqrt{3}$.

Example 21: A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution: Let r be the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h we use the fact that the volume is given at 1 L, which we take to be 1000 cm^3. Thus $\pi r^2 h = 1000$ which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore, the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r},\ r > 0$$

To find the critical points, we differentiate :

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $r^3 = 500/\pi$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

We can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$. So A is decreasing for all r to the left of the critical point and increasing for all r to the right. Thus, $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of A(r), which must occur at the critical point.

The value of h correponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely the diameter.

An **alternative method** for solving optimization problem is to use implicit differentiation. We work with the same equations

$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 100$

but instead of eliminating h, we differentiate both equations implicitly with respect to r :

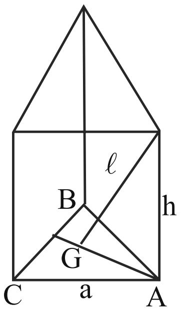
$A' = 4\pi r + 2\pi h + 2\pi h' \qquad 2\pi rh + \pi r^2 h' = 0$

The minimum occurs at a critical point, so we set $A' = 0$, simplify, and arrive at the equations

$2r + h + rh' = 0 \qquad 2h + rh' = 0$

and subtraction gives $2r - h = 0$, or $h = 2r$.

Example 22: In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is l. Find the altitude of the prism for which the volume is greatest.



Solution: $AG = \frac{2}{3} \cdot \frac{a\sqrt{3}}{2} = \frac{a}{\sqrt{3}}$

Now $l^2 = \frac{a^2}{3} + h^2$; $V = \frac{\sqrt{3}}{4} a^2 h$

$$V(h) = 3 \cdot \frac{\sqrt{3}}{4} h(l^2 - h^2),$$

$$V'(h) = 0$$

$$\Rightarrow h = \frac{\lambda}{\sqrt{3}} \Rightarrow V_{max} = \frac{\lambda^3}{2}.$$

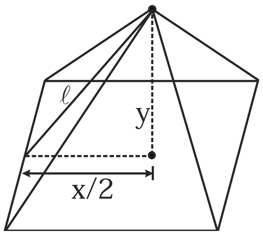
The altitude for maximum volume is $\frac{1}{\sqrt{3}}$.

Example 23: Among all regular square pyramids of volume $36\sqrt{2}$ cm³. Find dimensions of the pyramid having least lateral surface area.

Solution: Let the length of a side of base be x cm and y be the perpendicular height of the pyramid

$$V = \frac{1}{3} \text{ area of base x height}$$

$$\Rightarrow V = \frac{1}{3} x^2 y = 36\sqrt{2}$$



$$\Rightarrow y = \frac{108\sqrt{2}}{x^2}$$

and $S = \frac{1}{2}$ perimeter of base x slant height

$$= \frac{1}{2}(4x).\ \lambda$$

but $\lambda = \sqrt{\frac{x^2}{4} + y^2}$

$$\Rightarrow S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$$

$$\Rightarrow S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$$

$$\Rightarrow S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$$

Let $f(x) = x^4 + \frac{8.(108)^2}{x^2}$ [for minimizing S(x), we minimize f(x)]

$$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0$$

$\Rightarrow\quad f'(x) = 4\,\dfrac{(x^6 - 6^6)}{x^3} = 0$

$\Rightarrow\quad x = 6$, which a point of minima

Hence, $x = 6$ cm and $y = 3\sqrt{2}$.

Example 24: Find the dimension of a cylinder having greatest lateral surface area that can be inscribed in a given sphere.

Solution: Let x be the radius and 2y be the height of the cylinder.

We have $x^2 + y^2 = a^2$ ...(1)

where a denotes the radius of the given sphere.

The lateral surface area of the cylinder is given by

$$S = 2\pi x \cdot 2y = 4\pi x \cdot \sqrt{a^2 - x^2},\ 0 \le x \le a \qquad \text{[using (1)]}$$

Now, $\dfrac{dS}{dx} = 4\pi\left[\sqrt{a^2 - x^2} - \dfrac{x^2}{\sqrt{a^2 - x^2}}\right] = \dfrac{4\pi\,(a^2 - 2x^2)}{\sqrt{a^2 - x^2}}$

which exists everywhere in (0, a) and vanishes at $x = a/\sqrt{2}$ only. To find the greatest and least values, we need to check the values of f(x) at $x = 0, a/\sqrt{2}$, a. We have S(0) = S(a) = 0

and $S\left(\dfrac{a}{\sqrt{2}}\right) = 4\pi\,\dfrac{a}{\sqrt{2}}\sqrt{a^2 - \dfrac{a^2}{2}} = 2\pi a.$

Hence, the required maximum surface area, is $2\pi a$.

Example 25: Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

Solution: Let 'b' be the height of the cone and α be its semiverticle angle.

Let LD = x = radius of the inscribed cylinder and LM = h be its height

$LM = OM - OL = b - x\cot\alpha$

Now, $S = 2\pi rh$ = curved surface area

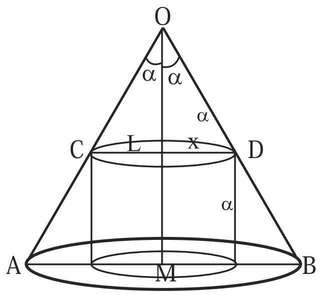
$S = 2\pi x\,(b - x\cot\alpha)$

or $S = 2\pi\,(bx - x^2\cot\alpha)$

$\therefore\quad dS/dx = 2\pi\,(b - 2x\cot\alpha) = 0$

$\Rightarrow\quad x = (b/2)\tan\alpha$

$\Rightarrow\quad x = \dfrac{1}{2}\,(b\tan\alpha) = \dfrac{1}{2}\,(r_1)$



Thus, radius of cylinder = (radius of cone)/2

Example 26: Find the maximum surface area of a closed cylinder of a given volume 8π m^3.

Solution: Let r be the radius and h be the height of the cylinder.

We have $\pi r^2 h = 8\pi$ ...(1)

The surface area of the closed cylinder is given by

$$S = 2\pi rh + \pi r^2$$

$$= 2\pi r\left(\frac{8\pi}{\pi r^2}\right) + \pi r^2 \qquad \text{[using (1)]}$$

$$= \frac{16\pi}{r} + \pi r^2,\ r > 0$$

Now, $\dfrac{dS}{dr} = \dfrac{-16\pi}{r^2} + 2\pi r$

which exists everywhere in (0, ∞) and vanishes at points given by

$2\pi r^3 = 16\pi$ i.e. r = 2.

To find the greatest and least values, we need to check the values of S(r) at r = 0, 2, ∞. We have

$S(0) = S(\infty) = \infty$

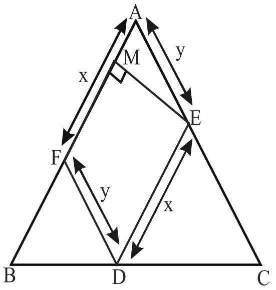
and $S(2) = \dfrac{16\pi}{2} + 4\pi = 12\pi.$

Hence, the required minimum surface area, is 12πm^2.

Co-ordinate Geometry Problems

Example 27: Let $A(p^2, -p)$, $B(q^2, -q)$ and $C(r^2, -r)$ be the vertices of the ΔABC. A parallelogram AFDE is drawn with vertices D, E, F on the line segments BC, CA and AB respectively, Show that maximum area of such a parallelogram is $\dfrac{1}{4}$ (p – q) (q – r) (r – p).

Solution:



Let AF = x = DE and AE = y = DF

Since, Δ's CAB and CED are similar, we have,

$$\frac{CE}{CA} = \frac{DE}{AB} \quad \text{(as shown in figure)}$$

$$\Rightarrow\quad \frac{b - y}{b} = \frac{x}{c} \qquad \text{...(1)}$$

(Here BC = a, AC = b and AB = c)

Now area of parallelogram, S = AF . EM = xy sin A

$$S = x.b\left(1-\frac{x}{c}\right)\sin A \qquad ...(2)$$

Differentiating w.r.t. x we have,

$$\frac{dS}{dx} = \frac{b}{c}(c-2x)\sin A \quad \text{[where sin A is constant]}$$

For extremum, $\frac{dS}{dx} = 0 \quad \Rightarrow \quad x = \frac{c}{2}$

Also $\frac{d^2S}{dx^2} = -\frac{2b}{c} < 0$ at $x = \frac{c}{2}$

Hence, S is maximum when $x = \frac{c}{2}$.

Now, $S_{\text{max.}} = \frac{1}{4}bc\sin A$ [from (2)]

$$= \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$$

$$= \frac{1}{2}(\text{area. } \Delta \text{ ABC})$$

$$= \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & -q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

$\therefore \quad S_{\max} = \frac{1}{4}(p-q)(q-r)(r-p).$

Example 28: The circle $x^2 + y^2 = 1$ cuts the x–axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x–axis and the line segment PQ at S. Find the maximum area of the triangle QSR.

Solution: The centre of the circle

$x^2 + y^2 = 1 \qquad ...(1)$

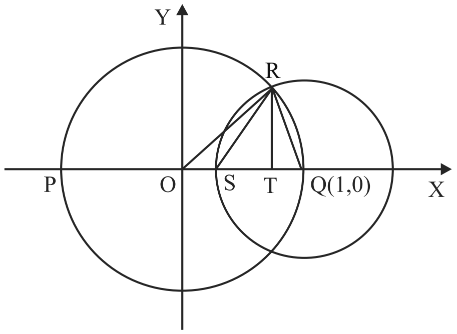
is (0, 0) and radius OP = 1 = OQ.

So, co–ordinates of Q are (1, 0).

Let the radius of the variable circle be r.

Hence, its equation is $(x-1)^2 + (y)^2 = r^2 \qquad ...(2)$

Subtracting (2) from (1) we get, $2x - 1 = 1 - r^2$



$$x = 1 - \frac{r^2}{2} = OT \qquad ...(3)$$

$$\text{Now, RT} = \sqrt{OR^2 - OT^2} = \sqrt{1-\left(1-\frac{r^2}{2}\right)^2} \qquad ...(4)$$

Now, the area of Δ QSR is,

$A = \frac{1}{2}.QS.RT$ i.e. $A^2 = \frac{1}{4}(QS^2).(RT^2)$

$A^2 = \frac{1}{4}r^2\left(r^2 - \frac{r^4}{4}\right)$ [using (2) and (4)]

$A^2 = \frac{1}{16}(4r^4 - r^6)$

Thus, $\frac{d(A^2)}{dr} = \frac{1}{16}(16r^3 - 6r^5) = 0$ (for extremum)

$\Rightarrow \quad r = 2\sqrt{\frac{2}{3}}$

Also, $\frac{d^2(A^2)}{dr^2} = \frac{1}{16}(48r^2 - 30r^4) = -\frac{16}{3} < 0$

when $r = 2\sqrt{\frac{2}{3}}$.

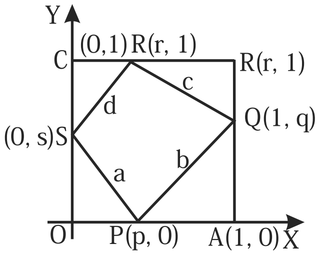
Hence, the area is maximum at $r = 2\sqrt{\frac{2}{3}}$ and $A_{\text{max.}} = \frac{4}{3\sqrt{3}}$ sq. units.

Example 29: Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \le a^2 + b^2 + c^2 + d^2 \le 4$

Solution: Consider a quadrilateral PQRS. Where co-ordinates P, Q, R, S are (p, 0), (1, q), (r, 1) and (0, s) respectively.

$\therefore \quad a^2 = p^2 + s^2, b^2 = (1-p)^2 + q^2$

$c^2 = (1-r)^2 + (1-q)^2, d^2 = r^2 + (1-s)^2$



So, $a^2 + b^2 + c^2 + d^2$

$= p^2 + (1-p)^2 + q^2 + (1-q)^2$

$+ r^2 + (1-r)^2 + s^2 + (1-s)^2$

Now we know that $0 \le p, q, r, s \le 1$

Consider the function $f(x) = x^2 + (1 - x)^2, 0 \le x \le 1$

$$f'(x) = 2x - 2(1 - x)$$

$$f'(x) = 0 \Rightarrow x = \frac{1}{2}$$

$$f(1/2) = 1/2, f(0) = 1, f(1) = 1.$$

$\therefore$ The minimum value of $f(x) = \frac{1}{2}$ and the maximum value of $f(x) = 1$.

$$\Rightarrow \frac{1}{2} \le p^2 + (1 - p)^2 \le 1; \quad \frac{1}{2} \le q^2 + (1 - q)^2 \le 1$$

$$\frac{1}{2} \le r^2 + (1 - r)^2 \le 1; \quad \frac{1}{2} \le s^2 + (1 - s)^2 \le 1$$

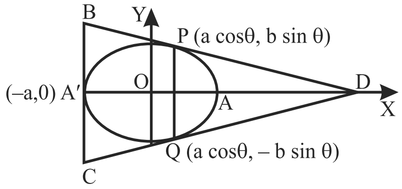
Adding these inequalities, we get

$$2 \le a^2 + b^2 + c^2 + d^2 \le 4.$$

Example 30: An ellipse is inscribed in an isosceles triangle of height h and base 2k, and having one axis lying along the perpendicular from the vertex of the triangle to the base. Show that the maximum area of the ellipse is $\sqrt{3}\,\pi hk / 9$.

Solution: Let BCD be the given isosceles triangle whose base is BC, vertex D and height DA'. Then BC = 2k and DA' = h.

An ellipse is inscribed in this triangle so that one axis of the ellipse is along A'A. Let A'A = 2a and let the length of the other axis of the ellipse be 2b. Let the equation of the ellipse be $x^2a^2 + y^2/b^2 = 1$.



where the x-axis is along OA and the y-axis is perpendicular to OA.

Here O is the middle point of A'A.

The sides BD, CD and BC of the triangle touch the ellipse at the points P, Q and A' respectively.

Here PQ is a double ordinate.

Let P be the point (a cos θ, b sin θ).

Then the equation of the tangent to the ellipse at the point (a cos θ, b sin θ) is

$$(x/a) \cos \theta + (y/b) \sin \theta = 1. \qquad \ldots(1)$$

The coordinates of the point B are (–a,k).

Since the point B lies on (1),

$$\text{therefore} -\frac{a}{a} \cos \theta + \frac{k}{b} \sin \theta = 1$$

$$\Rightarrow \frac{k}{b} \sin \theta = 1 + \cos \theta \Rightarrow b = k \sin \theta/(1 + \cos \theta). \qquad \ldots(2)$$

The coordinates of the point D are (h – a, 0).

[Note that DA' = h and A'O = a]

Since the point D also lies on (1), therefore

$\{(h - a)/a\} \cos \theta = 1 \Rightarrow h \cos \theta - a \cos \theta = a$

$$\Rightarrow a = h \cos \theta/(1 + \cos \theta) \qquad \ldots(3)$$

Let S be the area of the ellipse.

Then $S = \pi ab = \dfrac{\pi hk \sin\theta\cos\theta}{(1+\cos\theta)^2}$, from (2) and (3).

We have dS/dθ

$$= \pi hk \frac{(\cos^2\theta - \sin^2\theta)(1+\cos\theta)^2 + 2\sin^2\theta\cos\theta(1+\cos\theta)}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{(2\cos^2\theta - 1)(1+\cos\theta)^2 + 2(1-\cos^2\theta)\cos\theta(1+\cos\theta)}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{(1+\cos\theta)^2\{(2\cos^2\theta - 1) + 2\cos\theta(1-\cos\theta)\}}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{2\cos\theta - 1}{(1+\cos\theta)^2}, \text{ since } \cos\theta \ne -1.$$

Now, $dS/d\theta = 0 \quad \Rightarrow 2 \cos \theta - 1 = 0$

$\Rightarrow \cos \theta = 1/2 \quad \Rightarrow \theta = \pi/3.$

$$\frac{d^2S}{d\theta^2} = \pi hk\,(2 \cos \theta - 1)\frac{d}{d\theta}\left\{\frac{1}{(1+\cos\theta)^2}\right\}$$

$$+ \frac{\pi hk}{(1+\cos\theta)^2}(-2 \sin \theta),$$

which is negative when $\cos \theta = 1/2$ or $\theta = \frac{1}{3}\pi$.

Hence S is maximum when $\theta = \frac{1}{3}\pi$.

The maximum area of the ellipse

$= \pi hk \sin (\pi/3) \cos (\pi/3)/\{1 + \cos (\pi/3)\}^2$

$= \pi hk\,(\sqrt{3/4})/(1+1/2)^2 = \sqrt{3}\ \pi hk/9.$

Example 31: Prove that the minimum radius vector of the curve $a^2/x^2 + b^2/y^2 = 1$ is of length a + b.

Solution: Let r be the radius vector of the curve. Then changing the equation of the curve to polar coordinates by putting x = r cos θ and y = r sin θ, we get

$$a^2/(r^2 \cos^2\theta) + b^2/(r^2 \sin^2\theta) = 1$$

or $r^2 = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta = z$, say.

Now r is maximum or minimum according as r^2 i.e., z is maximum or minimum. We have

$$dz/d\theta = 2a^2 \sec^2 \theta \tan \theta - 2b^2 \operatorname{cosec}^2\theta \cot \theta.$$

$$dz/d\theta = 0$$

$\Rightarrow 2a^2 \sec^2 \theta \tan \theta - 2b^2 \operatorname{cosec}^2\theta \cot \theta = 0$

$\Rightarrow (\sec^2\theta \tan\theta)/(\operatorname{cosec}^2 \theta \cot \theta) = b^2/a^2.$

$\Rightarrow \tan^4\theta = b^2/a^2$

$\Rightarrow \tan^2\theta = b/a$ because $\tan^2\theta$ cannot be negative.

Now $d^2z/d\theta^2 = 4a^2\sec^2\theta\tan^2\theta + 2a^2\sec^4\theta + 4b^2\operatorname{cosec}^2\theta\cot^2\theta + 2b^2\operatorname{cosec}^4\theta$ = positive when $\tan^2\theta = b/a$. Therefore, z or r is minimum when $\tan^2\theta = b/a$.

Now $r = \sqrt{(a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta)}$

$$= \sqrt{\{a^2(1+\tan^2\theta) + b^2(1+\cot^2\theta)\}}$$

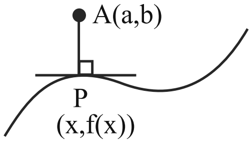
Therefore the minimum value of

$$r = \sqrt{\{a^2(1+b/a) + b^2(1+a/b)\}}$$

$$= \sqrt{(a^2+2ab+b^2)} = \sqrt{(a+b)^2} = a+b\cdot$$

Shortest Distance of a Point from a Curve

Given a fixed point A (a, b) and a moving point P(x, f(x)) on the curve y = f (x), the distance AP will be maximum or minimum if it is normal to the curve at P.



Proof : $F(x) = (x-a)^2 + (f(x)-b))^2$

$F'(x) = 2(x-a) + 2(f(x)-b)\cdot f'(x) = 0$

$\Rightarrow\ f'(x) = -\left(\dfrac{x-a}{f(x)-b}\right)$. Also $m_{AP} = \dfrac{f(x)-b}{x-a}$.

Hence, $f'(x)\cdot m_{AP} = -1$.

Example 32: A helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier placed at (3, 7) wants to shoot down the helicopter when it is nearest to him. Find the shortest distance.

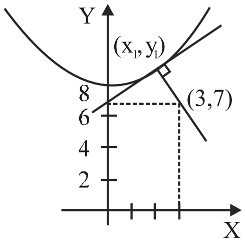
Solution: $\left.\dfrac{dy}{dx}\right|_{(x_1,y_1)} = 2x_1$

For finding the shortest distance,

$$(2x_1)\left(\frac{7-y_1}{3-x_1}\right) = -1$$

$2x_1(7-y_1) = x_1 - 3$

$14x_1 - 2x_1y_1 = x_1 - 3$



$13x_1 + 3 = 2x_1(x_1^2 + 7)$

$2x_1^3 + x_1 - 3 = 0 \Rightarrow 2x_1^3 + 3x_1 - 3 = 0$

$2x_1^2(x_1-1) + 2x_1(x_1-1) + 3(x_1-1)$

$(x_1-1)(2x_1^2 + 2x_1 + 3)$

$x_1 = 1.$

Hence, the point nearest to (3, 7) is (1, 8) and the shortest distance is $\sqrt{2^2+1^2} = \sqrt{5}$.

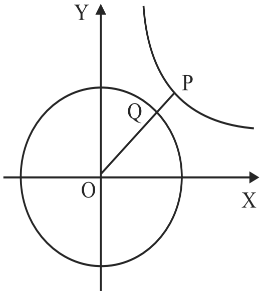
Example 33: Find the minimum distance between $x^2 + y^2 = 2$ and $xy = 9$.

Solution: $xy = 9$

$$\frac{dy}{dx} = -\frac{9}{x^2}$$

Slope of the normal at the point P(3t, 3t)

$$= \frac{x^2}{9} = t^2$$



Equation of normal at P is

$$y - \frac{3}{t} = t^2(x-3t) \qquad \text{...(1)}$$

The minimum distance PQ is found along the common normal.

∴ (1) must pass through origin (the centre of the circle)

$\Rightarrow\ -\dfrac{3}{t} = -3t^2 \ \Rightarrow\ t^4 = 1 \ \Rightarrow\ t = 1$ or -1.

(t = – 1 corresponds to a point xy = 9 in the 3rd quadrant)

∴ t = 1.

Hence, P is (3, 3) $\Rightarrow m_{OP} = 1$

Equation of OP is $y = x \Rightarrow Q(1, 1)$

Hence, $d_{\text{min.}} = \sqrt{4+4} = 2\sqrt{2}$.

Example 34: Find the shortest distance of the point (0, c) from the parabola $y = x^2$, where $0 \le c \le 5$.

Solution: Let P(x, y) be any point on the parabola $y = x^2$ ...(1)

and Q be the point (0, c).

Now $PQ^2 = x^2 + (y-c)^2 = y + (y-c)^2 = y^2 + (1-2c)y + c^2$ [from (1)]

Clearly PQ will be minimum when PQ^2 is minimum.

Let $z = PQ^2$, then

$$z = y^2 + (1-2c)y + c^2 \qquad \text{...(2)}$$

$$\therefore \quad \frac{dz}{dy} = 2y + (1-2c) \qquad \text{...(3)}$$

For maximum or minimum values of z, $\dfrac{dz}{dy} = 0$

$\therefore \quad 2y + (1 - 2c) = 0 \qquad \text{...(4)}$

or $\quad y = \dfrac{2c - 1}{2} = c - \dfrac{1}{2}$.

From (3), $\dfrac{d^2z}{dy^2} = 2 > 0$, therefore z is minimum when

$y = c - \dfrac{1}{2}$

From (2), the minimum value of

$$z = \left(c - \frac{1}{2}\right)^2 + (1 - 2c)\left(c - \frac{1}{2}\right) + c^2$$

$$= c^2 + \frac{1}{4} - c + 2c - \frac{1}{2} - 2c^2 + c^2 = c - \frac{1}{4}.$$

Hence, the minimum value of PQ = $\sqrt{c - \dfrac{1}{4}}$

For PQ to be real $c - \dfrac{1}{4} \geq 0 \Rightarrow c \geq \dfrac{1}{4}$

But it is given that $0 \leq c \leq 5$

Thus, for $\dfrac{1}{4} \leq c \leq 5$, the minimum value of

$$PQ = \sqrt{c - \frac{1}{4}}.$$

If $0 \leq c \leq \dfrac{1}{4}$, $\dfrac{dz}{dy} \geq 0$ using (3). Hence z is an increasing function of y. For minimum value of z, y should take its least value of y = 0. In that case, the minimum value of PQ is $\sqrt{(c - 0)^2 + 0^2} = c$.

Example 35: Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, –2).

Solution: Let the point on the curve be P(x, y),

then $4x^2 + a^2y^2 = 4a^2 \qquad \text{...(1)}$

Let the given point be Q(0, –2)

Let z = PQ, then $z^2 = PQ^2 = x^2 + (y + 2)^2$

Let $u = z^2$, then $u = x^2 + (y + 2)^2 \qquad \text{...(2)}$

z will be minimum when u is minimum.

Now from (2),

$$u = \frac{4a^2 - a^2y^2}{4} + (y + 2)^2 \text{ [from (1)]}$$

or $\quad u = a^2 - \dfrac{a^2}{4}y^2 + (y + 2)^2$

From (1), $4x^2 = a^2(4 - y^9) \qquad \therefore \; x = \pm\dfrac{a}{2}\sqrt{4 - y^2}$

For x to be real $4 - y^2 \geq 0 \Rightarrow 2 \leq y \leq 2$

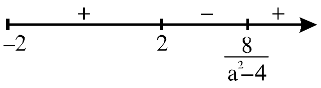
$$\therefore \quad \frac{du}{dy} = -\frac{a^2}{2}y + 2(y + 2)$$

$$= 4 + \left(2 - \frac{a^2}{2}\right)y = \frac{1}{2}[8 + (4 - a^2)y] \qquad \text{...(3)}$$

$\dfrac{du}{dy} = 0$ when $8 + (4 - a^2)y = 0$,

$y = \dfrac{8}{a^2 - 4} > 2 \quad [\because \; 8 > a^2 > 4]$

Sign scheme for $\dfrac{du}{dy} = 8 + (4 - a^2)y/2$ is as follows



$\therefore$ u has the maximum value at y = 2 and from (1), x = 0

$\therefore$ The farthest point on the curve from (0, –2) is (0, 2).

Alternative: On the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{4} = 1$, a point can be taken as $(a\cos\phi, 2\sin\phi)$.

Let the distance between (0, – 2) and the point be z. Then

$u = z^2 = (a\cos\phi)^2 + (2\sin\phi + 2)^2$

$\quad = a^2\cos^2\phi + 4\sin^2\phi + 4 + 8\sin\phi$

$$\frac{d(v)}{d\phi} = -2a^2\cos\phi\sin\phi + 8\sin\phi\cos\phi + 8\cos\phi \qquad \text{...(1)}$$

v is maximum or minimum $\Rightarrow \dfrac{du}{d\phi} = 0$

or $\quad (8 - 2a^2)\cos\phi\sin\phi + 8\cos\phi = 0$

or $\quad \cos\phi\,\{(8 - 2a^2)\sin\phi + 8\} = 0$

$\Rightarrow \quad \cos\phi = 0$ or $\sin\phi = \dfrac{4}{a^2 - 4}$.

But $4 < a^2 < 8$.

So, $\sin\phi = \dfrac{4}{a^2 - 4} > 1$ which is not possible.

$\therefore \quad \dfrac{du}{d\phi} = 0$ when $\cos\phi = 0$, i.e., $\phi = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$.

Differentiating (1) again,

$$\frac{d^2u}{d\phi^2} = (8 - 2a^2)(\cos^2\phi - \sin^2\phi) - 8\sin\phi$$

$\therefore \quad$ at $\phi = \dfrac{\pi}{2}$, $\dfrac{d^2u}{d\phi^2} = 2a^2 - 8 - 8$

$\quad = 2(a^2 - 8) < 0$ for $4 < a^2 < 8$

∴ $u = z^2$ is the maximum when $\phi = \frac{\pi}{2}$

Clearly, at $\phi = \frac{3\pi}{2}$, u is the minimum.

∴ z is the maximum when $\phi = \frac{\pi}{2}$,

Thus, the required point is $\left(a\cos\frac{\pi}{2},\ 2\sin\frac{\pi}{2}\right)$ i.e. (0, 2).

Example 36: What normal to the curve $y = x^2$ forms the shortest chord.

Solution: Let $A(t, t^2)$ be any point on the parabola $y = x^2$

Since $\frac{dy}{dx} = 2x$, $\left(\frac{dy}{dx}\right)_{(t,t^2)} = 2t$ is the slope of tangent. So, the slope of normal to $y = x^2$ at (t, t^2) is $\left(-\frac{1}{2t}\right)$.

∴ The equation of the normal to $y = x^2$ at (t, t^2) is

$$y - t^2 = \left(-\frac{1}{2t}\right)(x - t) \qquad ...(1)$$

Suppose equation (1) meets the curve again at B (t_1, t_1^2), then, $t_1^2 - t^2 = -\frac{1}{2t}(t_1 - t)$

$$\Rightarrow t_1 + t = -\frac{1}{2t}$$

$$\Rightarrow t_1 = -t - \frac{1}{2t} \qquad ...(2)$$

Let L be the square of the length of normal chord AB

$$L = AB^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$$

$$= \left(t + t + \frac{1}{2t}\right)^2\left[1 + \left(t - t - \frac{1}{2t}\right)^2\right] \quad \text{(using (2))}$$

$$= \left(2t + \frac{1}{2t}\right)^2\left(1 + \frac{1}{4t^2}\right)$$

$$\therefore\ L = 4t^2\left(1 + \frac{1}{4t^2}\right)^3$$

$$\Rightarrow \frac{dL}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^2 + 12t^2\left(1 + \frac{1}{4t^2}\right)^2\left(-\frac{2}{4t^3}\right)$$

$$\Rightarrow \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2\left[4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$$\Rightarrow \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2\left(4t - \frac{2}{t}\right) = 4\left(1 + \frac{1}{4t^2}\right)^2\left(2t - \frac{1}{t}\right)$$

For extremum, let $\frac{dL}{dt} = 0 \Rightarrow r = \pm\frac{1}{\sqrt{2}}$

$$\text{Again, } \frac{d^2L}{dt^2} = 8\left(1 + \frac{1}{4t^2}\right)\left(-\frac{1}{2t^2}\right)\left(2t - \frac{1}{t}\right) + 4\left(1 + \frac{1}{4t^2}\right)^2\left(2 + \frac{1}{t^2}\right)$$

$$\Rightarrow \left(\frac{d^2L}{dt^2}\right)_{t=\pm\frac{1}{\sqrt{2}}} > 0$$

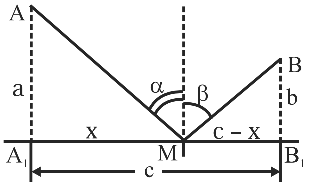
∴ L is minimum when $t = \pm\frac{1}{\sqrt{2}}$.

Thus, points are $A\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $B\left(\mp\sqrt{2},\ 2\right)$.

Hence, the equation of normal AB is $\sqrt{2}x + 2y - 2 = 0$ or $\sqrt{2}x - 2y + 2 = 0$.

Example 37: A man walking from A wishes to reach a river (the straight line A_1B_1 in the figure) and then go to B. How can he do this by traveling the shortest possible distance?

Solution: We have $AA_1 = a$, $BB_1 = b$, $A_1B_1 = c$, the numbers a, b, c are given. Let the broken line AMB be the path taken. Our aim is to find out for what position of the point M on the line A_1B_1 is this path the shortest. To determine the position of M it suffices to specify the distance from M to the point A_1, the foot of the perpendicular dropped from A onto the straight line representing the river. Denote this distance A_1M by x. Then



$$AM = \sqrt{a^2 + x^2},\ MB = \sqrt{b^2 + (c - x)^2}$$

The path traversed is then denoted by s(x),

$$s(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2} \qquad ...(1)$$

and we find

$$s'(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{c - x}{\sqrt{b^2 + (c - x)^2}}$$

Equating s′(x) to zero, we get

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{c - x}{\sqrt{b^2 + (c - x)^2}} \qquad ...(2)$$

It is easy to solve this equation. Squaring both sides, we have

$$\frac{x^2}{a^2+x^2} = \frac{(c-x)^2}{b^2+(c-x)^2}$$

or $x^2b^2 + x^2(c-x)^2 = a^2(c-x)^2 + x^2(c-x)^2$,

$$x^2b^2 = a^2(c-x)^2,\ \frac{x^2}{(c-x)^2} = \frac{a^2}{b^2}$$

Taking the root of both members, we find

$$\frac{x}{c-x} = \pm\frac{a}{b}$$

$$\Rightarrow\ x_1 = \frac{ac}{a+b},\ x_2 = \frac{ac}{a-b}$$

Substituting the values x_1 and x_2 into the original equation (2), we see that the second root does not satisfy the equation. This is an extraneous root generated by the squaring process. Thus, $x = \frac{ac}{a+b}$.

It is possible, however, to give a pictorial geometrical representation that will enable us to obtain the answer without solving the equation. Rewrite the condition (2) as

$$\frac{A_1M}{AM} = \frac{MB_1}{MB} \qquad ...(3)$$

But $\frac{A_1M}{AM} = \cos\angle A_1MA = \sin\alpha$.

Similarly, $\frac{MB_1}{MB} = \cos\angle B_1MB = \sin\beta$.

The condition (3) yields $\sin\alpha = \sin\beta$ ...(4)

$\Rightarrow\ \alpha = \beta$

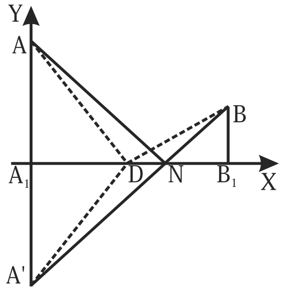
The man must take the path of a ray of light bounced off the river : the angle of incidence is equal to the angle of reflection. This is the Fermat Principle of least time for propagation of light.

For a complete solution to the problem it remains to demonstrate that for such a position of point M the distance is indeed minimum (and not maximum). This can be done by computing the second derivative of (1).

But it is also possible to reason differently. From the expression (1) for s(x) we see that s(x) is positive for any x. Then s(x) increases without bound together with the growth in the absolute value of x, irrespective of whether $x > 0$ or $x < 0$. And since $s'(x)$ vanishes only for one value of x, it is clear that at this value of x the function s(x) will then have a minimum.

This problem can be solved in a purely geometrical manner without resorting to methods of higher mathematics. Referring to the figure below extend the segment AA_1 to A' ($A_1A' = AA_1$) and join A' to B. Then $AM = A'M$ since triangle AA_1M is congruent to triangle $A_1A'M$.

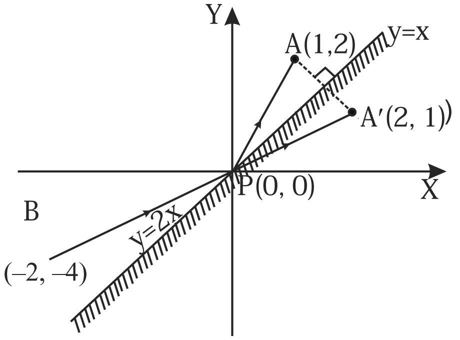
Therefore $AM + MB = A'M + MB = A'B$. For any other point D on the segment A_1B_1 we will have $AD + DB = A'D + DB$ and $A'D + DB > A'B$ since a polygonal line is longer than any segment of a straight line. Consequently, the desired point M is the point of intersection of the straight line $A'B$ and A_1B_1, from where it follows that $\alpha = \beta$.



This example shows that certain problems involving the finding of maxima and minima may be solved by the tools of elementary mathematics.

Example 38: Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

Solution: Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB).



The line y = 2x intersects the line y = x at the point P (0, 0).

The image of A is A′(2, 1).

The line A′B has equation $y - 1 = \frac{1+4}{2+2}(x-2)$

i.e. $y = 2x$.

Miscellaneous Problems

Example 39: Divide 20 into two parts such that the product of cube of first and square of second is maximum.

Solution: Let *a* be the first part and *b* be the second part. Then,

$a + b = 20$ ... (1)

Let P be the product of cube of first and square of second. Then,

$P = a^3b^2 \qquad = a^3(20-a)^2$ [using (1)]

$= a^3(400 - 40a + a^2)$

$= 400a^3 - 40a^4 + a^5,\quad 0 < a < 20$

$$\frac{dP}{da} = 1200\,a^2 - 160\,a^3 + 5\,a^4 = 0$$

$\Rightarrow$ a = 0, 12, 20. The only critical point is a = 12.

$$\frac{d^2P}{da^2} = 2400\,a - 480\,a^2 + 20\,a^3$$

$$\left(\frac{d^2P}{da^2}\right)_{a=12} = 2400 \times 12 - 480 \times 12^2 + 20 \times 12^3$$

$$= -5760 < 0$$

∴ a = 12 is a point of local maximum using Fork Extremum Theorem ; this is the absolute maximum.
So P is maximum when a = 12 and b = 20 – 12 = 8.

Example 40: The cost function at a factory is $C(x) = x^3 - 6x^2 + 15x$ (x in thousands of units and x > 0). Find the production level at which average cost is minimum.

Solution: C(x) = cost of producing x items > 0

$$\text{Average cost} = \frac{C(x)}{x} = x^2 - 6x + 15 = f(x)$$

$f'(x) = 2x - 3 = 0 \quad \Rightarrow x = 3.$

$f''(x) = 2 > 0$

Hence, the average cost is minimum when x = 3.

Example 41: A manufacturer can sell x items at a price of Rs. $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution: $P = S(x) - C(x) = \left(5 - \frac{x}{100}\right)x - \left(\frac{x}{5} + 500\right);$

$$P' = 5 - \frac{x}{50} - \frac{1}{5} \Rightarrow \frac{x}{50} = \frac{24}{5} \Rightarrow x = 240$$

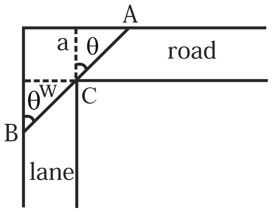
$P''(x) = -1/50 < 0.$

Hence, the maximum profit can be earned when x = 240.

Example 42: A lane runs at right angles out of a road a metres wide. Find how many metres wide the lane should be if it is just possible to carry a pole b metres long (b > a) from the road into the lane keeping it horizontal.

Solution: Let AB be the pole and AB = b m.
Let the inclination of the pole with the lane be θ when the pole is about to clear and the width of the lane = w m.
Then AB = BC + AC = w cosec θ + a sec θ

∴ b = w cosec θ + a sec θ
∴ w = b sin θ – a tan θ ...(1)



Here we have to find the largest w for $\theta \in \left(0, \frac{\pi}{2}\right)$ with ends of the pole on the lines as shown in the figure.

Now, $\frac{dw}{d\theta} = b\cos\theta - a\sec^2\theta$

∴ $\frac{dw}{d\theta} = 0$ if $b\cos\theta - a\sec^2\theta = 0$, i.e., $\cos^3\theta = \frac{a}{b}$

and $\frac{d^2w}{d\theta^2} = -b\sin\theta - a\sec^2\theta\tan\theta$

As $0 < \theta < \frac{\pi}{2}$, $\frac{d^2w}{d\theta^2} < 0$

∴ w is maximum when $\cos^3\theta = \frac{a}{b}$,

i.e., $\cos\theta = \left(\frac{a}{b}\right)^{1/3}$

∴ from (1), the maximum value of w

$$= b\sqrt{1-\cos^2\theta} - a\sqrt{\cos^2\theta - 1}$$

$$= b\sqrt{1-\left(\frac{a}{b}\right)^{2/3}} - a\sqrt{\left(\frac{a}{b}\right)^{2/3} - 1}$$

$$= b^{2/3}\sqrt{b^{2/3} - a^{2/3}} - a^{2/3}\sqrt{b^{2/3} - a^{2/3}}$$

$$= (b^{2/3} - a^{2/3})^{3/2}.$$

Note: Here b > a. If b ≤ a then whatever might be the width of the lane, the pole can be cleared. Also $\cos^3\theta = \frac{a}{b}$ will give no θ other than 90°, in which case the pole is cleared irrespective of the width of the lane.

Example 43: One corner of a long rectangular sheet of paper of width 1 unit is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

Solution: Let ABCD be the rectangular sheet whose corner C is folded over along EF so as to reach the edge AB at C′. Let $EF = x$, $\angle FEC = \theta = \angle FEC'$

$\therefore \quad EC = x\cos\theta = EC'$

From Δ BEC′, we have $BE = C'E\cos(\pi - 2\theta)$

$\Rightarrow \quad BE = -x\cos\theta \, . \cos 2\theta$

$\therefore \quad BC = BE + EC$

$1 = -x\cos\theta\cos 2\theta + x\cos\theta$

$$\Rightarrow \quad x = \frac{1}{\cos\theta(1-\cos 2\theta)} \qquad \text{....(1)}$$

For x to be minimum, $Z = 1/x$ has to be maximum.

$$\text{i.e., } Z = \frac{1}{x} = \cos\theta\,(1 - \cos 2\theta) \qquad \text{....(2)}$$

Differentiating (2) w.r.t. θ we get,

$$\frac{dZ}{d\theta} = \cos\theta\,(2\sin 2\theta) - \sin\theta\,(1 - \cos 2\theta)$$

$$\text{and } \frac{d^2Z}{d\theta^2} = \cos\theta\,(4\cos 4\theta) - 2\sin 2\theta \, . \sin\theta$$

$$- \sin\theta\,(2\sin 2\theta) - \cos\theta(1 - \cos 2\theta)$$

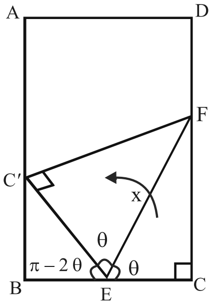
For maximum / minimum,

$$\frac{dZ}{d\theta} = 0 \Rightarrow 2\sin\theta(2 - 3\sin^2\theta) = 0$$

$$\therefore \quad \sin\theta = \pm\sqrt{2/3} \qquad (\because \quad \sin\theta \neq 0)$$

$$\text{When, } \sin\theta = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \frac{d^2Z}{d\theta^2} = -\frac{5}{5\sqrt{3}} - \frac{16}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{8}{\sqrt{3}} < 0$$



Hence, Z is maximum.

$$\Rightarrow \quad x = \frac{1}{Z} \text{ is minimum.} \qquad \text{[from (1)]}$$

the minimum length of the crease

$$\frac{1}{Z} = \frac{1}{(1/\sqrt{3})\left(1 + \frac{1}{3}\right)} = \frac{3\sqrt{3}}{4} \text{ units}$$

Example 44: The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance between source and the object. Two light bulbs, one 5 times as strong as the other, are 1 m apart. At what point on the line between the bulbs should a screen be placed so that the illumination it receives is minimal?

Solution: Let the screen be placed at a distance of x m from the weaker source. Even through one bulb is 5 times as strong as the other, the screen cannot be too close to the weaker bulb because of the inverse square rule. The illumination from the weaker bulb is $I_1 = \frac{k}{x^2}$, where the constant k depends on the units of measurement. The illumination from the stronger bulb is $I_2 = \frac{5k}{(1-x)^2}$

The problem is to minimize

$$I = \frac{k}{x^2} + \frac{5k}{(1-x)^2}, \text{ for } 0 < x < 1.$$

There are no end points in this problem since I is defined neither at x = 0 not at x = 1.

$$\frac{dI}{dx} = -\frac{2k}{x^3} + \frac{10k}{(1-x)^3}$$

This derivative is 0 for

$$\frac{2k}{x^3} = \frac{10k}{(1-x)^3} \quad \Rightarrow \quad 5x^3 = (1-x)^3.$$

$$\Rightarrow \quad (\sqrt[3]{5})x = 1 - x \quad \Rightarrow \quad x = \frac{1}{1 + \sqrt[3]{5}} \approx 0.369$$

The physics of the problem suggests that this x gives a minimum. As a quick check, take the second derivative:

$$\frac{d^2I}{dx^2} = \frac{6k}{x^4} + \frac{30k}{(1-x)^4}$$

which is always positive. Hence the screen should be $\frac{1}{1+\sqrt[3]{5}} \approx 0.369$ m from the weaker bulb.

Example 45: A bus contractor agrees to run special buses for the employees of ABC Co. Ltd . He agrees to run the buses if atleast 200 persons travel by his buses . The fare per person is to be Rs. 10/- per day if 200 travel and will be decreased for everybody by 2 paise per person over 200 that travels . How many passengers will give the contractor maximum daily revenue ?

Solution: Let number of passengers be x, which will yield maximum profit

$$f(x) = x\left[10 - (x - 200)\frac{2}{100}\right] \text{ for } x \ge 200$$

$f'(x) = 0 \qquad \Rightarrow x = 350$

Thus, 350 passengers will give the maximum revenue.

Example 46: Assuming that the petrol burnt in driving a motor boat varies as the cube of velocity, show that the most economical speed when going against a current of c kilometers per hour is (3/2)c kilometers per hour.

Solution: Let the speed of the boat be v kilometers per hour and the distance travelled in 't' hours be c kilometers. When the boat is going against the current of c kilometers per hour, its resultant speed is (v – c) kilometers per hour.

$\therefore$ t = distance/speed = a/(v – c) hours.

Now the petrol burnt per hour is kv^3, where k is a constant. If x is the petrol burnt in t hours, then,

$$x = \{a/(v-c)\}\, kv^3 = a\,kv^3/(v-c).$$

For the most economical speed, x should be minimum.

Now $dx/dv = ak\,\{3v^2(v-c) - v^3\}/(v-c)^2$

$= ak\,\{2v^2 - 3v^3c)/(v-c)^2$

$= ak \cdot v^2(2v-3c) \cdot \dfrac{1}{(v-c)^2}$.

$dx/dv = 0$

$\Rightarrow \quad v^2(2v-3c) = 0$

$\Rightarrow \quad 2v - 3c = 0, \qquad [\because v \ne 0]$

$\Rightarrow \quad v = 3c/2.$

Also

$$\frac{d^2x}{dv^2} = ak\left[-\frac{2}{(v-c)^3}.v^2(2v-3c) + \frac{1}{(v-c)^2}.6v(v-c)\right]$$

When $v = \dfrac{3c}{2}$, $\dfrac{d^2x}{dv^2} = ak\,.\dfrac{6.(3c/2)}{(3c/2)-c} = 18ak > 0.$

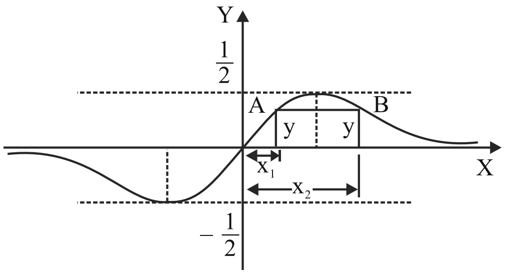
Using Fork extremum theorem, x is minimum when v = 3c/2 i.e., the most economical speed is c/2 kilometers per hour.

Example 47: A cylinder is obtained by revolving a rectangle about the x – axis , the base of the rectangle lying on the x – axis and the entire rectangle lying in the region between the curve $y = \dfrac{x}{x^2+1}$ and the x – axis . Find the maximum possible volume of the cylinder .

Solution: $y = \dfrac{x}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Now $\dfrac{dy}{dx} = 0 \Rightarrow x = 1$ or – 1. The graph of the function is shown below.



Now $V = \pi y^2 (x_2 - x_1)$

Since the ordinates at A & B have to be equal ,

$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$\Rightarrow \quad x_1x_2^2 + x_1 = x_2x_1^2 + x_2$

$\Rightarrow \quad x_1x_2(x_2 - x_1) - (x_2 - x_1) = 0$

$\Rightarrow \quad (x_2 - x_1)(x_1x_2 - 1) = 0$

$\Rightarrow \quad x_2 = \dfrac{1}{x_1}$ as $(x_1 \ne x_2)$

$$\text{Hence } V = \frac{\pi x_1^2}{(1+x_1^2)^2}\left(\frac{1}{x_1} - x_1\right) = \frac{\pi x_1(1-x_1^2)}{(1-x_1^2)^2}$$

Let $V(x) = \dfrac{\pi x}{(1+x^2)^2}$. $V'(x) = 0$

$\Rightarrow \quad \pi\,[(1+x^2)^2(1-3x^2) - (x-x^3)\,2(1+x^2)2x] = 0$

On simplifying we get $x^4 - 6x^2 + 1 = 0$

$\Rightarrow \quad x^2 = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

We rejected $x = \sqrt{3+2\sqrt{2}} = \sqrt{2} + 1$ since it is greater than 1 .

Hence, $x = \sqrt{3-2\sqrt{2}} = \sqrt{2} - 1.$

$$\therefore \quad V_{max} = \frac{\pi(\sqrt{2}-1)(1-((3-2\sqrt{2}))}{(1+3-2\sqrt{2})^2}$$

$$= \frac{\pi(\sqrt{2}-1)\,(2\sqrt{2}-2)}{(4-2\sqrt{2})^2} = \frac{2\pi(3-2\sqrt{2})}{4(2-\sqrt{2})^2}$$

$$= \frac{2\pi}{8}\,\frac{3-2\sqrt{2}}{(\sqrt{2}-1)^2} = \frac{\pi}{4}.$$

Concept Problems I

1. The total cost of making n pocket radios per day is Rs. $\left\{\frac{n^2}{4}+35n+25\right\}$ and the rate at which they may be sold to a distributor is Rs. $\left\{\frac{1}{2}(100-n)\right\}$.
For a maximum total profit, find the daily output.

2. A rectangle is inscribed in a sector of a circle of radius R, the central angle of the sector being a right angle. One of the vertices of the rectangle coincides with the centre of the circle and the opposite vertex lies on the circumference. Find the lengths of the sides of the rectangle that has the greatest area.

3. The base of a painting on a wall is a feet above the eye of an observer. The vertical side of a painting is b feet long. How far from the wall should the observer stand to maximize the angle that the painting substends?

4. Let f be a differentiable function and let A be a point not on the graph of f. Show that if B is the point on the graph closest to A, then the segment AB is perpendicular to the tangent line to the curve at B.

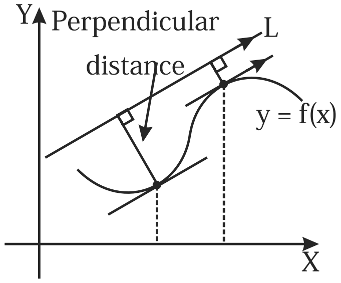
Practice Problems H

5. Show that the greatest chord through a point of intersection of two given circles is that which is parallel to the line of centres.

6. A truck is to be driven 300 miles at a constant speed of x miles per hour. Speed laws require $30 \le x \le 60$. Assume that fuel costs 30 rupees per litre and is consumed at the rate of $2 + x^2/600$ litres per hour. If the driver's wages are D rupees per hour and if he obeys all speed laws, find the most economical speed and the cost of the trip if (a) D = 0, (b) D = 10, (c) D = 20, (d) D = 30, (e) D = 40.

7. A log 12 feet long has the shape of a frustum of a right circular cone with diameters 4 feet and (4 + h) feet at its ends, where $h \ge 0$. Determine, as a function of h, the volume of the largest right circular cylinder that can be cut from the log, if its axis coincides with that of the log.

8. It can be proved that if f is differentiable on (a, b) and L is a line that does not intersect the curve y = f(x) over an interval (a, b), then the points at which the curve is closest to or farthest from the line L, if any, occur at points where the tangent line to the curve is parallel to L (see the accompanying figure). Use this result to find the points on the graph of $y = -x^2$, $-1 \le x \le 1.5$, that are closest to and farthest from the line $y = 2 - x$.



9. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are $x = t - 2\sin t$, $y = 2 - 2\cos t$, $(0 \le t \le 12)$. What are the highest and lowest points in the trajectory, and when is the airplane at those points ?

10. If 1200 sq. cm of material is available to make a box with a square base and an open top. Find the largest possible volume of the box (in cubic cm).

11. Find the equation of a line through (1, 8) cutting the positive semi axes at A and B if
   - (i) the area of ΔOAB is minimum
   - (ii) its intercept between the coordinate axes is minimum.
   - (ii) sum of its intercept on the coordinates axes is minimum.

12. Find the two positive numbers x and y whose sum is 35 and the product $x^2 y^5$ maximum.

13. Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point c and d respectively. The distance between C and D is C. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P.

14. A normal is drawn to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.
Find the maximum distance of this normal from the centre.

15. A line is drawn passing through point P(1, 2) to cut positive coordinates axes at A and B. Find minimum area of ΔPAB.

16. If $\theta + \Psi = \alpha$ (constant), show that $\sin\theta.\sin\psi$ has maximum value when $\theta = \psi$, $0 < \theta < \alpha$.

7.15 Asymptotes

A straight line is called an **asymptote** to a curve if the distance between a point tracing the curve and the straight line tends to zero as this point recedes to infinity.

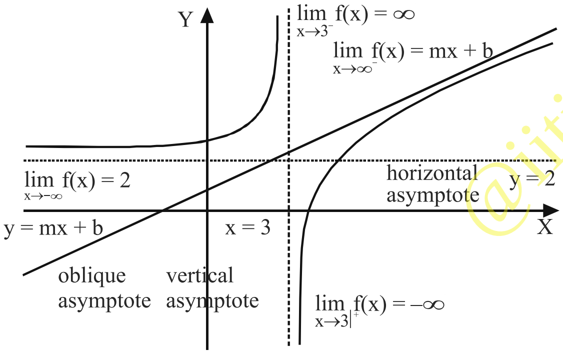
Note:

(i) A point is said to recede to infinity if the distance from this point to the origin (or some fixed point increases indefinitely.

(ii) Given the asymptotes of a curve, we can approximate to a good accuracy the coordinates of points along a curve far from the origin.

(iii) Not every curve has an asymptote while some curves have several asymptotes.

The figure shows a graph that approaches the horizontal line $y = 2$ as $x \to -\infty$, the vertical line $x = 3$ as x approaches 3 from either side, and the slanted line $x–2y = 2$ as $x \to \infty$. This is referred to as asymptotic behaviour, and the lines $y = 2$, $x = 3$ and $y = \frac{1}{2}x - 1$ are called asymptotes of the graph.


Y
$\lim_{x\to3^-} f(x) = \infty$
$\lim_{x\to\infty^-} f(x) = mx + b$
$\lim_{x\to-\infty} f(x) = 2$
horizontal asymptote
y = 2
y = mx + b
x = 3
X
oblique asymptote
vertical asymptote
$\lim_{x\to3|^+} f(x) = -\infty$

A typical graph with asymptotes

When finding aymptotes, we distinguish between horizontal, vertical and oblique (slanted or inclined) asymptotes, that is, between straight lines parallel to the x- axis or y-axis and those that are not. Now, we consider the formal definition of the three kinds of asymptotes.

Horizontal Asymptotes

If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still an useful information for sketching the curve.

The graph of the function $y = f(x)$ (we assume the function is single-valued) cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1: Using $\lim_{x\to\infty} e^{-x}\cos x$, find the horizontal asymptote to the curve $y = e^{-x}\cos x$.

Solution: We cannot use the product rule since $\lim_{x\to\infty} \cos x$ does not exist (it diverges by oscillation).

Note: However, that $e^{-x}\cos x = \frac{\cos x}{e^x}$ must become smaller and smaller as $x\to\infty$ since the numerator cos x lies between –1 and 1, while the denominator e^x grows larger and larger.

Thus, $\lim_{x\to\infty} e^{-x}\cos x = 0$ and $y = 0$ is a horizontal asymptote.

Example 2: Find the right horizontal asymptote of the function $f(x) = 4e^{2x}/(1 + e^x)^2$.

Solution: Upon dividing numerator and denominator by e^{2x}, we find that

$$f(x) = \frac{4e^{2x}}{(1+e^x)^2} = \frac{4}{(e^{-x}+1)^2} \to 4 \text{ as } x \to \infty.$$

Thus the curve $y = 4e^{2x}/(1 + e^x)^2$ has the line $y = 4$ as a horizontal asymptote.

Vertical Asymptotes

The line $x = a$ is a vertical asymptote if at least one of the following statements is true :

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

For rational functions we can locate the vertical asymptotes by equating the denominator to 0 after cancelling any common factors. But for other functions this method does not apply. Furthermore, in sketching the curve it is very useful to know exactly which of the statements above is true. If f(a) is not defined but a is an endpoint of the domain of f, then we should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

Oblique Asymptotes

If there are limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = m_1 \text{ and } \lim_{x\to\infty} [f(x) - m_1x] = c_1,$$

then the straight line $y = m_1x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m_2 \text{ and } \lim_{x\to-\infty} [f(x) - m_2x] = c_2,$$

then the straight line $y = m_2x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

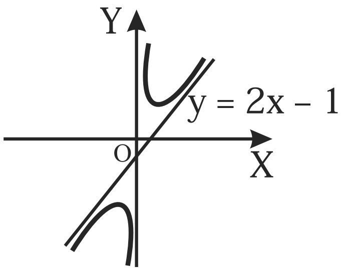
Example 3: Find the asymptotes of the curve $y = 2x + \frac{3}{x} - 1$. Also, illustrate the position of the curve relative to its asymptotes.

Solution: As $x \to 0^+$, $y \to \infty$ and as $x \to 0^-$, $y \to -\infty$. Hence, $x = 0$ is a vertical asymptote.

Now, we have $\lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{2x^2 - x + 3}{x^2} = 2$

and $\lim_{x\to\infty} y - 2x = \lim_{x\to\infty} \frac{2x^2 - x + 3}{x^2} - 2x$

$$= \lim_{x\to\infty} \frac{-x+3}{x} = -1$$



Hence, $y = 2x - 1$ is an oblique asymptote.
Now, we have

$$\left(2x + \frac{3}{x} - 1\right) - (2x - 1) = \frac{3}{x} > 0 \text{ as } x \to \infty$$

$$< 0 \text{ as } x \to -\infty$$

Hence, the curve lies above its asymptote $y = 2x - 1$ as $x \to \infty$ and below its asymptote $y = 2x - 1$ as $x \to -\infty$.

Example 4: Find the asymptotes of the curve

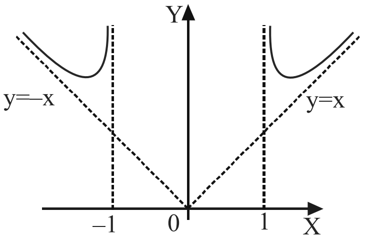
$$y = \frac{x^2}{\sqrt{x^2 - 1}}.$$

Solution: Equating the denominator to zero, we get two vertical asymptotes: $x = -1$ and $x = 1$.
We seek the inclined asymptotes.
For $x \to \infty$ we obtain

$$m_1 = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2 - 1}} = 1$$

$$c_1 = \lim_{x\to\infty} (y - x) = \lim_{x\to\infty} \frac{x^2 - x\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} = 0.$$

Hence, the straight line $y = x$ is the right asymptote.



Similarly, when $x \to -\infty$, we have

$$m_2 = \lim_{x\to-\infty} \frac{y}{x} = -1$$

$$c_2 = \lim_{x\to-\infty} (y + x) = 0.$$

Thus, the left asymptote is $y = -x$ (see figure). Testing a curve for asymptotes is simplified if we take into consideration the symmetry of the curve.

Example 5: Find the asymptotes of the curve $y = x + 2\tan^{-1}x$.

Solution: It is easy to see that the curve does not possess either vertical or horizontal aymptotes. Let us find inclined asymptotes:

(i) $m_1 = \lim_{x\to\infty} \frac{x + 2\tan^{-1}x}{x} = \lim_{x\to\infty}\left(1 + \frac{2\tan^{-1}x}{x}\right) = 1$

$c_1 = \lim_{x\to\infty} (x + 2\tan^{-1}x - x) = 2(\pi/2) = \pi.$

$y = x + \pi$ is the right inclined asymptote.

(ii) $m_2 = \lim_{x\to-\infty} \frac{x + 2\tan^{-1}x}{x} = \lim_{x\to-\infty}\left(1 + \frac{2\tan^{-1}x}{x}\right) = 1$

$c_2 = \lim_{x\to-\infty} (x + 2\tan^{-1}x - x) = 2(-\pi/2) = -\pi.$

$y = x - \pi$ is the left inclined asymptote.

Example 6: Find the asymptotes of the curve $y = x + \ln x$.

Solution: Since $\lim_{x\to 0^+} y = -\infty$, the straight line $x = 0$ is a vertical asymptote. Let us now test the curve only for the right inclined asymptote (since $x > 0$).
We have:

$$m = \lim_{x\to\infty} \frac{y}{x} = 1$$

$$c = \lim_{x\to\infty} (y - x) = \lim_{x\to\infty} \ln x = \infty$$

Hence the curve has no inclined asymptote.

Example 7: Let $f(x) = \frac{1}{2}(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x + 1})$

Solution: We compute the limits:

$$m = \lim_{x\to\infty}\frac{f(x)}{x} = \frac{1}{2}\lim_{x\to\infty}\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{x} = 1$$

$$c = \lim_{x\to\infty}[f(x)-mx]$$

$$= \frac{1}{2}\lim_{x\to\infty}[\sqrt{x^2+x+1}+\sqrt{x^2-x+1}-2x] = 0.$$

Thus, the straight line $y = x$ is an asymptote of the graph of the given function as $x \to \infty$

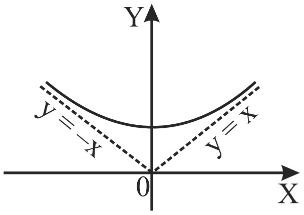
Let us now consider the limits as $x \to -\infty$

$$m = \lim_{x\to-\infty}\frac{f(x)}{x} = \frac{1}{2}\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{x} = 1$$

$$c = \lim_{x\to-\infty}[f(x)-kx]$$

$$= \frac{1}{2}\lim_{x\to-\infty}[\sqrt{x^2+x+1}+\sqrt{x^2-x+1}+2x] = 0$$

Consequently, the line $y = -x$ is an asymptote of the graph of the given function as $x \to -\infty$. The graph is shown in the figure.



Example 8: Find the asymptotes of the curve

$$y = \sqrt{x^3/(x-2)}.$$

Solution: The function is defined on the intervals $(-\infty, 0)$ and $(2, \infty)$. Since $\lim_{x\to2^+}\sqrt{x^3/(x-2)} = \infty$ the line $x = 2$ is a vertical asymptote of the curve.

The curve does not have horizontal asymptotes since $\lim_{x\to\infty}\sqrt{x^3/(x-2)}$ and $\lim_{x\to-\infty}\sqrt{x^3/(x-2)}$ are not finite values.

Let us determine whether there are inclined asymptotes.

(i) $$m_1 = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\sqrt{\frac{x^3/(x-2)}{x}}$$

$$= \lim_{x\to\infty}\sqrt{\frac{x}{x-2}} = \lim_{x\to\infty}\sqrt{\frac{1}{1-2/x}} = 1$$

$$c_1 = \lim_{x\to\infty}[f(x)-m_1x] = \lim_{x\to\infty}\left(\sqrt{\frac{x^3}{x-2}}-x\right)$$

$$= \lim_{x\to\infty}\frac{x(\sqrt{x}-\sqrt{x-2})}{\sqrt{x-2}} = \lim_{x\to\infty}\frac{x(x-x+2)}{\sqrt{x-2}(\sqrt{x}+\sqrt{x-2}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{1-\frac{2}{x}}\left(1+\sqrt{1-\frac{2}{x}}\right)} = 1.$$

Thus we see that there s a right inclined asymptote $y = x + 1$.

(ii) $$k_2 = \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\sqrt{\frac{x^3/(x-2)}{x}}$$

$$= \lim_{x\to-\infty}\frac{\sqrt{x/(x-2)}}{-1}$$

(we have divided the numerator and denominator by the positive value $-x$), i.e.

$$m_2 = -\lim_{x\to-\infty}\sqrt{\frac{1}{1-2/x}} = -1$$

$$c_2 = \lim_{x\to-\infty}[f(x)-m_2x] = \lim_{x\to-\infty}\left(\sqrt{\frac{x^3}{x-2}}+x\right)$$

$$= \lim_{x\to-\infty}\left(\sqrt{\frac{(-x)^3}{2-x}}+x\right) = \lim_{x\to-\infty}\frac{-x\sqrt{-x}+x\sqrt{2-x}}{\sqrt{2-x}}$$

$$= \lim_{x\to-\infty}\frac{x(\sqrt{-x}-\sqrt{2-x})}{\sqrt{2-x}}$$

$$= \lim_{x\to-\infty}\frac{x(-x-2+x)}{\sqrt{2-x}(\sqrt{x}+\sqrt{2}-x)} = -1$$

Thus, there exists a left inclined asymptote $y = -x - 1$.

Example 9: Find the asymptotes of the curve

$$y = \frac{x^2+2x-2^{-x^2}}{x}.$$

Solution: We first look for vertical asymptotes :

When $x \to 0^-$, $y \to \infty$

When $x \to 0^+$, $y \to -\infty$

Therefore, the straight line $x = 0$ is a vertical asymptote.

Now, we look for inclined asymptotes :

$$m = \lim_{x\to\pm\infty}\frac{y}{x} = \lim_{x\to\pm\infty}\frac{x^2+2x-2^{-x^2}}{x}$$

$$= \lim_{x\to\pm\infty}\left[1+\frac{2}{x}-\frac{2^{-x^2}}{x^2}\right] = 1$$

$$c = \lim_{x\to\pm\infty}[y-x] = \lim_{x\to\pm\infty}\left[\frac{x^2+2x-2^{-x^2}}{x}-x\right]$$

$$= \lim_{x\to\pm\infty}\left[\frac{x^2+2x-2^{-x^2}-x^2}{x}\right]$$

$$= \lim_{x\to\pm\infty}\left[2-\frac{2^{-x^2}}{x}\right] = 2$$

Finally, $c = 2$

Therefore, the straight line $y = x + 2$ is an inclined asymptote to the given curve.

To investigate the mutual positions of a curve and an asymptote, let us consider the difference of the ordinates of the curve and the asymptote for one and the same value of x :

$$\frac{x^2+2x-2^{-x^2}}{x} - (x+2) = -\frac{2^{-x^2}}{x}$$

This difference is negative for $x > 0$ and positive for $x < 0$; and so for $x > 0$ the curve lies below the asymptote, and for $x < 0$ it lies above the asymptote.

Example 10: Find the asymptotes of the curve $y = e^{-x}\sin x + x$.

Solution: It is obvious that there are no vertical asymptotes. Now we look for inclined asymptotes :

$$m = \lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty}\frac{e^{-x}\sin x + x}{x}$$

$$= \lim_{x\to\infty}\left(\frac{e^{-x}\sin x}{x}+1\right) = 1$$

$$c = \lim_{x\to\infty}\,[e^{-x}\sin x + x - x] = \lim_{x\to\infty} e^{-x}\sin x = 0$$

Hence, the straight line $y = x$ is an inclined asymptote as $x \to \infty$.

The given curve has no asymptote as $x \to -\infty$. Indeed, the limit $\lim_{x\to-\infty}\frac{y}{x}$ does not exist, since $\frac{y}{x} = \frac{e^{-x}}{x}\sin x + 1$. (Here, the first term increases without bound as $x \to -\infty$ and, therefore it has no limit.)

Oblique Asymptotes in Rational Functions

If a rational function P(x)/Q(x) is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of P(x)/Q(x) will have an oblique asymptote, that is, an asymptote that is neither vertical nor horizontal. To see this, we divide P(x) by Q(x) to obtain

$$\frac{P(x)}{Q(x)} = (ax+b) + \frac{R(x)}{Q(x)}$$

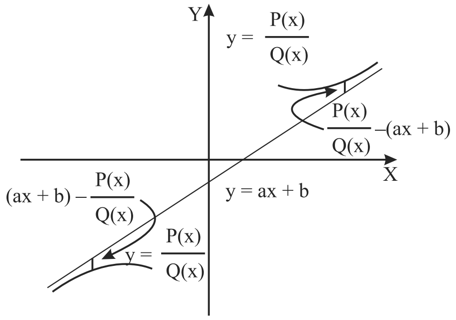
where $ax + b$ is the quotient and R(x) is the remainder.

We use the fact that the degree of the remainder R(x) is less than the degree of the divisor Q(x) to help prove

$$\lim_{x\to\infty}\left[\frac{P(x)}{Q(x)}-(ax+b)\right] = 0$$

$$\lim_{x\to-\infty}\left[\frac{P(x)}{Q(x)}-(ax+b)\right] = 0$$

As illustrated in the accompanying figure, these results tell us that the graph of the equation $y = P(x)/Q(x)$ "approaches" the line (an oblique asymptote) $y = ax + b$ as $x \to \infty$ or as $x \to -\infty$.


Y
$y = \frac{P(x)}{Q(x)}$
$\frac{P(x)}{Q(x)} - (ax + b)$
X
$(ax + b) - \frac{P(x)}{Q(x)}$
$y = ax + b$
$y = \frac{P(x)}{Q(x)}$

Example 11: Find the oblique asymptote of the curve with equation $y = \frac{x^2+4x+9}{x+2}$

Solution: By division $\frac{x^2+4x+9}{x+2} = x+2+\frac{5}{x+2}$

Because the term $\frac{5}{x+2}$ tends to zero as x approaches infinity, the oblique asymptote of the curve is the line $y = x + 2$.

Example 12: Find the asymptotes of the graph of

$$f(x) = \frac{x^2-3}{2x-4}.$$

Solution: We are interested in the behaviour as $x \to \pm\infty$ and also as $x \to 2$, where the denominator is zero. We divide $(x^2 - 3)$ by $(2x - 4)$:

$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\frac{x}{2}+1}_{\text{linear}}+\underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

Since $\lim_{x\to2^+} f(x) = \infty$ and $\lim_{x\to2^-} f(x) = -\infty$, the line $x = 2$ is a vertical asymptote.

As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to (x/2) + 1$. The line $y = (x/2) + 1$ is an asymptote both to the right and to the left.

Example 13: Show that the graph of $f(x) = (x^3 + 1)/x$ approaches the graph of $y = x^2$.

Solution: $\lim_{x\to\pm\infty}[f(x)-x^2] = 0$. This shows that the graph of f approaches the graph of $y = x^2$ and we say that the curve $y = f(x)$ is asymptotic to the parabola $y = x^2$.

Concept Problems J

1. Verify that each of the following functions has two horizontal asymptotes.

(i) $f(x) = \dfrac{|x|}{x+1}$ (ii) $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

2. Consider the function $f(x) = \dfrac{ax}{x-b}$.

(i) Determine the effect on the graph of f if $b \neq 0$ and a is varied. Consider cases where a is positive and a is negative.

(ii) Determine the effect on the graph of f if $a \neq 0$ and b is varied.

3. Find an equation of the oblique (slant) asymptotes.

(i) $y = \dfrac{x^2+1}{x+1}$ (ii) $y = \dfrac{4x^3-2x^2+5}{2x^2+x-3}$

4. Find the oblique asymptote of the curve with equation $y = \dfrac{x^2+3x+7}{x+2}$.

Solution 1 By division $\dfrac{x^2+3x+7}{x+2} = x+1+\dfrac{5}{x+2}$

Because the final term tends to zero as x grows, the asymptote is the line of the equation $y = x + 1$.

Solution 2 Follow a procedure frequently used in calculating limits at infinity:

$$\frac{x^2+3x+7}{x+2} = \frac{x+3+\frac{7}{x}}{1+\frac{2}{x}}$$

For large x, the value is approximately $x + 3$, so the asymptote should be the line of the equation $y = x + 3$. Which solution is wrong ? What is wrong ?

5. Find the asymptotes of the following curves:

(i) $y = 2x - \dfrac{\cos x}{x}$ (ii) $y = \dfrac{\ell n^2 x}{x} - 3x$

(iii) $y = \sqrt{x^3-6x^2}$

Practice Problems I

6. Find constants a and b that guarantee that the graph of the function defined by $f(x) = \dfrac{ax+5}{3-bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

7. Use the following information to find the values of a, b and c in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.

(i) The values of a, b and c are either 0 or 1.

(ii) The graph of f passes through the point $(-1, 0)$.

(iii) The line $y = 1$ is an asymptote of the graph of f.

8. Find the asymptotes of the following curves:

(i) $y = \dfrac{x^2-6x+3}{x-3}$ (ii) $y = x\tan^{-1}x$

(iii) $y = x + (\sin x)/x$ (iv) $y = \ln(4 - x^2)$

(v) $y = 2x - \cos^{-1}\dfrac{1}{x}$

9. Show that the curve $y = x - \tan^{-1}x$ has two slant asymptotes: $y = x + \pi/2$ and $y = x - \pi/2$. Use this fact to help sketch the curve.

10. Show that the curve $y = \sqrt{x^2+4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$.

11. Discuss the asymptotic behaviour of $f(x) = (x^4 + 1)/x$ w.r.t. the curve $y = x^3$. Then use this result to help sketch the graph of f.

12. Let $f(x) = (2 + 3x - x^3)/x$. Show that $y = f(x)$ approaches the curve $y = 3 - x^2$ asymptotically. Sketch the graph of $y = f(x)$ showing this asymptotic behaviour.

13. Show that, in general , the graph of the function $f(x) = \dfrac{ax^2+bx+c}{rx^2+sx+t}$ will have $y = \dfrac{a}{r}$ as a horizontal asymptote and that when $br \neq as$, the graph will cross this asymptote at the point where $x = \dfrac{at-cr}{br-as}$.

14. Consider the rational function

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0}$$

(i) If $m > n$ and $b_m \neq 0$, show that the x-axis is the only horizontal asymptote of f.

(ii) If $m = n$, show that the line $y = a_n/b_m$ is the only horizontal asymptote of f.

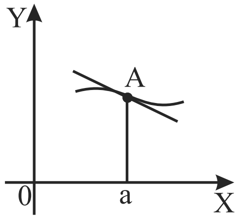
(iii) If $m < n$, is it possible for the graph to have a horizontal asymptote.

7.16 Points Of Inflection

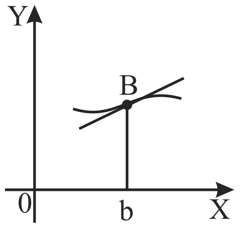
A point where the graph of function is continuous and has a tangent line and where the concavity changes is called point of inflection.

At the point of inflection either $f''(x) = 0$ or $f''(x)$ fails to exist, and $f''(x)$ changes sign about the point.

(1) Let $f''(x) < 0$ for $x < a$ and $f''(x) > 0$ for $x > a$. Then for $x < a$ the curve is concave down and for $x > a$, it is concave up. Hence, the point A of the curve with abscissa $x = a$ is a point of inflection.



(2) If $f''(x) > 0$ for $x < b$ and $f''(x) < 0$ for $x > b$, then for $x < b$ the curve is concave up, and for $x > b$, it is concave down. Hence the point B of the curve with abscissa $x = b$ is a point of inflection



Note:

(i) If $\frac{d^2y}{dx^2} > 0$ then y is concave up and if $\frac{d^2y}{dx^2} < 0$ the y is concave down.

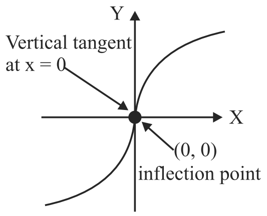
(ii) At the point of inflection, the curve crosses its tangent at that point.

(iii) A function can not have point of inflection and extrema at same point.

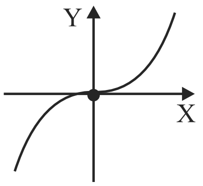
Consider the Following Examples:

(i) $f(x) = x^{1/5}$ at $x = 0$ has inflection point. y'' does not exist at $x = 0$.

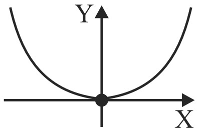
Note that $f(x)$ has a vertical tangent and the curve crosses its tangent line.



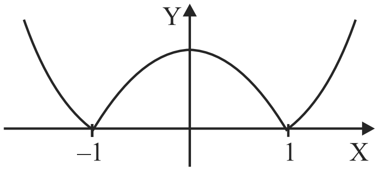
(ii) $f(x) = x^3$ at $x = 0$ has inflection point
$y'' = 0$ at $x = 0$ and changes sign



(iii) $f(x) = x^4$ at $x = 0$ has no inflection point though $y'' = 0$ at $x = 0$ as y'' does not change sign



(iv) $f(x) = |x^2 - 1|$ has no inflection point in its domain. as no tangent can be drawn at $x = \pm 1$.



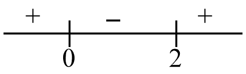
(v) $f(x) = x^4 - 4x^3 + 10$ $\quad f(1) > 0$ & $f(2) < 0$

$y' = 4x^2(x - 3) \Rightarrow y' = 0$ at $x = 0$ and $x = 3$

(change sign only at $x = 3$)

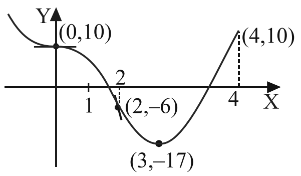
$y'' = 12x(x - 2) \Rightarrow y'' = 0$ at $x = 0$ and $x = 2$

The sign scheme of $\frac{d^2y}{dx^2}$ is shown below :



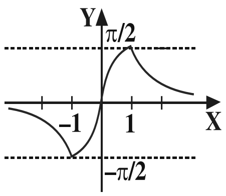
Thus, $x = 0$ and $x = 2$ are points of inflection.

Also, $x = 3$ is the point of minima as $y''(3) > 0$.



(vi) $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ has $x = 0$ as point inflection.

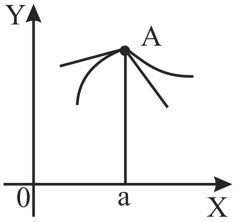
$x = \pm 1$ are not the points of inflection as no tangent line can be drawn.



Cusps

A point on a graph where the curve makes an abrupt change in direction is called a **corner**.

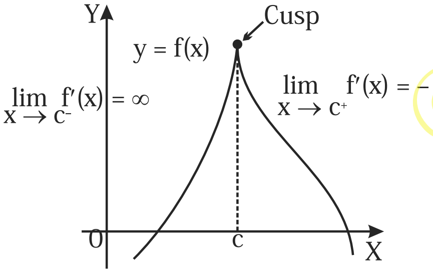
Corner points are such points of the curve, where the right-hand and the left-hand derivatives of the function f(x) have different values.



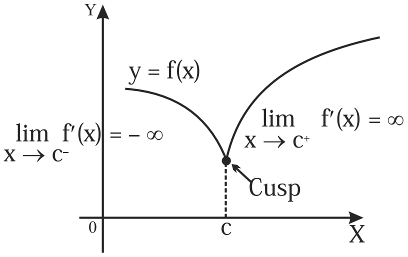
A special point of this type is called a cusp. A feature of such a point is that, as x approaches the point, the derivative f′(x) tends to ∞ on one side of the point and to – ∞ on the other side. The slope of the tangent changes in jump-like manner, and the graph does not have a definite direction of the tangent at these points.

Definition The graph of a continuous function y = f(x) has a cusp at a point x = c if the concavity is the same on both sides of c and either

(i) $\lim\limits_{x\to c^-} f'(x) = \infty$ and $\lim\limits_{x\to c^+} f'(x) = \infty$, or



(ii) $\lim\limits_{x\to c^-} f'(x) = -\infty$ and $\lim\limits_{x\to c^+} f'(x) = \infty$



A cusp can be either a local maximum as in (i) or a local minimum as in (ii).

For example, consider $y = x^{5/3} - 5x^{2/3}$.

$$y = x^{2/3}(x-5)$$

$$y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3} = (x-2)$$

$$y'' = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$$

(i) The function $y = x^{5/3} - 5x^{2/3}$ is continuous,

(ii) $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$ (see the formula for y′), and

(iii) The concavity does not change at x = 0.

This tells us that the graph has a cusp at x = 0.

Example 1: Sketch a possible graph of a differentiable function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$,
$f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$,
$f''(x) < 0$ on $(-2, 2)$

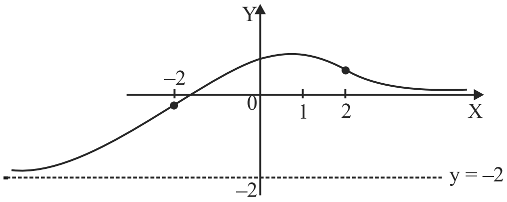
(iii) $\lim\limits_{x\to-\infty} f(x) = -2$, $\lim\limits_{x\to\infty} f(x) = 0$

Solution: Condition (i) tells us that f is increasing on (– ∞, 1) and decreasing on (1, ∞).

Condition (ii) says that f is concave upward on (– ∞, – 2) and (2, ∞), and concave downward on (– 2, 2).

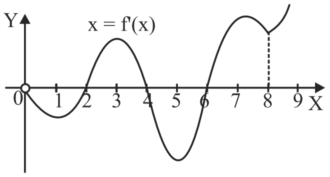
From condition (iii) we know that the graph of f has two horizontal asymptotes: y = – 2 and y = 0.

We first draw the horizontal asymptote y = – 2 and a dashed line. We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at x = 1 and decreasing toward the x-axis at the far right. We also make sure that the graph has inflection points when x = – 2 and 2. Notice that we made the curve bend upward for x < – 2 and x > 2, and bend downward when x is between – 2 and 2.



Concept Problems K

1. The graph of the first derivative f ' of a function f is shown.



(i) On what intervals is f increasing? Explain.

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is f concave upwards or concave downwards?

(iv) What are the x-coordinates of the inflection points of f ?

7.17 Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f(x). Not every item is relevant to every function (For instance, a given curve might not have an asymptote or possess symmetry.). But the guidelines provide all the information one needs to make a sketch that displays the most important aspects of the function.

A. Domain

It is often useful to start by determining the domain D of f, that is, the set of values of x for which f(x) is defined. Also find the values of the function at the points of discontinuity and the endpoints of the domain.

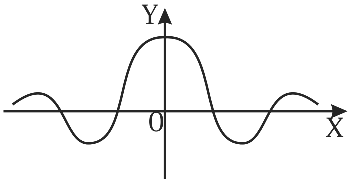
B. Intercepts

The y-intercept is f(0) and this tells us where the curve intersect the y-axis. To find the x-intercepts, we set y = 0 and solve for x. (We can omit this step if the equation is difficult to solve.)

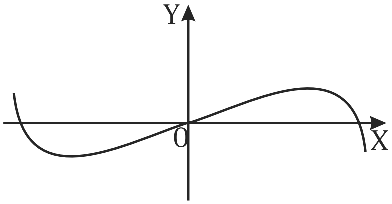
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by –x, then f is an even function and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need to only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples : $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an odd function and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$. [Rotate 180º about the origin; see Figure (b).] Some simple examples of odd functions are $y = x^3$, $y = x^5$, and $y = \sin x$.



(a) Even function



(b) Odd function

(iii) If $f(x + T) = f(x)$ for all x in D, where T is positive constant, then f is called a **periodic function** and the smallest such number T is called the period. For instance, y = sin x has period 2π and y = tan x has period π. If we know what the graph looks like in an interval of length T, then we can repeat the same segement of the graph to sketch the entire graph.

D. Asymptotes

Find the horizontal, vertical and oblique asymptotes, if available.

E. Intervals of Increase/Decrease

Compute f′(x) and find the intervals on which f′(x) is positive (f is increasing) and the intervals on which f′(x) is negative (f is decreasing).

F. Local Maximum and Minimum Value

Find the critical points of f [the number c where $f'(c) = 0$ or f′(c) does not exist]. Then use the First Derivative Test. If f′ changes from positive to negative at a critical number c, then f(c) is a local maximum. If f′ changes from negative to positive at c, then f(c) is a local minimum. We can also use the Second Derivative Test if c is a stationary point. $f''(c) > 0$ implies that f(c) is a local minimum, whereas $f''(c) < 0$ implies that f(c) is a local maximum.

G. Concavity and Points of Inflection

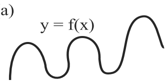
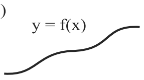
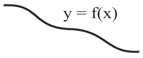
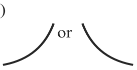
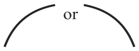
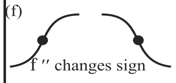
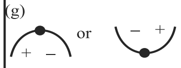
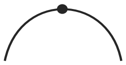
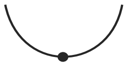
Compute f″(x). The curve is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Inflection points occur where the direction of concavity changes about a tangent.

H. Sketch the Curve

Using the infomation in items A – G, draw the graph. Sketch the asymptotes as dashed lines, Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

The following table contrasts the interpretations of the signs of f, f′, and f″. (It is assumed that f, f′, and f″ are continuous.)

Information on graphs based on derivatives :

(a)  differentiable ⇒ smooth,connected; may rise and fall	(b)  $f' > 0 \Rightarrow$ rises from left to right; may be wavy	(c)  $f' < 0 \Rightarrow$ falls from left to right; may be wavy
(d)  $f'' > 0 \Rightarrow$ concave up throughout; no waves may rise or fall	(e)  $f'' < 0 \Rightarrow$ concave down throughout; no waves; may rise or fall	(f)  ⇒ inflection point (if f is twice differentiable)
(g)  f ′ changes sign ⇒ local maximum or local minimum	(h)  $f' = 0$ and $f'' < 0$ at a point ⇒ local maximum	(i)  $f' = 0$ and $f'' > 0$ at a point ⇒ local minimum

Example 1: Use the guidelines to sketch the curve

$$y = \frac{2x^2}{x^2 - 1}.$$

Solution:

A. The domain is $x \in R - \{-1, 1\}$.

B. The x-and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2}{x^2-1} = \lim\limits_{x\to\pm\infty} \dfrac{2}{1-1/x^2} = 2.$

Therefore, the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits :

$$\lim_{x\to1^+} \frac{2x^2}{x^2-1} = \infty \qquad \lim_{x\to1^-} \frac{2x^2}{x^2-1} = \infty$$

$$\lim_{x\to-1^+} \frac{2x^2}{x^2-1} = -\infty \qquad \lim_{x\to1^-} \frac{2x^2}{x^2-1} = -\infty$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. This infomation about limits and asymptotes enables us to draw the preliminary sketch, showing the parts of the curve near the aymptotes.

E. $f'(x) = \dfrac{4x(x^2-1) - 2x^2.2x}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical point is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is local maximum by the First Derivative Test.

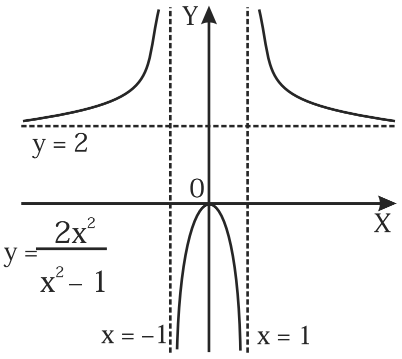
G. $f''(x) = \dfrac{-4(x^2-1)^2 - 4x.2(x^2-1)2x}{(x^2-1)^4} = \dfrac{12x^2+4}{(x^2-1)^3}$

Since $12x^2 + 4 > 0$ for all x, we have

$f''(x) > 0 \quad \Rightarrow x^2 - 1 > 0 \quad \Rightarrow |x| > 1$

and $f''(x) < 0 \Rightarrow |x| < 1$. Thus, the curve is concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave down on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the infomation in E – G, we finish the sketch.



Example 2: Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Solution:

A. Domain is $x \in (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since $\lim\limits_{x\to\infty} \dfrac{x^2}{\sqrt{x+1}} = \infty$ there is no horizontal asymptote.

Since $\sqrt{x+1} \to 0$ as $x \to 1^+$ and f(x) is always positive, we have $\lim\limits_{x\to-1^+} \dfrac{x^2}{\sqrt{x+1}} = \infty$,

and so the line $x = -1$ is a vertical asymptote.

E. $f'(x) = \dfrac{2x\sqrt{x+1} - x^2.1/(2\sqrt{x+1})}{x+1} = \dfrac{x(3x+4)}{2(x+1)^{3/2}}$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\dfrac{4}{3}$ is not in the domain of f), so the only critical point is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

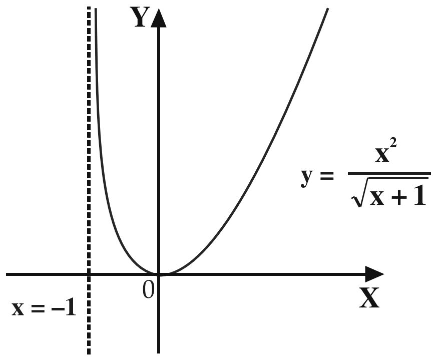
F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. $f''(x) = \dfrac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3}$

$= \dfrac{3x^2+8x+8}{4(x+1)^{5/2}}$.

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus, $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

H. The curve is sketched below.



Example 3: Sketch the graph of $f(x) = xe^x$

Solution:

A. The domain is R.

B. The x- and y-intercepts are both 0.

C. Symmetry : None

D. Because both x and e^x become large as $x \to \infty$, we have $\lim\limits_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminante product that requires the use of L'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus, the x-axis is a horizontal asymptote.

E. $f'(x) = xe^x + e^x = (x + 1)e^x$

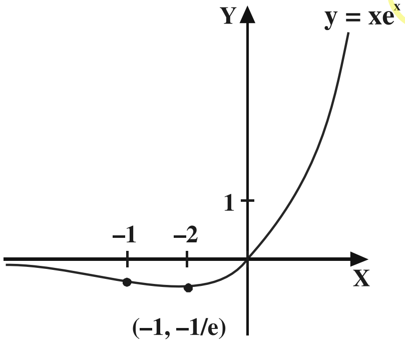
Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So, f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

F. Because $f'(-1) = 0$ and f changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum.

G. $f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.

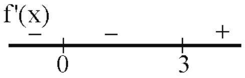
H. We use this information to sketch the curve as shown in the figure.



Example 4: Determine where the function $f(x) = x^4 - 4x^3 + 10$ is increasing and decreasing and where its graph is concave up and concave down. Find the relative extrema and inflection points and sketch the graph of f .

Solution: $f'(x) = 4x^3 - 12x^2 = 4x^2 = 4x^2(x - 3)$

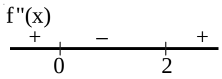
It is zero when $x = 0$ and 3. Because $4x^2 > 0$ for $x \neq 0$, we have $f'(x) < 0$ for $x < 3$ (except for $x = 0$) and $f'(x) > 0$ for $x > 3$. The pattern showing where f is increasing and where it is decreasing is displayed in the figure.



Next, to determine the concavity of the graph we compute

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

If $x < 0$ or $x > 2$, then $f''(x) > 0$ and the graph is concave up. It is concave down when $0 < x < 2$, because $f''(x) < 0$ on this interval. The concavity of the graph of f is shown in the figure.



The sign scheme tell us that there is a relative minimum at $x = 3$ and inflection points at $x = 0$ and $x = 2$(because the second derivative changes sign at these points).

To find the y-values of the critical points and the inflection points, evaluate f at x = 0, 2, and 3:

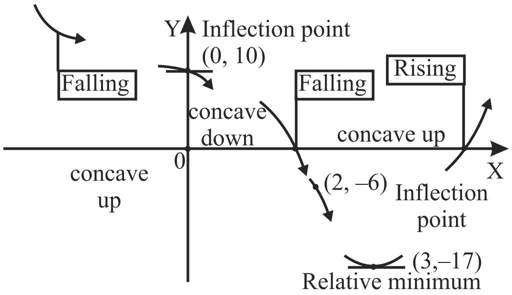
$f(0) = (0)^4 - 4(0)^3 + 10 = 10$

$f(2) = (2)^4 - 4(2)^3 + 10 = -6$

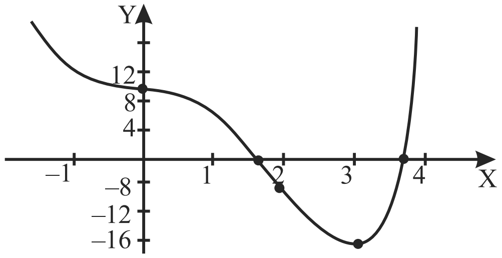
$f(3) = (3)^4 - 4(3)^3 + 10 = -17$

Finally, to sketch the graph of f, we first place a "cup" at the minimum point (3, –17) and note that (0, 10) and (2, –6) are inflection points; remember there is also a horizontal tangent at (0, 10). The preliminary graph is shown in the figure.

Now we complete the sketch by passing a smooth curve through these points, using the two sign schemes as a guide for determining where the graph is rising and falling and where it is concave up and down. The final graph is shown in the second figure.



Preliminary Sketch



Final Graph

Example 5: Sketch the graph of $f(x) = e^{1/x}$.

Solution: Notice that the domain of f is $x \in R - \{0\}$ so we check for vertical asymptotes by computing the left and right limits as $x \to 0$,

$$\lim_{x\to 0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty$$

and this shows that x = 0 is a vertical asymptote.

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{t\to\pm\infty} e^{1/x} = e^0 = 1$$

This shows that y = 1 is a horizontal asymptote.

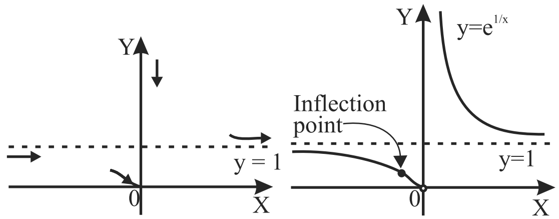
Now $f'(x) = -\dfrac{e^{1/x}}{x^2}$.

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and also on $(0, \infty)$. There is no critical point, so the function has no maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x+1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\dfrac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\dfrac{1}{2}$. So the curve is concave down on $(-\infty, -\dfrac{1}{2})$. and concave up on $(-\dfrac{1}{2}, 0)$ and on $(0, \infty)$. The inflection point is $(-\dfrac{1}{2}, e^{-2})$.

To sketch the graph of f we first draw the horizontal asymptote y = 1(as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch. These parts reflect the information concerning limits and the fact that f is decreasing in $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though f(0) does not exist. In the final graph we draw the graph by incorporating the information concerning concavity and the inflection point.



(a) Preliminary sketch (b) Final graph

Example 6: Sketch the graph of $f(x) = x^{2/3}(2x + 5)$.

Solution: We find the first and second derivatives and write them in factored form.

$f(x) = 2x^{5/3} + 5x^{2/3}$.

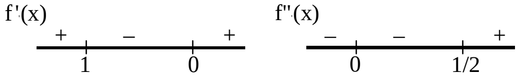
$$f'(x) = 2\left(\frac{5}{3}\right)x^{2/3} + 5\left(\frac{2}{3}\right)x^{-1/3} = \frac{10}{3}x^{-1/3}(x+1)$$

$$f''(x) = \frac{10}{3}\left(\frac{2}{3}\right)x^{-1/3} + \frac{10}{3}\left(-\frac{1}{3}\right)x^{-4/3} = \frac{10}{9}x^{-4/3}(2x-1)$$

We see $f'(x) = 0$ when $x = -1$; $f'(x)$ does not exist at x = 0. The critical points are –1 and 0.

$f''(x) = 0$ when $x = \dfrac{1}{2}$; $f''(x)$ does not exist at x = 0. The second-order (hyper) critical points are $\dfrac{1}{2}$ and 0.

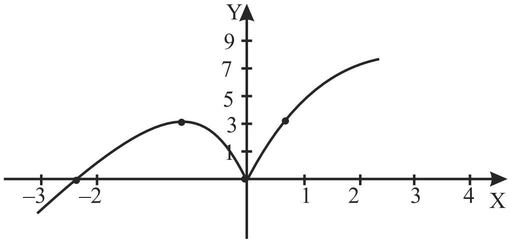
Check the intervals of increase and decrease and of concavity



These sign schemes suggest that the graph of f has a relative maximum at x = –1, a relative minimum at x = 0, and an inflection point at $x = \dfrac{1}{2}$. We must find the y-coordinates of these points by evaluating f :

$f(-1) = 3$, $\quad f(0) = 0$, $\quad f\left(\dfrac{1}{2}\right) = \dfrac{6}{\sqrt[3]{4}} \approx 3.78$.

Plot the points (–1, 3), (0, 0), and (0.5, 3.8) on the graph as shown in the figure. Note that the graph is concave down on both sides of x = 0 and that the slope f'(x) decreases without bound to the left of x = 0 and increases without bound to the right. This means the graph changes direction abruptly at x = 0, and we have a cusp at the origin. By plotting the relative extrema and the inflection point on a coordinate plane and passing a smooth curve through these points, we obtain the graph shown below.



Example 7: Plot the following curves :

(a) $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 6$ (b) $y = \dfrac{x}{\lambda n\, x}$

(c) $y = x\,\lambda n\, x$ (d) $y = \dfrac{\lambda n\, x}{x}$

(e) $y = \dfrac{x+1}{(x-1)(x-7)}$ (f) $2^{|x|}\,|y| + 2^{|x|-1} = 1$.

Solution:

(a) We have $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 6$ whose domain is $x \in R$, and $y' = x^2 - 3x + 2 = (x - 1)(x - 2)$

$y' > 0 \ \forall\ x \in (-\infty, 1)$ and $(2, \infty)$

$y' < 0 \ \forall\ x \in (1, 2)$

$\Rightarrow$ y strictly increases in $(-\infty, 1)$, strictly decreases in $(1, 2)$ and strictly increases in $(2, \infty)$.

Now we have

$$y(1) = \frac{1}{3} - \frac{3}{2} + 2 + 6 = \frac{41}{6}$$

$$y(2) = \frac{8}{3} - 6 + 4 + 6 = \frac{20}{3}$$

$y(-\infty) = -\infty$, $y(\infty) = \infty$

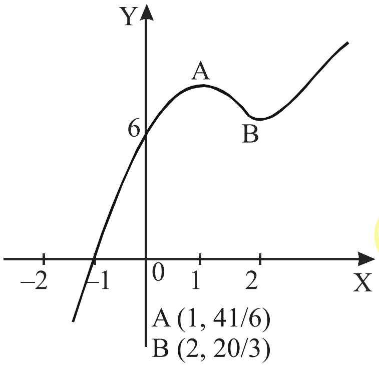
The curve cuts the Y-axis at (0, 6).

The curve cuts the –ve X-axis somewhere between –1 and –2, since

$$y(-1) = \frac{-1}{3} - \frac{3}{2} - 2 + 6 > 0$$

and $$y(-2) = \frac{-8}{3} - \frac{12}{2} - 4 + 6 < 0$$

The plot of the curve is shown below.



(b) We have $y = \frac{x}{\lambda n\, x}$

whose domain is $x \in (0, \infty) - \{1\}$, and

$$y' = \frac{\lambda n\, x - 1}{\lambda n^2\, x} < 0 \ \forall\ x \in (0, 1) \text{ and } (1, e)$$

$$> 0 \ \forall\ x \in (e, \infty)$$

$\Rightarrow$ y strictly decreases in (0, 1) and (1, e)
strictly increases in (e, ∞)

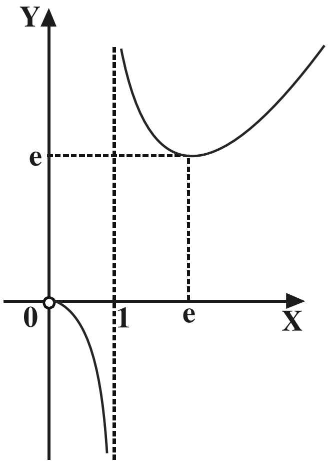
Now, we have

$$\lim_{x \to 0^+} \frac{x}{\lambda n\, x} = 0, \ y(e) = e$$

$$\lim_{x \to 0^-} \frac{x}{\lambda n\, x} = -\infty, \ \lim_{x \to 1^+} \frac{x}{\lambda n\, x} = \infty$$

$$\lim_{x \to \infty} \frac{x}{\lambda n\, x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{1}{1/x} = \infty$$

The plot of the curve is shown above.



(c) We have $y = x\, \lambda n\, x$ whose domain is $x \in (0, \infty)$, and

$$y' = 1 + \lambda n\, x < 0 \ \forall\ x \in (0, e^{-1})$$

$$> 0 \ \forall\ x \in (e^{-1}, \infty)$$

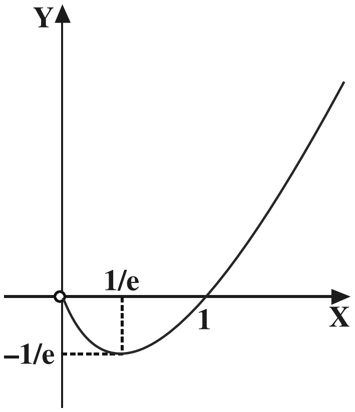
$\Rightarrow$ y strictly decreases in $(0, e^{-1})$
strictly increases in (e^{-1}, ∞).

Now, we have

$$\lim_{x \to 0^+} x\, \lambda n\, x = \lim_{x \to 0^+} \frac{\lambda n\, x}{1/x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to 0^+} \frac{1/x}{1/x^2} = 0$$

$$\lim_{x \to \infty} x\, \lambda n\, x = \infty, \ y(e^{-1}) = \frac{-1}{e}.$$

The curve cuts the X-axis at (1, 0). The plot of the curve is shown below.



(d) We have $y = \frac{\ell n\, x}{x}$

whose domain is $x \in (0, \infty)$, and

$$y' = \frac{1 - \ell n\, x}{x^2} > 0 \ \forall\ x \in (0, e)$$

$$< 0 \ \forall\ x \in (e, \infty)$$

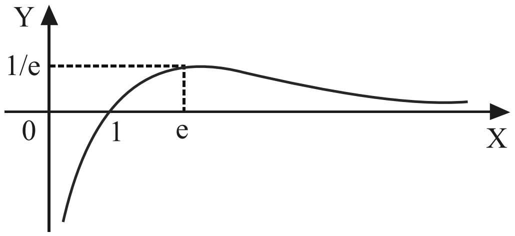
$\Rightarrow$ y strictly increases in (0, e)
strictly decreases in (e, ∞).

Now, we have

$$\lim_{x\to\infty}\frac{\ell n\,x}{x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

$$\lim_{x\to\infty}\frac{\ell n\,x}{x} - \infty, y(e) = \frac{1}{e}$$

The curve cuts the x-axis at (1, 0). The plot of the curve is shown below.

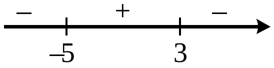


(e) We have $y = \frac{x+1}{(x-1)(x-7)}$

whose domain is $x \in R - \{1, 7\}$, and

$$y' = \frac{(x^2-8x+7)-(2x-8)(x+1)}{(x-1)^2(x-7)^2}$$

$$= \frac{-(x^2+2x-15)}{(x-1)^2(x-7)^2} = \frac{-(x+5)(x-3)}{(x-1)^2(x-7)^2}$$



⇒ y strictly decreases in $(-\infty, -5)$,
strictly increases in $(-5, 1)$ and $(1, 3)$
strictly decreases in $(3, 7)$ and $(7, \infty)$.

Now, we have

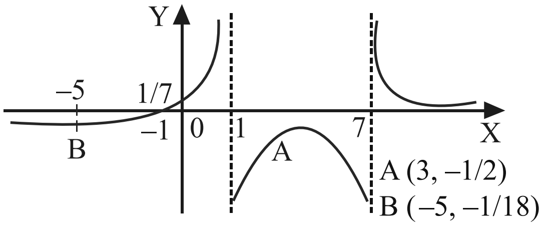
$$y(3) = \frac{4}{(2)(-4)} = \frac{-1}{2}, \; y(-5) = \frac{-4}{(-6)(-12)} = \frac{-1}{18}$$

$$\lim_{x\to1^-}\frac{x+1}{(x-1)(x-7)} = \infty, \; \lim_{x\to1^+}\frac{x+1}{(x-1)(x-7)} = -\infty$$

$$\lim_{x\to7^-}\frac{x+1}{(x-1)(x-7)} = -\infty, \; \lim_{x\to7^+}\frac{x+1}{(x-1)(x-7)} = \infty$$

$$\lim_{x\to\infty}\frac{x+1}{(x-1)(x-7)} = 0, \; \lim_{x\to-\infty}\frac{x+1}{(x-1)(x-7)} = 0$$

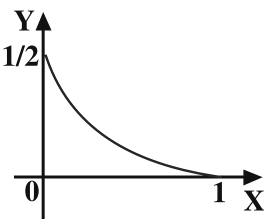
The curve cuts the y-axis at (0, 1/7). The curve cuts the x-axis at (–1, 0). The plot of the curve is shown below.

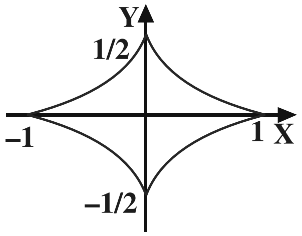


(f) We have $2^{|x|}|y| + 2^{|x|-1} = 1$

i.e. $|y| = 2^{-|x|} - \frac{1}{2}$

The curve is symmetrical about the x-axis as well as the y-axis. In the first quadrant the equation of the curve reduces to $y = 2^{-x} - 1/2$ whose plot is shown below





The complete curve is drawn by taking the mirror image of the above curve in the x-axis and the y-axis as shown alongside.

Example 8: Draw the graph of $y = \cos^{-1}(4x^3 - 3x)$.

Solution: For $y = \cos^{-1}(4x^3 - 3x)$.
Domain is $x \in [-1, 1]$ and range is $y \in [0, \pi]$

(i) If $\frac{1}{2} \le x \le 1$, $y = 3\cos^{-1}x$.

$$\Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2} \qquad ...(1)$$

$$\Rightarrow \frac{dy}{dx} < 0 \text{ if } x \in \left(\frac{1}{2}, 1\right)$$

⇒ y decreases if $x \in \left(\frac{1}{2}, 1\right)$

Now, $\frac{d^2y}{dx^2} = -\frac{3x}{(1-x^2)^{3/2}}$

$$\Rightarrow \frac{d^2y}{dx^2} < 0 \text{ if } x \in \left(\frac{1}{2}, 1\right)$$

⇒ the graph is concavity down if $x \in \left(\frac{1}{2}, 1\right)$.

(ii) If $-\frac{1}{2} \le x < \frac{1}{2}$, $y = 2\pi - 3\cos^{-1}x$.

$$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} > 0 \text{ if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

⇒ y increases if $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

and $\frac{d^2y}{dx^2} = \frac{3x}{(1-x^2)^{3/2}}$

(a) If $x \in \left(-\frac{1}{2}, 0\right)$ then $\frac{d^2y}{dx^2} < 0$

⇒ the graph is concavity down if

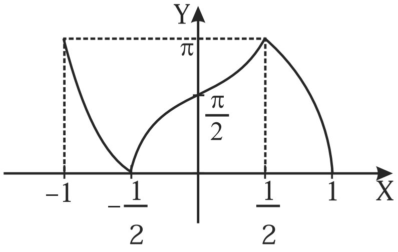
$x \in \left(-\frac{1}{2}, 0\right)$

(b) If $x \in \left(0, \frac{1}{2}\right)$ then $\frac{d^2y}{dx^2} > 0$

⇒ the graph is concavity up if $x \in \left(0, \frac{1}{2}\right)$

(iii) Similarly, if $-1 \le x < -\frac{1}{2}$ then $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$.

∴ The graph of $y = \cos^{-1}(4x^3 - 3x)$ is as follows:



Example 9: Draw the graph of $f(x) = \frac{6\sin x}{2+\cos x}$.

Solution: The function $f(x) = \frac{6\sin x}{2+\cos x}$ is defined for all $x \in R$. It is an odd function, and periodic with a period 2π. It is, therefore, sufficient to construct its graph on the interval $[0, \pi]$. Since $2 + \cos x \ge 1$, the function possesses no vertical asymptotes.

We find the derivative

$$f'(x) = 6\frac{\cos x(2+\cos x) - \sin x(-\sin x)}{(2+\cos x)^2} = 6$$

$$= \frac{2\cos x + 1}{(2+\cos x)^2}$$

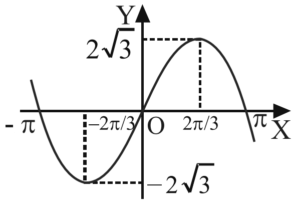
The point $x = 2\pi n \pm 2\pi/3$, $n \in I$, are critical.

On the interval $(0, 2\pi/3)$ the derivative is positive and, consequently, the function f(x) increases.

On the interval $(2\pi/3, \pi)$ the derivative is negative and the function f(x) decreases.

The point $x = 2\pi/3$ is a point of maximum. In fact at the points $x = 2\pi n + 2\pi/3$ $n \in I$, the function f(x) possesses maxima and at the points $x = 2\pi n - 2\pi/3$, $n \in I$, it possesses minima.

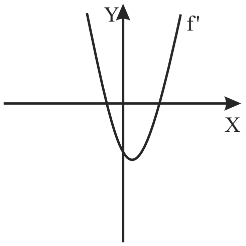
The graph of the function f(x) is shown below.



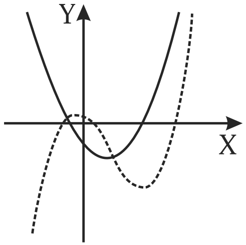
Example 10: Drawing the graph of a function, given the graph of its derivative.

Consider the graph of f', as shown in the figure below. Sketch a possible graph of f.

Solution: When the derivative is positive, the graph of f is rising, and when the derivative is negative, the graph of f is falling. Here, we also see that at places where the derivative is zero, the graph has a relative maximum to the left of the y-axis and a relative minimum to the right of the y-axis, as shown in the figure.



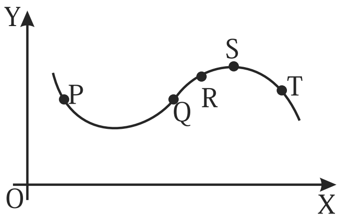
Graph of derivative of a function f



Graph of a function and its derivative

Concept Problems L

1. The figure shows a portion of the graph of a twice differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



2. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0, y'' > 0$
2	1	$y' = 0, y'' > 0$
$2 < x < 4$		$y' > 0, y'' > 0$
4	4	$y' > 0, y'' = 0$
$4 < x < 6$		$y' > 0, y'' < 0$
6	7	$y' = 0, y'' < 0$
$x > 6$		$y' < 0, y'' < 0$

3. Sketch the general appearance of the graph of the given function near (1,1) on the basis of the information given. Assume that f, f′ and f″ are continuous.

(i) $f(1) = 1, f'(1) = 0, f''(1) = 1$

(ii) $f(1) = 1, f'(1) = 0, f''(1) = 0$
(Sketch four possibilities)

(iii) $f(1) = 1, f'(1) = 0, f''(1) = 1$ and $f''(x) < 0$ for x near 1.

(iv) $f(1) = 1, f'(1) = 1, f''(1) = 0$ and $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.

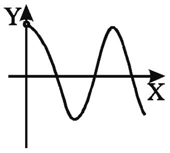
4. Find the relative extrema of the function and sketch the graph of $f(x) = (x^2 - 3)e^{-x}$.

5. Draw the graph of $f(x) = \dfrac{x^2 - 3}{2x - 4}$.

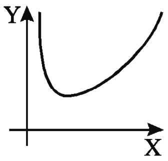
6. The graph of the given function f(x) for x > 0 is one of the six curves shown in the figure. In each case, match the function to a graph.

(i) $f(x) = x2^{-x}$ (ii) $f(x) = \dfrac{\ell n\sqrt{x}}{x}$

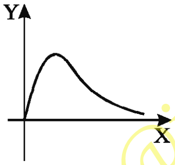
(iii) $f(x) = \dfrac{e^x}{x}$ (iv) $f(x) = e^{-x} \sin x$



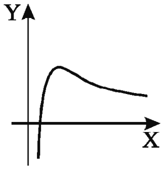
(a)



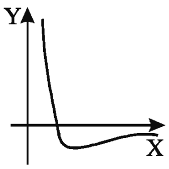
(b)



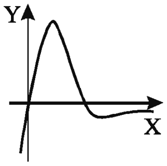
(c)



(d)

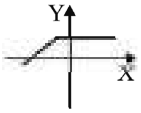


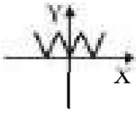
(e)

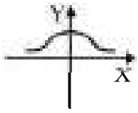


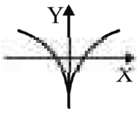
(f)

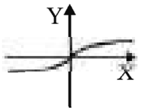
7. Match the graphs of the functions shown in (a)–(f) with the graphs of their derivatives in (A)–(F).

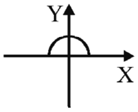
(a)


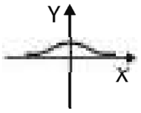
(b)


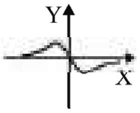
(c)


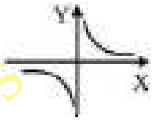
(d)


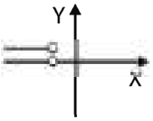
(e)


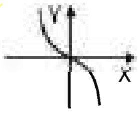
(f)


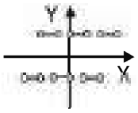
(A)


(B)


(C)


(D)


(E)


(F)


8. Prove that if $a \neq b$, then the function

$f(x) = \dfrac{1}{(x-a)(x-b)}$ is symmetrical about the line $x = (a + b)/2$.

9. Sketch the graph of a function f such that for all x

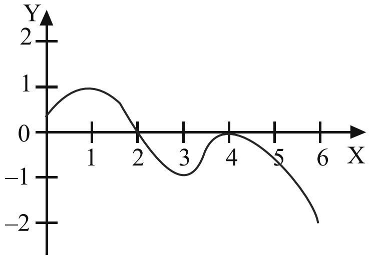
(a) $f(x) > 0, f'(x) > 0, f''(x) > 0$

(b) $f'(x) < 0, f''(x) < 0$

(c) Can there be a function such that for all x, $f(x) > 0$, $f'(x) < 0$ and $f''(x) < 0$? Explain.

Practice Problems J

10. The following questions refer to the graph below, which is the graph of the derivative f′ of a certain function f defined on [0, 6].



(i) Find the x-coordinate of the global maximum of f.

(ii) how many inflection points does the graph of f have ?

11. Find constants A, B and C that guarantee that the function $f(x) = Ax^3 + Bx^2 + C$ will have a relative extremum at (2, 11) and an inflection point at (1, 5). Sketch the graph of f.

12. Draw the graph of the following functions :

(i) $y = x^2 - (x^3/6)$ (ii) $y = x^3 - 3x + 2$.

13. Investigate the behaviour of the following functions with the aid of their derivatives and sketch their graphs.

(a) $y = (x+2)^2 (x-1)^2$ (b) $y = \dfrac{x}{1+x^2}$

(c) $y = \dfrac{x}{x^2-1}$ (d) $y = \dfrac{x^3}{3-x^2}$

14. Investigate and draw the graphs:

(i) $y = \sqrt{x^2+1} - x$ (ii) $y = \dfrac{x}{\sqrt{x^2+1}}$

(iii) $y = \dfrac{\sqrt{1-x^2}}{x}$ (iv) $y = x - 3x^{1/3}$

15. Investigate and draw the graphs:

(i) $y = x^4 + 4x^3$ (ii) $y = \dfrac{x}{x^2+9}$

(iii) $y = x\sqrt{5-x}$

16. Investigate and draw the graphs :

(i) $y = x + \sqrt{|x|}$

(ii) $y = x\tan x, -\pi/2 < x < \pi/2$

(iii) $y = \sin 2x - 2\sin x$

17. Draw the graph of the following functions :

(i) $f(x) = \dfrac{x^2-x+1}{x^2+x+1}$

(ii) $f(x) = \dfrac{(x+2)(x-1)}{x\,(x+1)}$

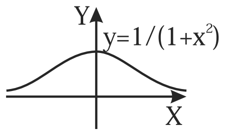
(iii) $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3}$ (iv) $f(x) = \dfrac{x^3-1}{x^2-1}$

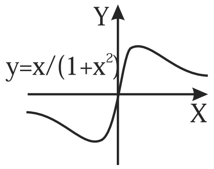
18. Draw the graph of $f(x) = 2 + \dfrac{\sin x}{x}$. Show that the slope of the curve approaches the slope of the asymptote as $x \to \infty$.

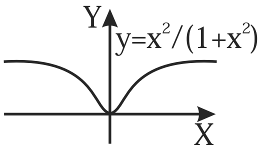
19. Draw the curves :

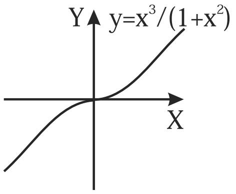
(i) $y = \sqrt[3]{x} - \sqrt[3]{x+1}$ (ii) $y = x + \ln(x^2 - 1)$

20. Show that the curves $y = \dfrac{x^n}{1+x^2}$ for n = 0, 1, 2 and 3 are as follows:



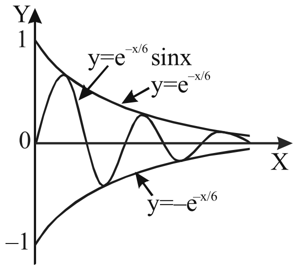






21. The figure shows the graph of the function $f(x) = e^{-x/6} \sin x$, together with the graphs of its "envelope curves" $y = e^{-x/6}$ and $y = -e^{-x/6}$.

(i) Find the first local maximum point and the first local minimum point on the graph of f for $x > 0$.



(ii) Find the first two points of tangency of the curve $y = e^{-x/6} \sin x$ with the two envelope curves shown in figure. Are these the same as the two local extreme points found above.

22. Draw the graphs of the following functions:

(i) $y = \dfrac{\sin x}{1+\cos x}$ (ii) $y = 1/(1+e^{-x})$

(iii) $y = \ell n(\sin x)$

23. Draw the graphs of the following functions:

(i) $y = \sin^{-1}(1/x)$ (ii) $y = e^x + e^{-3x}$

24. Sketch the graph of the following functions:

(i) $f(x) = \sin 2x - \sin x, [-\pi, \pi]$

(ii) $f(x) = x^2 \sqrt[n]{x}$ (iii) $f(x) = x(e^{-2x} + e^{-x})$

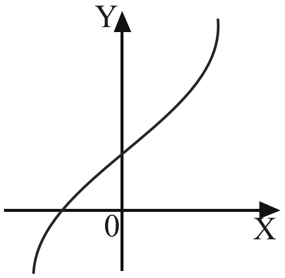
(iv) $f(x) = x^2 e^{-x}$.

7.18 Isolation of Roots

Nature of Roots of a Cubic Equation

Consider a cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a > 0$.
Let $f(x) = ax^3 + bx^2 + cx + d$, then $f'(x) = 3ax^3 + 2bx + c$

Case 1 The equation $f'(x) = 0$ has no real roots. Then f(x) always increases as x increases and the equation $f(x) = 0$ has one real root.

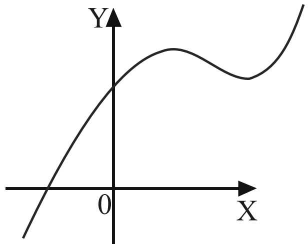


For example, $f(x) = x^3 - 2x^2 + 5x + 4$

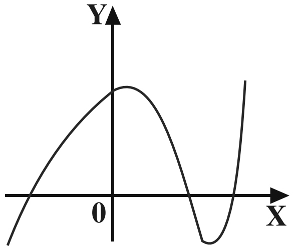
$f'(x) = 3x^2 - 4x + 5 > 0 \ (D < 0)$

Hence, f is monotonous and it has one real root.

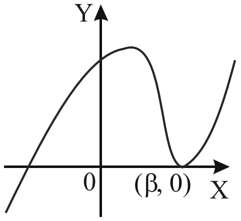
Case 2 The equation $f'(x) = 0$ has two distinct roots α and β $(\alpha < \beta)$. If $f(\alpha).\, f(\beta) > 0$, then the equation $f(x) = 0$ has one real root.



Case 3 The equation $f'(x) = 0$ has two distinct roots α and β. If $f(\alpha)\, f(\beta) < 0$ then the equation $f(x) = 0$ has three distinct real roots.



Case 4 The equation $f'(x) = 0$ has two distinct roots α and β and $f(\beta) = 0$ also. In this case $f(x) = 0$ has a repeated(double) real root and one other real root.



In general, the equation $g(x) = 0$ has a repeated root $x = \alpha$ if $g(\alpha) = g'(\alpha) = 0$.

Case 5 The equation $f'(x) = 0$ has two equal roots α and $f(\alpha) = 0$ also. In this case, $f(x) = 0$ has one repeated (treble) root i.e. there are three coincident real roots.

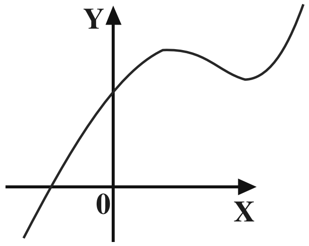
For example, $f(x) = (x - 2)^3$, $f'(x) = 3(x - 2)^2 = 0$

$\Rightarrow$ $x = 2, 2$ and $f(2) = 0$. Hence, $x = 2$ is a root repeated three times.

⊖ **Example 1:** Find all possible values of a for which the cubic $f(x) = x^3 + ax + 2$ is non-monotonic and has exactly one real root.

✓ **Solution:** For f to be non-monotonic with exactly one real root, a possible graph is shown below, where.

$f(\alpha)\,.\, f(\beta) > 0$, α and β being roots of $f'(x) = 0$.



Now $f(x) = x^3 + ax + 2$

$\Rightarrow$ $f'(x) = 3x^2 + a$

If $a \geq 0$, $f(x)$ is always increasing.

For f to be non-monotonic, $f'(x) = 0$ should have two distinct roots which is possible when $a < 0$

$$\Rightarrow \quad x = \pm\sqrt{-\frac{a}{3}} = \pm\sqrt{b} \quad \text{where } b = -\frac{a}{3}$$

For exactly one real root we should have

$$f\left(\sqrt{b}\right)\,.\, f\left(-\sqrt{b}\right) > 0$$

$$(b^{3/2} + a\,b^{1/2} + 2)\,(-b^{3/2} - a\,b^{1/2} + 2) > 0$$

or, $(b^{3/2} + a\,b^{1/2})^2 - 4 < 0$

or, $b^3 + a^2 b + 2\,a\,b^2 - 4 < 0$

Now substituting $b = -\dfrac{a}{3}$ we get $a^3 + 27 > 0$.

But $a < 0 \Rightarrow a \in (-3, 0)$.

⊖ **Example 2:** Let 'p' and 'q' be real numbers. Prove that the cubic $y = x^3 + p\,x + q$ has three distinct real roots, if $4p^3 + 27q^2 < 0$.

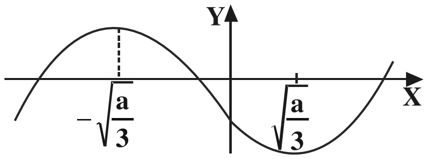
✓ **Solution:** Let $f(x) = x^3 + p\,x + q \Rightarrow f'(x) = 3x^2 + p$

If $p > 0$

$\Rightarrow$ f' has no real root $\Rightarrow$ $f(x)$ is monotonic and it cannot have three real roots.

$$\text{If } p < 0 \text{ then } x = \pm\sqrt{-\frac{p}{3}} = \pm\sqrt{\frac{a}{3}} \quad (a = -p)$$

For three distinct real roots $f(x)$ must have opposite signs at maxima and minima.



$$\text{Hence, } f\left(\sqrt{\frac{a}{3}}\right).\, f\left(-\sqrt{\frac{a}{3}}\right) < 0$$

$$\left[\left(\frac{a}{3}\right)^{3/2} + p\sqrt{\frac{a}{3}} + q\right] \times \left[-\left(\frac{a}{3}\right)^{3/2} - p\sqrt{\frac{a}{3}} + q\right] < 0$$

$$\Rightarrow \left(\left(\frac{a}{3}\right)^{3/2} + \sqrt{\frac{a}{3}}\,p + q\right).\left(\left(\frac{a}{3}\right)^{3/2} + \sqrt{\frac{a}{3}}\,p + q\right) > 0$$

$$\Rightarrow \left(\left(\frac{a}{3}\right)^{3/2} + p\sqrt{\frac{a}{3}}\right)^2 - q^2 > 0$$

$$\Rightarrow \left(\frac{a}{3}\right)^3 + p^2\frac{a}{3} + 2p\left(\frac{a}{3}\right)^2 - q^2 > 0$$

$$\Rightarrow \frac{a^3}{27} + \frac{p^2 a}{3} + \frac{2pa^2}{9} - q^2 > 0$$

$$\Rightarrow a^3 + 9p^2 a + 6pa^2 - 27q^2 > 0$$

$$\Rightarrow -p^3 - 9p^3 + 6p^3 - 27q^2 > 0 \ (a = -p)$$

$$\Rightarrow -4p^2 - 27q^2 > 0$$

$$\Rightarrow 4p^3 + 27q^2 < 0.$$

Example 3: In a triangle the area Δ and the semi-perimeter s are fixed. Show that any maximum or minimum of one of the sides is a root of the equation $s(x - s)x^2 + 4\Delta^2 = 0$. Discuss the reality of the roots of this equation and whether they correspond to maxima or minima.

Solution: The equations $a + b + c = 2s$ and $s(s - a)(s - b(s - c) = \Delta^2$ determine a and b as function of c.

We differentiate with respect to c, and suppose that $da/dc = 0$. It is found that $b = c$, $s - b = s$ $c = \frac{1}{2}a$, from which we deduce that $s(a - s)a^2 + 4\Delta^2 = 0$.

This equation has three real roots if $s^4 > 27\Delta^2$, and one if $s^4 < 27\Delta^2$. In an equilateral triangle (the triangle of minimum perimeter for a given area) $s^4 = 27\Delta^2$; thus it is impossible that $s^4 < 27\Delta^2$. Hence the equation in a has three real roots, and since their sum is positive and their product negative, two roots are positive and the third negative. Of the two positive roots one corresponds to a maximum and one to a minimum.

Geometric Implications of Multiplicity

A root $x = \alpha$ of a polynomial $p(x)$ has multiplicity m if $(x - \alpha)^m$ divides $p(x)$ but $(x - \alpha)^{m+1}$ does not. A root of mulitplicity 1 is called a simple root.

If α is a root of multiplicity m for a polynomial $P(x)$ then $P(\alpha) = 0$, $P'(\alpha) = 0, \ldots\ldots, P^{(m-1)}(\alpha) = 0$ and $P^{(m)}(\alpha) \neq 0$.

There is a close relationship between the multiplicity of a root of a polynomial and the behaviour of the graph in the vicinity of the root.

If α is a simple root of a polynomial equation $P(x) = 0$ the graph of the function $y = P(x)$ intersects the x-axis without tangency and that, if α is a two-fold root, the graph touches the x-axis and in a sufficiently small neighbourhood of the point of contact α it lies entirely on one side of the x-axis (this means that α is a point of extremum of the function $y = P(x)$). Similarly, for a three fold root α, the point $(\alpha, 0)$ on the x-axis is a point of inflection of the graph, the tangent line at this point coinciding with the x-axis. The case of a root α of multiplicity $k > 3$ can also be interpreted geometrically in a similar manner.

Suppose that $p(x)$ is a polynomial with a root of multiplicity m at $x = \alpha$.

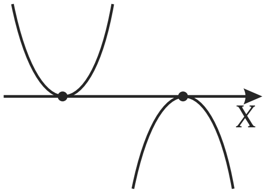
(a) If m is even, then the graph of $y = p(x)$ is tangent to the x-axis at $x = \alpha$, does not cross the x-axis there, and does not have an inflection point there.

(b) If m is odd and greater than 1, then the graph is tangent to the x-axis at $x = \alpha$, crosses the x-axis there, and also has an inflection point there.

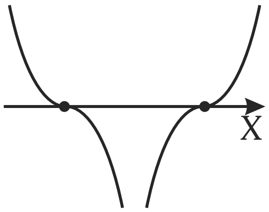
(c) If $m = 1$(so that the root is simple), then the graph is not tangent to the x-axis at $x = \alpha$, crosses the x-axis there, and may or may not have an inflection point there.

The relationship, stated above, is illustrated in the figures.

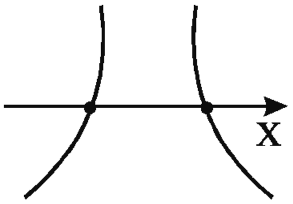
Roots of even multiplicity



Roots of odd multiplicity (>1)



Simple roots



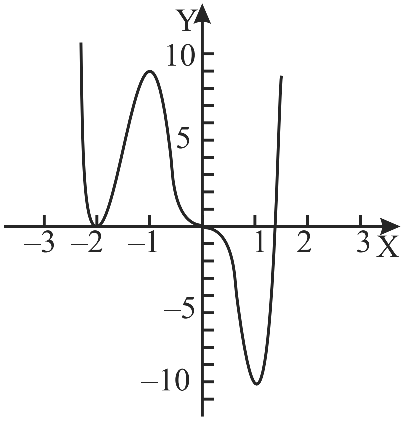
Example 4: Imagine about the behaviour of the graph of $y = x^3(3x - 4)(x + 2)^2$ in the vicinity of its x-intercepts, and test your imagination by generating the graph.

Solution: The x-intercepts occur at $x = 0$, $x = \frac{4}{3}$, and $x = -2$. The root $x = 0$ has multiplicity 3, which is odd, so at that point the graph should be tangent to the x-axis, cross the x-axis, and have an inflection point there.

The root $x = -2$ has multiplicity 2, which is even, so the graph should be tangent to but not cross the x-axis there.

The root $x = \frac{4}{3}$ is simple, so at that point the curve should cross the x-axis without being tangent to it.

Graph of $y = x^3(3x - 4)(x + 2)^2$



All of the points about the graph is consistent with the graph shown in the figure.

Example 5: Let f(x) be a function that has a continuous derivative on [a, b], f(a) and f(b) have opposite signs, and $f'(x) \neq 0$ for all numbers x between a and b, $(a < x < b)$. Find the number of solutions of the equationf(x) = 0.

Solution: As f(x) has continuous derivative $\forall\, x \in [a, b]$ f(x) is continuous and as f (a) and f (b) have opposite signs there exists atleast one value $x = c$, $a < c < b$ such that $f(c) = 0$. Also since $f'(x) \neq 0$, $f'(x) > 0$ or $f'(x) < 0$ for all x in the given interval which implies that the function is increasing or decreasing. Hence, it will have only one real root.

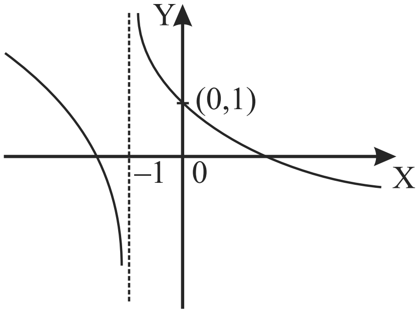
Example 6: Find the number of roots of the function $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x$.

Solution: $f'(x) = -\dfrac{3}{(x+1)^4} - 3 + \cos x < 0$

Hence f (x) is always decreasing.

$$\lim_{x\to -1^-} f(x) = -\infty,\ \lim_{x\to -1^+} f(x) = \infty.$$

Also as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$. The graph is shown in the figure.



Hence, f has one positive and one negative root.

Concept Problems

M

1. Prove that the equation $x \sin^{-1}x = 0$ has only one real root $x = 0$.
2. Show that the equation $x^3 - 3x^2 + 6x - 1 = 0$ has only one real simple root belonging to the interval (0, 1).
3. Examine whether the equation $x^3 - 12x + 16 = 0$ has double roots.
4. A polynomial p has a local maximum (–2, 4), a local minimum at (1, 1), a local maximum at (5, 7) and no other critical points. How many real zeroes does p have?
5. Suppose that g(x) is a function that is differentiable for all real numbers x and that g(x) has the following properties :
   - (i) $g(0) = 2$ and $g'(0) = -\dfrac{2}{3}$.
   - (ii) $g(4) = 3$ and $g'(4) = 3$.
   - (iii) g(x) is concave up for $x < 4$ and concave down for $x > 4$.
   - (iv) $g(x) \geq -10$ for all x.

   Now answer the questions:
   - (a) How many zeros does g have ?
   - (b) How many zeros does g′ have ?
   - (c) Exactly one of the following limits is possible:

   $$\lim_{x\to\infty} g'(x) = -5,\ \lim_{x\to\infty} g'(x) = 0,\ \lim_{x\to\infty} g'(x) = 5$$

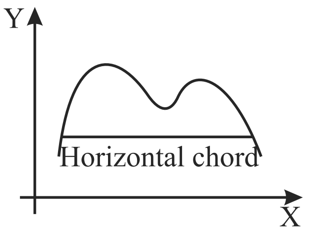
   Identify which of these results is possible and draw a rough sketch of the graph of such a function g(x). Explain why the other two results are impossible.
6. Prove that when x increases through a simple root of $\phi(x) = 0$, $\phi(x)$ and $\phi'(x)$ will have opposite signs just before, and the same sign just after, the passage. Does this hold in the case of a double root.
7. Let f and g be polynomials without a common root.
   - (a) Show that if the degree of g is odd, the graph of f/g has a vertical asymptote.
   - (b) Show that if f and g have the same degree, the graph of f/g has a horizontal asymptote.
   - (c) Show that if the degree of f is less than the degree of g, the graph of f/g has a horizontal asymptote.

Practice Problems K

8. Show that the equation $2x^5 - 10x + 5 = 0$ has exactly three real roots.
9. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.
10. Show that the equation $x^4 + 4x + c = 0$ has atmost two real roots.
11. Show that the equation $x^5 - 6x + 3 = 0$ has exactly three real roots.
12. Show that the equation $x^4 + 3x^2 - x - 2 = 0$ has two (and only two) real simple roots each belonging to one of the following intervals $(-1, 0)$ and $(0, 1)$.
13. Show that the equation $x^4 + 2x^2 - 6x + 2 = 0$ has two (and only two) real simple roots, each belonging to one of the two intervals $(0, 1)$ and $(1, 2)$.
14. Show that the equation $x^2 \tan^{-1}x = a$, where $a \neq 0$, has one real root.
15. Prove that the equation $e^x = ax + b$ has one real root if $a < 0$ or $a = 0, b > 0$. If $a > 0$ then it has two real roots or none, according as $a \ln a > b - a$ or $a \ln a < b - a$.
16. Show by graphical considerations that the equation $e^x = ax^2 + 2bx + c$ has one, two, or three real roots if $a > 0$, none, one, or two if $a < 0$.
17. Prove that the equation $a^2e^x = x^2$ has three real roots if $a^2 < 4e^{-2}$.

7.19 Rolle's Theorem

A line segment joining two points on the graph of a function f is called a chord of f. Assume that a certain differentiable function f has a chord parallel to the x-axis, as shown in the figure. It seems reasonable that the graph will then have atleast one horizontal tangent line. In the figure shown, there are three such lines tangent to the graph. This is substance of the next theorem.



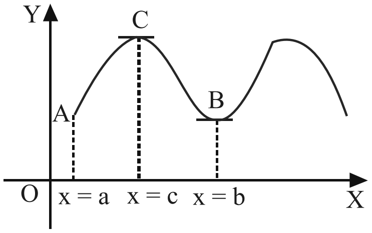
Rolle's Theorem

Let f be a function that satisfies the following three hypotheses :

1. f is continuous on the closed interval [a, b].
2. f is differentiable on the open interval (a, b).
3. f(a) = f(b)

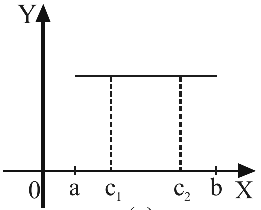
Then there is a number c in (a, b) such that f′(c) = 0.

Geometrical Meaning of Rolle's Theorem

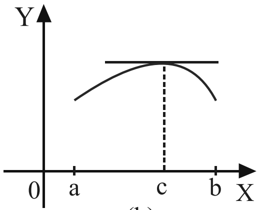


If the graph of y = f (x) has the equal ordinates at two points x = a and x = b, and if the graph be continuous throughout the interval from a to b and if the curve has a tangent at every point on it from a to b except possibly at the two extreme points a and b, then there must exist atleast one intervening point $c \in (a, b)$ at which the tangent is parallel to the x-axis.

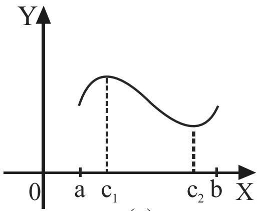
Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. The figure below shows the graph of four such functions. In each case it appears that there is at least one point (c, f(c)) on the graph where the tangent is horizontal and therefore f′(c) = 0. Thus, Rolle's Theorem is plausible.



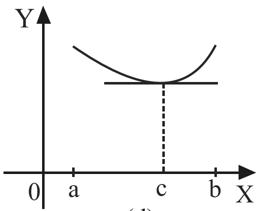
(a)



(b)



(c)



(d)

Proof There are three cases :

Case 1 f(x) = k, a constant

Then f′(x) = 0, so the number c can be taken to be any number in (a, b) [as in Figure (a)]

Case 2 f(x) > f(a) for some x in (a, b) [as in Figure (b) or (c)] By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in [a, b]. Since f(a) = f(b), it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable at c. Therefore, f′(c) = 0 by Fermat's Theorem.

Case 3 f(x) < f(a) for some x in (a, b) [as in Figure 1(c) or (d)] By the Extreme Value Theorem, f has minimum value in [a, b] and, since f(a) = f(b), it attains this minimum value at a number c in (a, b). Again f′(c) = 0 by Fermat's Theorem.

Physical Meaning of Rolle's Theorem

Assume that x is time and f(x) is the coordinate of a point, which moves along a straight line at the moment x. At the initial moment x = a the point has a coordinate f(a), then moves in a certain way with velocity $f'(x)$ and returns to the point with the coordinate f(a) at the moment x = b,[f(b) = f(a)]. It is clear that to return to the point f(a), it must stop at a certain moment (before "turning back"), i.e. at a certain moment x = c, the velocity $f'(c) = 0$.

Example 1: Verify Rolle's theorem for $f(x) = x(x + 3)e^{-x/2}$ in [–3, 0].

Solution: We have $f(x) = x(x + 3)e^{-x/2}$

$$\therefore \quad f'(x) = (2x + 3)e^{-x/2} + x(x + 3)e^{-x/2}\left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$$

which exists for every value of x in the interval [–3, 0]

Hence f(x) is differentiable and hence continuous in the interval [–3, 0]. Note that only differentiability over (–3, 0) is required. Also f(–3) = f(0) = 0.

Thus, all the three conditions of Rolle's theorem are satisfied. Hence, f '(x) = 0 for atleast one value of x lying in the open interval (–3, 0).

For $f'(x) = 0$, $-\frac{1}{2}((x^2 - x - 6)e^{-x/2} = 0$

$\Rightarrow \; x^2 - x - 6 = 0, \; e^{-x/2} \neq 0,$

$\Rightarrow \; (x - 3)(x + 2) = 0 \qquad \Rightarrow x = -2, 3$

Since the value x = –2 lies in the open interval (–3, 0), the theorem is verified.

Example 2: Suppose that $f(x) = x^{1/2} - x^{3/2}$ on [0, 1]. Find a number c that satisfies the conclusion of Rolle's theorem.

Solution: Note that f is continuous on [0, 1] and differentiable on (0, 1). Because the term $x^{1/2}$ is present, f is not differentiable at x = 0, but this is irrelevant. Also, f(0) = 0 = f(1), so all of the hypotheses of Rolle's theorem are satisfied. Finally,

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2}x^{-1/2}(1 - 3x).$$

So, we see f '(c) = 0 for $c = \frac{1}{3}$.

Example 3: Is Rolle's theorem valid for the function $f(x) = \sqrt[3]{8x - x^2}$ in [0, 8] ?

Solution: The function $f(x) = \sqrt[3]{8x - x^2}$ is continuous in [0, 8] and possesses a derivative

$$f'(x) = \frac{8 - 2x}{3\sqrt[3]{(8x - x^2)^2}}$$ for x ≠ 0, x ≠ 8, that is, differentiable on the open interval (0, 8).

In addition, f(0) = f(8) = 0.

Thus, on the interval [0, 8], Rolle's theorem is valid; indeed, f '(x) = 0 for x = c = 4 where 0 < 4 < 8.

Example 4: Let n ∈ N. If the value of c prescribed in the Rolle's theorem for the function $f(x) = 2x(x - 3)^n$ on [0, 3] is 3/4 then find n.

Solution: $f(x) = 2x(x - 3)^n$ is continuous and differentiable.

f (0) = f (3).

Hence, f′(c) = 0

$f'(x) = 2[(x - 3)^n + nx(x - 3)^{n-1}]$

$f'(c) = 2[(c - 3)^n + nc(c - 3)^{n-1}] = 0$

or $2(c - 3)^{n-1}[c - 3 + nc] = 0$; but c = 3/4

$$\therefore \quad (n + 1)\frac{3}{4} = 3 \;\Rightarrow \qquad n = 3.$$

Example 5: Find whether Rolle's Theorem is applicable to $f(x) = \ln\left\{\frac{x^2 + ab}{x(a + b)}\right\}$ in [a, b].

Solution: $f(x) = \ln\left\{\frac{x^2 + ab}{x(a + b)}\right\}$ in [a, b].

We know that logarithmic function is continuous and differentiable in its domain.

In the interval [a, b], f(x) is continuous and differentiable.

$$f(a) = \ell n\left\{\frac{a^2 + ab}{a(a + b)}\right\} = \ell n\left\{\frac{a(a + b)}{a(a + b)}\right\} = \ell n 1 = 0.$$

$$\text{and } f(b) = \ell n\left\{\frac{b^2 + ab}{b(a + b)}\right\} = \ell n\left\{\frac{b(a + b)}{b(a + b)}\right\} = \ell n 1 = 0.$$

Thus, f(a) = f(b). Hence, Rolle's theorem is applicable.

Example 6: If Rolle's Theorem is applicable to the function f defined by

$$f(x) = \begin{cases} ax^2 + b, & |x| < 1 \\ 1, & |x| = 1 \\ \frac{c}{|x|}, & |x| > 1 \end{cases}$$

in the interval [–3, 3], then show that

(i) | a | + | b | + | c | = 3 (ii) 2a + 4b + 3c = 8

Solution: As Rolle's theorem is applicable, the function should be continuous and differentiable in [–3, 3].

So, at x = 1, a + b = 1= c (continuity) and

$$2a = \lim_{h\to 0}\frac{\frac{c}{1+h}-1}{h} \quad \text{(differentiability)}$$

$$\Rightarrow \quad 2a = \lim_{h\to 0}\frac{1-(1+h)}{h(1+h)} \quad \text{(using } c = 1\text{)}$$

$$\Rightarrow \quad a = \frac{-1}{2},\ b = \frac{3}{2} \quad \text{and} \quad c = 1.$$

Now we can easily verify that (i) and (ii) are correct.

Example 7: If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3] satisfies the hypotheses of Rolle's theorem, then find the values of a and b.

Solution: Since f(x) is a polynomial function it is continuous and differentiable in [1, 3].

Now $\quad f(1) = f(3)$

$\therefore \quad 1 - 6 + a + b = 27 - 54 + 3a + b$

$\Rightarrow \quad 2a = 22$ or $a = 11$

Since f(1) = f(3) is independent of b

$\quad a = 11$ and $b \in R$.

Example 8: If the function $f(x) = x^3 - px^2 + qx$ defined on [1, 3] satisfies Rolle's theorem with c = 5/2, then find the values of p and q.

Solution: Since f(x) is a polynomial function it is continuous and differentiable in [1, 3].

$\therefore \quad f(1) = f(3)$

$\therefore \quad 1 - p + q = 27 - 9p + 3q$

$\Rightarrow \quad 4p - q = 13.$ ...(1)

$\quad f'(x) = 0$ has one root $x = 5/2$.

$\quad f'(x) = 3x^2 - 2px + q = 0$

$\Rightarrow \quad 3(5/2)^2 - 2p(5/2) + q = 0$ ...(2)

On solving (1) and (2) we get

$\quad p = 23/4$ and $q = 10$.

Example 9: Can Rolle's theorem be applied to f(x) = |x| in the interval [–2, 2]?

Solution: First, f(–2) = 2 = f(2); second, f is continuous. The function, however, fails to be differentiable at 0. Not all the conditions of Rolle's theorem hold; thus there is no need for the conclusion to hold. Indeed, there is no number c such that f ′(c) = 0.

Example 10: Given the function $f(x) = \sqrt[3]{(x-8)^2}$ in [0, 16], we have f(0) = f(16) = 4. However, the derivative $f'(x) = 2/(3\sqrt[3]{(x-8)^2})$ does not vanish at any point belonging to the interval (0, 16). Does it contradict Rolle's theorem.

Solution: No, it does not, since there is no derivative at the point x = 8 of the interval (0, 16) and the condtions of Rolle's theorem are violated.

Example 11: Check whether the following functions satisfy the conditions of Rolle's theorem :

(a) $f(x) = x^2 - \sqrt[3]{|x|},\ x \in [-1, 1]$

(b) $f(x) = \sec x,\ x \in \left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$

Solution:

(a) We have

$$f(x) = \begin{cases} x^2 + \sqrt[3]{x}, & -1 \le x < 0 \\ x^2 - \sqrt[3]{x}, & 0 < x \le 1 \end{cases}$$

which is continuous in [–1, 1]. Also, we have

$\quad f(-1) = 0 = f(1)$

Differentiating f(x) w.r.t. x, we have

$$f'(x) = 2x + \frac{1}{3x^{2/3}},\ -1 \le x < 0,$$

$$= 2x - \frac{1}{3x^{2/3}},\ 0 < x \le 1$$

which is not differentiable at x = 0

Hence, Rolle’s theorem is not applicable to the given function.

(b) We have $f(x) = \sec x,\ x \in \left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$

Differentiating w.r.t. x, we have

$$f'(x) = \sec x \tan x = \frac{\sin x}{\cos^2 x}$$

Thus, f(x) is continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$ and

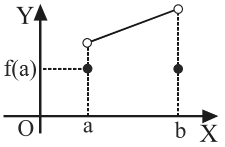
differentiable in $\left(\frac{-\pi}{4}, \frac{\pi}{3}\right)$,

but $f\left(\frac{-\pi}{4}\right) = \sqrt{2}$ is not equal to $f\left(\frac{\pi}{3}\right) = 2$.

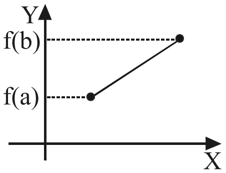
Hence, Rolle’s theorem is not applicable to the given function.

Note:

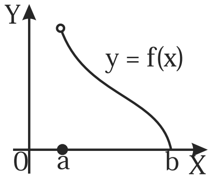
1. If the function $y = f(x)$ defined over $[a, b]$ does not satisfy even one of the three conditions, then Rolle's Theorem may fail i.e. there may or may not exist a point in (a, b) where $f'(x) = 0$.
   The following figures exhibit various cases where the conclusion does not hold, owing to the violation of one or other of the conditions.



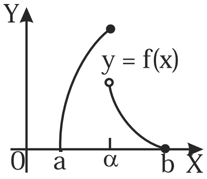
f is not continuous in [a, b] and Rolle's theorem failing



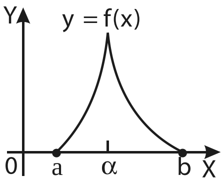
$f(a) \neq f(b)$



Discontinuous at an endpoint

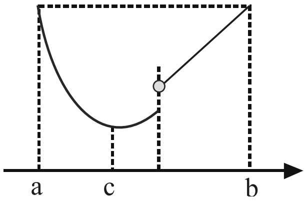


Discontinuous at an interior point



Continuous on [a, b] but not differentiable at some interior point

2. The three conditions are sufficient but not necessary for $f'(x) = 0$ for some x in (a, b). The converse of Rolle's theorem is not true i.e. $f'(x)$ may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.
   The following figure shows a discontinuous function whose derivative vanishes at an interior point.



For example, we consider the function $f(x) = x^3 - x^2 - x + 1$ in the interval $[-1, 2]$.
Here, $f'(x) = 3x^2 - 2x - 1$
$\therefore\ f'(1) = 3 - 2 - 1 = 0$ and $1 \in (-1, 2)$
But the third condition of Rolle's theorem is not satisfied since $f(-1) \neq f(2)$.

Also, $f(x) = \sin(1/x)$ is discontinuous at $x = 0$ in the interval $[-1, 3]$ but $f'(x)$ vanishes at an infinite number of points in the interval.

3. If f is a continuous function in the closed interval $[a, b]$ and if $f(a) = f(b)$, then f has atleast one critical point in the open interval (a, b).
   For example, if $f(x) = |x|$ then $f(-1) = 1$ and $f(1) = 1$. Hence, by the above statement, f has a critical point between -1 and 1. Such a point is 0, since $f'(0)$ does not exist.

Algebraic Meaning of Rolle's Theorem

Assume that f is continuous and differentiable. Rolle's Theorem implies that between two roots of the equation $f(x) = 0$, there is a root of $f'(x) = 0$. Thus, this theorem helps us to locate the roots of the equation $f'(x) = 0$ if we know already the roots of the equation $f(x) = 0$.

Example 12: Show that the derivative of the polynomial $f(x) = x^3 - x^2 - x + 1$ possesses a real root on the open interval $(-1, 1)$.

Solution: We find the roots of the given polynomial $x^3 - x^2 - x + 1 = 0$, or $(x - 1)^2(x + 1) = 0$, i.e. $x_1 = x_2 = 1$, $x_3 = -1$. Since $f(-1) = f(1) = 0$, it follows that in accordance with Rolle's theorem $f'(x)$ possesses a root on the interval $(-1, 1)$.
Next we find the roots of derivative :

$$f'(x) = 3x^2 - 2x - 1 = 0,\ \text{i.e. } x_1 = -1/3,\ x_2 = 1.$$

Thus, we see that between the roots -1 and 1 of the function there is a root of the derivative equal to $-1/3$.

Example 13: Verify Rolle's theorem when $f(x) = (x - a)^m(x - b)^n$, where m, n are positive integers and $a < b$.

Solution: The first function vanishes for $x = a$ and $x = b$. And $f'(x) = (x - a)^{m-1}(x - b)^{n-1}\{(m + n)x - mb - na\}$ vanishes for $x = (mb + na)/(m + n)$, which lies between a and b.

Example 14: Taking the function $f(x) = (x - 3) \ln x$, prove that there is atleast one value of x in $(1, 3)$ which satisfies $x \ln x = 3 - x$.

Solution: Given $f(x) = (x - 3) \ln x$ ...(1)

$$\therefore\quad f'(x) = (x - 3) \cdot \frac{1}{x} + 1 \cdot \ln x \qquad ...(2)$$

Clearly $f(x)$ is finite for all positive values of x and hence $f(x)$ is differentiable for all $x > 0$
$\therefore$ $f(x)$ is differentiable in $[1, 3]$
$\therefore$ $f(x)$ is also continuous in $[1, 3]$
Also $f(1) = (1 - 3) \ln 1 = 0$ and $f(3) = (3 - 3) \ln 3 = 0$
$\therefore$ $f(1) = f(3)$.
Therefore, by Rolle's theorem, there will be atleast one value of x in $(1, 3)$ such that $f'(x) = 0$.

Hence, $\frac{x-3}{x} + \ln x = 0$ or $x \ln x = 3 - x$ has atleast one root in (1, 3).

Example 15: If $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\ldots+\frac{a_{n-1}}{2}+a_1=0$, show that equation $a_1x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_1 = 0$ will have at least one root between 0 and 1.

Solution: Let $f(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + \ldots + a_nx$ ...(1)

Then $f'(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ ...(2)

From (2), it is clear that f(x) is differentiable at all x and hence also continuous at all x.

$\therefore$ f(x) is continuous in [0, 1] and differentiable in (0, 1).

Also from (1), f(0) = 0

and $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + a_n = 0$ (given)

$\therefore$ f(0) = f(1).

Thus, the conditions of Rolle's theorem are satisfied for f(x) in [0, 1]. Therefore, there exists atleast one value of x in (0, 1) such that f '(x) = 0.

Example 16: Let f(x) = (x – a) (x – b)(x – c), a < b < c, show that $f'(x) = 0$ has two roots one belonging to (a, b) and other belonging to (b, c).

Solution: Here, f(x) being a polynomial is continuous and differentiable for all real values of x. We also have f(a) = f(b) = f(c).

If we apply Rolle's theorem to f(x) in [a, b] and [b,c] we would observe that $f'(x) = 0$ would have at least one root in (a, b) and at least one root in (b, c).

But $f'(x)$ is a polynomial of degree two, hence $f'(x) = 0$ cannot have more than two roots. It implies that exactly one root of $f'(x) = 0$ would lie in (a, b) and exactly one root of $f'(x) = 0$ would lie in (b, c).

Example 17: Show that the equation xcos x = sin x has a root between π and 2π.

Solution: We show that the equation xcos x – sin x = 0 has a root between π and 2π.

Consider $f(x) = \frac{\sin x}{x}$,

$$f'(x) = \frac{x\cos x - \sin x}{x^2}.$$

Here, f(x) = 0 for x = π and x = 2π.

Hence, by Rolle's Theorem, f′(x) must vanish for some intermediate value of x.

$\Rightarrow$ xcos x – sin x = 0 has a root between π and 2π.

Example 18: Consider the function $f(x) = \sin^5x + \cos^5x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$. Prove the following :

(i) The equation f (x) = 0 has two roots in $\left[0, \frac{\pi}{2}\right]$.

(ii) There exists some $c \in \left(0, \frac{\pi}{2}\right)$ for which f ' (c) = 0.

(iii) f ''(x) = 0 has atleast two roots in $\left[0, \frac{\pi}{2}\right]$.

Solution: (i) $f(0) = f\left(\frac{\pi}{2}\right) = 0 \Rightarrow f(x) = 0$ has atleast two roots. We have

$f'(x) = 5\sin^4x\cos x - 5\cos^4x\sin x$

$= 5\sin x\cos x(\sin x - \cos x)(1 + \sin x\cos x)$ ...(1)

$\Rightarrow f'(x) = 0$ at $x = \frac{\pi}{4}$.

Also in $\left(0, \frac{\pi}{4}\right)$, f is decreasing and in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ f is increasing

$\Rightarrow$ f has a local minimum at $x = \frac{\pi}{4}$.

Hence, f (x) = 0 has only two roots in $\left[0, \frac{\pi}{2}\right]$

(ii) From (1), we have $f'\left(\frac{\pi}{4}\right) = 0$. Hence, $c = \frac{\pi}{4}$.

(iii) We notice that f is twice differentiable.

Also $f'(0) = f'\left(\frac{\pi}{4}\right) = f'\left(\frac{\pi}{2}\right) = 0$.

Hence, by Rolle's Theorem, there exists some

$x \in \left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ for which f ''(c) = 0 .

Example 19: Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution: First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that f(c) = 0. Thus, the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then f(a) = 0 = f(b) and, since f is a polynomial, it is differentiable on (a, b) and continuous on [a, b]. Thus, by Rolle's Theorem, there is a number c between a and b such that f′(c) = 0. But

$f'(x) = 3x^2 + 1 > 0$ for all x

So $f'(x)$ can never be 0. This gives a contradiction. Therefore, the equation cannot have two real roots.

Example 20: Let f (x) and g (x) be differentiable functions such that $f'(x)\, g(x) \neq f(x)\, g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

Solution: Let a, b be the solutions of $f(x) = 0$.

Suppose g(x) is not equal to zero for any x belonging to [a, b].

Now consider $h(x) = \dfrac{f(x)}{g(x)}$.

Since g(x) not equal to zero, h(x) is differentiable and continuous in [a, b]

$h(a) = h(b) = 0$

(as $f(a) = 0$ and $f(b) = 0$ but $g(a)$ or $g(b) \neq 0$)

Applying Rolle's theorem to h (x) in [a, b],

$h'(c)=0$ for some c belonging to (a, b)

$\Rightarrow$ $f(x)g'(x) = f'(x)g(x)$

This gives a contradiction and hence the desired result is proved.

Example 21: Let the function f(x) satisfy the following conditions :

(i) f(x) possesses a continuous derivative on [a, b],

(ii) f(x) possesses a second derivative in (a, b),

(iii) $f(a) = f'(a) = 0$, $f(b) = 0$. Prove that there is a point $c \in (a, b)$ such that $f''(c) = 0$.

Solution: All conditions of Rolle's theorem are evidently satisfied for the function f(x) on the closed interval [a, b]. Therefore there is a point $d \in (a, b)$ such that $f'(d) = 0$. We consider the function $f'(x)$ on the interval [a,d]. We have

(i) $f'(x)$ is continuous on [a, d],

(ii) $f'(x)$ possesses a derivative $f'(x))' = f''(x)$ in (a, d),

(iii) $f'(a) = f'(d) = 0$.

By virtue of Rolle's theorem, there is a point $c \in (a, d)$ [and, consequently, $c \in (a,b)$] such that

$(f'(x))'\big|_{x=c} = f''(c) = 0$.

Example 22: If f(x) is continuous in [a, b] and differentiable in (a, b), prove that there is atleast one $c \in (a, b)$, such that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b)-f(a)}{b^3-a^3}$.

Solution: Let us consider a function,

$h(x) = f(x) - f(a) + A\,(x^3 - a^3)$

where A is obtained from the relation $h(b) = 0$.

So that, $0 = h(b) = f(b) - f(a) + A(b^3 - a^3)$ ...(1)

Also, $h(a) = 0$

Since, h(x) is continuous in [a, b], differentiable in (a, b) and $h(a) = 0 = h(b)$, by Rolle's theorem,

there must exist some $c \in (a, b)$ such that $f'(c) = 0$.

$\Rightarrow$ $f'(c) + A\,(3c^2) = 0$

or $f'(c) = 3c^2\, \dfrac{f(b)-f(a)}{b^3-a^3}$ [using (1)]

$\Rightarrow$ $\dfrac{f'(c)}{3c^2} = \dfrac{f(b)-f(a)}{b^3-a^3}$.

Example 23: Let P(x) be a polynomial with real coefficients. let a and b ($a < b$), be two consecutive real roots of P(x). Show that there exists c such that $a < c < b$ and $P'(c) + 10\,P(c) = 0$.

Solution: Consider $f(x) = e^{10x}P(x)$.

Now $f(a) = f(b) = 0$ (as $P(a) = P(b) = 0$)

Also as P(x) is a polynomial, f(x) is continuous and differentiable in [a, b].

$\Rightarrow$ Rolle's theorem can be applied to f(x)

$\Rightarrow$ There exists $c \in (a, b)$ such that $f'(c) = 0$.

Now $f'(x) = e^{10x}\,[P'(x) + 10\,P(x)]$

$\Rightarrow$ $e^{10c}\,[P'(c) + 10\,P(c)] = 0$

$\Rightarrow$ $P'(c) + 10\,P(c) = 0$. (as $e^{10c} \neq 0$)

Example 24: Let f be a continuous function on [0, 1], differentiable in (0, 1) such that $f(1) = 0$, then prove that there exists some $c \in (0, 1)$ such that $c\,f'(c) + f(c) = 0$

Solution: Consider a function $g(x) = x\,f(x)$.

Obviously g is continuous in [0, 1] and differentiable in (0, 1).

As $f(1) = 0$, $g(0) = 0 = g(1)$

Hence Rolle's theorem is applicable to g

$\therefore$ There exists some $c \in (0, 1)$ such that $g'(c) = 0$

$\Rightarrow$ $x\,f'(x) + f(x) = 0$ for $x = c$

$\Rightarrow$ $c\,f'(c) + f(c) = 0$.

Concept Problems N

1. Let f be continuous on [a, b] and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$?
2. Find all numbers c that satisfy the conclusion of Rolle's theorem.
   (i) $x^4 - 2x^2 + 1$ on $[-2,2]$
   (ii) $\sin x + \cos x$ on $[0, 2\pi]$
3. Let $f(x) = x^3 - 3x^2 + 2x + 5$. Verify Rolle's theorem in [0, 2].
4. If $2a + 3b + 6c = 0$, then prove that the equation $ax^2 + bx + c = 0$ has at least one real root in (0, 1).

5. Find whether Rolle's theorem is applicable to the following functions :
(i) $f(x) = 1 - x^{2/3}$ in $[-1, 1]$
(ii) $f(x) = |x|^3$ in $[-1, 1]$.

6. Let $f(x) = x^{2/3}$ for x in $[-1,1]$
(a) Show that $f(-1) = f(1)$.
(b) Is there a number c in $(-1,1)$ such that $f'(c) = 0$?
(c) Why does this function not contradict Rolle's theorem?

7. If $f(x) = (x-1)(x-2)(x-3)(x-4)$, find the number of real roots of $f'(x) = 0$ and indicate the intervals in which they lie.

8. If $f(x) = x^2(1-x)^3$ then prove that the equation $f'(x) = 0$ has atleast one root in $(0, 1)$.

9. Suppose two race cars begin at the same time and finish at the same time. Show that at some point in the race they had the same speed.

10. Show that if an object moves along a straight line in such a way that its velocity is the same at two different times (that is, for a differentiable function v, we are given $v(t_1) = v(t_2)$ for $t_1 \neq t_2$), then there is some intermediate time when the acceleration is zero.

11. If the graph of a function has three x-intercepts, then is it true that it must have atleast two points at which its tangent is horizontal.

12. Let p be a nonconstant polynomial function.
(i) Prove that between any two consecutive zeros of p', there is atmost one zero of p.
(ii) If p has three distinct zeros in the interval $[a, b]$, prove that $p''(c) = 0$ for some real number c in (a, b).

13. If $f'(x) = 0$ for all x in an interval (a, b), then is it true that f is constant on $[a, b]$.

14. Show that any twice continuously differentiable function has atleast one abscissa of the point of inflection on the graph of the function between two points of extremum.

15. Taking the function $y = x^4 + 8x^3 + 18x^2 + 8$ as an example, ascertain that there may be no points of extremum between the abscissas of the points of inflection on the graph of a function.

16. Prove that any polynomial of an odd degree $n \geq 3$ has atleast one point of inflection.

17. If $(y + 4)^3 = x^2$, then $y = 0$ when $x = 8$ or -8. Does Rolle's theorem justify the conclusion that $dy/dx = 0$ for some number x in the interval $(-8, 8)$?

Practice Problems L

18. Show that the equation $x = \tan x$ has a root between π and $3/2\ \pi$.

19. Use Rolle's theorem with $f(x) = (x-1)\sin x$ to show that the equation $\tan x = 1 - x$ has atleast one solution for $0 < x < 1$.

20. Show, by means of Rolle's Theorem, that $5x^4 - 4x + 1 = 0$ has a solution in $[0, 1]$.

21. Suppose that $f : [0, 1] \rightarrow R$ is differentiable, $f(0) = 0$ and $f(x) > 0$ for $x \in (0, 1)$. Is there a number

$c \in (0, 1)$ such that $\dfrac{2f'(c)}{f(c)} = \dfrac{f'(1-c)}{f(1-c)}$?

22. Use Rolle's theorem to show that there is no number a for which the equation $x^3 - 3x + a = 0$ has two distinct solutions in the interval $[-1, 1]$.

23. If $a > 0$ is a constant, show that the equation $x^3 + ax - 1 = 0$ has exactly one real solution.

24. Suppose $a > 0$ and that f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b) = 0$. Prove that there exists $c \in (a, b)$ such that $f'(c) = f(c)/c$. Interpret graphically.

25. Assume that f and g are continuous on $[a, b]$ and differentiable on (a, b) Prove that there exists $c \in (a,b)$

such that $\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$

26. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point.

Hence, show that $f(x) = \dfrac{1}{2}\cos x$ has atmost one fixed point.

27. Let $f(x) = \tan x$
(a) Show that there is no number c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
(b) Explain why the result in part (a) does not violate Rolle's Theorem.

28. Use the fact that $\dfrac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$ to show that the equation $6x^5 - 4x + 1 = 0$ has atleast one solution in the interval $(0, 1)$.

7.20 Deductions of Rolle's Theorem

Let f(x) be a polynomial having roots α and β where $\alpha < \beta$ so that we have $f(\alpha) = f(\beta) = 0$. Also, a polynomial function is continuous and differentiable everywhere. Thus f(x) satisfies the conditions of Rolle's theorem. Consequently, there exists atleast one number $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$. In other words $x = \gamma$ is a root of $f'(x) = 0$.

Thus, Rolle's theorem can be interpreted algebraically as follows : between any two roots of a polynomial f(x), there is a always a root of its derivative f '(x).

Important Deductions

(a) If all the roots of $f(x) = 0$ are real, then all the roots of $f'(x) = 0$ are also real, and the roots of the latter equation separate those of the former.
For if f(x) is of degree n, f′(x) is of degree $n - 1$, and a root of $f'(x) = 0$ exists in each of the $n - 1$ intervals between the n roots of $f(x) = 0$.

(b) If all the roots of $f(x) = 0$ are real, so also are those of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$, ..., and the roots of any one of these equations separate those of the preceding equation. This follows from (a).

(c) Not more than one root of $f(x) = 0$ can
(i) lie between two consecutive roots of $f'(x) = 0$, or
(ii) be less than the least of these, or
(iii) be greater than the greatest of these.
For let $\beta_1, \beta_2, \ldots\ldots, \beta_r$ be the real roots of f′(x), any of which may be multiple roots, and suppose that $\beta_1 < \beta_2 < \ldots. < \beta_r$.
Let α_1, α_2 be real roots of $f(x) = 0$.
If $\alpha_1 = \alpha_2$, then α_1 is one of the set $\beta_1, \beta_2, \ldots.. \beta_r$
If $\alpha_1 \neq \alpha_2$, by Rolle's theorem, a root of f′(x) lies between α_1 and α_2.
Hence
(i) if $\beta_1 < \alpha_1 < \alpha_2 < \beta_2$, then β_1 and β_2 cannot be consecutive roots,
(ii) if $\alpha_1 < \alpha_2 < \beta_1$, then β_1 cannot be the least root of $f'(x) = 0$;
(iii) if $\beta_r < \alpha_1 < \alpha_2$, then β_r cannot be the greatest.
Thus, not more than one root of $f(x) = 0$ can lie in any one of the open intervals
$(-\infty, \beta_1), (\beta_1, \beta_2), \ldots\ldots (\beta_{r-1}, \beta_r), (\beta_r, \infty)$.

(d) If $f'(x) = 0$ has r real roots, then $f(x) = 0$ cannot have more than $(r + 1)$ real roots.
If $f(x) = 0$ has no multiple root, none of the roots of $f'(x) = 0$ is a root of $f(x) = 0$, and the theorem follows from (c).
If $f(x) = 0$ has an m-multiple root, we regard this as the limiting case in which m roots tend to equality. Thus the theorem is true in all cases.

(e) If $f^{(r)}(x)$ is the r^{th} derivative of f(x) and the equation $f^{(r)}(x) = 0$ has some imaginary roots, then $f(x) = 0$ has atleast as many imaginary roots.
If follows from (d) that $f(x) = 0$ has at least as many imaginary roots as $f'(x) = 0$.

(f) If all the real roots $\beta_1, \beta_2, \ldots.$ of $f'(x) = 0$ are known, we can find the number of real roots of $f(x) = 0$ by considering the signs of $f(\beta_1), f(\beta_2), \ldots..$. A single root of $f(x) = 0$, or no root, lies between β_1 and β_2, according as $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, or the same sign.

Example 1: Prove that the equation $3x^5 + 15x - 8 = 0$ has only one real root.

Solution: The existence of atleast one real root follows from the fact that the polynomial $f(x) = 3x^5 + 15x - 8$ is of odd degree. Let us prove the uniqueness of such a root.

Suppose there exist two roots $\alpha < \beta$. Then in the interval $[\alpha, \beta]$ the function $f(x) = 3x^5 + 15x - 8$ satisfies all conditions of the Rolle's theorem ; it is continuous, vanishes at the end-points and has derivative at all points.

Consequently, at some point c, $\alpha < c < \beta$, $f'(c) = 0$. But $f'(x) = 15(x^4 + 1) > 0$. This contradiction proves that the equation in question has only one real root.

Example 2: Use Rolle's theorem to determine how many real roots there are for the equation
$$x^3 - 6x^2 + 15x + 3 = 0.$$

Solution: Since $f(x) = x^3 - 6x^2 + 15x + 3$ is a polynomial of odd degree, there is atleast one real number r such that $f(r) = 0$ (the argument is based on the intermediate value theorem). Could there be another root, s? If so, by Rolle's theorem, there would be a number c (between r and s) at which $f'(c) = 0$.

To check, we compute the derivative of f(x) and see if it is ever equal to 0.

We have $f'(x) = 3x^2 - 12x + 15$.

To find when f′(x) is 0, we solve the equation $3x^2 - 12x + 15 = 0$. Since discriminant is negative, the equation has no real roots. Since f ′(x) is never 0, it follows that the polynomial $x^3 - 6x^2 + 15x + 3$ has only one real root.

Example 3: Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in (0, 1).

Solution: Suppose that there are two distinct roots α, β in (0, 1) such that $0 < \alpha < \beta < 1$.

Let $f(x) = x^3 - 3x + k$. Since f(x) is a polynomial, it is continuous in $[\alpha, \beta]$ and differentiable in (α, β). Also we have $f(\alpha) = f(\beta) = 0$.

Thus f satisfies all the conditions of Rolle's theorem in $[\alpha, \beta]$.

Hence there is a value c of x in (α, β) such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 3c^2 - 3 = 0 \Rightarrow c = \pm 1$ which contradicts the fact that $\alpha < c < \beta$ and $0 < \alpha < \beta < 1$.

Hence our assumption is wrong. So there cannot be two distinct roots of $f(x) = 0$ in (0,1) for any value of k.

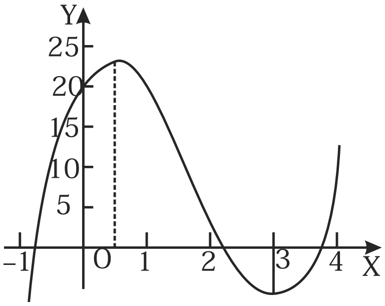
Example 4: Prove that $4x^3 - 21x^2 + 18x + 20 = 0$ has three real roots.

Solution: Let $\phi(x) = 4x^3 - 21x^2 + 18x + 20$

We have $\phi'(x) = 12x^2 - 42x + 18 = 6(2x - 1)(x - 3)$.

Hence the real roots of $\phi(x) = 0$, if any, will lie in the intervals between $-\infty$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and ∞, respectively.

Now, for $x = -\infty, \frac{1}{2}, 3, \infty$, the signs of $\phi(x)$ are –, +, –, +, respectively, so that $\phi(x)$ must in fact vanish once in each of the above intervals. Hence there are three real roots. The graph of $\phi(x)$ is shown in the figure.



Example 5: Find the character of the roots of

$f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0.$

Solution: $f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^2 - 1)(x - 2).$

The roots of $f'(x) = 0$ are $-1, 1, 2$.

When $x =$	$-\infty$	-1	1	2	∞
$f(x) =$	∞	$-$	$+$	$+$	∞

Therefore $f(x) = 0$ has two real roots, one lying between $-\infty$ and -1 and the other between -1 and 1. The other two roots are imaginary.

Example 6: If $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots, prove that k must lie between 8 and 11.

Solution: Let $f(x) = x^4 - 14x^2 + 24x - k$.

Then $f'(x) = 4x^3 - 28x + 24 = 4(x - 1)(x - 2)(x + 3)$.

$\therefore$ $f'(x) = 0$ has the roots $-3, 1, 2$.

By Rolle's theorem the position and sign of the roots of $f(x) = 0$ can be found out as follows.

$x =$	$-\infty$	-3	1	2	∞
$f(x) =$	∞	$-117 - k$	$11 - k$	$8 - k$	∞

If $f(x) = 0$ has four real and unequal roots, we must have $-117 - k < 0$, $11 - k > 0$ and $8 - k < 0$, or, $k > -117$, $k < 11$, $k > 8$. Therefore $8 < k < 11$.

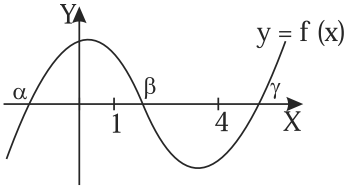
Example 7: Given the cubic equation $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one root of the equation is less than 1, other root is in the interval (1, 4) and the third root is greater than 4, then the value of k lies in the interval $(a + \sqrt{b}, b(a + \sqrt{b}))$ where $a, b \in N$. Find the value of a and b.

Solution: $f(x) = x^3 - 2kx^2 - 4kx + k^2 = 0$

note that $f(0) = k^2 \geq 0$

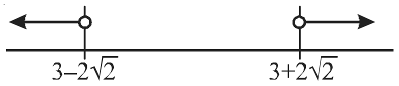
$f(1) > 0$

$\Rightarrow$ $1 - 2k - 4k + k^2 > 0$



$k^2 - 6k + 1 > 0$

$[k - (3 + 2\sqrt{2})][k - (3 - 2\sqrt{2})] > 0$



...(1)

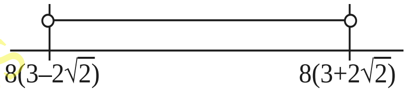
Also $f(4) < 0$

$\Rightarrow$ $64 - 32k - 16k + k^2 < 0$ $k^2 - 48k + 64 < 0$

$(k - 24)^2 < 512$

$(k - 24 + 16\sqrt{2})(k - 24 - 16\sqrt{2}) < 0$

$[k - 8(3 - 2\sqrt{2})][k - 8(3 + 2\sqrt{2})] < 0$



...(2)

(1) $\cap$ (2) $\Rightarrow$ $3 + 2\sqrt{2} < k < 8(3 + 2\sqrt{2})$ $3 + \sqrt{8} < k < 8(3 + \sqrt{8})$

$\therefore$ $a = 3;\ b = 8$

Example 8: Let f (x) be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6 - x)$ and $f'(0) = 0 = f'(2) = f'(5)$. Determine the minimum number of zeroes of $g(x) = (f''(x))^2 + f'(x) f'''(x)$ in the interval [0, 6].

Solution: $f(x) = f(6 - x)$ ...(1)

On differentiating (1) w.r.t. x, we get

$f'(x) = -f'(6 - x)$ ...(2)

Putting x = 0, 2, 3, 5 in (2), we get

$f'(0) = -f'(6) = 0$

$f'(2) = -f'(4) = 0$

$f'(3) = 0$

$f'(5) = -f'(1) = 0$

$f'(0) = 0 = f'(2) = f'(3) = f'(5) = f'(1) = f'(4) = f'(6)$

$\therefore$ $f'(x) = 0$ has minimum 7 roots in [0, 6].

Now, consider a function $y = f'(x)$

As $f'(x)$ satisfies the conditions of Rolle's theorem in intervals [0, 1], [1, 2], [2, 3], [3, 4], [4, 5] and [5, 6] respectively, the equation $f''(x) = 0$ has minimum 6 roots.

Now $g(x) = (f''(x))^2 + f'(x) f'''(x) = \dfrac{d}{dx}(f'(x)f''(x)) = h'(x)$,

where $h(x) = f'(x) f''(x)$

Clearly $h(x) = 0$ has minimum 13 roots in [0, 6]

Hence again by Rolle's theorem, $g(x) = h'(x)$ has minimum 12 zeroes in [0, 6].

Concept Problems O

1. Let $f(x) = x(x + 1)(x + 2)(x + 3)$. Show that the equation $f'(x) = 0$ has three real roots.
2. If $3a + 4b + 6c + 12d = 0$, then prove that the equation $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in R$ has atleast one root in $x \in (0, 1)$.
3. Can f have three distinct roots and f ′ have no roots? Can f have no roots and f ′ have roots?
4. The equation $e^x = 1 + x$ obviously has a root $x = 0$. Show that this equation cannot have any other real root.
5. If f(x) and f"(x) are continuous and have the same sign at every point of an interval (a, b), then prove that this interval can include atmost one root of either of the equations $f(x) = 0$, $f'(x) = 0$.
6. Let g be a continuous function on [a, b] and suppose that g″ exists for all x in the open interval [a, b]. Prove that if there are three values of x in [a, b] for which $g(x) = 0$, then there is at least one value of x in [a, b] where $g''(x) = 0$.
7. If $a, b, c \in R$ and $a + b + c = 0$ then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has atleast one root in (0, 1).
8. Prove that the equation $x^4 - 4x - 2 = 0$ cannot have more than two real roots.
9. Let f be a polynomial such that $f(a) = f(b) = 0$, but $f(x) \neq 0$ for all $x \in (a, b)$. Prove that the number of roots of f′ on (a, b), counting multiplicities, is odd.
   Further, let k be a constant. Prove that the number of zeros of $f' + kf$ on (a, b) is odd.

Practice Problems M

10. If equation $ax^3 + bx^2 + cx + d = 0$ has only one real root then prove that the equation $\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + e = 0$ cannot have more than 2 distinct real roots.
11. If equation $ax^4 + bx^2 + cx^2 + dx + k = 0$ has three distinct real roots then prove that the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ will have atleast two distinct real roots.
12. Prove that if the equation
    $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x = 0$
    has a positive root x = r, then the equation
    $nx^{n-1} + (n - 1)a_1x^{n-2} + (n - 2)a_2x^{n-3} + ... + a^{n-1} = 0$
    has a positive root less than r.
13. Prove that the polynomial $P(x) = x^n + ax + b$ of degree n cannot have more than two real roots if n is even or more than three real roots if n is odd.
14. Prove that the equation $x^4 - 4x - 1 = 0$ has two different real roots.
15. Prove that the function $f(x) = x^n + px + q$ cannot have more than two real roots for n even and more than three for n odd.
16. Prove that all roots of the derivative of the given polynomial $f(x) = (x + 1)(x - 1)(x - 2)(x - 3)$ are real.
17. Let $f(x) = x^4 - x^3 + 7x^2 + 3x - 11$. Prove that the graph of f has atleast one horizontal tangent line.
18. Prove that the equation $x^5 + x^3 + 4x + 1 = 0$ has atleast four imaginary roots.

7.21 Lagrange's Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

(i) f is continuous on the closed interval [a, b].

(ii) f is differentiable on the open interval (a, b).

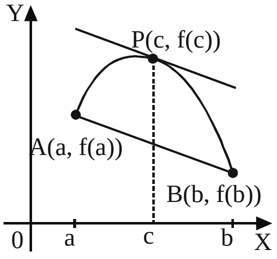
Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad ...(1)$$

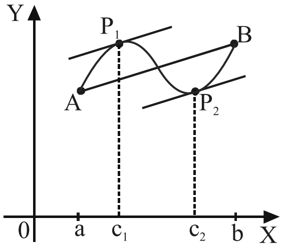
Geometrical Meaning of LMVT

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures (a) and (b) show that points A(a, f(a)) and B(b, f(b)) on the graphs of two differentiable functions. The slope of the secant line AB is

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$



(a)



(b)

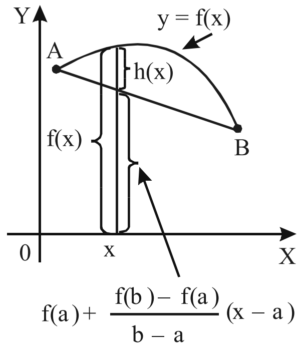
which is the same expression as on the left side of Equation 1. Since f′(c) is the slope of the tangent line at the point (c, f(c)), the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P(c, f(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB.

Proof We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB.

The equation of the line AB can be written as

$$y - f(x) = \frac{f(b)-f(a)}{b-a}(x-a)$$

or, as $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ ...(2)



So, as shown in the figure,

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) \qquad ...(3)$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

(i) The function h is continuous on [a, b] because it is the sum of f and a first-degree polynomial, both of which are continuous.

(ii) The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact we can compute h′ directly from Equation 3:

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

(Note that f(a) and [f(b) – f(a)]/(b – a) are constants.)

(iii) $h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a)$$

$$= f(b) - f(a) - [f(b) - f(a)] = 0$$

Therefore, h(a) = h(b).

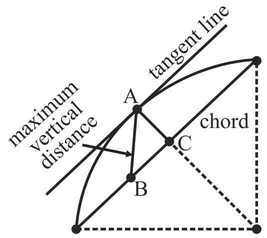
Since h satisfies the hypotheses of Rolle's Theorem, there is a number c in (a, b) such that h′(c) = 0. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

and so $\frac{f(b)-f(a)}{b-a} = f'(c)$

The function h measures the vertical displacement of the graph of f from the chord with extremities (a, f(a)), (b, f(b)). The crux of the proof is that the slope f '(x) of the curve is expected to agree with the slope (f(b) – f(a))/(b – a) of the extreme chord at a point c where h(c) is a local maximum.

This idea is hit upon when one considers a circular segment (see figure) in which one can prove at once that the point on the bounding arc of a circular segment at which the tangent line is parallel to the bounding chord has the property that the vertical distance between the chord and the arc is a maximum.



To illustrate the theorem with a specific function, let us consider $f(x) = x^3 - x$, a = 0, b = 2. Since f is a polynomial, it is continuous and differentiable for x, so it is certainly continuous on [0, 2] and differentiable on (0, 2) such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

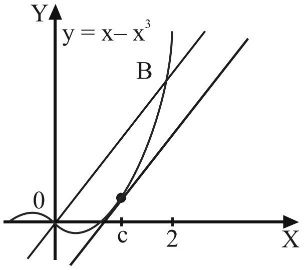
Now f(2) = 6, f(0) = 0, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is , $c = \pm 2/\sqrt{3}$.

But c must lie in (0, 2), so $c = 2/\sqrt{3}$.

The figure give below illustrates this calculation. The tangent line at this value of c is parallel to the secant line OB.



The adjective "mean" carries both the notions "between" and "average", each of which gives a significant clue to the basic idea in the theorem. For what LMVT does is to single out a derivative value which plays the role of an average derivative value, and this derivative value is attained at a point strictly between the end points of the interval domain of the function. We must understand that the location of c is not really pinpointed; we only know that it lies somewhere inside an open interval. But the interesting fact is that the mere knowledge that c is a mean point (i.e., lies strictly between the end points of the interval) is the real power behind the theorem and its applications.

Physical Meaning of LMVT

Assume that x is time and f(x) is the coordinate of a point, which moves along a straight line at the moment x. We write Lagrange's theorem in the form $\frac{f(b)-f(a)}{b-a} = f'(c)$.

The quantity on the left hand side of the relation is, evidently, the average velocity of movement of the point along the straight line during the time interval from a to b. LMVT shows that there is a moment $x = c$ at which the instantaneous velocity is equal to the average velocity on the time interval $[a, b]$.

For instance, if a car travelled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

Thus, the theorem says that the displacement $f(b) - f(a)$ is obtainable as a mean velocity $f'(c)$ times the length $b - a$ of the time interval. That is, during motion, a velocity is attained which if it had been applied as a uniform (constant) velocity for the same time interval would have achieved the same displacement.

Now $[f(b) - f(a)]$ is the change in the value of function f as x changes from a to b so that $[f(b) - f(a)] / (b - a)$ is the average rate of change of the function over the interval $[a, b]$. Also $f'(c)$ is the instantaneous rate of change of the function at $x = c$. Thus, the theorem states that the average rate of change of a function over an interval is equal to the actual rate of change of the function at some point of the interval.

Note:

1. Rolle's theorem is a special case of LMVT since $f(a) = f(b)$ gives

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \Rightarrow \quad f'(c) = \frac{0}{b-a} = 0.$$

2. There are several ways of writing the mean value theorem. For example, the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

is equivalent to $f(b) - f(a) = (b - a) f'(c)$. ...(1)

hence to $f(b) = f(a) + (b - a) f'(c)$.

In this form, the Mean-Value Theorem asserts that f(b) is equal to f(a) plus a quantity that involves the derivative f′.

The relation (1) is known as the **formula of finite increment**. It states that the increment of a differentiable function on an interval is equal to the product of the derivative of the function at an intermediate point by the increment of the independent variable.

3. If a function f is continuous in a closed interval $[a, a + h]$ and derivable in the open interval $(a, a + h)$, then there exists at least one number $\theta \in (0, 1)$ such that $f(a + h) = f(a) + h f'(a + \theta h)$. $\theta \in (0, 1)$

Proof Putting $b = a + h$ in the above theorem, there will be atleast one c, $a < c < a + h$ such that

$$f'(c) = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h} \qquad ...(1)$$

Let $c = a + \theta h$

Then, $a < c < a + h \Rightarrow a < a + \theta h < a + h$

$\Rightarrow \; 0 < \theta h < h \Rightarrow 0 < \theta < 1$

$\therefore$ From (1), $f'(a + \theta h) = \dfrac{f(a+h)-f(a)}{h}$

or, $f(a + h) = f(a) + hf'(a + \theta h)$, where $0 < \theta < 1$.

Example 1: Show that the function $f(x) = x^3 + x^2$ satisfies the hypotheses of LMVT on the closed interval [1, 2], and find a number c between 1 and 2 so that $f'(c) = \dfrac{f(2)-f(1)}{2-1}$.

Solution: Because f is a polynomial function, it is differentiable and hence also continuous on the entire interval [1, 2]. Thus, the hypotheses of LMVT are satisfied.

By differentiating f, we find that $f'(x) = 3x^2 + 2x$ for all x. Therefore, we have $f'(c) = 3c^2 + 2c$, and the equation $f'(c) = \dfrac{f(2)-f(1)}{2-1}$ is satisfied when

$$3c^2 + 2c = \frac{f(2)-f(1)}{2-1} = \frac{12-2}{1} = 10$$

Solving the resulting equation $3c^2 + 3c - 10 = 0$ by the quadratic formula, we obtain

$$c = \frac{-1 \pm \sqrt{31}}{3}$$

The negative value is not in the open interval (1, 2), but the positive value

$c = \dfrac{-1 \pm \sqrt{31}}{3} \approx 1.52$ satisfies the requirements of the theorem.

Example 2: On the arc AB of the curve specified by the parametric equations $x = t^2$, $y = t^3$ find the point M at which the tangent is parallel to the chord AB if the points A and B are associated with the values $t = 1$ and $t = 3$.

Solution: The slope of the chord AB is equal to $\dfrac{y(3)-y(1)}{x(3)-x(1)}$ and the slope of the tangent at the point M (for $t = \xi$) is equal to $\dfrac{y_t'(\xi)}{x_t'(\xi)}$, or $\dfrac{27-1}{9-1} = \dfrac{3\xi^2}{2\xi}$, or $\dfrac{13}{4} = \dfrac{3}{2}\xi$.

$\therefore \quad \xi = \dfrac{13}{8}$. Hence, the point M is (t^2, t^3)

where, $t = \dfrac{13}{8}$.

Example 3: If $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the curve $y = ax^2 + bx + c$ then using Lagrange's Mean value theorem show that there will be at least one point $C(x_3, y_3)$ where tangent will be parallel to chord AB. Also show that

$$x_3 = \frac{x_1 + x_2}{2}.$$

Solution: Given $y = f(x) = ax^2 + bx + c$ ...(1)

$\therefore \quad f'(x) = 2ax + b$ ...(2)

Since $f'(x)$ is finite for all x, therefore it is differentiable and hence continuous at all x. Hence by geometrical meaning of mean value theorem, there will be at least one point $C(x_3, y_3)$ between $A(x_1, y_1)$ and $B(x_2, y_2)$ where tangent will be parallel to chord AB and x

coordinate x_3 of C is given by $f'(x_3) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

or $\quad 2ax_3 + 3 = \dfrac{ax_2^2+bx_2+c-ax_1^2-bx_1-c}{x_2-x_1}$

$\therefore \quad 2ax_3 + 3 = \dfrac{a(x_2^2+x_1^2)+b(x_2-x_1)}{x_2-x_1}$

or $\quad 2ax_3 + 3 = \dfrac{(x_2-x_1)[a(x_2+x_1)+b]}{x_2-x_1}$

or $\quad 2ax_3 + b = a(x_2 + x_1) + b$

$\therefore \quad x_3 = \dfrac{x_1+x_3}{2}$

Example 4: On the arc AB of the curve $f(x) = 2x - x^2$ find the point M at which the tangent is parallel to the chord AB if A (1, 1) and B(3, –3).

Solution: On the arc AB of the curve $y = 2x - x^2$ is continuous and differentiable for all values of x. In accordance with Lagrange's theorem, between the two values $a = 1$ and $b = 3$ there is a value $x = c$, satisfying the equality $f(b) - f(a) = (b - a)f'(c)$, where $y' = 2 - 2x$. Substituting the requisite values, we obtain.

$$f(3) - f(1) = (3 - 1)f'(c) \Rightarrow (2 \,.\, 3 - 3^2) - (2 \,.\, 1 - 1^2)$$
$$= (3 - 1) \,.\, (2 - 2c) \Rightarrow -4 = 4(1 - c)$$

Hence $c = 2$, $f(2) = 0$. Thus, the point M has the coordinates (2, 0).

Example 5: Let f be defined and continuous on [a – h, a + h], and derivable on (a – h, a + h). Prove that there is a real number θ between 0 and 1 for which

$f(a + h) - 2f(a) + f(a - h) = h(f'(a + \theta h) - f'(a - \theta h)$

Solution: Let F be the function defined on [0, 1] by setting $F(t) = f(a + ht) + f(a - ht)$, for all $t \in [0, 1]$.

Then F is continuous on [0, 1] and derivable on (0, 1). By LMVT, there exists a number θ between 0 and 1 such that $F(1) - F(0) = (1 - 0) F'(\theta)$.

i.e. $f(a + h) + f(a - h) - 2f(a) = h\{f'(a + \theta h) - f'(a - \theta h)\}$

Example 6: If $a = -1$, $b \geq 1$, and $f(x) = 1/|x|$, show that the conditions of Lagrange's Mean Value Theorem are not satisfied in the interval [a, b], but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.

Solution: The function is not defined at $x = 0$. It has an infinite discontinuity at $x = 0$. Hence it is not differentiable at $x = 0$. Thus the conditions of the Mean Value Theorem are not satisfied in an interval [a, b] which includes the origin. Now the conclusion of the Mean Value Theorem is

$\dfrac{f(b)-f(a)}{b-a} = f'(c)$ where $a < c < b$.

Assuming this result to be true, we have

$$\frac{1}{|b|} - \frac{1}{|a|} = (b-a)\left\{\frac{d}{dx}\frac{1}{|x|}\right\}_{x=c} = (b-a)\left\{-\frac{1}{|c|^2}\right\}$$

or $\quad \dfrac{1}{b} - 1 = (b+1)\left(-\dfrac{1}{|c|^2}\right) = -\dfrac{b+1}{c^2}$, since $a = -1$

or $\quad c^2 = \dfrac{b^2+b}{b-1}$ or $\dfrac{b^2+b}{b-1} < b^2 \quad [\because b^2 > c^2]$

or $\quad b + 1 < b^2 - b$ or $b^2 - 2b - 1 > 0$

or $\quad (b - 1)^2 > 2$ or $b - 1 > \sqrt{2}$.

Hence $b > 1 + \sqrt{2}$.

Under this condition the conclusion of the Mean Value Theorem is true, although the conditions for the validity of the theorem are not satisfied.

Example 7: Let $f : [0, \infty) \to [0, \infty)$ be a continuous and differentiable function. Then show that

$$(f(4) - f(2))\frac{(f(4))^2 + (f(2))^2 + f(2)f(4)}{3} = 2f^2(c)f'(c),$$

where $c \in (2, 4)$

Solution: We have to prove that

$$(f(4) - f(2))\frac{(f(4))^2 + (f(2))^2 + f(2)f(4)}{3} = 2f^2(c)f'(c)$$

i.e. $\quad \dfrac{(f(4))^3 - (f(2))^3}{4-2} = 3f^2(c)f'(c)$.

From here, we can guess the function on which LMVT can be applied. Consider the function $g(x) = (f(x))^3$ on [2, 4]. Now g(x) is continuous and differentiable in [2, 4] because cube of a differentiable function is also a differentiable function. By Lagrange's Mean Value Theorem there exists atleast one $c \in (2, 4)$ such that

$$g'(c) = \frac{g(4)-g(2)}{4-2}$$

$$\Rightarrow \quad 3f^2(c)f'(c) = \frac{(f(4))^3 - (f(2))^3}{4-2}.$$

Example 8: Show that there is no cubic curve for which the tangent lines at two distinct points coincide.

Solution: Suppose $y = ax^3 + bx^2 + cx + d$, $(a \neq 0)$ be a cubic curve.

We assume that (x_1, y_1) and (x_2, y_2), $(x_1 < x_2)$ are two distinct points on the curve at which the tangents coincide.

Then, by Lagrange's Mean Value Theorem there exists x_3 $(x_1 < x_3 < x_2)$ such that $\dfrac{y_2 - y_1}{x_2 - x_1} = y'(x_3) = m$

Since, tangent x_1, x_2, x_3 are solutions of the equation $3ax^2 + 2bx + c = m$, and it being a quadratic equation, cannot have more than two roots. Therefore, no such cubic curve is possible.

Example 9: If the function f : [0, 4] → R is differentiable, then show that, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$ for some a, b ∈ (0, 4)

Solution: Since, f is differentiable, it is continuous also.

Thus by Lagrange's Mean Value Theorem, $a \in (0, 4)$ such that

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - f(0)}{4} \quad \text{... (1)}$$

Also, by Intermediate Value Theorem there exists $b \in (0, 4)$ such that

$$f(b) = \frac{f(4) + f(0)}{2} \quad \text{... (2)}$$

$$f'(a) f(b) = \frac{(f(4))^2 - (f(0))^2}{8}$$

$\Rightarrow$ $(f(4))^2 - (f(0))^2 = 8f'(a) f(b)$ for some a, b $\in (0, 4)$.

Example 10: Let f(x) and g(x) be differentiable function in (a, b), continuous at a and b and $g(x) \neq 0$ in [a, b] then prove that

$$\frac{g(a)f(b) - f(a)g(b)}{g(c)f'(c) - f(c)g'(c)} = \frac{(b-a)g(a)g(b)}{(g(c))^2}$$

for at least one c ∈ (a, b).

Solution: We have to prove that

$$\frac{g(a)f(b) - f(a)g(b)}{g(c)f'(c) - f(c)g'(c)} = \frac{(b-a)g(a)g(b)}{(g(c))^2}$$

i.e. $$\frac{\dfrac{f(b)}{g(b)} - \dfrac{f(a)}{g(a)}}{(b-a)} = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$$

Let $h(x) = \dfrac{f(x)}{g(x)}$ in [a, b].

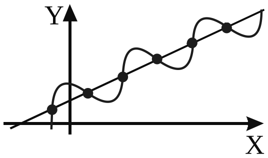
As f(x) and g(x) are differentiable functions in (a, b), h(x) will also be differentiable in (a, b).

Further, h is also continuous in [a, b]. So according to Lagrange's Mean Value Theorem, there exists atleast one $c \in (a, b)$ such that $h'(c) = \dfrac{h(b) - h(a)}{b - a}$,

which proves the required result.

Example 11: A continuous and differentiable function y = f (x) is such that its graph cuts the line y = px + q at m distinct points. Find the minimum number of points at which f '' (x) = 0.

Solution: If line cuts the graph at m points then using LMVT there exists atleast (m – 1) points where f ' (x) = p (slope of the line)



∴ Using Rolle's Theorem, there exists atleast (m – 2) points where f '' (x) = 0.

Example 12: Suppose f'' exists on (a, b) and is continuous at an interior point c. Then prove that

$$\lim_{h \to 0} \left[\frac{f(c+h) - 2f(c) + f(c-h)}{h^2}\right] = f''(c)$$

Solution: Let g(x) = f(x) – f(x – h).

Then f(c + h) – f(c) + f(c – h) = g(c + h) – g(c).

By Lagrange's Mean Value Theorem

$g(c + h) - g(c) = hg'(z)$,

where z is between c and c + h.

But $g'(z) = f'(z) - f'(z - h) = hf''(w)$,

where w is between z – h and z,

hence between c – h and c + h. The result is

$$\frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(w)$$

where $|c - w| < h$. Let h → 0. Then w → c, so $f''(w) \to f''(c)$ since we assumed f″ is continuous at c.

Concept Problems P

1. Verify the mean value theorem for

$$f(x) = \begin{cases} x + x^2, & x \geq 0 \\ 1 - x^2, & x < 0 \end{cases} ; a = -1, b = 1.$$

2. Find c of LMVT

(i) $f(x) = \sqrt{x - 1}$ in [1, 3]

(ii) $f(x) = \dfrac{x}{x + 2}$ on [1, 4]

3. Find a point on the curve $f(x) = \sqrt{x - 2}$ in [2, 3] when the tangent is parallel to the chord joining the end points.

4. Let f have a derivative for all x.

(a) Is every chord of the graph of f parallel to some tangent to the graph of f ?

(b) Is every tangent to the graph of f parallel to some chord of the graph of f ?

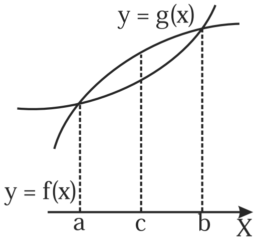
5. Find 'θ' in the LMVT expressed as $f(a + h) = f(a) + h f'(a + \theta h)$

(i) f (x) = λn x, a = 1, h = e – 1

(ii) $f(x) = 2x^2 - 7x + 10$ in [2, 5]

(iii) $f(x) = 3x^2 - 5x + 12$ in [0, 1]

6. Let f(x) and g(x) be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



7. Using the Lagrange's Mean Value Theorem, prove that $\sin(x + h) - \sin x = h \cos \alpha$, where $x < \alpha < x + h$.

8. If a < b, show that a real number c can be found in (a, b) such that $3c^2 = a^2 + ab + b^2$.

9. Show that the function $f(x) = x^{2/3}$ does not satisfy the hypotheses of the Mean Value Theorem on [–1, 27] but that nevertheless there is a number c in (–1, 27) such that $f'(c) = \frac{f(27) - f(1)}{27 - (-1)}$.

Practice Problems N

10. Find a point c in the Lagrange's Mean Value Theorem for the function

$$f(x) = \begin{cases} 0.5\,(3 - x^2) & \text{for} \quad 0 \le x \le 1, \\ 1/x & \text{for} \quad 1 < x < \infty \end{cases}$$

on the interval [0, 2].

11. Two radar patrol cars are located at fixed position 6 km apart on a long, straight road where the speed limit is 55 km/h. A sports car passes the first patrol car travelling at 53 km/h, and then 5 min later, it passes the second patrol car going 48 km/h. Show that at some time between the two clockings, the sports car exceeded the speed limit.

12. During the time T s the point covered the distance of S m moving along a straight line. At the initial and the final moment the velocity of the point is zero. Prove that the some moment the absolute value of the acceleration of the point was not smaller than $4S/T^2$ m/s^2.

13. (i) Let n be a positive integer. Show that there is a number c between 0 and x for which

$$\frac{(1+x)^n - 1}{x} = n(1 + c)^{n-1}.$$

(ii) Use part (i) to evaluate $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x}$.

14. (i) Show that there is a number between 0 and x for which

$$\frac{\cos x - 1}{x} = -\sin w.$$

(ii) Use part (i) to evaluate $\lim\limits_{k \to 0} \frac{\cos x + 1}{x - \pi}$.

15. Let f and g be continuous on [a, b] and differentiable on (a, b). Prove that if f(a) = g(a) and f(b) = g(b), then there is a number c in (a, b) such that f ′(c) = g′(c). Illustrate this result by drawing an appropriate picture.

16. Let f be differentiable everywhere. Assume that f′(a) = 0 and f ′(b) = 1. Must there be a number c, with a < c < b, such that f ′(c) = 1/2 ? Note that f ′ need not be continuous.

17. (a) Define a new function g on [a, b] as follows :

$$g(x) = \frac{f(x) - f(a)}{x - a} \text{ if } x \ne a,\ g(a) = f'(a).$$

Prove that g takes on every values between f′(a) and g(b) in the open interval (a, b). Use the Mean Value Theorem for derivatives to show that f′ takes on every value between f′(a) and g(b) in the open interval (a, b).

(b) Define a new function h on [a,b] as follows :

$$h(x) = \frac{f(x) - f(a)}{x - a} \text{ if } x \ne b,\ h(b) = f'(b).$$

By an argument similar to that in part (a), show that f′ takes on every value between f′(b) and h(a) in (a, b). Since h(a) = g(b), this proves the Intermediate Value Theorem for Derivatives.

18. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in (–b, b) such that f ′(c) = f(b)/b.

19. A number a is called a fixed point of a function f if f(a) = a. Prove that if f′(x) ≠ 1 for all real numbers x, then f has atmost one fixed point.

20. If f ′ is continuous on [a, a + h] and derivable on (a, a + h), then prove that there exists a real number c between a and a + h such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c).$$

21. If a < x < b, prove that

$$f(x) = \frac{(x-a)f(b) - (x-b)f(a)}{b-a} - \frac{1}{2}(b-x)(x-a)f''(x_1)$$

where $a < x_1 < b$, stating the conditions under which the result is valid.

22. If f(x) and g(x) are continuous functions in [a, b] and they are differentiable in (a, b), then prove that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

23. Use the Mean Value Theorem to prove the following result: let f be continuous at x_0 and suppose that $\lim\limits_{x \to x_0} f'(x)$ exists.

Then f is differentiable at x_0, and

$$f'(x_0) = \lim_{x \to x_0} f'(x).$$

24. Use the Mean Value Theorem to prove the following result: The graph of a function f has a vertical tangent line at $(x_0, f(x_0))$ if f is continuous at x_0 and $f'(x)$ approaches either $+\infty$ or $-\infty$ as $x \to x_0^+$ and as $x \to x_0^-$.

25. Given the function f(x) continuous together with its derivatives up to the (n – 1)th order inclusive on the closed interval [a,b] and possessing the nth-order derivative on the interval (a, b), with the equalities $f(a) = f(x_1) = f(x_2) = ... = f(x_{n-1}) = f(b)$, where $a < x_1 < x_2 < ... < x_{n-1} < b$, holding true for that function. prove that there is at least one point ξ on the interval (a, b) for which $f^{(n)}(\xi) = 0$.

7.22 Corollaries of LMVT

Constant Function Theorem (Zero Derivative Theorem)

Suppose f is a continuous function on the closed interval [a, b] and is differentiable on the open interval (a, b), with $f'(x) = 0$ for all x on (a, b). Then the function f is constant on [a, b]

Proof Let x_1 and x_2 be two distinct numbers $(x_1 \neq x_2)$ chosen arbitrarily from the closed interval [a, b]. The function f satisfies the requirements of LMVT on the interval with endpoints x_1 and x_2, which means that there exists a number c between x_1 and x_2 such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

By hypothesis, $f'(x) = 0$ throughout the open interval (a, b), and because c lies within this interval, we have $f'(c) = 0$. Thus, by substitution we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$$f(x_2) = f(x_1)$$

Because x_1 and x_2 were chosen arbitrarily from [a, b], we conclude that f(x) = k, a constant, for all, x as required.

The constant function theorem has far-reaching consequences. Physically, it merely confirms our intuitive notion that a body with zero velocity over an interval does not get beyond its initial position; that a body with zero acceleration over an interval has constant velocity; and that a body with zero jerk over an interval has constant acceleration.

Example 1: Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Solution: Although calculus is not needed to prove this identity, the proof using calculus is quite simple.

If $f(x) = \tan^{-1} x + \cot^{-1} x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x. Therefore, f(x) = C, a constant. To determine the value of C, we put x = 1 [because we can evaluate f(1) easily].

Then $C = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

Thus $\tan^{-1} x + \cot^{-1} x = \pi/2$.

However, the most significant result implied by the above theorem is the following theorem, which gives the structure of functions having the same derivative over an interval. Geometrically we learn from the theorem that continuous graph having the same slopes are vertical transitions of each other.

Constant Difference Theorem

Suppose the functions f and g are continuous on the closed interval [a, b] and differentiable on the open interval (a, b). Then if $f'(x) = g'(x)$ for all x in (a, b), there exists a constant C such that $f(x) = g(x) + C$ for all x on [a, b].

Proof Let h(x) = f(x), then

$$h'(x) = f'(x) - g'(x) = 0 \quad \text{because } f'(x) = g'(x)$$

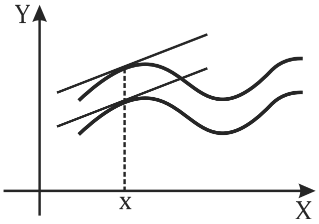
Thus, by the constant function theorem, h (x) = C for some constant C and all x on [a, b], and because h(x) = f(x) – g(x), it follows that

$$f(x) - g(x) = C$$

$$f(x) = g(x) + C.$$

This theorem tells that two functions with equal derivatives on an open interval differ by a constant on the interval.

The corollary asserts that if the graphs of two functions have the property that their tangent lines at points with the same x coordinate are parallel, then one graph can be obtained from the other by raising (or lowering) it by an amount c.



$$\text{Let } f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x | x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict the theorem because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

Example 2: What functions have a derivative equal to 2x everywhere?

Solution: One such function is x^2; another is $x^2 + 2$. For any constant c, $\frac{d}{dx}(x^2 + c) = 2x$. Are there any other possibilities? Corollary 2 tells us there are not, for if f is a function such that $f'(x) = 2x$, then $f'(x) = (x^2)'$ for all x. Thus, the functions f and x^2 differ by a constant, say c, that is. $f(x) = x^2 + c$.

The only functions whose derivatives are 2x are of the form $x^2 + c$.

This example shows that every antiderivative of the function 2x must be of the form $x^2 + c$ for some constant c. More generally, if F(x) is a particular antiderivative of the function f(x) on an interval, then any other antiderivative of f(x) there must be of the form F(x) + c for some constant c.

Example 3: If $f(x) = x^2 + 2x$, find the set of all functions with derivative equal to f.

Solution: Here the interval is the set of all real numbers, and it is easy to see that one function in the set is $\frac{x^3}{3} + x^2$, since

$$\frac{d}{dx}\left(\frac{x^3}{3} + x^2\right) = x^2 + 2x = f(x).$$

Hence, each function G in the set is defined by

$$G(x) = \frac{x^3}{3} + x^2 + c,$$

for some real number c. As c takes on all real values, we get all members of the set. There are no other possibilities.

Example 4: A particle moves along a line with velocity function given by $v(t) = \sin\frac{1}{2}t$. If the particle's initial position is at the origin, find the position at any instant t.

Solution: We are to solve for p(t) given that

$p'(t) = \sin\frac{1}{2}t$ and $p(0) = 0$.

By "hindsight" we know that $(-2\cos\frac{1}{2}t)' = \sin\frac{1}{2}t$.

Therefore by the Constant Difference Theorem

$p(t) - (-2\cos\frac{1}{2}t) = C$ for some constant C.

Setting t = 0, since p(0) = 0 is known,

we have $0 + 2\cos\frac{1}{2}(0) = C$, which gives C = 2.

Therefore $p(t) = -2\cos\frac{1}{2}t + 2$ is our solution.

Monotonicity Theorem

If f is continuous on [a, b] and has a positive derivative on the open interval (a, b), then f is increasing on the interval [a, b]. If f is continuous on [a, b] and has a negative derivative on the open interval (a, b), then f is decreasing on the interval [a, b].

Proof We prove the "increasing" case. Take two numbers x_1 and x_2 such that $a \le x_2 < x_1 \le b$.

By the Mean Value Theorem, there is some number c between x_1 and x_2 such that

$$f(x_1) = f(x_2) + (x_1 - x_2)f'(c)$$

Since, $x_1 - x_2$ is positive, and since $f'(c)$ is assumed to be positive, it follows that $(x_1 - x_2)f'(c) > 0$.

Thus $f(x_1) > f(x_2)$, and the corollary is proved.

(The "decreasing" case is proved similarly.)

As an illustration, consider $3x + 2\sin x$; its derivative is $3 + 2\cos x$. Since $\cos x \ge -1$, $3 + 2\cos x$ is positive. Thus, $3x + 2\sin x$ is an increasing function.

Example 5: Show that the function f, defined on R by $f(x) = -x^3 + 6x^2 - 13x + 4$, is strictly decreasing everywhere.

Solution: $f(x) = -x^3 + 6x^2 - 13x + 4$,

$f'(x) = -3x^2 + 12x - 13$,

$= -3(x-2)^2 - 1$

Here $f'(x) < 0\ \forall\ x \in R$. Hence, the function is strictly decreasing everywhere.

Concept Problems

Q

1. (a) Differentiate $\sec^2 x$ and $\tan^2 x$.
   (b) The derivatives in (a) are equal. Corollary 2 then asserts that there exists a constant c such that $\sec^2 x = \tan^2 x + c$. Find that constant.

2. Which of the corollaries to the Mean Value Theorem implies that (a) if two cars on a straight road have the same velocity at every instant, they remain a fixed distance apart? (b) if all the tangent to a curve are horizontal, the curve is a horizontal straight line?

3. (a) Use the Constant Difference Theorem to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, \infty)$, and if f and g have the same value at some number x_0, then $f(x) = g(x)$ for all x in $(-\infty, \infty)$.
   (b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

4. (a) Use the Constant Difference Theorem to show that if $f'(x) = g'(x)$ for all x in $(-\infty, \infty)$, and if $f(x_0) - g(x_0) = c$ at some number x_0, then $f(x) - g(x) = c$ for all x in $(-\infty, \infty)$.
   (b) Show that the function $h(x) = (x-1)^3 - (x^2+3)(x-3)$ is constant for all x in $(-\infty, \infty)$, and find the constant.

(c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

5. If the graphs of two functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Given reasons for your answer.

6. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

7. Show that $f(x) = (x-2)^3$ and $g(x) = (x^2+12)(x-6)$ differ by a constant. Use f and g to demonstrate the Constant Difference Theorem.

8. Consider $f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$

Does this example contradict the Zero Derivative Theorem? Why or why not ?

9. Suppose that f and g are continuous on [a, b] and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$.

Practice Problems O

10. Calculate the first derivatives of $f(x) = x^2/(x^2+1)$ and $g(x) = -1/(x^2+1)$. What can you conclude about the graphs of these functions?

11. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

12. Let $f(x) = 1/x$ and $g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ 1+\dfrac{1}{x} & \text{if } x < 0 \end{cases}$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude that $f - g$ is constant ?

13. Show that $f(x) = \dfrac{x+4}{5-x}$ and $g(x) = \dfrac{-9}{x-5}$ differ by a constant. Are the conditions of the Constant Difference Theorem satisfied ? Does $f'(x) = g'(x)$?

14. Show that $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right) = \dfrac{d}{dx}\left(-\dfrac{1}{x+1}\right)$ even though $\dfrac{x}{x+1} \ne -\dfrac{1}{x+1}$. Does't this contradict Corollary 2 of the Mean Value Theorem? Given reasons for your answer.

15. Show that if $f''(x) = 0$ for all x, then f is a linear function. (that is, $f(x) = ax + b$ for constants $a \ne 0$ and b.)

16. Show that if $f''(x) = Ax + B$ for constants $A \ne 0$ and B, then f(x) is a quadratic function. (that is, $f(x) = ax^2 + bx + c$ for constants a, b and c, where $a \ne 0$).

17. Show that the function $f(x) = 9 - 12x + 6x^2 - x^3$, $x \in R$, is decreasing in every interval.

18. (a) Show that if f and g are function for which $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x, then $f^2(x) + g^2(x)$ is a constant.

(b) Give an example of functions f and g with this property.

19. Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = f(x)$ for all x, then $f^2(x) - g^2(x)$ is a constant.

7.23 Related Inequalities

1. If f is continuous on [a, b] and $m \le f'(x) \le M$ for all $x \in (a, b)$, then $m(b-a) \le f(b) - f(a) \le M(b-a)$.

Using LMVT, we have $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

Now, if M and m be the maximum and minimum values respectively of the derivative of f(x) in (a, b), then we have $m \le f'(c) \le M$

i.e. $m \le \dfrac{f(b)-f(a)}{b-a} \le M$

i.e. $m(b-a) \le f(b) - f(a) \le M(b-a)$

i.e. $m(b-a) + f(a) \le f(b) \le f(a) + M(b-a)$

The above result can be used to estimate the value of f(b) if the value of f(a) is known.

2. If f is continuous on [a, b] and $|f'(x)| \le M$ for all $x \in (a, b)$, then $|f(b) - f(a)| \le M|b-a|$.

Using LMVT, we have $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$\Rightarrow \left|\dfrac{f(b)-f(a)}{b-a}\right| = |f'(c)|$

Since $|f'(c)| \le M$, $\left|\dfrac{f(b)-f(a)}{b-a}\right| \le M$

$\Rightarrow |f(b) - f(a)| \le M|b-a|$.

3. If f is continuous on [a, b] and $|f'(x)| \ge m$ for all $x \in (a, b)$, then $|f(b) - f(a)| \ge m|b-a|$.

Using LMVT, we have $\left|\frac{f(b)-f(a)}{b-a}\right| = |f'(c)|$

Since $|f'(c)| \geq m$, $\left|\frac{f(b)-f(a)}{b-a}\right| \geq m$

$\Rightarrow$ $|f(b) - f(a)| \geq m\,|b - a|$.

4. Appromaximation of a function

The functional value f(a) may be taken as an approximation for f(b) with error measured by a mean derivative multiple of the deviation of b from a.

If, in particular, a constant (say) M bounds derivative values numerically on (a, b), then the product M(b – a) is an error bound for the difference f(b) – f(a) :

that is, if $|f'(x)| \leq M$ for all $x \in (a, b)$, then

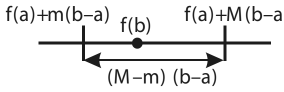
$$|f(b) - f(a)| = |f'(c)(b - a)| \leq M(b - a).$$

In fact, if $m \leq f'(x) \leq M$ for all $x \in (a, b)$ then

$$f(a) + m(b - a) \leq f(a) + f'(c)(b - a) \leq f(a) + M(b - a)$$

$$f(a) + m(b - a) \leq f(b) \leq f(a) + M(b - a)$$

from which f(b) may be approximated by either of the extremes of the above inequalities, within an error bound of (M – m) (b – a), obtained as the difference of the extremes. See figure.



5. Suppose that f is continuous on [a, b], that f″ exists on (a, b), and that $f''(x) > 0$ for all $x \in (a, b)$. Let $c \in (a, b)$. Then for all $x \in [a, b]$,

$$f(x) \geq f(c) + f'(c)(x - c)$$

with equality only for x = c. If $f''(x) > 0$ is replaced by $f''(x) \geq 0$, there may be equality for other values of x.

Proof Define $g(x) = f(x) - f(c) - f'(c)(x - c)$.

Then $g(c) = 0$, $g'(c) = 0$,

$g''(x) = f''(x) > 0$ for $a < x < b$.

Then function $g'(x) > 0$ for $c < x < b$ and

$g'(x) < 0$ for $a < x < c$.

Suppose $c < x < b$.

Then $g(x) - g(c) = (x - c)\,g'(z)$, where $c < z < x$.

Therefore $g(x) - g(c) > 0$, $g(x) > g(c)$.

Likewise $g(x) > g(c)$ for $a < x < c$.

This proof is easily modified to cover the case $f''(x) \geq 0$.

Example 1: Let f be differentiable for all x. If f(1) = –2 and $f'(x) \geq 2$ for all $x \in [1, 6]$, then find the least possible value of f(6).

Solution: By Lagrange's Mean Value Theorem, there exists $c \in (1,6)$ such that

$$\frac{f(6)-f(1)}{6-1} = f'(c) \geq 2 \quad (\because f'(x) \geq 2 \text{ for all } x \in [1, 6])$$

$$\Rightarrow \frac{f(6)+2}{5} \geq 2$$

$\Rightarrow$ $f(6) + 2 \geq 10 \Rightarrow f(6) \geq 8$.

Hence, the least possible value of f(6) is 8.

Example 2: Suppose that f(0) = –3 and $f'(x) \leq 5$ for all values of x. How large can f(2) possible by ?

Solution: We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval [0, 2]. There exists a number c such that

$$f(2) - f(0) = f'(2)(2 - 0)$$

so $f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$

We are given that $f'(x) \leq 5$ for all x, so in particular we know that $f'(c) \leq 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10$. so

$$f(2) = -3 + 2f'(c) \leq -3 + 10 = 7$$

The largest possible value for f(2) is 7.

Example 3: If f (x) is continuous on [0, 2], differentiable on (0, 2), f (0) = 2, f (2) = 8, and $f'(x) \leq 3$ for all x in (0, 2), then find the value of f (1).

Solution: Applying LMVT to f in [0, 1] and again in [1, 2] There exists $c_1 \in (0, 1)$, such that

$$\frac{f(1)-f(0)}{1-0} = f'(c_1) \leq 3 \Rightarrow f(1) \leq 5 \qquad \text{...(1)}$$

There exists $c_2 \in (1, 2)$, such that

$$\frac{f(2)-f(1)}{2-1} = f'(c_2) \leq 3 \Rightarrow f(1) \geq 5 \qquad \text{...(2)}$$

Hence, (1) and (2) imply that f (1) = 5.

Example 4: Prove that there exists no such function f (x) such that f (1) = – 1, f (4) = 7 and $f'(x) > 3$ for all $x \in R$.

Solution: Using LMVT for f in [1, 4], there must exist some $c \in (1, 4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1} = \frac{7+1}{3} = \frac{8}{3}$.

But it is given that $f'(x) > 3 \ \forall x \in R$.

Hence, no such function exists.

Example 5: Let $f'(x) = e^{x^2}$ and f (0) = 10. If A < f (1) < B can be concluded from the Mean Value Theorem, then find the largest value of (A – B).

Solution: Applying LMVT in [0, 1] to the function y = f(x), we get

$$f'(c) = \frac{f(1)-f(0)}{1-0}, \text{ for some } c \in (0, 1)$$

$$\Rightarrow e^{c^2} = \frac{f(1)-f(0)}{1}$$

$\Rightarrow$ $f(1) - 10 = e^{c^2}$ for some $c \in (0, 1)$

but $1 < e^{c^2} < e$ in (0, 1)

$1 < f(1) - 10 < e$

$\Rightarrow$ $11 < f(1) < 10 + e$

$\therefore$ $A = 11, B = 10 + e$

$\Rightarrow$ $A - B = 1 - e.$

Example 6: Let $f(x)$ be twice differentiable function such that $f''(x) < 0$ in $[0, 2]$. Then prove that

(i) $f(0) + f(2) = 2f(c),\ 0 < c < 2,$

(ii) $f(0) + f(2) < 2f(1).$

Solution: By Intermediate Value Theorem

$$\frac{f(0)+f(2)}{2} = f(c), \quad 0 < c < 2$$

By Mean Value Theorem,

$$f(1) - f(0) = f'(c_1), \quad 0 < c_1 < 1$$
$$f(2) - f(1) = f'(c_2), \quad 1 < c_2 < 2$$

By subtraction,

$$f(0) + f(2) - 2f(1) = f'(c_2) - f'(c_1)$$
$$= (c_2 - c_1) f''(c),\ c_1 < c < c_2$$

$\Rightarrow$ $f(0) + f(2) - 2f(1) < 0$

$\Rightarrow$ $f(0) + f(2) < 2f(1).$

Example 7: If $f'(x) = \dfrac{1}{1+x^2}$ for all x and $f(0) = 0$, show that $0.4 < f(2) < 2$.

Solution: Given $f'(x) = \dfrac{1}{1+x^2}$ for all x ...(1)

$\therefore$ $f'(x) > 0$ for all x $[\because\ 1 + x^2 > 0]$

Also given $f(0) = 0$ ...(2)

From (1), it follows that f(x) is differentiable at all x, therefore f(x) is also continuous at all x

$\therefore$ By Lagrange's Mean Value Theorem in [0, 2]

$$\frac{f(2)-f(0)}{2-0} = f'(c) = \frac{1}{1+c^2}, \text{ where } 0 < c < 2$$

or, $\dfrac{f(2)-0}{2-0} = \dfrac{1}{1+c^2}$ or, $f(2) = \dfrac{2}{1+c^2}$. ..(3)

Now $0 < c < 2$

$\therefore$ $\dfrac{2}{1+c^2} < \dfrac{2}{1+0^2}$ or, $\dfrac{2}{1+c^2} < 2$ ...(4)

and $\dfrac{2}{1+c^2} > \dfrac{2}{1+2^2} = \dfrac{2}{5} = 0.4$

or, $\dfrac{2}{1+c^2} > 0.4$ ...(5)

From (3), (4) and (5) it follows that $0.4 < f(2) < 2$.

Example 8: Use LMVT to prove that $\tan x > x$ for $x \in \left(0, \dfrac{\pi}{2}\right)$.

Solution: Consider $f(x) = \tan x$ in $[0, x]$, $0 < x < \pi/2$.

Using LMVT, $\dfrac{\tan x - 0}{x - 0} = \sec^2 x > 1$

$\Rightarrow$ $\dfrac{\tan x}{x} > 1$

$\Rightarrow$ $\tan x > x.$

Example 9: Prove that if n is an even natural number then $na^{n-1} \le \dfrac{b^n - a^n}{b - a} < nb^{n-1}$ for $b > a$.

Solution: Consider the function

$$f(x) = x^n,\ x \in R$$

and $f'(x) = nx^{n-1}$.

Since, f is continuous and differentiable on R, therefore applying Lagrange's Theorem on f(x) in (a, b) we have

$$f'(c) = \frac{f(b)-f(a)}{b-a} \text{ for some c in (a, b)}$$

Since, $n - 1$ is odd, therefore $f'(x)$ is strictly increasing in (a, b) irrespective of whether a, b are positive or negative number. Thus, we have

$$f'(a) < f'(c) < f'(b)$$

i.e. $na^{n-1} < \dfrac{f(b)-f(a)}{b-a} < nb^{n-1}$.

Example 10: Prove that $(\beta - \alpha)\sec^2\alpha < \tan\beta - \tan\alpha < (\beta - \alpha)\sec^2\beta$, where $0 < \alpha < \beta < \dfrac{\pi}{2}$.

Solution: Let $f(x) = \tan x$

We know that tanx is continuous and differentiable function in $(0, \pi/2)$, so according to LMVT,

there exists a point 'γ' in (α, β) where

$$f'(\gamma) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \frac{\tan\beta - \tan\alpha}{\beta - \alpha}$$

Also, $f'(x) = \sec^2 x,\ f''(x) = 2\sec^2 x \tan x > 0$

So, $f'(x)$ is increasing in $(0, \pi/2)$

$\Rightarrow$ $f'(\alpha) < f'(\gamma) < f'(\beta)$

$\Rightarrow$ $\sec^2\alpha < \dfrac{\tan(\beta) - \tan(\alpha)}{(\beta - \alpha)} < \sec^2\beta$

which proves the required result.

Example 11: Using Mean Value Theorem show that

$a^\alpha < \{a\alpha + b(1 - \alpha)\}\, b^{\alpha-1}$, ...(1)

for $0 < \alpha < 1$ and a, b are positive real numbers.

Solution: To prove that inequality (1), define a function f by $f(t) = t^\alpha,\ t > 0,\ 0 < \alpha < 1$.

Then, evidently, f is continuous on [a,b]. Applying the Mean Value Theorem to f, we obtain

$$\frac{f(b)-f(a)}{b-a} = f'(\eta)$$

for some η in the open interval (a, b). This yields

$\dfrac{b^\alpha - a^\alpha}{b-a} = \alpha\eta^{\alpha-1}$. ...(2)

Since $\eta \in (a, b)$, we obtain

$$\eta^{\alpha-1} > b^{\alpha-1}.$$

Hence, since $\alpha > 0$, we have

$$\alpha\eta^{\alpha-1} > \alpha b^{\alpha-1}$$

Using (2) in the above inequality, we see that

$$b^{\alpha} - a^{\alpha} > (b - a)\,\alpha b^{\alpha-1}$$

which after some simplification yields the inequality (1), that is

$$a^{\alpha} < \{a\alpha + b(1 - \alpha)\}\, b^{\alpha-1}.$$

Example 12: Show that the square roots of two successive natural numbers greater than N^2 differ by less than $1/2N$.

Solution: Consider $f(x) = \sqrt{x}$ on the interval $[n + n + 1]$. Applying Lagrange's Mean Value Theorem, we get $\xi \in (n, n + 1)$, so that

$$\frac{f(n+1)-f(n)}{n+1-n} = f'(\xi)$$

$$\Rightarrow \quad \frac{1}{2\sqrt{\xi}} = f'(\xi) = \frac{f(n+1)-f(n)}{n+1-n} = \sqrt{n+1} - \sqrt{n}$$

In $n > N^2$, then $\xi > N^2$

$$\Rightarrow \quad \frac{1}{2\sqrt{\xi}} < \frac{1}{2N} \;\Rightarrow\; \sqrt{n+1} - \sqrt{n} < \frac{1}{2N}.$$

Example 13: Using Lagrange's Mean Value Theorem prove that $|\cos a - \cos b| \le |a - b|$.

Solution: Consider $f(x) = \cos x$ in $[a, b]$.

It is continuous and differentiable in $[a, b]$.

Hence, according to Lagrange's Mean Value Theorem there exists atleast one $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \quad \frac{\cos b - \cos a}{b-a} = -\sin c$$

$$\Rightarrow \quad \left|\frac{\cos b - \cos a}{b-a}\right| = |-\sin c| \le 1$$

$$\Rightarrow \quad |\cos b - \cos a| \le |b - a|$$

$$\Rightarrow \quad |\cos a - \cos b| \le |a - b|.$$

Example 14: Show that $|\sin x - \sin y| \le |x - y|$ for numbers x and y by applying the Mean Value Theorem.

Solution: The inequality is true if $x = y$. Suppose $x \ne y$; then $f(\theta) = \sin\theta$ is differentiable and hence continuous for all θ, with $f'(\theta) = \cos\theta$. By applying the LMVT to f on the closed interval with endpoints x and y, we see that $\dfrac{f(x)-f(y)}{x-y} = f'(c)$ for some c between x and y. Because $f'(c) = \cos c$, it follows that $\dfrac{\sin x - \sin y}{x-y} = \cos c$

Finally, we take the absolute value of the expression on each side, remembering that $|\cos c| \le 1$ for any number c :

$$\left|\frac{\sin x - \sin y}{x-y}\right| = |\cos c| \le 1$$

Thus, $|\sin x - \sin y| \le |x - y|$.

Example 15: Prove that $|\tan^{-1} x| \le |x|,\ x \in R$

Solution: Consider the function

$$f(x) = \tan^{-1} x - x,\ x \in R$$

and $f'(x) = \dfrac{1}{1+x^2} - 1 = \dfrac{-x^2}{1+x^2}$

Applying Lagrange's Theorem on f(x) in the interval [0, x], $(x \ge 0)$, we have

$$f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} \quad \text{for some c in (0, x).}$$

Since, $1 + x^2$ increases in [0, x], therefore $\dfrac{1}{1+x^2} - 1$ decreases in [0, x],

Thus, we have $f'(x) < f'(c) < f'(0)$

i.e. $\dfrac{-x^2}{1+x^2} < \dfrac{f(x)}{x} < 0$

From the right hand inequality, we have

$f(x) \le 0$, since $x \ge 0$

i.e. $\tan^{-1} x \le x,\ x \ge 0.$ ...(1)

Similarly, applying Lagrange's Theorem on f(x) in the interval [x, 0], $(x \le 0)$, we have

$$f'(0) \le f'(c) \le f'(x)$$

i.e. $0 \le \dfrac{f(x)}{x} \le \dfrac{-x^2}{1+x^2}$

From the left hand inequality, we have

$f(x) \ge 0$, since $x \le 0$

i.e. $\tan^{-1} x \ge x,\ x \le 0$

i.e. $\tan^{-1}(-x) \ge -x,\ x \le 0$ ...(2)

Hence, from inequalities (1) and (2), we have the desired result

$$|\tan^{-1} x| \le |x|,\ x \in R.$$

Example 16: Using Lagrange's Mean Value Theorem prove that $\dfrac{x}{1+x} < \ln(1 + x) < x \ \ \forall\, x > 0.$

Hence show that $0 < [\ln(1 + x)]^{-1} - x^{-1} < 1\ \forall\, x > 0$

Solution: Consider $f(x) = \ln(1 + x)$ in [0, x].

Since f(x) satisfies the condition of Lagrange's Mean Value Theorem in [0, x], there exists θ $(0 < \theta < 1)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$$

$$\Rightarrow \quad \frac{1}{2}\ln(1 + x) = \frac{1}{1 + \theta x}$$

Now $\quad 0 < \theta < 1,\ x > 0 \Rightarrow \theta x < x$

$$\Rightarrow \quad 1 + \theta x < 1 + x$$

$$\Rightarrow \quad \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \frac{x}{1+\theta x} > \frac{x}{1+x} \qquad \text{...(1)}$$

Again $0 < \theta < 1,\ x > 0$

$$\Rightarrow \quad \theta x > 0$$

$$\Rightarrow \quad 1 + \theta x > 1$$

$$\Rightarrow \quad \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+\theta x} < x \qquad \text{...(2)}$$

(1) and (2) give

$$\frac{x}{1+x} < \ln(1 + x) < x$$

$$\Rightarrow \frac{1+x}{x} > [\ln(1+x)]^{-1} > \frac{1}{x}$$

$$\Rightarrow \frac{1}{x} + 1 > [\ln(1+x)]^{-1} > \frac{1}{x}$$

$$\Rightarrow 1 > [\ln(1+x)]^{-1} - x^{-1} > 0.$$

Example 17: Show that $\sqrt{1+h} < 1 + \frac{1}{2}h$ if $h > 0$.

Solution: In $f(b) - f(a) = f'(c)(b - a)$
let $f(x) = \sqrt{1+x}$, $a = 0$, and $b = h$.
Then for some $c \in (0, h)$,

$$\sqrt{1+h} - 1 = \frac{1}{2\sqrt{1+c}}h < \frac{1}{2}h$$

so that $\sqrt{1+h} < 1 + \frac{1}{2}h$

Example 18: Let $0 < x < y$. Prove that

$$\frac{\ln y - \ln x}{y - x} < \frac{1}{\sqrt{xy}}.$$

Solution: Let $f(x) = \ln x$ and $g(x) = \sqrt{x} - 1/\sqrt{x}$ for $x \geq 1$. We shall prove $f(x) < g(x)$ for $x > 1$.
First $f(1) = g(1) = 0$.
It suffices to prove $f'(x) < g'(x)$ for $x > 1$,

that is $x^{-1} < \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$,

$2x^{1/2} < x + 1$. This is true because $0 < (\sqrt{x} - 1)^2$.

Thus, $\ln x < \sqrt{x} - 1/\sqrt{x}$ for $x > 1$.

We now simply replace x by y/x to get the above inequality.

Example 19: Assume that f is differentiable on $[0, \infty)$. If $f'(x) \to \infty$ as $x \to \infty$, then prove that $f(x) \to \infty$ as $x \to \infty$.

Solution: Let there be a number x_0 such that
$f'(x) \geq 1$ for $x \geq x_0$.
Then by LMVT,
$f(x) - f(x_0) = f'(y)(x - x_0) \geq x - x_0$,
so $f(x) \geq f(x_0) - x_0 + x$ ...(1)
Now as $x \to \infty$, (1) implies that $f(x) \to \infty$.

Example 20: If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0$ $\forall\ x \in (a, b)$, then prove that $f''(c) < 0$ for some $c \in (a, b)$.

Solution: Applying LMVT to $f(x)$ for $x \in \left[a, \frac{a+b}{2}\right]$

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = \frac{2}{(b-a)} \cdot f\left(\frac{a+b}{2}\right), \text{ for}$$

some $c_1 \in \left(a, \frac{a+b}{2}\right)$.

Applying LMVT again on $f(x)$ for $x \in \left[\frac{a+b}{2}, b\right]$

$$f'(c_2) = -\frac{2f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}}, \text{ for some } c_2 \in \left(\frac{a+b}{2}, b\right)$$

Applying LMVT to $y = f'(x)$ in $[c_1, c_2]$,

$$f''(x) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1}$$

$$= \frac{-\frac{4}{(b-a)} \cdot f\left(\frac{a+b}{2}\right)}{c_2 - c_1} < 0, \text{ where } x \in (c_1, c_2).$$

Example 21: The second derivative $f''(x)$ of the function $f(x)$ exists for all x in [0, 1] and satisfies $|f''(x)| \leq 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in [0, 1].

Solution: The first derivative $f'(x)$ exists for all x in [0, 1] which implies that f(x) is continuous for all x in [0, 1]. Also , it is given that $f(0) = f(1)$.

Thus, applying Rolle's Theorem on f(x) in the interval [0, 1], we have $f'(c) = 0$ for some c in [0, 1].

The second derivative $f''(x)$ exists for all x in [0, 1] which implies that $f'(x)$ is continuous for all x in [0, 1].

Thus, applying Lagrange's Theorem on $f'(x)$ in the interval $[c, x]$ $(c < x \leq 1)$, we have

$$f''(\alpha) = \frac{f'(x) - f'(c)}{x - c} \text{ for some } \alpha \text{ in } [c, x]$$

i.e. $f'(x) = (x - c) f''(\alpha)$ $[\because f'(c) = 0]$
i.e. $|f'(x)| = |x - c|\,|f''(\alpha)| < 1$
[since $|x - c| < 1$ and $|f''(x)| \leq 1$ for all x]

Similarly, applying Lagrange's Theorem on $f'(x)$ in the interval $[x, c]$ $(0 \leq x < c)$, we have

$$f''(\beta) = \frac{f'(c) - f'(x)}{c - x} \text{ for some } \beta \text{ in } [x, c]$$

i.e. $f'(x) = (x - c) f''(\beta)$
i.e. $|f'(x)| = |x - c|\,|f''(\beta) < 1$
When $x = c$, $f'(x) = f'(c) = 0$
$\therefore$ $|f'(x)| = |\,0\,| = 0 < 1$.
Hence, we have the desired result $|f'(x)| < 1$ for all x in [0, 1].

Example 22: Suppose $f^{(n)}$ and $g^{(n)}$ exist in [a, b] and $f(a) = g(a)$, $f'(a)$, ..., $f^{(n-1)}(a)$, and $f^{(n)}(x) \leq g^{(n)}(x)$ on (a, b). Prove $f(x) \leq g(x)$ on [a, b].

Solution: Let $h(x) = g(x) - f(x)$.
Then $h(a) = h'(a) = \ldots = h^{(n-1)}(a) = 0$
and $h^{(n)}(x) \geq 0$ for $a < x < b$.

For n = 1, the Mean Value Theorem yields $h(x) \geq 0$. In general, $h'(x)$ satisfies the given conditions for $n - 1$. Hence, by induction $h'(x) \geq 0$, so $h(x) \geq 0$.

Concept Problems R

1. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be ?
2. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.
3. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.
4. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x ?
5. Show that for any $x > 4$, there is a number w between 4 and x such that $\frac{\sqrt{x}-2}{x-4} = \frac{1}{2\sqrt{w}}$.

   Use this fact to show that if $x > 4$, then $\sqrt{x} < 1 + \frac{x}{4}$.
6. Use the LMVT to show that

   $\sqrt{1+x} - 4 < \frac{1}{8}(x-15)$, if $x > 15$.
7. Use the Mean Value Theorem to show that if f is differentiable on an interval I and if $|f'(x)| \leq M$ for all values of x in I, then $|f(x) - f(y)| \leq M|x - y|$ for all values of x and y in I.
8. (a) Use the Mean Value Theorem to show that if f is differentiable on an open interval I, and if $|f'(x)| \geq M$ for all values of x in I, then $|f(x) - f(y)| \geq M|x - y|$ for all values of x and y in I.

   (b) Use the result in part (a) to show that $|\tan x - \tan y| \geq |x - y|$ for all values of x and y in the interval $(-\pi/2, \pi/2)$.

   (c) Use the result in part (b) to show that $|\tan x + \tan y| \geq |x + y|$ for all values of x and y in the interval $(-\pi/2, \pi/2)$.
9. If a, b, $\in [0, \pi/2)$, prove that $|\cos a - \cos b| \leq |a - b|$.

Practice Problems P

10. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for all $x \in [1, 6]$. Then prove that $f(6) \geq 8$.
11. If $f'(x) \leq 2$ for all x, what is the most the values of f can increase on [0, 6]? give reasons for your answer.
12. Suppose f is continuous on [0, b], $f(0) = 0$, and $f'' \geq 0$ on (0, b). Prove that $f(x)/x \leq f(x)$ for all $x \in (0, b]$.
13. Using Lagrange's Theorem, prove the validity of the following inequalities :

    (i) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y| \ \forall$ x, y,

    (ii) $\frac{x-y}{x} < \ln\frac{x}{y} < \frac{x-y}{x}$ for $0 < y < x$.
14. Suppose that f is differentiable on [0, 1] and that its derivative is never zero. Show that $f(0) \neq f(1)$.

    What conditions could you place on f to guarantee that $\min f' \leq \frac{f(b)-f(a)}{b-a} \leq \max f'$, where min f′ and max f′ refer to the minimum and maximum value of f′ on [a, b]? Give reasons for your answer.
15. Show that if f is concave up on an interval I, then the graph of f lies above all its tangents on I.
16. Use the Mean Value Theorem to show that if c is a point in I, then $f(x) > f(c) + f'(c)(x - c)$ for all x in I. This property is sometimes used as the definition of concave up.
17. (a) Use the Mean Value Theorem to show that

    $\sqrt{y} - \sqrt{x} < \frac{y-x}{2\sqrt{x}}$ if $0 < x < y$.

    (b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{yx} < \frac{1}{2}(x + y)$.
18. If f″ exists on [a, b] and if $|f''(x)| \leq M$, for all x in [a, b], then prove that

    $|f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\}| \leq \frac{1}{2}(b - a)^2 M$.
19. If $f''(x)$ exists for all x in [a, b], and $|f''(x)| \leq M$, for all x in [a, b], and if c be any real number between a and b, then prove that

    $$\left|f(c) - f(a) - (c-a)\left\{\frac{f(b)-f(a)}{b-a}\right\}\right| \leq \frac{1}{8}(b-a)^2 M .$$
20. Use the LMVT to show that

    $\frac{1}{2x+1} > \frac{1}{5} + \frac{2}{25}(2-x)$ if $0 < x < 2$.
21. Prove that $x - x^2/3 < \tan^{-1}x$, if $x > 0$.
22. Use the Mean Value Theorem to establish the following inequalities :

    (i) $\frac{x}{1+x^2} < \tan^{-1}x < x$, if $x > 0$

    (ii) $x < \sin^{-1}x < \frac{x}{\sqrt{(1-x^2)}}$, if $0 < x < 1$.

    (iii) $e^a(x - a) < e^x - e^a < e^x(x - a)$, if $a < x$.

    (iv) $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2}e^x$, if $x > 0$.
23. Prove that $x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^3}{2(1+x)}$, if $x > 0$
24. Prove that e^{-x} lies between $1 - x$ and $1 - x + \frac{x^2}{2}$, $\forall x \in R$.

25. Using Mean Value Theorem, show that
$$\frac{\beta-\alpha}{1+\beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2}, \beta > \alpha > 0 .$$

26. Use the inequality $\frac{1}{6}\sqrt{3} < 0.29$ to prove that $1.71 < \sqrt{3} < 1.75$.

27. Prove that $\pi/36 < \sin 55° - \sin 45° < \pi\sqrt{2}/36$.

28. Suppose $|f'(x)| \le g'(x)$ on (a, b). Prove that $|f(b) - f(a)| \le g(b) - g(a)$. What do you conclude if $|f(b) - f(a)| = g(b) - g(a)$?

29. Suppose $f'(x) \le B$ on (a, b) and $f(b) - f(a) = B(b - a)$. Prove that f(x) is linear.

30. Suppose $f'(x) \ge A$ on (a, b) and $f(b) - f(a) = A(b - a)$. Prove that f(x) is linear.

31. Assume that f is differentiable on $[0, \infty)$. If $f(x) + f'(x) \to 0$ as $x \to \infty$, then prove that $f(x) \to 0$.

32. Assume that f is differentiable on $[0, \infty)$. If $f(x) + f'(x) \to L$ as $x \to \infty$, then prove that $f(x) \to L$.

33. Let $0 < x < y$ and $0 < p < 1$. Prove that $x^p y^{1-p} < px + (1 - p)y$ with equality only if $x = y$.

34. Suppose f is differentiable on (0, b) and $f(x) > 0$, $f'(x) > 0$ for all x. Suppose $f(x) \to 0$, $f'(x) \to 0$, and $f(x)/f'(x) \to L$ as $x \to 0^+$. Prove that $L = 0$.
Also, prove that $x^{f(x)} \to 1$.

35. Suppose $f(x) > 0$, $f(x) \to 0$ as $x \to 0^+$. Does it follow that $x^{f(x)} \to 1$ as $x \to 0^+$?

36. Let $0 < x < y$. Prove that $\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}$.

37. Prove that $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$.

38. Suppose f is continuous on [a, b], $f(a) = f(b) = 0$, and f'' exists on [a, b). Suppose $f(c) > 0$ for some c. Prove $f''(c) < 0$ for some $c \in (a, b)$.

7.24 Cauchy's Mean Value Theorem

Let f and g be functions defined on [a, b] such that

(i) f and g are continuous on [a, b],
(ii) f and g are derivable on (a, b), and
(iii) g'(x) does not vanish at any point of (a, b).

then there exists a real number $c \in (a, b)$ such that
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof Let us first observe that as a consequence of condition (iii), $g(a) \ne g(b)$. For, if g(a) were equal to g(b), then the function g would satisfy all the conditions of Rolle's theorem, and consequently for some x in (a, b), we would have $g'(x) = 0$.

The main idea behind the proof of Cauchy's Theorem is the function
$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a)),$$
for every x in [a, b]
which meets the requirements of Rolle's Theorem. Rolle's Theorem is applicable to this function, thus implying Cauchy's Theorem.

The function h is continuous on [a, b] differentiable on (a, b), and, in addition, $h(a) = h(b) = 0$. Hence, by Rolle's Theorem, there exists a real number c in (a, b) such that $h'(c) = 0$. Since
$$h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(x),$$
we obtain
$$0 = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(c),$$
and from this equation the conclusion of the theorem follows at once.

Alternative proof :

Let $\phi(x) = A\,g(x) - f(x)$ in [a, b],
$$\text{where } A = \frac{f(b)-f(a)}{g(b)-g(a)}$$
$$\phi(a) = A\,g(a) - f(a)$$
$$\phi(b) = A\,g(b) - f(b)$$
Now $\quad A\,g(a) - f(a) = A\,g(b) - f(b)$
$$f(b) - f(a) = A\big(g(b) - g(a)\big)$$
$\Rightarrow \quad \phi(a) = \phi(b)$

Using Rolle's Theorem, $\phi'(c) = 0$

$\Rightarrow \quad A\,g'(c) - f'(c) = 0.$
$$\frac{f'(c)}{g'(c)} = A = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Note:

(i) If we put $b = a + h$, then c can be written as $a + \theta h$ where θ is some real number such that $0 < \theta < 1$. The above theorem then reads as follows:

If f and g are continuous on [a, a + h] and are derivable on (a, a + h), and if g'(x) does not vanish for any x in (a, a + h), then there exists a real number θ between 0 and 1, such that
$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

(ii) If we take $g(x) = x$, for all x in [a, b], then Cauchy's Mean Value Theorem yields Lagrange's Mean Value Theorem as a particular case.

(iii) The reader might be tempted to prove Cauchy's Mean Value Theorem by applying Lagrange's Mean Value Theorem to the functions f and g. It can be easily seen that the desired result cannot be obtained in this manner. In fact, we would thus obtain that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c_1)}{g'(c_2)}$$

where $a < c_1 < b$, $a < c_2 < b$. Note that there c_1 is not necessarily equal to c_2.

Let us show that Cauchy's Theorem can be given the same **geometrical interpretation** as Lagrange's Theorem. For the sake of convenience, let us denote the independent variable by t and consider the functions $y = f(t)$ and $x = g(t)$ as representing parametric equations of a plane curve in the xy-plane. As the parameter t runs through the corresponding interval $[t_1, t_2]$ the variable point (x, y) describes a curve whose initial and terminal points are, respectively, $[g(t_1), f(t_1)]$ and $[g(t_2), f(t_2)]$.

The slope of the chord connecting these points is equal to the ratio $\frac{f(t_2)-f(t_1)}{g(t_2)-g(t_1)}$.

The derivative of y (regarded as a parametrically represented function of x) with respect to x is $\frac{dy}{dx}=\frac{f'(t)}{g'(t)}$. Consequently, the formula

$$\frac{f(t_2)-f(t_1)}{g(t_2)-g(t_1)}=\frac{f'(\xi)}{g'(\xi)} \quad (t_1 < \xi < t_2)$$

again expresses the equality of the slopes of the chord subtending an arc and of the tangent line drawn to that arc at an intermediate point.

The theorem has the following **physical interpretation**. Let the functions f(x) and g(x) govern the motion of two particles such that f(x) is the distance of one particle from the origin at the instant of time x, the function g(x) having a similar meaning for the other particle. From time a to time b the first particle covers the distance f(b) – f(a), while the second covers the distance g(b) – g(a). The ratio of the distances covered by the particle during this interval of time is $\frac{f(a)-f(b)}{g(b)-g(a)}$.

The theorem states that at some instant in time the ratio of the velocities of the particles is equal to the ratio of the distances covered. If this were not so, then

$\frac{f(b)-f(a)}{g(b)-g(a)}-\frac{f'(x)}{g'(x)}$ would be nonzero for all $x \in (a, b)$ and its sign would be unchanged. Suppose

$$\frac{f(b)-f(a)}{g(b)-g(a)}>\frac{f'(x)}{g'(x)} \qquad ...(1)$$

This would lead to a contradiction since the ratio of the distances covered cannot be greater than the ratio of the velocities at every instant in time.

It should be noted that this interpretation of Cauchy's Theorem uses somewhat more than the hypothesis. When we proposed inequality (1), we implied the continuity of the function f '(x)/g'(x), but this is not stipulated by the theorem.

Example 1: Find whether the functions $f(x) = x^2 - 2x + 3$ and $g(x) = x^3 - 7x^2 + 20x - 5$ satisfy the conditions of the Cauchy's Mean Value Theorem in the interval [1, 4] and find the corresponding value of c.

Solution: The given functions f(x) and g(x) are continuous everywhere in the interval [1, 4] , and their derivatives $f'(x) = 2x - 2$ and $g'(x) = 3x^2 - 14x + 20$ exist everywhere. Further g'(x) does not vanish at any real x.

Consequently, the Cauchy's Mean Value Theorem is applicable to the given functions :

$$\frac{f(4)-f(1)}{g(4)-g(1)}=\frac{f'(c)}{g'(c)},$$

$$\Rightarrow \frac{11-2}{27-9}=\frac{2c-2}{2c^2-14c+20}, \quad (1 < c < 4).$$

Solving the latter equation, we find two values of c : $c_1 = 2$ and $c_2 = 4$.

Of these two values only $c_1 = 2$ is an interior point of the interval.

Example 2: If, in the Cauchy's Mean Value Theorem, we have $g(x) = \sqrt{x}$ and $f(x) = 1/\sqrt{x}$ then, prove that c is the geometric mean between a and b and if we write $g(x) = (1/x^2)$, $f(x) = (1/x)$, then, c is the harmonic mean between a and b.

Solution: When $g(x) = \sqrt{x}$ and $f(x) = (1/\sqrt{x})$,

we have $\frac{g(b)-g(a)}{f(b)-f(a)}=\frac{g'(c)}{f'(c)}$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{(1/\sqrt{(b)}-(1/\sqrt{a})}=\frac{\frac{1}{2}c^{-1/2}}{-\frac{1}{2}c^{-3/2}}$$

Thus $-\sqrt{(ab)} = -c$ or $c = \sqrt{(ab)}$.

that is, c is the geometric mean between a and b.

And when $g(x) = (1/x^2)$, $f(x) = (1/x)$,

we have $\frac{g(b)-g(a)}{f(b)-f(a)}=\frac{g'(c)}{f'(c)}$

$$\Rightarrow \frac{(1/b^2)-(1/a^2)}{(1/b)-(1/a)}=\frac{-2c^{-3}}{-c^{-2}}$$

Thus, $\frac{a+b}{ab}=\frac{2}{c}$ or $c=\frac{2ab}{a+b}$

That is, c is the harmonic mean between a and b.

Example 3: Let f(x) and g(x) be two differentiable functions and f(x) = 8, g(2) = 0, f(4) = 10 and g(4) = 8, then prove that g'(x) = 4 f '(x) for atleast one x∈ (2, 4)

Solution: Consider h(x) = g(x) – 4f(x) in [2, 4].
h(2) = g(2) – 4f(2) = –32, also h(4) = –32.
Thus, h'(x) = 0 for atleast one x ∈ (2, 4) using Rolle's Theorem.
⇒ g'(x) – 4f'(x) = 0 for atleast one x∈ (2, 4)

Alternative:
Using Cauchy's Mean Value Theorem, there exists atleast one c ∈ (2, 4) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(4)-f(2)}{g(4)-g(2)}$$

$$= \frac{10-8}{8-0} = \frac{1}{4}$$

⇒ 4f'(c) = g'(c)
⇒ 4f'(x) = g'(x) holds true for atleast one x ∈ (2, 4).

Example 4: Suppose α, β and θ are angles satisfying $0 < \alpha < \theta < \beta < \frac{\pi}{2}$, then prove that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = -\cot\theta.$$

Solution: Let f(x) = sinx and g(x) = cosx, then f and g are continuous and derivable.

Also, sin x ≠ 0 for any $x \in \left(0, \frac{\pi}{2}\right)$.

So by Cauchy's Mean Value Theorem,

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)},\ 0 < \alpha < \theta < \beta < \frac{\pi}{2}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}.$$

Generalized Mean Value Theorem

If f, g and h are continuous on [a, b] and derivable on (a, b), then there exists a number c in (a, b) such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Proof Consider the function F defined as

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \quad ...(1)$$

Since each of the functions, f, g and h is continuous on [a, b] and derivable on (a, b), F is also continuous on [a, b] and derivable on (a, b). Also, F(a) = F(b) = 0. Thus F satisfies all the conditions of Rolle's Theorem on [a, b]. Consequently, there exists c in (a, b) such that F'(c) = 0.

$$\text{Since } F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

therefore, the result follows .

Corollaries:

(i) Taking the function h to be the constant function h(x) = 1, for all x in [a, b], the above theorem reduces to Cauchy's Mean Value Theorem.

(ii) Taking the functions g and h as g(x) = x, h(x) = 1, for all x in [a, b], the above theorem reduces to Lagrange's Mean Value Theorem.

Concept Problems S

1. (i) For the function f(x) = $x^2 + 2$ and F(x) = $x^2 - 1$ test whether the Cauchy's Mean Value Theorem holds on the interval [1, 2] and find c.

(ii) Do the same with respect to f(x) = sin x and F(x) = cos x on the interval $\left[0, \frac{\pi}{2}\right]$.

2. Is Cauchy's Theorem valid for the function f(x) = x^2 and g(x) = x^3 on the interval [–1, 1]? What condition of Cauchy's Theorem is not satisfied for these functions?

3. Do the function f(x) = e^x and $g(x) = \frac{x^2}{1+x^2}$ satisfy the conditions of the Cauchy's Mean Value Theorem in the interval [–3, 3] ?

4. Show that $\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(x_1)}{g''(x_1)}$ where $a < x_1 < b$, stating the conditions under which the result is valid.

5. Assume that f is differentiable on [0, ∞). If $f'(x)/e^x \to 0$ as $x \to \infty$, then prove that $f(x)/e^x \to 0$ as $x \to \infty$.

6. By considering the function F, defined by
F(x) = Af(x) + Bg(x) + Cx + D, for all x in [a – h, a + h], where the constants are chosen so that
F(a – h) = F(a) = F(a + h) = 0,
prove that there exists a real number c between a – h and a + h, such that

$$\frac{f(a+h)-2f(a)+f(a-h)}{g(a+h)-2g(a)+g(a-h)} = \frac{f''(c)}{g''(c)},$$

stating the conditions under which the result is valid.

Target Problems for JEE Advanced

Problem 1: If the function $f(x) = (x^2 - 4)^n(x^2 - x + 1)$, $n \in N$ assumes a local minimum value at $x = 2$, then find all possible values of n.

Solution: $f(x) = (x^2 - 4)^n (x^2 - x + 1)$

$= (x - 2)^n (x + 2)^n(x^2 - x + 1)$

$f(2) = 0$.

For $x = 0$ to be a point of minima, we must have

$f(2 + h) > 0$ and $f(2 - h) > 0$

Now $x^2 - x + 1 > 0$ for $\forall$ x.

$f(2 + h) = (2 + h - 2)^n (2 + h + 2)^n((2 + h)^2 - (2 + h) + 1)$

$= (h)^n (4 + h)^n((2 + h)^2 - (2 + h) + 1)$

> 0 for any $n \in N$.

$f(2 - h) = (2 - h - 2)^n (2 - h + 2)^n((2 - h)^2 - (2 - h) + 1)$

$= (-h)^n (4 - h)^n((2 - h)^2 - (2 - h) + 1)$

$= (-h)^n$ (a positive quantity)

> 0 for any even $n \in N$.

For $x = 0$ to be a point of minima, n must be an even natural number.

Problem 2: Suppose f is differentiable on [a, b], and that $f'(a)$ and $f'(b)$ have opposite signs. Prove there exists $c \in (a, b)$ such that $f'(c) = 0$. [Note that continuity of f' is not assumed.]

Solution: Suppose $f'(a) < 0$ and $f'(b) > 0$ to be definite.

There is $x_0 > a$ with $f(x_0) < f(a)$

and $x_1 < b$ with $f(x_1) < f(b)$.

Let f(c) be the minimum of f on [a, b].

Then $f(c) \le f(x_0) < f(a)$, $f(c) \le f(x_1) < f(b)$, so $a < c < b$. By Fermat Theorem, $f'(c) = 0$.

Problem 3: Determine the points of maxima and minima of the function, $f(x) = (1/8) \ln x - bx + x^2$, $x > 0$ when $b \ge 0$ is a constant.

Solution: Here $f(x) = (1/8) \ln x - bx + x^2$ is defined and continuous for all $x > 0$.

Then $f'(x) = \frac{1}{8x} - b + 2x$

or $f'(x) = \frac{16x^2 - 8bx + 1}{8x}$

For extrema, let $f'(x) = 0$

$\Rightarrow$ $16x^2 - 8bx + 1 = 0$

so, $x = \frac{8b \pm \sqrt{64(b^2 - 1)}}{2 \times 16}$ or $x = \frac{b \pm \sqrt{b^2 - 1}}{4}$

Obviously the roots are real if $b^2 - 1 \ge 0$

$\Rightarrow$ $b \ge 1$ [as $b > 0$]

When $b > 1$, then using sign scheme for a quadratic function $f'(x)$, we find that $f'(x)$ changes sign from positive to negative at $x = \frac{b - \sqrt{b^2 - 1}}{4}$

$\therefore$ f(x) has a local maximum at $x = \frac{b - \sqrt{b^2 - 1}}{4}$

and $f'(x)$ changes sign from negative to positive at

$x = \frac{b + \sqrt{b^2 - 1}}{4}$

$\therefore$ f(x) has a local minimum at $x = \frac{b + \sqrt{b^2 - 1}}{4}$

If $b = 1$, $f'(x) = \frac{16x^2 - 8x + 1}{x} = \frac{(4x - 1)^2}{x}$

It does not change sign.

$\therefore$ f(x) has neither maximum nor minimum if $b = 1$.

Thus,

$f(x)_{max}$ when $x = \frac{b - \sqrt{b^2 - 1}}{4}$ and $b > 1$

$f(x)_{min}$ when $x = \frac{b + \sqrt{b^2 - 1}}{4}$ and $b > 1$

f(x) has neither maximum nor minimum when $b = 1$

Problem 4: Let $f(x) = (x - 1)^m (x - 2)^n$, $x \in R$. Then which of the following cases ensures that each critical point of f(x) is either local maximum or local minimum :

(i) $m = 2, n = 3$ (ii) $m = 2, n = 4$

(iii) $m = 3, n = 4$

Solution: We have $f(x) = (x - 1)^m (x - 2)^n$

$f'(x) = (x - 1)^m\, n(x - 2)^{n-1} + (x - 2)^n\, m(x - 1)^{m-1}$

$\Rightarrow$ $f'(x) = (x - 1)^{m-1} (x - 2)^{n-1} (nx - n + mx - 2m)$

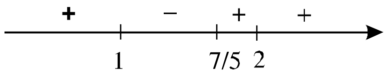
$= (x - 1)^{m-1} (x - 2)^{n-1} ((m + n)x - (2m + n))$

For maxima or minima $f'(x) = 0$

$\therefore$ $x = 1, 2, \frac{2m + n}{m + n}$

(i) If $m = 2$, $n = 3$, then $x = 1, 2, 7/5$

$\therefore$ $f'(x) = (x - 1)(x - 2)^2(5x - 7)$



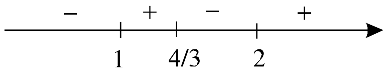
Hence at $x = 1$, f(x) is maximum,

at $x = 2$, f(x) is neither maximum nor minimum

at $x = 7/5$, f(x) is minimum .

(ii) If $m = 2$, $n = 4$, then $x = 1, 2, 4/3$

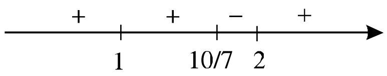
$\therefore$ $f'(x) = (x - 1)(x - 2)^3(6x - 8)$



Hence at $x = 1$, $f(x)$ is minimum
at $x = 2$, $f(x)$ is minimum
at $x = 4/3$, $f(x)$ is maximum .

(iii) If $m = 3$, $n = 4$, then $x = 1, 2, 10/7$

$\therefore$ $f'(x) = (x-1)^2 (x-2)^3 (7x-10)$



Hence at $x = 1$, $f(x)$ is neither maximum nor minimum
at $x = 10/7$, $f(x)$ is maximum.
at $x = 2$, $f(x)$ is minimum .
Here at each critical point of $f(x)$ is either maximum or minimum when $m = 2$, $n = 4$.

Problem 5: Let $F(x)$ be a cubic polynomial defined by $F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$. Find the sum of all possible integral value(s) of a for which $F(x)$ has negative point of local minimum in the interval [1, 100].

Solution: We have $F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$

$\therefore$ For $F(x)$ to have negative point of local minimum, the equation $F'(x) = 0$ must have two distinct negative roots.

Now, $F'(x) = x^2 + 2(a-3)x + 1$

$\therefore$ The following condition(s) must be satisfied simultaneously.

(i) Discriminant > 0; (ii) Sum of roots < 0 ;
(iii) Product of roots > 0

Now, $D > 0$

$\Rightarrow$ $4(a-3)^2 > 4$ $\Rightarrow$ $(a-3)^2 - 1 > 0$

$\Rightarrow$ $(a-2)(a-4) > 0$

$\therefore$ $a \in (-\infty, 2) \cup (4, \infty)$ ...(1)

Also $-2(a-3) < 0$ $\Rightarrow$ $a - 3 > 0$

$\Rightarrow$ $a > 3$ ...(2)

And product of root(s) $= 1 > 0 \ \forall \ a \in R$ ...(3)

$\therefore$ $(1) \cap (2) \cap (3)$

$\Rightarrow$ $a \in (4, \infty)$

Hence, the sum of value(s) of a
$= 5 + 6 + 7 + \ldots\ldots\ldots + 100 = 5040$.

Problem 6: Find the values of a and b for which all the extrema of the function $f(x) = a^2x^3 - ax^2/2 - 2x - b$, is positive and the minimum is at the point $x = 1/3$.

Solution: Since $f(x)$ has minimum at $x = 1/3$,

$f'(x) = 0$ at $x = 1/3$

$\Rightarrow$ $3a^2x^2 - ax - 2 = 0$ at $x = 1/3$

$\therefore$ $3a^2(1/3)^2 - a(1/3) - 2 = 0$

$$\Rightarrow \frac{a^2}{3} - \frac{a}{3} - 2 = 0$$

$\Rightarrow$ $a^2 - a - 6 = 0$ or $a = -2, 3$

Now there arise two cases:

Case 1 $a = 3$

Since the local minimum value is positive, we have

$f(1/3) > 0$

$\therefore$ $9(1/3)^3 - 3(1/3)^2/2 - 2(1/3) - b > 0$

$\Rightarrow$ $b < -1/2$

Case 2 $a = -2$.

Since the local minimum value is positive, we have

$f(1/3) > 0$

$\therefore$ $(-2)^2(1/3)^3 - (-2)(1/3)^2/2 - 2(1/3) - b > 0$

$$\Rightarrow b < \frac{4}{27} + \frac{1}{9} - \frac{2}{3} \text{ or } b < -\frac{11}{27}$$

Finally, when $a = 3$, $b < -\frac{1}{2}$ and

when $a = -2$, $b < -\frac{11}{27}$.

Problem 7: Let $f(x)$ be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then find the distance between its two horizontal tangents.

Solution: Let $f''(x) = 6a(x-1)$ $(a > 0)$

On integrating both sides, we get

$$f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b.$$

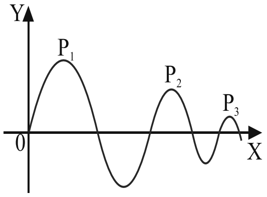
Now $f'(-1) = 0 \Rightarrow 9a + b = 0 \Rightarrow b = -9a$.

$\therefore$ $f'(x) = 3a(x^2 - 2x - 3) = 0 \Rightarrow x = -1$ and 3.

So, $y = f(-1)$ and $y = f(3)$ are two horizontal tangents.

Hence, distance between its two horizontal tangents
$= |f(3) - f(-1)| = |-22 - 10| = 32$.

Problem 8: A rough sketch of a part of the graph of $y = e^{-x} \sin x$ for $x \geq 0$ is as shown



If $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, ... , $P_n(x_n, y_n)$ are the successive points at which maxima occurs, find the values of y_1 and y_n. Hence deduce that

$$y_1 + y_2 + \ldots\ldots + y_n < \frac{e^{2y} \cdot y_1}{e^{2\pi} - 1}.$$

Solution: $\frac{dy}{dx} = 0 \Rightarrow \tan x = 1$, $x = n\pi + \frac{\pi}{4}$

$$\Rightarrow x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, \ldots.$$

The alternate points are $x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, \ldots$

$$\Rightarrow \quad P_1(x_1, y_1) = P_1\left(\frac{\pi}{4}, \frac{e^{-\pi/4}}{\sqrt{2}}\right)$$

$$P_2\,(x_2, y_2) = P_2\left(2\pi + \frac{\pi}{4}, e^{-2\pi}\,y_1\right)$$

Hence $y_n = e^{-\left[2(n-1)\pi - \frac{\pi}{4}\right]} \cdot \frac{1}{\sqrt{2}} = e^{-(2n-1)\pi} \cdot y_1$

$$\therefore \quad y_1 + y_2 + y_3 + \ldots\ldots + y_n$$

$$= y_1\,(1 + e^{-2\pi} + e^{-4\pi} + \ldots\ldots + e^{-2(n-1)\pi})$$

$$= y_1\,\frac{(1 - e^{-2\pi n})}{1 - e^{-2\pi}} < \frac{y_1}{1 - e^{-2\pi}} = \frac{y_1 e^{2\pi}}{e^{2\pi} - 1}.$$

Problem 9: Let $P(x)$ be a polynomial of degree 5 having extremum at $x = -1, 1$ and $\lim\limits_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$.

If M and m are the maximum and minimum value of the function $y = P'(x)$ on the set $A = \{x \mid x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

Solution: We have $\lim\limits_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$

$$\therefore \quad \lim_{x\to 0}\frac{P(x)}{x^3} = 6$$

Consider $P(x) = ax^5 + bx^4 + 6x^3$

$\Rightarrow \quad P'(x) = 5ax^4 + 4bx^3 + 18x^2$

Now, $P'(-1) = 0$ gives $5a - 4b = -18$

and $P'(1) = 0$ gives $5a + 4b = -18$

$\therefore$ On solving, we get

$$a = \frac{-18}{5}, \; b = 0.$$

Hence, $P(x) = \frac{-18}{5}x^5 + 6x^3$

$\Rightarrow \quad P'(x) = -18x^4 + 18x^2 = 18(x^2 - x^4)$

and $P''(x) = 18(2x - 4x^3) = 36(x - 2x^3)$

$\Rightarrow \quad P''(x) = 36x(1 - 2x^2)$

Also $A = \{x \mid x^2 + 6 \le 5x\}$ gives $x \in [2, 3]$

Clearly $P''(x) < 0 \; \forall \; x \in [2, 3]$

So, $y = P'(x)$ is decreasing function in [2, 3]

$$\therefore \quad M = P'_{max}\,(x = 2) = 18(4 - 16) = -18 \times 12$$

$$\text{and } m = P'_{min}\,(x = 3) = 18(9 - 81) = -18 \times 72$$

$$\text{Hence } \frac{m}{M} = \frac{-18 \times 72}{-18 \times 12} = 6.$$

Problem 10: Let two non-collinear vectors $\vec{a}$ and $\vec{b}$ inclined at an angle $\frac{2\pi}{3}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = 4$.

A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given as

$\overrightarrow{OP} = (e^t + e^{-t})\,\vec{a} + (e^t - e^{-t})\,\vec{b}$. If the least distance of P from origin is $\sqrt{2}\sqrt{\sqrt{a} - b}$ where $a, b \in N$ then find the value of $(a + b)$.

Solution: We have $(\overrightarrow{OP})^2 = (e^t + e^{-t})^2\,(\vec{a})^2$

$$+ (e^t - e^{-t})^2\,(\vec{b})^2 + 2\,(e^t + e^{-t})\,(e^t - e^{-t})\,(\vec{a}\cdot\vec{b})$$

$$\left((\vec{a})^2 = |\vec{a}|^2 = 9, \; (\vec{b})^2 = |\vec{b}|^2 = 16\right.$$

$$\left.\text{and } \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \left|\overrightarrow{OP}\right|^2 = 9(e^t + e^{-t})^2 + 16(e^t - e^{-t})^2$$

$$+ 2(e^{2t} - e^{-2t}) \cdot 3 \cdot 4 \cdot \left(-\frac{1}{2}\right)$$

$$= 13e^{2t} + 5e^{-2t} - 14.$$

Now $\frac{d}{dt}\left|\overrightarrow{OP}\right|^2 = 0 \Rightarrow \quad 26e^{2t} - 10e^{-2t} = 0$

$$\Rightarrow \quad e^{4t} = \frac{5}{13}, \text{ so } e^{2t} = \frac{\sqrt{5}}{\sqrt{13}}$$

$$\therefore \quad \left|\overrightarrow{OP}\right|^2_{min} = 13\left(\frac{\sqrt{5}}{\sqrt{13}}\right) + 5\left(\frac{\sqrt{13}}{\sqrt{5}}\right) - 14$$

$$\Rightarrow \quad \left|\overrightarrow{OP}\right|^2_{min} = 2\sqrt{65} - 14$$

$$\Rightarrow \quad \left|\overrightarrow{OP}\right|_{min} = \sqrt{2}\sqrt{\sqrt{65} - 7} = \sqrt{2}\sqrt{\sqrt{a} - b},$$

so $a = 65, b = 7$

Hence, $(a + b) = 72$.

Problem 11: Let $x, y \in R$ in the interval $(0, 1)$ and $x + y = 1$. Find the minimum value of the expression $x^x + y^y$.

Solution: Let $Z = x^x + (1 - x)^{1-x}$

$Z = f\,(x) + f\,(1 - x)$ where $f\,(x) = x^x$

$$\frac{dZ}{dx} = f'(x) - f'(1 - x) = 0$$

For maxima and minima $\frac{dZ}{dx} = 0$

$$\Rightarrow \quad f'(x) = f'(1 - x) \qquad \Rightarrow \quad x = \frac{1}{2}$$

$$Z\Big|_{x=1/2} = f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = 2\,f\left(\frac{1}{2}\right)$$

$$= 2\cdot\left(\frac{1}{2}\right)^{\frac{1}{2}} = \sqrt{2}$$

$$\frac{d^2Z}{dx^2} = f''(x) + f''(1-x)$$

$$\frac{d^2Z}{dx^2}\Bigg|_{x=1/2} = 2\,f''\left(\frac{1}{2}\right) > 0,$$

as $\left[\begin{array}{l} f'(x) = x^x(1+\ln x) \\ f''(x) = x^x\cdot\dfrac{1}{x} + (1+\ln x)^2 x^x \\ f''(1/2) > 0 \end{array}\right.$

∴ minimum value of Z = $\sqrt{2}$.

Problem 12: Consider a polynomial P(x) of the least degree that has a maximum equal to 6 at x = 1, and a minimum equal to 2 at x = 3. Compute the value of P(2) + P'(0).

Solution: The polynomial is an everywhere differentiable function. Therefore, the points of extremum can only be roots of the derivative. Furthermore, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x_1 = 1$ and $x_2 = 3$ has the form a(x – 1) (x – 3).

Hence P'(x) = a(x – 1) (x – 3) = a(x^2 – 4x + 3) since at the point x = 1, there must be P(1) = 6, we have

$$P(x) = \int_1^x P'(x)dx + 6 = a\int_1^x (x^2 - 4x + 3)dx + 6$$

$$= a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$$

The coefficient 'a' is determined from the condition

P(3) = 2, whence a = 3.

Hence P(x) = $x^3 - 6x^2 + 9x + 2$

Now P(2) = 8 – 24 + 18 + 2 = 28 – 24 = 4

Also P'(x) = 3(x^2 – 4x + 3) ⇒ P'(0) = 9

∴ P(2) + P'(0) = 4 + 9 = 13.

Problem 13:

(i) Investigate for the points of maxima and minima of the function defined by

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$$

(ii) Find all possible values of the parameter 'a' so that the function f(x) = x^3 – 3 (7 – a) x^2 – 3 (9 – a^2) x + 2, has a negative point of local minimum.

Solution:

(i) We have

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$$

$$= \int_1^x (t-1)(t-2)^2[2(t-2) + 3(t-1)]dt$$

$$= \int_1^x (t-1)(t-2)^2(5t-7)dt$$

⇒ f′(x) = (x – 1) (x – 2)^2 (5x – 7)

Now, for maximum or minimum,

f′(x) = 0 ⇒ x = 1, 7/5, 2

$$f'(x) = \frac{dy}{dx} = 5[(x-1)(x-7/5)](x-2)^2$$

(a) Consider x = 1

$$\left.\frac{dy}{dx}\right]_{x=1^+} > 0 \text{ and } \left.\frac{dy}{dx}\right]_{x=1^-} < 0$$

⇒ f(x) is maximum at x = 1

(b) Consider x = 7/5

$$\left.\frac{dy}{dx}\right]_{x=(7/5)^-} < 0 \text{ and } \left.\frac{dy}{dx}\right]_{x=(7/5)^+} > 0$$

⇒ f(x) is minimum at x = 7/5

(c) Consider x = 2

$$\left.\frac{dy}{dx}\right]_{x=2^-} > 0 \text{ and } \left.\frac{dy}{dx}\right]_{x=2^+} > 0$$

⇒ f(x) is neither maximum nor minimum at x = 2.

(ii) f(x) = x^3 – 3 (7 – a) x^2 – 3 (9 – a^2)x + 2

f′(x) = 3x^2 – 6(7 – a) x – 3(9 – a^2)

For distinct real roots D > 0

36(7–a)^2 + 4 × 3 × 3 (9 – a^2) > 0

⇒ 49 + a^2 – 14a + 9 – a^2 > 0

14 a < 58 ⇒ $a < \frac{29}{7}$

For local minima,

f″(x) = 6x – 6(7 – a) > 0 ⇒ x – 7 + a > 0

7 – a < x, as x must be negative

⇒ 7 – a < 0 ⇒ a > 7

Thus, it is contradictory i.e., for real roots $a < \frac{29}{7}$ and for negative point of local minimum a > 7. So there is no possible value of a.

Problem 14: If the function f : [0, 4] → R is differentiable, then show that

(a) for some a, b ∈ [0, 4], $\{f(4)\}^2 - \{f(0)\}^2 = 8f'(a)\,f(b)$.

(b) for some α, β ∈ [0, 2], $\int_0^4 f(t)\,dt = 2\alpha f(\alpha^2) + 2\beta f(\beta^2)$.

Solution: (a) From Lagrange's Mean Value Theorem, we have

$$\frac{f(4)-f(0)}{4-0} = f'(a) \text{ for some } a \in [0, 4] \qquad ...(1)$$

From Intermediate Value Theorem we have

$$\frac{f(4)+f(0)}{2} = f(b) \text{ for some } b \in [0, 4] \qquad ...(2)$$

By multiplying (1) and (2) we get

$$\frac{\{f(4)\}^2-\{f(0)\}^2}{8} = f'(a)\, f(b) \text{ for some } a, b \in [0, 4] \text{ for}$$

some $b \in [0, 4]$.

(b) Putting $t = z^2$ and $dt = 2z\,dz$, we have

$$\int_0^4 f(t)\ dt = \int_0^{\sqrt{x}} f(z^2)\cdot 2z\,dz$$

From Lagrange's Mean Value Theorem, we have

$$\frac{\int_0^{\sqrt{4}} f(z^2)\cdot 2z\,dz - \int_0^{\sqrt{0}} f(z^2)\cdot 2z\,dz}{\sqrt{4}-\sqrt{0}} = f(y^2)\cdot 2y \text{ for some}$$

$y \in [0, 2]$

i.e. $\int_0^2 f(z^2)\cdot 2z\,dz = 4y\,f(y^2)$ for any 0 , $\alpha < y < \beta < 2$

Together, we have

$$\int_0^2 f(z^2)\cdot 2z\,dz = 2\alpha f(\alpha^2) + 2\beta f(\beta^2) \text{ for some}$$

$\alpha, \beta \in [0, 2]$, that is

$$\int_0^4 f(t)\cdot dt = 2\alpha f(\alpha^2) + 2\beta f(\beta^2) \text{ for some } \alpha, \beta \in [0, 2]$$

Problem 15: Find the shortest distance between the curves $9x^2 + 9y^2 - 30y + 16 = 0$ and $y^2 = x^3$.

Solution: $9x^2 + 9y^2 - 30y + 16 = 0$ can be rewritten as

$$x^2 + \left(y - \frac{5}{3}\right)^2 = 1.$$

Any point on the curve $y^2 = x^3$ can be taken as (t^2, t^3).

Let L be the distance between the centre of the given circle and the point (t^2, t^3), then

$K = L^2 = t^4 + (t^3 - 5/3)^2$

Now, we calculate the minimum value of L.

Required distance = L – radius of given circle.

Now, $\frac{dK}{dt} = 4t^3 + 2\left(t^3 - \frac{5}{3}\right)3t^2 = 0$

For maximum or minimum, t = 0 or 1

Now, $\frac{d^2K}{dt^2} = 12t^2 + 30t^4 - 20t$, $\left.\frac{d^2K}{dt^2}\right|_{t=0} = 0$

But, $\left.\frac{d^3K}{dt^3}\right|_{t=0} \neq 0 \Rightarrow$ There is neither maxima nor minima at t = 0.

Also, $\frac{d^2K}{dt^2} > 0$ at t = 1 ⇒ L^2 is minimum at t = 1

i.e., L is minimum at t = 1

So, shortest distance = (value of L at t = 1) – (Radius of the circle) = $\frac{\sqrt{13}}{3} - 1$.

Problem 16: If f(x) is differentiable in [0, 1] such that f(0) = f(1) = 0, then prove that for any $\lambda \in R$, there exists c such that $f'(c) = \lambda f(c)$, $0 < c < 1$.

Solution: Consider, $F(x) = e^{-\lambda x} f(x)$, $\lambda \in R$

$F(0) = f(0) = 0$

$F(1) = e^{-\lambda} f(1) = 0$

∴ By Rolle's theorem, $F'(c) = 0$

$F'(x) = e^{-\lambda x}(f'(x) - \lambda f(x))$

$F'(c) = 0 \Rightarrow e^{-\lambda c}(f'(c) - \lambda f(c)) = 0$

⇒ $f'(c) = \lambda f(c)$, $0 < c < 1$.

Problem 17: Let a_i, i = 1, 2, 3, 4, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers b_i, i = 1, 2, 3 the equation

$$a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6 = 0$$

has at least one real root in the interval $-1 \le x \le 1$.

Solution: Let f(x)

$$= a_1x + b_1\frac{x^2}{2} + a_2x^3 + b_2\frac{x^4}{4} + a_3x^5 + b_3\frac{x^6}{6} + a_4x^7$$

f (x) is continuous in [–1, 1] and differentiable in (–1, 1)

$$f(-1) = \left(\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}\right) - (a_1 + a_2 + a_3 + a_4)$$

$$= \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}$$

$$f(1) = \left(\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}\right) + (a_1 + a_2 + a_3 + a_4)$$

$$= \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}$$

⇒ f (–1) = f (1)

Applying Rolle's Theorem, there exists some $c \in (-1, 1)$ where $f'(c) = 0$

i.e. $a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6 = 0$.

Problem 18: Using Rolle's Theorem, prove that the derivative of the function, $f(x) = x \sin \frac{\pi}{x}$, $x > 0$ and $f(0) = 0$, vanishes at infinite number of points in the interval (0, 1)

Solution: The function f(x) is continuous everywhere in [0, 1] and vanishes at points where

$$\sin \frac{\pi}{x} = 0$$

i.e. $\frac{\pi}{x} = k\pi,\ k \in I$

i.e. $x = \frac{1}{k} = 1, \frac{1}{2}, \frac{1}{3}, \ldots..$

Differentiating w.r.t.x, we have

$f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$

Thus, f(x) has a derivative at every point in (0, 1). Hence, according to Rolle's Theorem, the derivative f '(x) of the function vanishes atleast once in each of the intervals

$$\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{3}\right), \ldots.., \left(\frac{1}{k}, \frac{1}{k+1}\right)$$

which gives an infinite number of points.

We now show that it is necessary to assume in Rolle's Theorem that f is differentiable throughout (a,b).

Problem 19:

(i) Find all possible values of the parameter 'a' so that $x^3 - 3x + a = 0$ has three real and distinct roots.

(ii) Prove that there exists exactly two non-similar isosceles triangles ABC such that tanA + tanB + tanC = 100.

Solution:

(i) Let $f(x) = x^3 - 3x + a$

$\Rightarrow f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

Clearly $x = -1$ is the point of maxima and $x = 1$ is the point of minima.

Now, $f(1) = a - 2$, $f(-1) = a + 2$

The roots of f(x) = 0 would be real and distinct if $f(1)\, f(-1) < 0$

$\Rightarrow (a - 2)(a + 2) < 0 \Rightarrow -2 < a < 2$

$\Rightarrow$ Thus given equation would have real and distinct roots if $a \in (-2, 2)$.

(ii) Let A = B, then 2A + C = 180° and 2tanA + tanC = 100

Now 2A + C = 180° $\Rightarrow$ tan2A = –tanC ...(1)

Also 2tanA + tanC = 100

$\Rightarrow$ 2tanA – 100 = – tanC ...(2)

From (1) and (2), $2\tan A - 100 = \frac{2\tan A}{1 - \tan^2 A}$

Let tanA = x, then $\frac{2x}{1 - x^2} = 2x - 100$

$\Rightarrow x^3 - 50x^2 + 50 = 0$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus $f'(x) = 0$ has roots $0, \frac{100}{3}$.

Also $f(0).f\left(\frac{100}{3}\right) < 0$. Thus f(x) = 0 has exactly three distinct real roots. Therefore tanA and hence A has three distinct values. But one of them will be obtuse angle. Hence there exists exactly two non similar isosceles triangles.

Problem 20: Using Mean Value Theorem show that $\left(1 + \frac{1}{x}\right)^x$ is an increasing function of x while $\left(1 + \frac{1}{x}\right)^{x+1}$ is a decreasing function of x for x > 0.

Solution: Let us define a function f by

$f(t) = \ln t,\ t > 0.$

Applying the Mean Value Theorem to f, we get

$f(x + 1) - f(x) = f'(\eta)$

for some $\eta \in (x, x + 1)$. This yields

$$\ln(x + 1) - \ln(x) = \frac{1}{\eta},\ x > 0. \qquad \ldots(1)$$

Since $\frac{d}{dx}\left[\ln\left(1 + \frac{1}{x}\right)^x\right] = \frac{d}{dx}[x(\ln x + 1) - \ln(x))]$

$= \ln(x + 1) - \ln(x) + x\left[\frac{1}{x+1} - \frac{1}{x}\right]$

$= \ln(x + 1) - \ln(x) - \frac{1}{x+1}$

$= \frac{1}{\eta} - \frac{1}{x+1} > 0$, by (1),

and ln(x) is an increasing function, we conclude that $\left(1 + \frac{1}{x}\right)^x$ is an increasing function of x.

To show that $\left(1 + \frac{1}{x}\right)^{x+1}$ is a decreasing function of x we proceed in a similar manner and show

$\frac{d}{dx}\left[\ln\left(1 + \frac{1}{x}\right)^{x+1}\right] = \frac{d}{dx}[(x + 1)(\ln(x + 1) - \ln(x))]$

$= \ln(x + 1) - \ln(x) + (x + 1)\left[\frac{1}{x+1} - \frac{1}{x}\right]$

$= \ln(x + 1) - \ln(x) - \frac{1}{x}$

$= \frac{1}{\eta} - \frac{1}{x} < 0$, by (1).

Hence $\left(1 + \frac{1}{x}\right)^{x+1}$ is a decreasing function of the variable x.

Problem 21: If f(x) is continuous and differentiable on R and its derivative vanishes for two values of x only, then show that f(x) cannot vanish at more than three points.

Solution: The function f(x) is continuous and differentiable on R. Suppose that f(x) vanishes at four points (more than three points), so that the equation f(x) = 0 has four different roots, say α, β, γ and δ in increasing order.

Then, we have $f(\alpha) = f(\beta) = f(\gamma) = f(\delta) = 0$

Thus, according to Rolle's Theorem, $f'(x) = 0$ for some x in $[\alpha, \beta]$, $[\beta, \gamma]$ and $[\gamma, \delta]$, which implies that $f'(x)$ will vanish for atleast three values of x. According to the given condition $f'(x)$ vanishes for two values of x only. Hence, by contradiction, f(x) cannot vanish at more than three points.

Problem 22: If $f(x) = \tan x$, $x \in \left[0, \frac{\pi}{7}\right]$, then show that $\frac{\pi}{7} \le f\left(\frac{\pi}{7}\right) \le \frac{2\pi}{7}$.

Solution: We have $f(x) = \tan x$, $x \in \left[0, \frac{\pi}{7}\right]$

and $f'(x) = \sec^2 x$, $x \in \left[0, \frac{\pi}{7}\right]$.

Applying Lagrange's Theorem on f(x) in the interval $\left[0, \frac{\pi}{7}\right]$, we have

$$f'(c) = \frac{f(\pi/7) - f(0)}{(\pi/7) - 0} \text{ for some c in } \left(0, \frac{\pi}{7}\right).$$

Since, $\sec^2 x$ is strictly increasing in $\left[0, \frac{\pi}{7}\right]$, therefore we have

$$f'(0) \le f'(c) \le f'\left(\frac{\pi}{7}\right)$$

i.e. $1 \le \frac{f(\pi/7)}{(\pi/7)} \le \sec^2\left(\frac{\pi}{7}\right) \le \sec^2\left(\frac{\pi}{4}\right) = 2$

i.e. $\frac{\pi}{7} \le f\left(\frac{\pi}{7}\right) \le \frac{2\pi}{7}$

Problem 23: Let the function f(x) satisfy the following conditions : (i) f(x) is continuous on [a, b], (ii) f(x) is differentiable in (a, b), (iii) f(x) is not a linear function. Prove that there is a point c ∈ (a,b) such that $|f(b) - f(a)| < |f'(c)| \cdot |b - a|$.

Solution: Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}.(x - a).$$

It is continuous on [a, b] and differentiable in (a, b), with $g(a) = g(b) = 0$. Since f(x) is not a linear function, it follows that $g(x) \not\equiv 0$ in (a, b).

Hence, there exists $c_1, c_2 \in (a, b)$ such that $g'(c_1) > 0$ but $g'(c_2) < 0$ (explain why),

$$\Rightarrow f'(c_1) > \frac{f(b) - f(a)}{b - a}, \text{ but } f'(c_2) < \frac{f(b) - f(a)}{b - a}.$$

This means that at one of the points c_i we have

$$|f'(c_1)| > \left|\frac{f(b) - f(a)}{b - a}\right|,$$

i.e., $|f(b) - f(a)| < |f'(c_1)| \cdot |b - a|$.

Problem 24: Prove that for b > a > 1

$$\frac{1}{b \ln b} < \frac{f(b) - f(a)}{b - a} < \frac{1}{a \ln a}$$

where $f(x) = \ln(\ln x)$, x > 1.

Solution: We have $f(x) = \ln(\ln x)$, x > 1

and $f'(x) = \frac{1}{x \ln x}$, x > 1

Since, f is continuous and differentiable in (1, ∞), therefore applying Lagrange's Theorem on f(x) in (a,b), we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ for some c in (a, b)}$$

Since $\frac{1}{x \ln x}$, is strictly decreasing in (a, b), therefore we have $f'(b) < f'(c) < f'(a)$

i.e. $\frac{1}{b \ln b} < \frac{f(b) - f(a)}{b - a} < \frac{1}{a \ln a}$

which is the desired result.

Problem 25: If f(x) and g(x) are continuous in [a,b] and differentiable in (a,b), show that there exists some c in [a,b] such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a)\begin{vmatrix} f(a) & f'(b) \\ g(a) & g'(b) \end{vmatrix}$$

Solution: Consider the function

$$h(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix}, x \in [a,b]$$

which is continuous in [a, b] and differentiable in (a, b). Thus, applying Lagrange's Theorem on h(x) in the interval [a, b], we have

$$h'(c) = \frac{h(b) - h(a)}{b - a} \text{ for some c in [a, b]} \qquad ...(1)$$

Now, we have $h'(x) = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix}$

Hence, we have from equation (1)

$$h(b) - h(a) = (b - a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

i.e. $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} - \begin{vmatrix} f(a) & f(a) \\ g(a) & g(a) \end{vmatrix}$

$$= (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

i.e. $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

which is the desired result.

Problem 26: Assuming that f''(x) exists for all x in [a, b]. Show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} - \frac{1}{2}(c-a)(c-b)\, f''(\xi) = 0$$

where c and ξ both lie in [a, b].

Solution: Let us first observe that if the result to be proved be multiplied throughout by (b – a) and the terms be rearranged, then it can be shown to be equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2} f''(\xi) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \qquad \text{...(1)}$$

(1) suggests that we must consider the function F defined as

$$F(x) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ f(a) & f(b) & f(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{vmatrix}, \qquad \text{...(2)}$$

where A is a constant to be chosen so that

$F(c) = 0.$ ...(3)

Since F(a) = F(b) = 0 and since F'(x) exists in [a, b], therefore F satisfies the conditions of Rolle's Theorem in each of the intervals [a, c] and [c, b]. Consequently, there exists real number ξ_1 and ξ_2 such that $a < \xi_1 < c < \xi_2 < b$, and $F'(\xi_1) = 0$, $F'(\xi_2) = 0$

$$\text{Now, } F'(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ f(a) & f(b) & f'(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ a^2 & b^2 & 2x \end{vmatrix} \qquad \text{...(4)}$$

The function F' satisfied the conditions of Rolle's Theorem in $[\xi_1, \xi_2]$ and consequently, there exists a real number ξ such that $\xi_1 < \xi < \xi_2$ and $F''(\xi) = 0$.

$$\text{Now, } F''(\xi) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ f(a) & f(b) & f''(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 2 \end{vmatrix}$$

so that $F''(\xi) = 0$ yields $A = \frac{1}{2} f''(\xi)$ ...(5)

By (3), F(c) = 0, therefore,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2} f''(\xi) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \qquad \text{...(6)}$$

where we have substituted the value of A as obtained in (5). We have thus shown that (1) hold and this is equivalent to the relation desired to be proved.

Problem 27: If the function f, g, h are continuous on [a, b] and twice differentiable on (a, b), prove that there exists $\xi, \eta \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{\frac{1}{2}(b-c)}{(c-a)(a-b)} \begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix}$$

where a < c < b.

Solution: Consider the function F(x)

$$= \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} - \frac{(x-a)(x-b)}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

Then F(a) = F(b) = F(c) = 0. Applying Rolle's Theorem on [a,c] and [c,b], we get

$F'(x_1) = F'(x_2) = 0$

where $a < x_1 < c$ and $c < x_2 < b$.

Therefore a second application of Rolle's Theorem on $[x_1, x_2]$ gives $F''(\eta) = 0$ where $x_1 < \eta < x_2$.

$$\text{But } F''(\eta) = 0 \Rightarrow \begin{vmatrix} f(a) & f(b) & f''(\eta) \\ g(a) & g(b) & g''(\eta) \\ h(a) & h(b) & h''(\eta) \end{vmatrix}$$

$$= \frac{2}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} \qquad \text{...(1)}$$

Again consider the function ϕ defined by

$$\phi(x) = \begin{vmatrix} f(a) & f(x) & f''(\eta) \\ g(a) & g(x) & g''(\eta) \\ h(a) & h(x) & h''(\eta) \end{vmatrix}$$

$$- \frac{2(x-a)}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}.$$

Now $\phi(a) = 0$ and using (1), we see that $\phi(b) = 0$.

Hence applying Rolle's Theorem for ϕ on [a, b], we get

$$\begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix}$$

$$= \frac{2}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}.$$

Problem 28: If f'' be continuous on [a, b] and derivable on (a, b), then prove that

$$f(b) - f(a) - \left(\frac{b-a}{2}\right)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d)$$

for some real number d between a and b.

Solution: Let g be the function defined on [a, b] as

$$g(x) = f(x) - f(a) - \frac{1}{2}(x-a)\{f'(a) + f'(x)\} + A(x-a)^3,$$

for all x in [a, b], where A is a constant to be suitably chosen.

Now (i) g is continuous on [a, b],

(ii) g is derivable on (a, b),

(iii) Let A be so chosen that g(a) = g(b).

Since g(b) = 0, it means that A is such that

$$f(b) - f(a) - \frac{1}{2}(b-a)\{f'(a) + f'(b)\} + A(b-a)^3 = 0 \qquad \text{...(1)}$$

The function g now satisfies all the conditions of Rolle's Theorem in [a, b] and therefore, there exists a real number c between a and b, such that

$g'(c) = 0$,

i.e. $$\frac{1}{2}\{f'(c) - f'(a)\} - \frac{1}{2}(c-a) f''(c) + 3A(c-a)^2 = 0 \qquad \text{...(2)}$$

Let h be the function defined on [a, c] by setting

$$h(x) = \frac{1}{2}\{f'(x) - f'(a)\} - \frac{1}{2}(x-a) f''(x) + 3A(x-a)^2$$

for all x in [a, c]. ...(3)

Now

(a) h is continuous on [a, c],

(b) h is derivable on (a, c),

(c) h(c) = 0 by (2), so that h(c) = h(a).

The function h now satisfies all the conditions of Rolle's Theorem in [a, c] and therefore, there exists a real number d (a < d < c < b), such that h'(d) = 0

From (3),

$$h'(d) = \frac{1}{2} f''(d) - \frac{1}{2} f''(d) - \frac{1}{2}(d-a) f'''(d) + 6A(d-a) = 0,$$

i.e. $A = f'''(d)/12$, since $d - a \neq 0$. ...(4)

From (1) and (4), we have

$$f(b) - f(a) - \frac{1}{2}(b-a)\{f'(a) + f'(b)\} = -(b-a)^3 f'''(d)/12.$$

Problem 29: If f(0) = 0 and f''(x) exists on [0, ∞], show that

$$f'(x) - \frac{f(x)}{x} = \frac{1}{2} x f''(\xi),\ 0 < \xi < x,$$

and deduce that if f''(x) is positive for positive values of x, then f(x)/x strictly increases in (0, ∞).

Solution: $$f'(x) - \frac{f(x)}{x} = \frac{1}{2} x f''(\xi),\ 0 < \xi < x \qquad \text{...(1)}$$

The relation (1) can be rearranged in the form

$$f(x) - xf'(x) + \frac{1}{2} x^2 f''(\xi) = 0 \qquad \text{...(2)}$$

We may, therefore, consider the function F, defined as

$$F(x) = f(x) - xf'(x) + \frac{1}{2} Ax^2, \qquad \text{...(3)}$$

where A is a constant to be suitably chosen.

Let c be any positive real number.

Choosing A in (3) so that F(c) = 0, we find that F satisfies the hypothesis of Rolle's Theorem on [0, c]. Therefore, there exists ξ such that 0 < ξ < c and F'(ξ) = 0.

Since $F''(x) = -xf''(x) + Ax$,

therefore F'(ξ) = 0 yields

$$A = f''(\xi) \qquad \text{...(4)}$$

Also, F(c) = 0 yields

$$f(c) - cf'(c) + \frac{1}{2} Ac^2 = 0. \qquad \text{...(5)}$$

From (4) and (5), we have

$$f(c) - cf'(c) + \frac{1}{2} c^2 f''(\xi) = 0$$

or $$f'(c) - \frac{f(c)}{c} = \frac{1}{2} c f''(\xi) \qquad \text{...(6)}$$

Since (6) is true for each c > 0, therefore, (1) is established.

Also, if G be defined by setting

G(x) = f(x)/x, whenever x < 0, then

$$G'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{1}{2} f''(\xi), \text{ by (1).}$$

Assuming that f''(x) > 0 whenever x > 0, it follows that G'(x) > 0, whenever x > 0.

If x_1 and x_2 be any two positive real numbers such that $x_2 < x_1$, then by applying the Mean Value Theorem to G in $[x_1, x_2]$ it follows that

$G(x_2) - G(x_1) = (x_2 - x_1)G'(\gamma)$,

where γ is some real number in (x_1, x_2).

Since G'(γ) > 0, therefore it follows that

$G(x_2) < G(x_1)$.

Hence f(x) / x is strictly increasing in (0, ∞).

Problem 30: If 0 < x < 1, show that

$$2x < \ell n\frac{1+x}{1-x} < 2x\left(1 + \frac{1}{3} \cdot \frac{x^2}{1-x^2}\right).$$

Hence, deduce that

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e.e^{\frac{1}{12n(n+1)}}.$$

Solution: To show that

$$2x < \ell n\frac{1+x}{1-x},$$

consider the function f defined on [0, 1] as

$$f(x) = 2x - \ell n\frac{1+x}{1-x}.$$

If c be any real number in [0, 1], then f is continuous in [0, c] and derivable in (0, c).

Also $f'(x) = -\frac{2x^2}{1-x^2}$,

so that f'(x) < 0 in [0, c). This shows that f is strictly decreasing in [0, c]. In particular, f(c) < f(0).
Since f(0) = 0, this means that

$$2x - \ell n\frac{1+x}{1-x} < 0\text{, when } x = c.$$

Since c is any real number in (0, 1), it follows that

$$2x < \ell n\frac{1+x}{1-x}\text{, whenever } 0 < x < 1. \qquad ...(1)$$

To show that

$$\ell n\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}.\frac{x^2}{1-x^2}\right),$$

consider the function g defined by setting

$$g(x) = \ell n\frac{1+x}{1-x} - 2x\left(1+\frac{1}{3}.\frac{x^2}{1-x^2}\right)$$

Then $g'(x) = -\frac{4}{3}.\frac{x^4}{(1-x^2)^2}$

Since g'(x) < 0 for all x in (0, 1) and since g(0) = 0, therefore, it follows in the same manner as above that g(x) < 0 whenever 0 < x < 1,

that is $\ell n\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}.\frac{x^2}{1-x^2}\right)$. ...(2)

Whenever 0 < x < 1.
From (1) and (2), we have

$$2x < \ell n\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}.\frac{x^2}{1-x^2}\right) \qquad ...(3)$$

whenever 0 < x < 1.

Putting, $x = \frac{1}{2n+1}$ in (3), we have

$$\frac{2}{2n+1} < \ell n\frac{n+1}{n} < \frac{2}{2n+1}\left(1+\frac{1}{12n(n+1)}\right),$$

or $$1 < \left(n+\frac{1}{2}\right)\ell n\left(1+\frac{1}{n}\right) < 1+\frac{1}{12n(n+1)},$$

so that $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$

which is the desired inequality

Problem 31: Prove that if f : R → R is three times differentiable, then there exists a real number

$\zeta \in (-1, 1)$ such that $\frac{f'''(\zeta)}{6} = \frac{f(1)-f(-1)}{2} - f'(0)$.

Solution: Let $g(x) = \frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1)$

$$+\frac{f(1)}{2}x^2(x+1) - f'(0)x(x-1)(x+1)$$

It is easy to check g(±1) = f(±1), g(0) = f(0) and g'(0) = f'(0).
Apply Rolle's Theorem for the function
h(x) = f(x) – g(x) and its derivatives.
Since h(–1) = h(0) = h(1) = 0, there exist η ∈ (–1, 0) and ν ∈ (0, 1) such that h'(η) = h'(ν) = 0. We also have h'(0) = 0, so there exist a ∈ (η, 0) and σ ∈ (0, ν) such that h"(a) = h"(σ) = 0.
Finally, there exists
a ζ ∈ (a, σ) ⊂ (–1, 1) where h'''(ζ) = 0. Then

$$f'''(\zeta) = g'''(\zeta) = -\frac{f(-1)}{2}$$

$$.\,6 - f(0)\,.\,0 + \frac{f(1)}{2}\,.\,6 - f'(0)\,.\,6$$

$$= \frac{f(1)-f(-1)}{2} - f'(0).$$

Problem 32: Let f be a real-values function with n + 1 derivatives at each point of R. Show that for each pair of real numbers a, b, a < b, such that

$$\ln\left(\frac{f(b)+f'(b)+f^{(n)}(b)}{f(a)+f'(a)+....+f^{(n)}(a)}\right) = b - a$$

there is a number c in the open interval (a, b) for which
$f^{(n+1)}(c) = f(c)$.
Note that ln denotes the natural logarithm

Solution: Set $g(x) = (f(x) + f'(x) + ... f^{(n)}(x))e^{-x}$. From the assumption one get g(a) g(b). Then there exists c ∈ (a, b) such that g'(c) = 0. Replacing in the last equality $g'(x) = (f^{(n+1)}(x) - f(x))\,e^{-x}$ we finish the proof.

Problem 33: Let n ≥ 1 be an integer and let f : [0, 1] → R be differentiable and such that f(0) = 0 and f(1) = 1. Prove that there exist distinct points $0 < a_0 < a_2 < ..... < a_{n-1} < 1$ such that

$$\sum_{k=0}^{n-1} f'(a_k) = n.$$

Solution: For 0 ≤ k n – 1, consider the interval $\left[\frac{k}{n};\frac{k+1}{n}\right]$. By the Mean Value Theorem, there are

$a_k \in \left[\frac{k}{n}; \frac{k+1}{n}\right]$ such that

$$f'(a_k) = \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{1}{n}} = n\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right).$$

Summing from k = 0 to k = n – 1 and noting that the right hand side telescopes,

$$\sum_{k=0}^{n-1} f'(a_k) = n\sum_{k=0}^{n-1}\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right)$$

$$= n(f(1) - f(0)) = n.$$

Problem 34: Let $n \ge 1$ be an integer and let $f : [0; 1] \to R$ be differentiable and such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $0 < a_0 < a_2 < .... < a_{n-1} < 1$ such that $\sum_{k=0}^{n-1} \frac{1}{f'(a_k)} = n$.

Solution: Let $k_i \in [0, 1]$ be the smalest number such that $f(k_i) = \frac{i}{n}$, $1 \le i \le n - 1$. Put $k_0 = 0$, $k_n = 1$. The existence of the k_i is guaranteed by the Intermediate Value Theorem we must have

$$0 < k_1 < k_2 < .... < k_{n-1} < 1.$$

Hence, by the Mean Value Theorem, there exists $a_i \in (k_i, k_{i+1})$, $0 \le i \le n - 1$, such that

$$f'(a_i) = \frac{f(k_{i+1}) - f(k_i)}{k_{i+1} - k_i} = \frac{1}{n(k_{i+1} - k_i)}$$

$$\Rightarrow \frac{1}{f'(a_i)} = n(k_{i+1} - k_i).$$

Summing,

$$\sum_{k=0}^{n-1} \frac{1}{f'(a_k)} = n\sum_{k=0}^{n-1}(k_{i+1} - k_i) = n(k_n - k_0) = n.$$

Problem 35: Let f be twice continuously differentiable on $(0, \infty)$ such that $\lim_{x \to 0^+} f'(x) = -\infty$. Show that

$$\lim_{x \to 0^+} \frac{f(x)}{f'(x)} = 0.$$

Solution: Since f' tends to $-\infty$ and f'' tends to $+\infty$ as x tends to 0^+, there exists an interval (0, r) such that $f'(x) < 0$ and $f''(x) > 0$ for all $x \in (0, r)$. Hence f is decreasing and f' is increasing on (0, r). By the Mean Value Theorem for every $0 < x < x_0 < r$ we obtain

$$f(x) - f(x_0) = f'(\zeta)(x - x_0) > 0,$$

for some $\zeta \in (x, x_0)$. Taking into account that f' is increasing $f'(x) < f'(\zeta) < 0$, we get

$$x - x_0 < \frac{f'(\zeta)}{f'(x)}(x - x_0) = \frac{f(x) - f(x_0)}{f'(x)} < 0.$$

Taking limits as x tends to 0^+ we obtain

$$-x_0 \le \liminf_{x \to 0^+} \frac{f(x)}{f'(x)} \le \limsup_{x \to 0^+} \frac{f(x)}{f'(x)} \le 0.$$

Since this happens for all $x_0 \in (0, r)$ we deduce that $\lim_{x \to 0^+} \frac{f(x)}{f'(x)}$ exists and $\lim_{x \to 0^+} \frac{f(x)}{f'(x)} = 0$.

Things to Remember

1. A function f(x) is said to have a local maximum at x = a if $f(a) > f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$, where h is a small positive arbitrary number.
   A function f(x) is said to have a local minimum at x = a if $f(a) < f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$, where h is a small positive arbitrary number.
2. A function is said to have a local maximum at the left end point x = a if f(a) > f(a + h), where h is a small positive arbitrary number. It is said to have a local minimum at the left end point x = a if f(a) < f(a + h).
   The function is said to have a local maximum at the right end point x = b if f(b – h) < f(b). It is said to have a local minimum at the right end point x = b if f(b – h) > f(b).
3. Fermat Theorem : If a belongs to an open interval in the domain of f, if f′(a) exists, and if (a, f(a)) is a point of local extremum (either a maximum or a minimum), then f′(a) = 0.
4. The First Derivative Test : Suppose that x = a is a critical point of a continuous function y = f(x).
   (i) If f′(x) changes from positive to negative at x = a, then f has a local maximum at x = a.
   (ii) If f′(x) changes from negative to positive at x = a, then f has a local minimum at x = a.
   (iii) If f′(x) does not change sign at x = a (that is, if f′(x) is positive on both sides of x = a or negative on both sides), then f has no local maximum or minimum at x = a.
5. Test for Local Maximum/Minimum when f(x) is not differentiable at x = a.
   Case 1 When f(x) is continuous at x = a and $f'(a^-)$ and $f'(a^+)$ exist and are non-zero, then f(x) has a local maximum or minimum at x = a if $f'(a^-)$ and $f'(a^+)$ are of opposite signs.

If $f'(a^-) > 0$ and $f'(a^+) < 0$, then $x = a$ is a point of local maximum . If $f'(a^-) < 0$ and $f'(a^+) > 0$, then $x = a$ is a point of local minimum.

Case 2 When f(x) is continuous and one or both of $f'(a^-)$ and $f'(a^+)$ are either zero or does not exist, then we should consider the signs of $f'(a - h)$ and $f'(a + h)$, where h is a small positive arbitrary number.
If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ is a point of local maximum . If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ is a point of local minimum.

Case 3 If f(x) is discontinuous at $x = a$, then we should find about the existence of local maxima/minima using the basic definition of local maxima/minima i.e. compare the values of f(x) at the neighbouring points of $x = a$.

6. Extremum at Endpoints :
At a left endpoint a :
If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.
At a right endpoint b :
If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.

7. The Second Derivative Test
Let $x = a$ be a stationary point of a function f (i.e. $f'(a) = 0$) in an open interval (α, β), that is, assume $\alpha < a < \beta$. Assume also that the second derivative f'' exists in (α, β). Then we have the following :
(i) If f'' is negative in (α, β), f has a local maximum at a.
(ii) If f'' is positive in (α, β), f has a relative minimum at a.
Special case of the second derivative test :
If f'' is continuous at a, and if $f''(a) \neq 0$, there will be a neighbourhood of a in which f'' has the same sign as $f''(a)$. Therefore, if $f'(a) = 0$, the function f has a local maximum at a if $f''(a)$ is negative, and a relative minimum if $f''(a)$ is positive. This test suffices for many examples that occur in practice.

8. The Higher Order Derivative Test
If at $x = a$ we have $f'(a) = f''(a) = \ldots = f^{(n)}(a) = 0$ and the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of even order, then at the point a
f(x) has a maximum if $f^{(n+1)}(a) < 0$
f(x) has a minimum if $f^{(n+1)}(a) > 0$
But if the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of odd order, then the function has neither maximum nor minimum at the point a. Here,
f(x) increases if $f^{(n+1)}(a) > 0$
f(x) decreases if $f^{(n+1)}(a) < 0$.

9. First Derivative Test for parametric functions
Assume that the function is continuous.
Calculate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and find values of t, where dy/dx is zero or does not exist. Then we find the sign scheme of dy/dx on the number line of t. Now, we convert the sign scheme of dy/dx on the number line of x. If $x = x(t)$ is a strictly increasing function of t, then the sign scheme in x is same as the sign scheme in t. If $x = x(t)$ is a strictly decreasing function of t, then the sign scheme in x is obtained by reversing the number line in t.

10. Second Derivative Test for parametric functions
Assume that the function is differentiable.
Stationary points : we find the values of $t = t_c$
where $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$

If $\frac{dx}{dt} = 0$, then this test is not applicable.

If $\left.\frac{d^2y}{dt^2}\right|_{t=t_c} > 0$, then $x = x(t_c)$ is a point of local minimum.

If $\left.\frac{d^2y}{dt^2}\right|_{t=t_c} < 0$, then $x = x(t_c)$ is a point of local maximum.

11. **(i)** If $y = f(x)$ has a local maximum at $x = a$ then $y = -f(x)$ has a local minimum at $x = a$ and vice-versa.

(ii) If f(x) and g(x) both have a local maximum (minimum) at $x = a$ then $y = f(x) + g(x)$ also has a local maximum (minimum) at $x = a$.
We have a similar result in case of local maximum.

(iii) We have no comments for product and quotient of functions, in general.
Let f and g be twice differentiable functions each having a point of local maximum at $x = a$. If $f(a) > 0$ and $g(a) > 0$, then prove that $y = f(x).\,g(x)$ has a local maximum at $x = a$.

(iv) Let f and g be twice differentiable functions.
(a) If g has a local maximum (minimum) at $x = f(a)$ then gof also has a local maximum (minimum) at $x = a$.
(b) If f has a local maximum (minimum) at $x = a$ and g is increasing at $x = f(a)$, then gof has a local maximum (minimum) at $x = a$.
(c) If f has a local maximum (minimum) at $x = a$ and g is decreasing at $x = f(a)$ then gof has a local minimum (maximum) at $x = a$.

(v) (a) An even function has an extremum at $x = 0$.
Note that odd function cannot have an extremum at $x = 0$.
(b) If an even function f has a local maximum (minimum) at $x = a$ then it also has a local maximum (minimum) at $x = -a$.

12. Extreme Value Theorem : If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value of f(d) at some numbers c and d in [a, b].

13. In order to find the global maximum and minimum of a continuous function f(x) in [a, b] : Find out all the critical points of f(x) in (a, b).

Let $c_1, c_2, \ldots, c_n$ be the different critical points.

Find the value of the function at these critical points. Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at critical points.

Let $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$

Then M_1 is the global maximum (greatest value) of f(x) in [a, b] and M_2 is the global minimum (least value) of f(x) in [a, b].

14. The first derivative test for global extrema

Let $f : [a, b] \to R$ be continuous on [a, b] and derivable on $(a, c) \cup (c, b)$, then

(i) if $f' > 0$ on (a, c) and $f' < 0$ on (c, b)

$\Rightarrow f(c) = \max f$ on [a, b]; and

(ii) if $f' < 0$ on (a, c) and $f' > 0$ on (c, b)

$\Rightarrow f(c) = \min f$ on [a, b]

15. Asymptotes :

(i) If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line y = L is a horizontal asymptote of the curve y = f(x).

(ii) The line x = a is a vertical asymptote if at least one of the following statements is true :

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(iii) If there are limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = m_1 \text{ and } \lim_{x\to\infty} [f(x) - m_1 x] = c_1,$$

then the straight line $y = m_1 x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m_2 \text{ and } \lim_{x\to-\infty} [f(x) - m_2 x] = c_2,$$

then the straight line $y = m_2 x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

16. Rolle's Theorem: Let f be a function that satisfies the following three hypotheses :

(i) f is continuous on the closed interval [a, b].

(ii) f is differentiable on the open interval (a, b).

(iii) f(a) = f(b)

Then there is a number c in (a, b) such that $f'(c) = 0$.

17. **(a)** If all the roots of f(x) = 0 are real, then all the roots of $f'(x) = 0$ are also real, and the roots of the latter equation separate those of the former.

For if f(x) is of degree n, $f'(x)$ is of degree n – 1, and a root of $f'(x) = 0$ exists in each of the n – 1 intervals between the n roots of f(x) = 0.

(b) If all the roots of f(x) = 0 are real, so also are those of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$, ..., and the roots of any one of these equations separate those of the preceding equation. This follows from (a).

(c) Not more than one root of f(x) = 0 can

(i) lie between two consecutive roots of $f'(x) = 0$, or

(ii) be less than the least of these, or

(iii) be greater than the greatest of these.

(d) If $f'(x) = 0$ has r real roots, then f(x) = 0 cannot have more than (r + 1) real roots.

(e) If $f^{(r)}(x)$ is the r^{th} derivative of f(x) and the equation $f^{(r)}(x) = 0$ has some imaginary roots, then f(x) = 0 has atleast as many imaginary roots.

(f) If all the real roots $\beta_1, \beta_2, \ldots$ of $f'(x) = 0$ are known, we can find the number of real roots of f(x) = 0 by considering the signs of $f(\beta_1), f(\beta_2), \ldots$. A single root of f(x) = 0, or no root, lies between β_1 and β_2, according as $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, or the same sign.

18. Lagrange's Mean Value Theorem :

Let f be a function that satisfies the following hypotheses :

(i) f is continuous on the closed interval [a, b].

(ii) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

19. Constant Function Theorem : Suppose f is a continuous function on the closed interval [a, b] and is differentiable on the open interval (a, b), with $f'(x) = 0$ for all x on (a, b). Then the function f is constant on [a, b]

20. Constant Difference Theorem : Suppose the functions f and g are continuous on the closed interval [a, b] and differentiable on the open interval (a, b). Then if $f'(x) = g'(x)$ for all x in (a, b), there exists a constant C such that f(x) = g(x) + C for all x on [a, b].

21. **(i)** If f is continuous on [a, b] and $m \le f'(x) \le M$ for all $x \in (a, b)$, then $m(b - a) \le f(b) - f(a) \le M(b - a)$.

(ii) If f is continuous on [a, b] and $|f'(x)| \le M$ for all $x \in (a, b)$, then $|f(b) - f(a)| \le M|b - a|$.

(iii) If f is continuous on [a, b] and $|f'(x)| \ge m$ for all $x \in (a, b)$, then $|f(b) - f(a)| \ge m|b - a|$.

22. Cauchy's Mean Value Theorem :

Let f and g be functions defined on [a, b] such that

(i) f and g are continuous on [a, b],

(ii) f and g are derivable on (a, b), and

(iii) g'(x) does not vanish at any point of (a, b).

then there exists a real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

23. Generalized Mean Value Theorem : If f, g and h are continuous on [a, b] and derivable on (a, b), then there exists a number c in (a, b) such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. $f(x) = \begin{cases} 2-|x^2+5x+6| & , \ x \neq -2 \\ a^2+1 & , \ x = -2 \end{cases}$, then the range of a so that f(x) has maxima at x = – 2 is

(A) $|a| \geq 1$ (B) $|a| < 1$

(C) $a > 1$ (D) $a < 1$

2. A triangle has one vertex at (0, 0) and the other two on the graph of $y = -2x^2 + 54$ at (x, y) and (–x, y) where $0 < x < \sqrt{27}$. The value of x so that the corresponding triangle has maximum area is

(A) $\dfrac{\sqrt{27}}{2}$ (B) 3

(C) $2\sqrt{3}$ (D) None

3. The set of all values of the parameters a for which the points of minimum of the function $y = 1 + a^2 x - x^3$ satisfy the inequality $\dfrac{x^2+x+2}{x^2+5x+6} \leq 0$ is

(A) an empty set

(B) $(-3\sqrt{3}, -2\sqrt{3})$

(C) $(2\sqrt{3}, 3\sqrt{3})$

(D) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

4. If $f(x) = \dfrac{1+x}{[x]}$ $x \in [1, 3]$ then which of following statements about f(x) is correct

(A) f(x) has local maxima at x = 2

(B) f(x) has no point of local maxima

(C) f(x) hasneither maxima nor minima at x = 2

(D) f(x) has 1 point of local minimum and 2 points of local maxima

5. Which of the following statement is true for the function

function $f(x) = \begin{cases} \sqrt{x} & x \geq 1 \\ x^3 & 0 \leq x \leq 1 \\ \dfrac{x^3}{3} - 4x & x < 0 \end{cases}$

(A) It is monotonic increasing $\forall \ x \in R$

(B) $f'(x)$ fails to exist for 3 distinct real values of x

(C) $f'(x)$ changes its sign twice as x varies from $(-\infty, \infty)$

(D) function attains its extreme values at x_1 & x_2, such that $x_1 x_2 > 0$

6. Consider the function for $x \in [-2, 3]$,

$f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x-1} & \text{if } \ x \neq 1 \\ -6 & \text{if } \ x = 1 \end{cases}$ then

(A) f is discontinuous at x = 1 ⇒ Rolle's theorem is not applicable in [–2, 3]

(B) f(–2) ≠ f(3) ⇒ Rolle's theorem is not applicable in [–2, 3]

(C) f is not derivable in (–2, 3) ⇒ Rolle's theorem is not applicable

(D) Rolle's theorem is applicable as f satisfies all the conditions and c of Rolle's theorem is 1/2

7. The total number of values of x, where $f(x) = 2^{-|x|}(\cos x + \cos\sqrt{3}\, x)$ attains its maximum value is

(A) 1 (B) 2

(C) 4 (D) None

8. Let $f(x) = \begin{cases} |x^3+x^2+3x+\sin x|\,(3+\sin 1/x), & x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

then number of points (where f(x) attains its minimum value) is

(A) 1 (B) 2

(C) 3 (D) infinite many

9. In which of the following functions Rolle's theorem is applicable -

(A) $f(x) = \begin{cases} x & , 0 \leq x < 1 \\ 0 & , x = 1 \end{cases}$

(B) $f(x) = \begin{cases} \dfrac{\sin x}{x} & , -\pi \leq x < 0 \\ 0 & , x = 0 \end{cases}$

(C) $f(x) = \dfrac{x^2 - x - 6}{x - 1}$ on [–2, 3]

(D) $f(x) = \begin{cases} \dfrac{x^3-2x^2-5x+6}{x-1} & \text{if } \ x \neq 1, \ x \in [-2, 3] \\ -6 & \text{if } \ x = 1 \end{cases}$

10. The function in which the conditions of Rolle's theorem are satisfied, is

(A) $f(x) = 2x^3 + x^2 - 4x - 2$, [–root 2, root 2]

(B) tanx in $[0, \pi]$

(C) $f(x) = \begin{cases} x^2 + 1 \; ; \; 0 \le x \le 1 \\ 3 - x \; ; 1 \le x \le 2 \end{cases}$

(D) None of these

11. If $a > b > 0$ and $f(\theta) = \dfrac{(a^2 - b^2)\cos\theta}{a - b\sin\theta}$, then the maximum value of $f(\theta)$ is

(A) $\sqrt{a^2 + b^2}$ (B) $\sqrt{a^2 - b^2}$

(C) $a^2 - b^2$ (D) $\dfrac{a-b}{a+b}$

12. Let $f(x) = \sin(x^2 - 3x)$, if $x \le 0$; and $6x + 5x^2$, if $x > 0$, then at $x = 0$, f(x)

(A) has a local maximum

(B) has a local minimum

(C) is discontinuous

(D) None of these

13. The greatest value of $f(x) = \cos(xe^{[x]} + 7x^2 - 3x)$, $x \in [-1, \infty)$ is

(A) –1 (B) 1

(C) 0 (D) None of these

14. The function $f(x) = (4\sin^2x - 1)^n(e^x - x + 1)$, $n \in N$, has a local minimum at $x = \dfrac{\pi}{6}$, then

(A) n can be any even natural number

(B) n can be an odd natural number

(C) n can be odd prime number

(D) n can be any natural number

15. Let $f(x) = \cos\pi x + 10x + 3x^2 + x^3, -2 \le x \le 3$. The absolute minimum value of f(x) is

(A) 0 (B) –15

(C) $3 - 2\pi$ (D) None of these

16. The global minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is

(A) e (B) 1/3

(C) 1 (D) 0

17. The global maximum value of $f(x) = \log_{10}(4x^3 - 12x^2 + 11x - 3)$, $x \in [2, 3]$ is

(A) $-\dfrac{3}{2}\log_{10} 3$ (B) $1 + \log_{10}3$

(C) $\log_{10}3$ (D) $\dfrac{3}{2}\log_{10} 3$

18. The least natural number a for which $x + ax^{-2} > 2, \forall x \in (0, \infty)$ is

(A) 1 (B) 2

(C) 5 (D) None of these

19. Let $f(x) = \begin{cases} x^3 + x^2 - 10x, & -1 \le x < 0 \\ \cos x, & 0 \le x < \pi/2 \\ 1 + \sin x, & \pi/2 \le x \le \pi \end{cases}$

Then f(x) has

(A) a local minimum at $x = \pi/2$

(B) a global maximum at $x = -1$

(C) an absolute minimum at $x = -1$

(D) an absolute maximum at $x = \pi$

20. If $a > b > 0$ then the maximum value of $\dfrac{ab(a^2 - b^2)\sin x\cos x}{a^2\sin^2 x + b^2\cos^2 x}$ in $\left(0, \dfrac{\pi}{2}\right)$ is

(A) $a^2 - b^2$ (B) $\dfrac{a^2 - b^2}{2}$

(C) $\dfrac{a^2 + b^2}{2}$ (D) none of these

21. Let $f : [2, 7] \to [0, \infty)$ be a continuous and differentiable function. Then the value of

$$(f(7) - f(2))\,\frac{(f(7))^2 + (f(2))^2 + f(2)\cdot(7)}{3}, \text{ is}$$

(where $c \in (2,7)$).

(A) $3f^2(c) \cdot f'(c)$ (B) $4f^2(c) \cdot f'(c)$

(C) $5f^2(c) \cdot f'(c)$ (D) None of these

22. Let $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex$, where $a, b, c, d, e \in R$ and $f(x) = 0$ has a positive root α, then

(A) $f'(x) = 0$ has root a1 such that $0 < \alpha_1 < \alpha$

(B) $f''(x) = 0$ has at least one real root

(C) $f'(x) = 0$ has at least two real roots

(D) All of the above

23. Between any two real roots of the equation $e^x \sin x - 1 = 0$, the equation $e^x \cos x + 1 = 0$ has

(A) Atleast one root

(B) Atmost one root

(C) Exactly one root

(D) No root

24. A differentiable function f(x) has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(A) all a and all b (B) all b if a = 0

(C) all b > 0 (D) all a > 0

25. Suppose that f is a polynomial of degree 4 and that $f''(x) \ne 0$, then

(A) f has exactly one stationary point

(B) f must have no stationary point

(C) f must have exactly 3 stationary points

(D) f has either 1 or 3 stationary points

26. If $f : R \to R$ and $g : R \to R$ are two functions such that $f(x) + f''(x) = -x\, g(x)\, f'(x)$ and $g(x) > 0\ \forall\ x \in R$, then the function
$f^2(x) + (f'(x))^2$ has
(A) a maxima at $x = 0$
(B) a minima at $x = 0$
(C) a point of inflection at $x = 0$
(D) none of these

27. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is
(A) 1
(B) $1 + \sin^2 1$
(C) $1 + \cos^2 1$
(D) Does not exist

28. If the function $f(x) = \dfrac{t+3x-x^2}{x-4}$, where t is a parameter, has a minimum and maximum, then the range of values of t is
(A) $(0, 4)$
(B) $(0, \infty)$
(C) $(-\infty, 4)$
(D) $(4, \infty)$

29. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha)f(\beta) < 0$. Then the equation $f(x) = 0$ has
(A) three equal real roots
(B) one negative root if $f(\alpha) < 0$ and $f(\beta) > 0$
(C) one positive root if $f(\alpha) > 0$ and $f(\beta) < 0$
(D) none of these

30. A bell tent consists of a conical portion above a cylindrical portion placed on the ground. For a given volume and a circular base of a given radius, the amount of the canvas used is a minimum when the semi-vertical angle of the cone is
(A) $\cos^{-1}2/3$
(B) $\sin^{-1}2/3$
(C) $\cos^{-1}1/3$
(D) None of these

31. A cylindrical gas container is closed at the top and open at the botton. If the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides, the ratio of the radius to the height of the cylinder using minimum material for the same capacity is
(A) 3 : 4
(B) 5 : 6
(C) 4 : 5
(D) None of these

32. A given right circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then p : q is
(A) 9 : 4
(B) 8 : 3
(C) 7 : 2
(D) None of these

33. A wire of length a is cut into two parts which are bent, respectively, in the form of a square and a circle. The least value of the sum of the areas so formed is
(A) $\dfrac{a^2}{\pi+4}$
(B) $\dfrac{a}{\pi+4}$
(C) $\dfrac{a}{4(\pi+4)}$
(D) $\dfrac{a^2}{4(\pi+4)}$

34. The largest term of the sequance $< t_n >$ where $t_n = \dfrac{n^2}{n^4+300}$ is
(A) t_3
(B) t_4
(C) t_5
(D) t_6

35. The set of value of c for which $\sin\{\ln(\cos x + c)\} = 1$ has at most one solution in $[0, \pi]$ is
(A) $(2\pi, \infty)$
(B) $(e^{2\pi}, \infty)$
(C) $\left(\dfrac{e^{2\pi}+1}{e^{2\pi}-1}, \infty\right)$
(D) null set

36. For $\alpha \neq \beta$, $n \in N$ which of the following is not always true
(A) $|\sin n\alpha - \sin n\beta| < n|\alpha - \beta|$
(B) $|\cos n\alpha - \cos n\beta| > n|\alpha - \beta|$
(C) $|\sin n\alpha + \sin n\beta| < n|\alpha + \beta|$
(D) $|\cos n\alpha - \cos n\beta| < n|\alpha - \beta|$

37. The maximum value of the expression $\left|\sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 1 - \cos^2 x}\right|$, where a and x are real numbers, is
(A) $\sqrt{2}$
(B) 1]
(C) $\sqrt{3}$
(D) None of these

38. Which of the following statements are true ?

(i) If $p(x) = Ax^2 + Bx + C$, $x \in [a, b]$, then a value of 'c' given by Lagrange's Mean Value Theorem is the mid point of the interval.

(ii) A differentiable function f has one critical point at $x = 5$. If $f'(4) = -2$ and $f'(6) = 3$, then $x = 5$ is a point of local minimum.

(iii) If f(x) is differentiable on R snd $f'(x) < 1\ \forall\ x \in R$, then atmost one value of x (say $x = c$) exists such that $f(c) = c$.

(A) TFT
(B) FTF
(C) TTT
(D) TTF

39. If $a, b, c\ d \in R$ such that $\dfrac{a+2c}{b+3d} + \dfrac{4}{3} = 0$, then the equation $ax^2 + cx + d = 0$ has
(A) atleast one root in $(-1, 0)$
(B) atleast one root in $(0, 1)$
(C) no root in$(-1, 1)$
(D) no root in $(0, 2)$

40. A box, constructed from a rectangular metal sheet, is 21 cm by 16 cm by cutting equal squares of sides x from the corners of the sheet and then turning up the projected portions. The value of x so that volume of the box is maximum is
(A) 1
(B) 2
(C) 3
(D) 4

41. If $A\left(\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$, $B\left(-\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$, $C\left(-\dfrac{3}{\sqrt{2}}, -\sqrt{2}\right)$ and $D(3\cos\theta, 2\sin\theta)$ are four points, then the value of θ for which the area of quadrilateral ABCD is maximum, $\left(\frac{3\pi}{2} \le \theta \le 2\pi\right)$ is

(A) $2\pi - \sin^{-1}\frac{1}{3}$ (B) $\frac{7\pi}{4}$

(C) $2\pi - \cos^{-1}\frac{3}{\sqrt{85}}$ (D) None of these

42. A rod of fixed length k slides along the coordinate axes. If it meets the axes at A(a,0) and B(0,b), then the minimum value of $\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2$ is

(A) 0 (B) 8

(C) $k^2 - 4 + \frac{4}{k^2}$ (D) $k^2 + 4 + \frac{4}{k^2}$

43. Let $f(x) = \begin{cases} |x^3+x^2+3x+\sin x|(3+\sin 1/x) & , x \ne 0 \\ 0 & , x = 0 \end{cases}$

then number of points (where f(x) attains its minimum value) is

(A) 1 (B) 2

(C) 3 (D) infinite many

44. The function $f(x) = \left|\frac{x^2-2}{x^2-4}\right|$ has

(A) no point of local minima

(B) no point of local maxima

(C) exactly one point of local minima

(D) exactly one point of local maxima

45. Which of the follwoing is always correct ?

(A) If f’(x) > 0 ∀ x wherever f’(x) exists, then f(x) must be one-one

(B) If f’(x) < 0 ∀ x wherever f’(x) exists, then f(x) must be one-one

(C) If | f(x)| be continuous at x = a, then f(x) is also continuous at x = a

(D) If f(x) is continuous at x = a, f(A) = 2 and x = a is the point of local minimum of f(x), then [f(x)], where [.] denotes gresatest integer function, is also continuous at x = a,

46. f(x) is a continuous function having 4 critical points a, b, c, d where a < b < c < d. If a and d are points of maxima, then f(x)

(A) two points of minima

(B) has two points of inflexion

(C) can have no point of inflection

(D) has exactly one point of inflection

47. Given $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$; g(x) = {x}

$h(x) = \begin{cases} \frac{\tan[x]}{x}, & x \ne 0 \\ 1 \quad , & x = 0 \end{cases}$ and $p(x) = 5^{\ln(x+3)}$ defined

in [0, 1], the functions on which LMVT is applicable is

(A) f (B) g

(C) p (D) h

48. The set of critical points of the function

$f(x) = x - \log x + \int_2^x (1/z - 2 - 2\cos 4z)\, dz$ is

(A) $\left\{\frac{\pi}{6}, \frac{n\pi}{2} + \frac{\pi}{6}\right\}$, n ∈ N

(B) {nπ} , n ∈ N

(C) $\left\{\frac{\pi}{2}, n\pi + \frac{\pi}{6}\right\}$, n ∈ N

(D) None of these

49. If $f(\theta) = \dfrac{\sin^4\frac{\theta}{n} - \sin^2\frac{\theta}{n} + 1}{\sin^4\frac{\theta}{n} + \sin^2\frac{\theta}{n} + 1}$, where θ ≠ k n π ,

k ∈ I, then range of f(θ) is

(A) (1/3, 1] (B) [1/3, 1]

(C) [1/3, 1) (D) (1/3, 1)

50. Consider f(x) = |1 – x| 1 < x < 2   1 ≤ x ≤ 2

and g(x) = f(x) + b sin π/2 x, 1 ≤ x ≤ 2

then which of the following is correct ?

(A) Rolles Theorem is applicable to both f, g and b = 3/2

(B) LMVT is not applicable to f and Rolle’s Theorem if applicable to g with b = 1/2

(C) LMVT is applicable to f and Rolle’s Theorem is applicable to g with b = 1

(D) Rolle’s Theorem is not applicable to both f, g for any real b

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. If $f(x) = \begin{cases} 3x^2 + 12x - 1 ; & -1 \le x \le 2 \\ 37 - x & ; 2 < x \le 3 \end{cases}$ then

(A) f(x) is increasing in [–1, 2]

(B) f(x) is continuous in [–1,3]

(C) f '(2) does not exist

(D) f(x) has the maximum value at x = 2

52. Which of the statements are necessarily true ?

(A) If f is differentiable and f(–1) = f(1), then there is a number c such that |c| < 1 and f'(C) = 0.

(B) If f"(2) = 0, then (2, f(2)) is an inflection point of the curve y = f(x).

(C) There exists a function f such that f(x) > 0, f'(x) < 0, and f"(x) > 0 for all x.

(D) If f'(x) exists and is nonzero for all x, then f(1) ≠ f(0).

53. Let $f(x) = (x-1)^4(x-2)^n$, $n \in N$. Then f(x) has
(A) a maximum at $x = 1$ if n is odd
(B) a maximum at $x = 1$ if n is even
(C) a minimum at $x = 1$ if n is even
(D) a minima at $x = 2$ if n is even

54. Identify the correct statements :
(A) If $f(x) = ax^3 + b$ and f is strictly increasing on $(-1, 1)$, then $a > 0$.
(B) An *n*th-degree polynomial has atmost $(n-1)$ critical points.
(C) If $f'(x) > 0$ for all real numbers x, then f increases without bound.
(D) The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

55. Which of the given equation has exactly one solution in the indicated interval ?
(A) $x^5 + 2x - 3 = 0$; $[0, 1]$
(B) $e^{-x} = x - 1$; $[1, 2]$
(C) $x \ln x = 3$; $[2, 4]$
(D) $\sin x = 3x - 1$; $[-1, 1]$

56. If f(x) is a differentiable function and $\phi(x)$ is twice differentiable function and α and β are roots of the equation $f(x) = 0$ and $\phi'(x) = 0$ respectively, then which of the following statement is true ? $(\alpha < \beta)$
(A) there exists exactly one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ on (α, β)
(B) there exists atleast one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ on (α, β)
(C) there exists odd number of roots of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ on (α, β)
(D) None of these

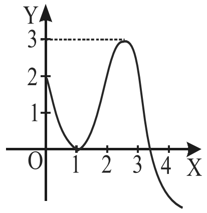
57. An even polynomial function f(x) satisfies a relation

$$f(2x)\left(1-f\left(\frac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy) \ \forall\ x, y \in$$

$R - \{0\}$ and $f(4) = -255$, $f(0) = 1$.
Which of the following holds good?
(A) f(x) has local maximum at $x = 1$.
(B) $f(x)f\left(\frac{1}{x}\right) \le 0$
(C) Range of values of k for which $|f(x)| = k - 2$ has exactly four distinct solutions is (2, 3).
(D) $\int_0^1 f(x)\,dx = \frac{3}{4}$.

58. Let $f(x) = ax^3 + bx^2 + cx + d$ $(a, b, c, d, \alpha, \beta \in R)$ and $f'(\alpha) = f'(\beta) = 0$, $(\alpha \ne \beta)$, then
(A) if $f(\alpha) . f(\beta) < 0$ then all the roots of $f(x) = 0$ are real
(B) if $f(\alpha) . f(\beta) > 0$ then all the roots of $f(x) = 0$ is real
(C) if α is a point of local maxima and β is point of local minima for the function f(x) then $\alpha < \beta$
(D) If $a > 0$ then f(x) is decreasing on (α, β), where $\beta > \alpha$

59. The diagram shows the graph of the derivative of a function f(x) for $0 \le x \le 4$ with $f(0) = 0$. Which of the following could be correct statements for $y = f(x)$?



(A) Tangent line to $y = f(x)$ at $x = 0$ makes an angle of $\sec^{-1}\sqrt{5}$ with the x-axis.
(B) f is strictly increasing in (0, 3).
(C) $x = 1$ is both an inflection point as well as point of local extremum.
(D) Number of critical point on $y = f(x)$ is two.

60. Which of the functions f satisfies neither the hypotheses nor the conclusions of LMVT
(A) $f(x) = |x-2|$; $[1, 4]$
(B) $f(x) = 1 + |x-1|$; $[0, 3]$
(C) $f(x) = [x]$ (the greatest integer function); $[-1, 1]$
(D) $f(x) = 3x^{2/3}$; $[-1, 1]$

61. If $x + y = 60$, $x > 0$, $y > 0$, then the expression $x^2(30-y)^2$ has
(A) least value = 0
(B) greatest value = 15^4
(C) two extrema
(D) no greatest value

62. Let $f(x) = \begin{cases} x^{\alpha}\sin^2\dfrac{1}{nx} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$ where $n \in I$, $n \ne 0$.

If Rolle's Theorem is applicable to f(x) in the interval [0, 1], then
(A) $\alpha > 0$, greatest value of n is $\frac{1}{\pi}$
(B) $\alpha > 2$, greatest value of n is $\frac{1}{\pi}$
(C) $\alpha > 0$, least value of n is $-\frac{1}{\pi}$
(D) α can not be < 1

63. The function $f(x) = 3 + 2(a+1)x + (a^2+1)x^2 - x^3$ has a local minimum at $x = x_1$ and local maximum at $x = x_2$ such that $x_1 < 2 < x_2$ then a belongs to the interval(s)
(A) $\left(-\infty, -\frac{3}{2}\right)$
(B) $\left(-\frac{3}{2}, 1\right)$
(C) $(0, \infty)$
(D) $(1, \infty)$

64. If $y = g(x)$ is a curve which is obtained by the reflection of $y = f(x) = \frac{e^x - e^{-x}}{2}$ by the line $y = x$ then
(A) $y = g(x)$ has exactly one tangent parallel to x-axis
(B) $y = g(x)$ has no tangent parallel to y-axis
(C) The tangent to $y = g(x)$ at (0, 0) is $y = x$
(D) g(x) has no extremum

65. An extreme value of $4\sin^2 x + 3\cos^2 x - 24\sin\frac{x}{2} - 24\cos\frac{x}{2}$, where $0 \le x \le \frac{\pi}{2}$, is

(A) $4 + \sqrt{2}$ (B) $4(1 - 6\sqrt{2})$
(C) –21 (D) 4

66. If $f(x) = (x-3)^9 + (x-3^2)^9 + \ldots\ldots + (x-3^9)^9$, then

(A) f(x) is always increasing
(B) f(x) = 0 has one real & eight imaginary roots
(C) $x = 3, 3^2, \ldots\ldots\ldots\ldots 3^9$ are the roots of f(x) = 0
(D) f(x) = 0 has a negative real root

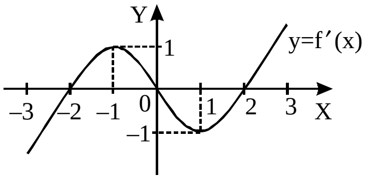
67. If $f(x) = -\frac{1}{3}x^3 + t^2 x$, where t is a real parameter. Let m(t) denote the minimum of f(x) over [0, 1] then

(A) $m(t) = 0$ if $t^2 \ge \frac{1}{3}$
(B) $m(t) = 0$ for all t
(C) $m(t) = t^2 - \frac{1}{3}$ if $t^2 < \frac{1}{3}$
(D) $m(t) = \frac{1}{3} - t^2$ for all t.

68. If f : [–1, 1] → R is a continuously differetiable function such that $f(1) > f(-1)$ and $|f'(y)| \le 1$ for all $y \in [-1, 1]$ then

(A) there exists an $x \in [-1, 1]$ such that $f'(x) > 0$
(B) there exists an $x \in [-1, 1]$ such that $f'(x) < 0$
(C) $f(1) \le f(-1) + 2$
(D) $f(-1) \cdot f(1) < 0$

69. The figure shows the graph of f′(x). Then the correct statement(s) is/are



(A) f is decreasing in (–1, 1)
(B) f has a local minimum at x = 2
(C) f is an odd function
(D) f has atmost 4 zeros.

70. Let $g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)\,x(x-1)(x+1)$ where f is a thrice differentiable function. Then the correct statements are

(A) there exists $x \in (-1, 0)$ such that $f'(x) = g'(x)$
(B) there exists $x \in (0, 1)$ such that $f''(x) = g''(x)$
(C) there exists $x \in (-1, 1)$ such that $f'''(x) = g'''(x)$
(D) there exists $x \in (-1, 1)$ such that $f'''(x) = 3f(1) - 3f(-1) - 6f'(0)$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

71. **Assertion (A) :** A tangent parallel to x-axis can be drawn for f(x) = (x – 1) (x – 2) (x – 3) in the interval [1,3]
Reason (R) : A horizontal tangent can be drawn to any cubic function.

72. **Assertion (A) :** The least value of the function $f(x) = -x^2 + 4x + 1 + \sin^{-1}\left(\frac{x}{2}\right)$ on the interval [–1, 1] is $-4 - \frac{\pi}{6}$
Reason (R) : The least value of f(x) in $[-1, 1] = \min\{f(-1), f(1)\} = \min\left\{-4 - 4\frac{\pi}{6}, 4 + \frac{\pi}{6}\right\} = -4 - \frac{\pi}{6}$

73. Let f (x) be a twice differentiable function.
Assertion (A) : If a < b < c < d and f (a) = 0, f (b) = 1, f (c) = – 1, f (d) = 0, then the minimum number of zeroes $g(x) = (f'(x))^2 + f(x)\,f''(x)$ in [a, d] is 4.
Reason (R) : If $f(\alpha)\,f(\beta) < 0$ then $f(\gamma) = 0$ for some $\alpha < \gamma < \beta$ and if $f(\alpha) = f(\beta) = 0$ then $f'(\gamma) = 0$ for some $\alpha < \gamma < \beta$.

74. **Assertion (A) :** Let $f(x) = 5 - 4(x-2)^{2/3}$, then at x = 2 the function f(x) attains neither the least value nor the greatest value.
Reason (R) : At x = 2, the first derivative does not exist.

75. **Assertion (A) :** $f(x) = \frac{x^3}{3} + \frac{ax^2}{2} + x + 5$ has positive point of maxima for a < –2.
Reason (R) : $x^2 + ax + 1 = 0$ has both roots positive for a < –2.

76. **Assertion (A) :** For all a, b ∈ R, the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has exactly one extremum.
Reason (R) : If a cubic function is monotonic, then its graph cuts x -axis only once.

77. **Assertion (A) :** Conditions of Lagrange's Mean Value Theorem are not satisfied when $f(x) = |x-1|(x-1)$ in [0, 2].
Reason (R) : |x – 1| is not differentiable at x = 1.

78. **Assertion (A) :** If 27a + 9b + 3c + d = 0, then the equation $f(x) = 4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one real root lying between (0, 3).
Reason (R) : If f(x) is continuous in [a, b], derivable in (a, b) such that f(a) = f(b), then ther exists atleast one point c ∈ (a, b) such that f'(c) = 0.

79. **Assertion (A) :** The maximum value of $(\sqrt{-3+4x-x^2}+4)^2 + (x-5)^2$ (where $1 \le x \le 3$) is 36.
Reason (R) : The maximum distance between the point (5, –4) and the point on the circle $(x-2)^2 + y^2 = 1$ is 6.

80. **Assertion (A) :** If g(x) is a differentiable function g(2) ≠ 0, g(–2) ≠ 0 and Rolle's Theorem is not applicable to $f(x) = \frac{x^2-4}{g(x)}$ in [–2, 2], then g(x) has atleast one root in (–2, 2)

Reason (R) : If a function f is differentiable in (a, b) and f(a) = f(b), then Rolle's Theorem is applicable to f(x) for $x \in (a, b)$.

Comprehension – 1

Consider f, g and h be three real valued differentiable functions defined on R.

Let $g(x) = x^3 + g''(1)\,x^2 + \left(3g'(1) - g''(1) - 1\right)x + 3g'(1)$,
$f(x) = x\,g(x) - 12x + 1$
and $f(x) = (h(x))^2$ where $h(0) = 1$.

81. The function y = f(x) has
(A) Exactly one local minima and no local maxima
(B) Exactly one local maxima and no local minima
(C) Exactly one local maxima and two local minima
(D) Exactly two local maxima and one local minima

82. Which of the following is/are true for the function y = g(x)?
(A) g(x) monotonically decreases in $\left(-\infty, 2-\frac{1}{\sqrt{3}}\right)$ and $\left(2+\frac{1}{\sqrt{3}}, \infty\right)$
(B) g(x) monotonically increases in $\left(2-\frac{1}{\sqrt{3}}, 2+\frac{1}{\sqrt{3}}\right)$
(C) There exists exactly one tangent to y = g(x) which is parallel to the chord joining the points (1, g(1)) and (3, g(3))
(D) There exists exactly two distinct Lagrange's Mean Value in (0, 4) for the function y = g(x).

83. Which one of the following does not hold good for y = h(x)?
(A) Exactly one critical point
(B) No point of inflection
(C) Exactly one real zero in (0, 3)
(D) Exactly one tangent parallel to x-axis

Comprehension – 2

Let $f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ and the global minimum value of f(x) for $x \in [0, 2]$ is equal to 3.

84. The number of values of a for which the global minimum for $x \in [0, 2]$ occurs at the end point of interval [0, 2] is
(A) 1 (B) 2
(C) 3 (D) 0

85. The number of values of a for which the global minimum for $x \in [0, 2]$ occurs at a value of x lying in (0, 2) is
(A) 1 (B) 2
(C) 3 (D) 0

86. The values of a for which f(x) is monotonic for $x \in [0, 2]$ are
(A) $a \le 0$ or $a \ge 4$ (B) $0 \le a \le 4$
(C) $a > 0$ (D) None of these

Comprehension – 3

Consider a polynomial y = P(x) of the least degree passing through A(– 1, 1) and whose graph has two points of inflexion B(1, 2) and C with abscissa 0 at which the curve is inclined to the positive axis of abscissas at an angle of $\sec^{-1}\sqrt{2}$.

87. If $\int_{-2}^{2} P(x)dx = \frac{k}{5}$, $k \in N$ then k equals
(A) 17 (B) 24
(C) 32 (D) 42

88. The value of P(–1) equals
(A) – 1 (B) 0
(C) 1 (D) 2

89. The area of ΔABC equals
(A) $\frac{1}{2}$ (B) 1
(C) $\frac{1}{8}$ (D) $\frac{1}{12}$

Comprehension – 4

Let $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

90. The values of a, if f(x) has a negative point of local minimum, are
(A) ϕ (B) (–3, 3)
(C) $\left(-\infty, \frac{58}{14}\right)$ (D) None of these

91. The values of a, if f(x) has a positive point of local maxima, are
(A) ϕ (B) $(-\infty, -3) \cup (3, \infty)$
(C) $\left(-\infty, \frac{58}{14}\right)$ (D) None of these

92. The values of a if f(x) has points of extrema which are of opposite signs, are
(A) ϕ (B) (–3, 3)
(C) $\left(-\infty, \frac{58}{14}\right)$ (D) None of these

Comprehension – 5

Let $f(x) = e^{(p+1)x} - e^x$ for real number p > 0.

93. The value of $x = s_p$ for which f (x) is minimum, is
(A) $-\frac{\ln(p+1)}{p}$ (B) $-\ln(p+1)$
(C) $-\ln p$ (D) $\ln\left(\frac{p+1}{p}\right)$

94. Let $g(t) = \int_{t}^{t+1} f(x)e^{t-x}\,dx$. The value of $t = t_p$, for which g (t) is minimum is

(A) $-\ln\left(\frac{e^p-1}{p}\right)$ (B) $-\frac{1}{p}\ln\left(\frac{e^p-1}{p}\right)$ (C) $-\frac{1}{p}\ln\left(\frac{(p+1)(e^p-1)}{p}\right)$ (D) $-\ln\left((p+1)(e^p-1)\right)$

MATCH THE COLUMNS FOR JEE ADVANCED

95.	Column - I	Column - II
(A)	If $x^2 + y^2 = 1$, then minimum value $x + y$ is	(P) -3
(B)	If the maximum value of $y = a\cos x - \frac{1}{3}\cos 3x$ occurs when $x = \frac{\pi}{6}$, then the value of ‘a’ is	(Q) $-\sqrt{2}$
(C)	If $f(x) = x - 2\sin x$, $0 \le x \le 2\pi$ is increasing in the interval $[a\pi, b\pi]$, then $a + b$ is	(R) 3
(D)	If equation of the tangent to the curve $y = -e^{-x/2}$ where it crosses the y-axis is $\frac{x}{p} + \frac{y}{q} = 1$, then $p - q$ is	(S) 2

96.	Column - I	Column - II
(A)	The greatest value of $f(x) = \frac{x}{4+x+x^2}$ on $[0, \infty)$ is	(P) $\frac{18}{e}$
(B)	The maximum value of $\frac{\ln x}{x}$ in $[2, \infty)$ is	(Q) $\frac{1}{e}$
(C)	Let $x > 0$, $y > 0$ and $xy = 1$, then minimum value of $\frac{3}{e^3}x + 27ey$	(R) e
(D)	The perimeter of a sector is 4e. The area of the sector is maximum when its radius is	(S) $\frac{1}{5}$

97.	Column - I	Column - II
(A)	The intercept of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ on the axis of y is equal to	(P) -1
(B)	Let f be a real function whose derivates upto third order exist and for some pair $a, b \in R$, $a < b$ $\log\frac{f(a)+f'(a)+f''(a)}{f(b)+f'(b)+f'')b)} = a - b$, then there exists c $(a, \in b)$ for which $\frac{f'''(c)}{f(c)}$ is equal to	(Q) 0
(C)	Let $f(x) = (x^2 - 1)(x^2 - 4)$, and α, β, γ be the roots of the equation $f'(x) = 0$ then $[\alpha] + [\beta] + [\gamma]$ is equal to ([t] represents the integral part of t)	(R) 1
(D)	If three normals can be drawn to the curve $y^2 = x$ from the point (c ,0) then c can be equal to	(S) 5/4
		(T) 2

98.	Column - I	Column - II
(A)	If the maximum and minimum value of the function $h(y) = y^3 - 6y^2 + 9y + 1$ on [0,2] are M and m respectively, then (M + m) is equal to	(P) 1
(B)	If the maximum and minimum value of the function $f(x) = \tan^{-1}x - \frac{1}{2}\ell nx$ on $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$ are M and m respectively, then [M + m] is equal to (where [.] denotes greatest integer function)	(Q) 6

(C) If the maximum and minimum value of the function $h(y) = \begin{cases} 2y^2 + \frac{2}{y^2}, & \lvert y\rvert \in [1,2] \\ 1 & , \lvert y\rvert < 1 \end{cases}$ are M and m respectively, then M + m is equal to	(R) 0
(D) For a given n ∈ N, number of real solutions of the equation $\min(e^x, x^2) = n$ is	(S) 19/2

99.

Column-I	**Column-II**
(A) The value of the sum $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + ... + \frac{29}{14^2.15^2}$ is	(P) 1
(B) If the tangents to the graph of $f(x) = x^3 + ax + b$ at x = a and x = b(a ≠ b) are parallel then f(1) is equal to	(Q) 0
(C) If the sum of x coordinates of points of extrema and points of inflection of $f(x) = 3x^5 - 250x^3 + 735x$ is 75k then k equals	(R) $\frac{224}{225}$
(D) If a right triangle is drawn in a semi circle of radius $\frac{1}{2}$ with one leg (not the hypotenuse) along the diameter, the maximum area of the triangle is	(S) $\frac{3\sqrt{3}}{32}$

Review Exercises for JEE Advanced

1. Show that the function $f(x) = \sin^m x \sin mx + \cos^m x \cos mx$, has a minimum at x = π/4 when m = 2 and a maximum at x = π/4 when m = 4 or 6.

2. Find all the values of ϕ between 0 and $\frac{1}{2}\pi$ for which $\cos^9\phi \sec^9\phi$ is a minimum.

3. Show that $\frac{\cos x}{(1+\cos^2 x)^{3/2}}$ has $\frac{2}{9}\sqrt{3}$ and $-\frac{1}{4}\sqrt{2}$ as local maximum values and $\frac{1}{4}\sqrt{2}$ and $-\frac{2}{9}\sqrt{3}$ as local minimum values.

4. Show that the function $f(x) = x^4(x - \frac{1}{2}\pi)^2 + \sin^4 x$, has extreme values at x = 0 and $x = \frac{1}{2}\pi$ and determine whether they are maxima or minima.

5. Let $f(x) = \frac{x^4 - x^2 - 2(x-3)}{x^4 - x^2 - 2(x-2)}$, x ∈ R. Find the range of f(x).

6. If p and q are integers and $f(x) = (x-1)^p (x+1)^q$, (p ≥ 2, q ≥ 2), show that f has the three critical points, – 1, $\frac{q-p}{q+p}$, 1.

Find the extrema of f for the following cases :

(a) p and q are both even

(b) p is even and q is odd.

(c) p is odd and q is even

(d) p and q are both odd.

7. Investigate the following functions and draw their graphs:

(i) $y = \frac{1}{x(x-3)^2}$

(ii) $y = x^2/(x + 8)$

(iii) $y = x\sqrt{2+x}$

8. Prove or give a counterexample :

(a) If f '(c)=0 then f has a minimum or a maximum at c

(b) if f has a maximum at c and if f is differentiable at c, then f'(c) = 0.

9. Suppose that the second derivative of f exists everywhere and that $f(x_1) = f(x_2) = f(x_3) = 0$, where $x_1 < x_2 < x_3$. Show that f"(c) = 0 for some number c with $x_1 < c < x_3$.

10. Find extrema of the following functions :

(i) $f(x) = \begin{cases} \min\{3t^4 - 8t^3 - 6t^2 + 24t, 1 \le t \le x\}, & 1 \le x < 2 \\ \max\{3t + 1/4 \sin^2 \pi t + 2, 2 \le t \le x\}, & 2 \le x \le 4 \end{cases}$

(ii) $f(x) = \begin{cases} (x+2)^{2/3}(x+4), & x \le 0 \\ \frac{|x-1|}{x^2}, & x > 0 \end{cases}$

11. (i) Check whether the following functions satisfy the condtions of Rolle's Theorem.

(a) $f(x) = x^2 - \sqrt[3]{|x|}$, $x \in [-1, 1]$

(b) $f(x) = \sec x$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$

(ii) $f(x) = (x - a)(x - b)(x - c)$, $a < b < c$, show that $f'(x) = 0$ has roots one belonging to (a, b) and other belonging to (b, c)

12. If the graph of $f(x) = 2x^3 + ax^2 + bx$, $a \in N$, $b \in N$, cuts the x-axis at three distinct points then find the minimum possible value of $a + b$. Also prove that there is only one such function for that value of $a + b$ and find it.

13. If $xy(y - x) = 2a^3$, $(a > 0)$ then show that y has a minimum when $x = a$. Determine the minimum value. Also, show that y has a second value at $x = a$ which is less than the minimum. How do you explain this paradox?

14. (i) Show that $-1/2 \le x/(1+x^2) \le 1/2$ for every value of x.

(ii) Suppose that f is a function whose derivative is $f'(x) = x/(1+x^2)$. Use the result in (i) to show that $|f(b) - f(a)| \le \frac{1}{2}|b - a|$, for any a and b.

15. The function $\frac{x}{(a+x)(b+x)}$ is a maximum when $x = \sqrt{(ab)}$, and a minimum when $x = -\sqrt{(ab)}$.

16. Prove that the function $m_1(x - x_1)^2 + m_2(x - x_2)^2 + \ldots + m_n(x - x_n)^2$ is a minimum when $x = \dfrac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$

17. Let the function f be differentiable at all real numbers. What can be said about the number of solutions of the equation $f(x) = 3$?

(a) $f'(x) < 0$ for all x.

(b) $f'(x) < 0$ for $x < 5$; $f'(x) > 0$ for $x > 5$.

(c) $f'(x) > 0$ for $x < -2$; $f'(x) < 0$ for x in $(-2,1)$; $f'(x) > 0$ for $x > 1$.

18. Prove the following inequalities :

(i) $\frac{a - b}{a} < \ln\frac{a}{b} < \frac{a - b}{b}$ if $0 < b < a$,

(ii) $py^{p-1}(x - y) \le x^p - y^p \le px^{p-1}(x - y)$ if $0 < y < x$ and $p > 1$.

19. Prove that $\tan^{-1} x > \dfrac{x}{1 + \frac{1}{3}x^2}$ if $x \in (0, \infty)$.

20. Use the function $f(x) = \frac{\ell nx}{x}$, and show that $x^y > y^x$ when $e \le x < y$.

21. If y be given as a function of x by the equation $4x^2 + 6xy + 9y^2 - 8x - 24y + 4 = 0$,
find the maximum and minimum values of y.

22. Show that the function $e^{ax}/(1 + x^2)$, where a is real, has a maximum and a minimum if $|a| < 1$, but that there are no turning points if $|a| \ge 1$.

23. The illumination of an area by a source of light is proportional to $\dfrac{x}{(x^2+a^2)^{\frac{1}{2}}} - \dfrac{x}{(x^2+b^2)^{\frac{1}{2}}}$ where a and b (> a) are constants. Find the value of x which gives maximum illumination.

24. Find the minimum value of xe^{ax}, where a is a constant, and show that the equation $xe^{a/x} = \lambda$, has two or no positive roots, according as λ is greater or less than ae.

25. Suppose that f is continuous on [0, 4], $f(0) = 1$, and $2 \le f'(x) \le 5$ for all x in (0, 4). Show that $9 \le f(4) \le 21$.

26. In ΔABC, D lies on AB, $CD \perp AB$, $AD = BD = 4$ cm, and $CD = 5$ cm. Where should a point P be chosen on CD so that the sum $PA + PB + PC$ is minimum?

27. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim\limits_{x \pm \infty}[f(x) + x] = 0$.

28. Show that, for all positive values of x and y, $\dfrac{e^{x+y}}{xy} \ge e^2$.

29. Show that $x^2y^2(4 - x^2)(4 - y^2) \le 16$ for all numbers x and y such that $|x| \le 2$ and $|y| \le 2$.

30. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

31. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$ and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

Target Exercises for JEE Advanced

1. Show that the function

$$f(x) = \frac{\sin^2 x}{\sin(x+a)\sin(x+b)}, (0 < a < b < \pi),$$

has an infinity of minima equal to 0 and of maxima equal to $-4\dfrac{\sin a \sin b}{\sin^2(a-b)}$.

2. Show that the maxima and minima of $f(x) = \sin mx \operatorname{cosec} x$, where m is an integer, are given by $\tan mx = m \tan x$ and deduce that $\sin^2 mx \le m^2 \sin^2 x$.

3. Prove that the greatest value of $ax + by$, where x and y are positive and $x^2 + xy + y^2 = 3k^2$, is

$$2k\sqrt{(a^2 - ab + b^2)}\,.$$

4. If θ and ϕ are acute angles connected by the relation $a\sec\theta + b\sec\phi = c$, where a, b, c are positive, then prove that $a\cos\theta + b\cos\phi$ is a minimum when $\theta = \phi$.

5. Investigate the maxima and minima of $f(x) = x - \sin x - \tan\alpha(1 - \cos x)$ and find the real roots of $f(x) = 0$, and α being an angle between 0 and π. Show that the condition for a double root is that $\tan\alpha - \alpha$ should be a multiple of π .

6. Show that the condition that $x^4 + 4px^3 - 4qx - 1 = 0$ should have equal roots may be expressed in the form $(p + q)^{2/3} - (p - q)^{2/3} = 1$.

7. If f(x) is a polynomial and λ is real, then prove that there is a root of $f'(x) + \lambda f(x) = 0$ between any pair of roots of $f(x) = 0$.

8. If α and β are successive roots of $f = 0$, then prove that the number of roots of $f' + \lambda f = 0$ between α and β (each counted according to its multiplicity) is odd.

9. Prove that the polynomial

$$(a - x)^6 - 3a(a - x)^5 + \frac{5}{2}a^2(a - x)^4 - \frac{1}{2}a^4(a - x)^2$$

has only negative values for $0 < x < a$.

10. Prove that $\pi < \dfrac{\sin \pi x}{x(1-x)} \le 4$, when $0 < x < 1$.

11. If f is a continuous and differentiable function and $f(0) = 1$, $f(1) = 2$. Then prove that there exists atleast one $c \in (0, 1)$ for which $f'(c)(f(c))^{n-1} > \sqrt{2^{n-1}}$, where $n \in N$.

12. Let a, b, c be three real numbers such that $a < b < c$, f(x) is continuous in [a, c] and differentiable in (a, c). Also f '(x) is strictly increasing in (a, c). prove that $(b - c)f(a) + (c - a)f(b) + (a - b)f(c) < 0$.

13. This problem shows why df generally differs from Δf by an amount proportional to $(dx)^2$. Let f be differentiable and have a continuous second derivative. Then $f(b) = f(a) + f'(a)(b - a) + R$ (Here $b - a$ plays the role of dx.) We shall express the "discrepancy" R in terms of f″. (Note the $R = \Delta f - df$.) We accomplish this by introducing a new function $g(x) = f(a) + f'(a)(x - a) + k(x - a)^2$, where the constant k is chosen so that $g(b) = f(b)$.

(a) Show that $g(a) = f(a)$.

(b) Deduce that there is a number c_1, between a and b, such that $g'(c_1) = f'(c_1)$.

(*Hint :* consider g – f.)

(c) Show that $g'(a) = f'(a)$

(d) Deduce that there is a number c_2, between a and c_1, such that $g''(c_2) = f''(c_2)$.

(e) From (d) conclude that $R = f'(c_2)(b - a)^2/2$. Since $dx = b - a$, this shows that the discrepancy is on the order of $(dx)^2$.

14. Using Rolle's Theorem, prove that if all the roots of the polynomial $P_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ $(a_0 \ne 0)$ with real coefficients a_k $(k = 0, 1, \ldots., n)$ are real, then its derivatives $P_n'(x), P_n''(x), \ldots, P_n^{(n-1)}(x)$ also have only real roots.

15. Let the function f(x) satisfy the following conditions:

(i) f(x) has a continuous (n – 1)th derivative on $[x_0, x_n]$.

(ii) f(x) has an n^{th} derivative in (x_0, x_n),

(iii) $f(x_0) = f(x_1) = \ldots. = f(x_n)$, where $x_0 < x_1 \ldots < x_n$.

Prove that there is a point $\xi \in (x_0, x_n)$ such that $f^{(n)}(\xi) = 0$.

16. Prove that all the roots of polynomial

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\{(x^2 - 1)^n\}$$

are real and lie in the interval (–1, 1).

17. Assume that the functions f(x) and g(x) are defined and n times differentiable for $x \ge x_0$, and $f^{(k)}(x_0) = g^{(k)}(x_0)$ for $k = 0, 1, \ldots n - 1$; $f^{(n)}(x) > g^{(n)}(x)$ for $x > x_0$. Prove that $f(x) > g(x)$ for $x > x_0$.

18. Show that the function $f(x) = |x|^m |x - 1|^n$, has a maximum value $m^m n^n/(m + n)^{m+n}$, m and n being positive.

19. Prove that if the function f(x) satisfies the following conditions :

(i) It is defined and has a continuous derivative of the (n – 1)th order $f^{(n-1)}(x)$ on the interval $[x_0, x_n]$;

(ii) It has a derivative of the nth order $f^{(n)}(x)$ in the interval (x_0, x_n) ;

(iii) $f(x_0) = f(x_1) = \ldots = f(x_n)$ $(x_0 < x_1 < \ldots < x_n)$,

then inside the interval $[x_0, x_n]$ there is atleast one point c such that $f^{(n)}(c) = 0$.

20. Draw the graph of the following functions :

(i) $y = \dfrac{x^2 + 3x + 2}{x - 3}$ (ii) $f(x) = \dfrac{2x^2 - 3}{7x + 4}$

(iii) $f(x) = \dfrac{11x + 2}{2x^3 - 1}$ (iv) $f(x) = \dfrac{x^3}{x^2 - 1}$

21. Find the equation giving the values of x for which $y = Ae^{-x^2} + Be^{-(x-c)^2}$ is stationary, and prove that the value of y corresponding to such a value x_1 of x is $\dfrac{Ac}{c - x_1}e^{-x_1^2}$. Show also that, when A, B, c are positive, the equation has just two roots, one greater than c and the other negative; and that they correspond to a minimum and a maximum respectively.

22. Trace the curve $y = \frac{1}{x}\ln\left(\frac{e^x - 1}{x}\right)$, showing that the point $\left(0, \frac{1}{2}\right)$ is a centre of symmetry, and that as x increases through all real values, y steadily increases from 0 to 1. Deduce that the equation $\frac{1}{x}\ln\left(\frac{e^{x-1}}{x}\right) = x$ has no real root unless $0 < \alpha < 1$, and then one, whose sign is the same as that of $\alpha - 1/2$.

23. Trace the curve $y = e^{1/x}\sqrt{(x^2 + 2x)}$, and show that the equation $e^{1/x}\sqrt{(x^2 + 2x)} = \alpha$ has no real roots if α is negative, one negative root if
$0 < \alpha < a = e^{1/\sqrt{2}}\sqrt{(2 + 2\sqrt{2})}$, and two positive roots and one negative if $\alpha > a$.

24. Prove that the maximum and minimum values of $y = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C}$ themselves are the values of λ for which $ax^2 + 2bx + c - \lambda(Ax^2 + 2Bx + C)$ is a perfect square.

25. Using the function $f(x) = (x - a)^m(x - b)^n(x - c)^p$, where m, n, p are positive integers and $a < b < c$ prove that the quadratic equation $(m + n + p)x^2 - \{m(b + c) + n(c + a) + p(a + b)\}x + mbc + ca + pab = 0$ has roots between a and b and between b and c.

26. Suppose f is differentiable in [a,b] and $f'(a) = f'(b) = 0$. Prove there exists $c \in (a, b)$ such that $\frac{f(c) - f(a)}{c - a} = f'(c)$.
Further prove the conclusion assuming only $f'(a) = f'(b)$. Interpret graphically.

27. Find the greatest value of $y = x^4 - 8cx^2 + c^2$ on the interval [–1, 2] depending on the value of the parameter c.

28. If $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, where A is a positive constant. Find the smallest A such that $f(x) \geq 25$ for all $x > 0$.

29. Prove that $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \ln(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$, if $x > 0$.

30. $f : [-5,5] \to [3, 17]$ is a surjective differentiable function and $g(x) = f(x)\, f'(x)$ is a twice differentiable function. Given that $g(0) = k$, find least integral value of k such that there always exist some $c \in (-5, 5)$ for all functions f(x) to satisfy $(f'(c))^2 = -f(c)\, f''(c)$ and $(f(c))^2 f'''(c) < 3(f'(c))^3$.

31. If PN, P′N′ be two neighbouring ordinates of a curve $y = \phi(x)$, and if QH, any intermediate ordinate, meet the chord PP′ in V, prove that
$QV = 1/2NH \,.\, HN' \,.\, \phi''(c)$,
ultimately, where c is the abscissa of some point between N and N'.

32. Let A_n and G_n be the arithmetic and geometric means of a set of n positive numbers.
Show that the maximum value of the ratio

$$\frac{\sqrt[n+1]{a_1 a_2 \ldots a_n x}}{\frac{1}{n+1}(a_1 + a_2 + \ldots + a_n + x)}$$

occurs for $x = A_n$, and compute the maximum.

Also conclude that $\frac{G_{n+1}}{A_{n+1}} \leq \left(\frac{G_n}{A_n}\right)^{n/(n+1)}$

Further, show that $\frac{G_n}{A_n} \leq \left(\frac{G_1}{A_1}\right)^{n/(n+1)} = 1,$

and therefore that $G_n \leq A_n$. Explain why $G_n = A_n$ if and only if $a_1 = a_2 = \ldots = a_n$.

Previous Year's Questions (JEE Advanced)

A. FILL IN THE BLANKS

1. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2 then the maximum value of A is................... **[IIT - 1994]**

B. TRUE / FALSE

2. For $0 < a < x$, the minimum value of the function $\log_a x + \log_x a$ is 2. **[IIT - 1984]**

C. MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER

3. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has **[IIT - 1983]**
(A) at least one root in [0, 1]
(B) one root in [2, 3] and the other in [–2, –1]
(C) imaginary roots
(D) none of these

4. AB is a diameter of a circle and C is any point on the circumference of the circle. Then **[IIT - 1983]**

(A) the area of Δ ABC is maximum when it is isosceles
(B) the area of Δ ABC is minimum when it is isosceles
(C) the perimeter of Δ ABC is minimum when it is isosceles
(D) none of these

5. If $y = a \ln |x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then **[IIT - 1983]**

(A) $a = 2, b = -1$
(B) $a = 2, b = -\frac{1}{2}$
(C) $a = -2, b = \frac{1}{2}$
(D) none of these

6. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$ be a polynomial in a real variable x with
$0 < a_0 < a_1 < a_2 < \ldots\ldots\ldots < a_n$. The function P(x) has **[IIT - 1986]**

(A) neither a maximum nor a minimum
(B) only one maximum
(C) only one minimum
(D) only one maximm and only one minimum
(E) none of these

7. On the interval [0, 1] then function $x^{25}(1 - x)^{75}$ takes its maximum value at the point **[IIT - 1995]**

(A) 0
(B) $\frac{1}{4}$
(C) $\frac{1}{2}$
(D) $\frac{1}{3}$

8. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f **[IIT - 1998]**

(A) does not exist because f is unbounded
(B) is not attained even though f is bounded
(C) is equal to 1
(D) is equal to –1

9. The number of values of x where the function f(x) = cos x + cos $(\sqrt{2}x)$ attains its maximum is **[IIT - 1998]**

(A) 0
(B) 1
(C) 2
(D) infinte

10. Let $f(x) = \begin{cases} |x| & \text{for} \quad 0 < |x| \le 2 \\ 1 & \text{for} \quad x = 0 \end{cases}$, then at x = 0, f has **[IIT - 2000]**

(A) a local maximum
(B) no local maximum
(C) a local minimum
(D) no extremum

11. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and m(B) is minimum value of f(x). As b varies, the range of m(B) is **[IIT - 2001]**

(A) [0, 1]
(B) (0, 1/2]
(C) [1/2,1]
(D) (0, 1]

12. In [0, 1] Lagrange's Mean Value theorem is not applicable to **[IIT - 2003]**

(A) $f(x) = \begin{cases} \frac{1}{2} - x \;, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 , & x \ge \frac{1}{2} \end{cases}$

(B) $f(x) = \begin{cases} \frac{\sin x}{x} \;, & x \ne 0 \\ 1 \;, & x = 0 \end{cases}$

(C) $f(x) = x|x|$
(D) $f(x) = |x|$

13. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta \in (0, \pi/2)$). Then the value of θ such that sum of intercepts on axes made by this made by this tangent is minimum, is **[IIT - 2003]**

(A) $\pi/3$
(B) $\pi/6$
(C) $\pi/8$
(D) $\pi/4$

14. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$ **[IIT - 2004]**

(A) f(x) is strictly increasing function
(B) f(x) has a local maxima
(C) f(x) is strictly decreasing function
(D) f(x) is bounded

15. If $f(x) = x^{\alpha} \log x$ and f(0) = 0, then the value of α for which Rolle's Theorem can be applied in [0, 1] is **[IIT - 2004]**

(A) –2
(B) – 1
(C) 0
(D) 1/2

16. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{2/3} \;, & -1 < x < 2 \end{cases}$

(A) 0
(B) 1
(C) 2
(D) 3 **[IIT - 2008]**

17. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then : **[IIT - 2010]**

(A) a = b and c ≠ b
(B) a = c and a ≠ b
(C) a ≠ b and c ≠ d
(D) a = b = c

D. COMPREHENSION

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by : $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$ **[IIT - 2008]**

18. What of the following is true ?
(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$
(B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$
(C) $f'(1) f'(-1) = (2 - a)^2$
(D) $f'(1) f'(-1) = (2 + a)^2$

19. Which of the following is true ?
(A) f(x) is decreasing on (–1, 1) and has a local minimum at x = 1
(B) f(x) is increasing on (–1, 1) and has a local maximum at x = 1
(C) f(x) is increasing on (–1, 1) and has neither a local maximum nor a local minimum at x = 1
(D) f(x) is decreasing on (–1, 1) and has neither a local maximum nor a local minimum at x = 1

20. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$

which of the following is true ?
(A) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(B) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(C) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
(D) $g'(x)$ does not change sign on $(-\infty, \infty)$

E. MULTIPLE CHOICE QUESTIONS WITH ONE OR MORE THAN ONE CORRECT ANSWER :

21. The function $f(x) = \int_{-1}^{x} t(e^t - 1)\ (t-1)\ t-2)^3\ (t-3)^5\ dt$ has a local minimum at x = **[IIT - 1999]**
(A) 0 (B) 1
(C) 2 (D) 3

22. f(x) is cubic polynomial with f(3) = 18 and f(–1) = 2. Also f(x) has local maxima at x = –1 and f'(x) has local minima at x = 0, then **[IIT - 2006]**
(A) the distance between (–1, 2) and (a(f(a)), where x = a is the point of local minima is $2\sqrt{5}$
(B) f(x) is increasing for $x \in [1, 2\sqrt{5}]$
(C) f(x) has local minima at x = 1
(D) the value of f(0) = 5

23. Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$,

$x \in [1, 3]$ then g(x) has **[IIT - 2006]**
(A) local maxima at x = 1 + ln 2
(B) local minima at x = e
(C) no local maxima
(D) no local minima

24. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$.

Then which of the following statement (s) is (are) true ? **[IIT - 2010]**
(A) f''(x) exists for all $x \in (0, \infty)$
(B) f'(x) exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$
(C) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$
(D) There exists $\beta > 1$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

25. If $f(x) = \int_0^x e^{t^2} (t-2)(t-3)dt$ for all $x \in (0, \infty)$, then
(A) f has a local maximum at x = 2 **[IIT - 2012]**
(B) f is decreasing on (2, 3)
(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$
(D) f has a local minimum at x = 3

F. SUBJECTIVE PROBLEMS

26. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$, a, b > c, x > – c is $(\sqrt{a-c} + \sqrt{b-c})^2$ **[IIT - 1979]**

27. Let x and y be two real variable such that x > 0 and xy = 1. Find the minimum value of x + y. **[IIT - 1981]**

28. For all x in [0, 1], let the second derivative f(x) of a function f(x) exist and satisfy |f"(x)| < 1. If f(0) = f(1), then show that |f'(x)| < 1 for all x in [0, 1]. **[IIT - 1981]**

29. If f(x) and g(x) are differentiable function for $0 \le x \le 1$ such that f(0) = 2, g(0) = 0, f(1) = 6; g(1) = 2, then show that there exist c satisfying 0 < c < 1 and f'(c) = 2g'(c). **[IIT - 1982]**

30. Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$. **[IIT - 1982]**

31. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of U km/hr and walk at a speed of V km/hr (V > U). At what point on the shore should he land so that he reaches hes house in the shortest possible time? **[IIT - 1983]**

32. Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. **[IIT - 1984]**

33. Let $f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lies in order that f(x) has exactly one minimum and exactly one maximum. **[IIT - 1985]**

34. Let $A(p^2, -p)$, $B(q^2, -q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is

$\frac{1}{4}(p + q)(q + r)(p + r)$ **[IIT - 1986]**

35. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. **[IIT - 1987]**

36. Investigate for maxima and minima the function

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$$

[IIT - 1988]

37. Find all maxima and minima of the function $y = x(x-1)^2$, $0 \le x \le 2$ **[IIT - 1989]**

Also determine the area bounded by the curve $y = x(x-1)^2$, the y-axis and the line $y = 2$.

38. Show that $2\sin x + \tan x \ge 3x$ where $0 \le x < \frac{\pi}{2}$.

[IIT - 1990]

39. A point P is given on the circumference of the a circle of radius r, chord QR is parallel to the tangent at P. Determine the maximum possible area of triangle PQR.

[IIT - 1990]

40. A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light? **[IIT - 1991]**

41. A cubic f(x) vanishes at $x = -2$ and has relative minimum / maximum at $x = -1$ and $x = \frac{1}{3}$ if $\int_{-1}^{1} f\,dx = \frac{14}{3}$, find the cubic f(x). **[IIT - 1992]**

42. $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} & ;\ 0 \le x < 1 \\ 2x - 3 & ;\ 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at $x = 1$. **[IIT - 1993]**

43. Find the values of x for which the function $f(x) = 1 + 2\sin x + 3\cos^2 x$, $(0, \le x \le 2\pi/3)$ is maximum or minimum. Also find these values of the function.

[IIT - 1993]

44. If A > O, B > O and $A + B = \pi/3$, then find the maximum value of tan A tan B. **[IIT - 1993]**

45. The circle $x^2 + y^2 = 1$ cuts the x axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x axis and line segment PQ at S. Find the maximum area of the triangle QSR.

[IIT - 1994]

46. Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the postive direction of the coordinates axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.

[IIT - 1995]

47. Determine the points of maxima and minima of the function $f(x) = \frac{1}{8}\log x - bx + x^2$, $x > 0$, where $b \ge 0$ is a constant. **[IIT - 1996]**

48. Suppose f(x) is a function satisfying the following conditions.

(a) f (0) = 2, f (1) = 1

(b) f has minimum value at x = 5/2 and

(c) for all x

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where, a and b are some constant, Determine the constants a, b and the function f(x). **[IIT - 1998]**

49. Find the point on the curve $ax^2 + 2bxy + ay^2 = c$; $c > b > a > 0$, whose distance from the origin is minimum.

[IIT - 1998]

50. Find the coordinate of all the points P on the ellipse $\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N is the foot of perpendicular from O to the tangent at P.

[IIT - 1999]

51. Find the normals of the ellipse $\left(\frac{x^2}{9}\right) + \left(\frac{y^2}{4}\right) = 1$ which are farthest from its centre. **[IIT - 1999]**

52. Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval [1/2, 1] and identify it.

[IIT - 2000]

53. A straight line L with negative slope pases through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin.

[IIT - 2002]

54. Let $x^2 + y^2 = r^2$ be a circle of variable radius. From a point P(6, 8) tangents are drawn to the circle. Find r such that the triangle formed by chord of contact & two tangents is maximum. **[IIT - 2003]**

55. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

[IIT - 2003]

56. Using Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the polynomial

$P(x) = 51x^{101} - 2323(x)^{100} - 45x + 1035$. **[IIT - 2004]**

57. If p(x) be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and p(x) has maxima at $x = -1$ and p'(x) has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve.

[IIT - 2005]

J. INTEGER ANSWER TYPE

58. For a twice differentiable function f(x), g(x) is defined as $g(x) = (f'(x))^2 + f''(x) f(x)$ on [a, e]. If for $a < b < c < d < e$, f(a) = 0, f(b) = 2, f(c) = –1, f(d)= 2, f(e) = 0 then find the minimum number of zeros of g(x).
[IIT - 2006]

59. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x|x^2 + 20 \le 9x|$ is
[IIT - 2009]

60. Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is
[IIT - 2009]

61. Let f be a function defined on R(the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$, for all $x \in R$.
If g is a function on R with values in the interval $(0, \infty)$ such that $f(x) = \ell n(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is
[IIT - 2010]

62. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT - 2011]**

63. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT - 2012]**

64. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, the p'(0) is **[IIT - 2012]**

Previous Year's Questions (JEE Main Papers)

1. The cost of running a bus from A to B is ₹$\left(av+\frac{b}{v}\right)$, where v km/h is the average speed of bus. When the bus travels at 30 km/h, the cost comes out to be ₹ 75 while at 40 km/h, it is ₹ 65. Then the most economical speed in km/h of the bus is

(a) 45 (b) 50
(c) 60 (d) 40 **[2013, online]**

2. The maximum area of a right-angled triangle with hypotenuse, h is

(A) $\frac{h^2}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$
(c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$ **[2013, online]**

3. If x = –1 and x = 2 are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then

(a) $\alpha = -6, \beta = \frac{1}{2}$ (b) $\alpha = -6, \beta = -\frac{1}{2}$
(c) $\alpha = 2, \beta = -\frac{1}{2}$ (d) $\alpha = 2, \beta = \frac{1}{2}$ **[2014]**

4. Let f(x) be a polynomial of degree four having extreme value at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then f(2) is equal to

(a) –8 (b) –4
(c) 0 (d) 4 **[2015]**

5. Let k and K be the minimum and maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1] respectively, then the order pair (k, K) is equal to

(a) $(1, 2^{0.6})$ (b) $(2^{-0.4}, 2^{0.6})$
(c) $(2^{-0.6}, 1)$ (d) $(2^{-0.4}, 1)$ **[2015, online]**

6. Let the tangents drawn to the circle, $x^2 + y^2 = 16$ from the point P(0, h) meet the x-axis at points A and B. If the area of ΔAPB is minimum, then h is equal to

(a) $4\sqrt{3}$ (b) $3\sqrt{3}$
(c) $3\sqrt{2}$ (d) $4\sqrt{2}$ **[2015]**

7. If m and M are are the minimum and the maximum values of $4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$, $x \in \mathbf{R}$, then M – m is equal to

(a) $\frac{9}{4}$ (b) $\frac{15}{4}$
(c) $\frac{7}{4}$ (d) $\frac{1}{4}$ **[2016, online]**

8. If a point P has coordinates (0, –2) and Q is any point on the circle, $x^2 + y^2 - 5x - y + 5 = 0$, then the maximum value of $(PQ)^2$ is

(a) $\frac{25+\sqrt{6}}{2}$ (b) $14+5\sqrt{3}$
(c) $\frac{47+10\sqrt{6}}{2}$ (d) $8+5\sqrt{3}$

9. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed is

(a) 30 (b) 12.5
(c) 10 (d) 25 **[2017]**

10. Let f(x) be a polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left(\frac{f(x)}{x^2}+1\right) = 3$ then f(–1) is equal to

(a) 5/2 (b) 9/2
(c) 1/2 (d) 3/2 **[2018, online]**

11. If a right circular cone, having maximum volume is inscribed in a sphere of radius 3 cm, then the curved surface area (in cm²) of this cone is:

(a) $6\sqrt{3}\pi$ (b) $6\sqrt{2}\pi$
(c) $8\sqrt{2}\pi$ (d) $8\sqrt{3}\pi$ **[2018, online]**

12. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R \sim \{-1, 0, 1\}$.

If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is

(a) –3 (b) $-2\sqrt{2}$
(c) $2\sqrt{2}$ (d) 3 **[2018]**

13. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is

(a) $20\sqrt{2}$ (b) $18\sqrt{3}$
(c) 32 (d) 36 **[2019]**

14. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is

(a) $\sqrt{19}$ (b) $\sqrt{\frac{79}{2}}$
(c) $\sqrt{31}$ (d) $\sqrt{34}$ **[2019]**

15. The maximum value of the function $f(x) = 3x^2 - 18x^2 + 27x - 40$ on the set $S = \{x \in R, x^2 + 30 \le 11x\}$ is

(a) 122 (b) –222
(c) –122 (d) 222 **[2019]**

16. The maximum volume (in cu.m) of the right circular cone having height 3 m is

(a) $3\sqrt{3}\pi$ (b) 6π
(c) $2\sqrt{3}\pi$ (d) $\frac{4}{3}\pi$ **[2019]**

ANSWERS

Concept Problems—A

1. (i) Local maxima
(ii) Neither maxima nor minima
(iii) Local minima
(iv) Neither maxima nor minima

2. (i) Local minima at $x = \pm 1$, local maxima of $x = 0$
(ii) none
(iii) Local minima at $x = 0$

3. Local minima at $x = 3$

5. Local minima at $x = 2$ if $a = 2$ both maxima and minima between $x = 2$ and $x = a$ if $a \ne 2$

6. $\lambda \in [-1, 1]$

8. local minimum value of 0 at each point of (0, 1)

10. Consider $g(x) = f(x) - kx$.

Concept Problems—B

2. local maxima at $x = 2 - \frac{1}{\sqrt{3}}$, max. value = $\frac{2}{3\sqrt{3}}$, local minima at $x = 2 + \frac{1}{\sqrt{3}}$, min. value = $-\frac{2}{3\sqrt{3}}$

3. local maxima at $x = 1$, max. value = –3, local minima at $x = 6$, min. value = –128

4. (i) increasing on (1, 5) ; decreasing on (0, 1) and (5, 6)
(ii) local maxima at $x = 5$, local minima at $x = 1$.

7. (i) Local maxima at $x = -2$, local minima at $x = 0$
(ii) Local maxima at $x = n\pi + \frac{\pi}{6}$; local minima at $x = n\pi - \frac{\pi}{6}$
(iii) No point of local maxima or minima.

8. $x = 1$ is a point of local maximum; $x = 3$is a point of local minimum; $x = 2$ and $x = 4$ are points of inflection.

10. $x = 1/3$ is a point of local minimum, $x = 1$ is a point of local maximum.

11. Local minima

Practice Problems—A

13. (i) $y_{max} = 0$ at $x = 0$, $y_{min} = -1$ at $x = 1$.
(ii) $y_{max} = 0$ at $x = 0$
(iii) strictly increases.

14. (i) Local minima at $x = 3$
(ii) Local maxima at $x = 0, 1$

15. (i) increasing on (2, 4) and (6, 9)
(ii) local maxima at x = 0, 4, 9 ; local minima at x = 2, 6
(iii) concave up on (1, 3), (5, 7) and (8, 9)
(iv) x = 1, 3, 5, 7, 8

19. x = 4/3 is a point of local maximum; x = 2 is a point of local minimum; x = 5/4 and x = 3/2 are points of inflection.

20. 3 critical point $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Local maxima at $x = \frac{\pi}{3}$, Local minima at $x = \frac{5\pi}{3}$.

21. Local maximum

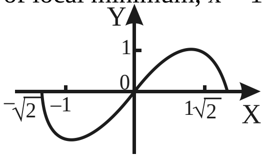
25. x = 1/2 is neither a local maximum nor a local minimum, x = 1 is a local minimum, x = 2 is neither.

26. A = –9.5, B = 4.5, C = 0, D = 7.

27. x = –2B/5A is a relative minimum; x = 0 is neither.

PRACTICE PROBLEMS—B

1. x = –1 is a point of local minimum, x = 1 is a point of local maximum.



2. x = 1/3

3. 1/e < a < 1

4. Local minima at x = 1/2 (b + a)

6. a = 2

7. $a = 1,\ b \in (-\infty, -1)\ ;\ a = -\frac{1}{3},\ b \in \left(-\infty, -\frac{5}{9}\right)$

8. $a = 81/25$ then $b \in (400/243, \infty)$

9. Local maxima at x = – 2, f(–2) = – 2
Local minima at x = 2, f(2) = 2.

10. Non-Diff. at x = 1 , Local maxima at x = 0, Local minima for $x \in [1/2, 1] \cup \{2\}$

11. f(x) = 2.

CONCEPT PROBLEMS—C

1. (i) $y_{max} = \frac{a^4}{16}$ at $x = \frac{a}{2}$, $y_{min} = 0$ at x = 0 and at x = a.
(ii) $y_{max} = \frac{5}{4}$ at $x = \frac{3}{4}$ (iii) $y_{max} = \sqrt[e]{e}$ at x = e.

2. (i) Local maxima at x = 1; Local minima at x = 3
(ii) Local maxima at x = 0 ; Local minima at x = 2/5

3. (i) Local maxima at x = π/6 , 5π/6 : Local minima at x = π/2 , 3π/2
(ii) Local minima at x = 1/8

4. Local minimum at x = 2, inflection points at x = 1 and x = 5/3.

5. (ii) nothing
(iii) f has a relative minimum at x = 1, g has no relative extremum at x = 1.

6. Yes ; f has a point of local maximum and g has a point of local minimum

8. Yes. Since f''(x) changes sign when passing through the point x_0, the latter is a point of extremum for the function f'(x).

PRACTICE PROBLEMS—C

11. (i) $x_{min} = -\frac{1}{2}$, $y_{min} = -\frac{1}{2}e^{-3/4}$; $x_{max} = 1$, $y_{max} = 1$.
(ii) $x_{max} = 3$, $y_{max} = \frac{1}{3}$; $x_{min} = -3$, $y_{min} = -\frac{1}{3}$.
(iii) $x_{min} = e$, $y_{min} = e$. (iv) $x_{max} = 0$, $y_{max} = -2$, $x_{min} = 2$, $y_{min} = 2$.

12. (i) $\begin{cases} \text{L.Max at } x = \dfrac{3+\sqrt{17}}{4} \\ \text{L.Min at } x = \dfrac{3-\sqrt{17}}{4}, 3 \end{cases}$
(ii) Local maxima at x = 0, π, Local minima at $x = \frac{2\pi}{3}$,
(iii) Local minima at $x = e^{-1/2}$
(iv) Local minima at x = π/2; Local maxima at x = 3π/2

13. (i) Local minima at x = 7/5
(ii) Local minima at $x = \pm \tan^{-1}\sqrt{b/a}$
(iii) Local maxima at x = – 1, 1 ; L. Min. at x = 0
(iv) Local maxima at x = 0 ; Local minima at ± 1

15. Local maxima if a = 2.

17. (i) increasing in $(0, e^2)$; decreasing in (e^2, ∞)
(ii) local maxima $f(e^2) = 2/e$
(iii) concave up on $(e^{8/3}, \infty)$, concave down on $(0, -e^{8/3})$; point of inflection $(e^{8/3}, \frac{8}{3}e^{-4/3})$

18. (i) f has a local maximum at x = 2.
(ii) f has a horizontal tangent at x = 6.

19. $\frac{1}{4}(3 - \sqrt{17})$

20. Local maxima at x = 2nπ + π/2, local min. at x = 2nπ + 3π/2, n ∈I,

21. (i) 4x – 3y – 7 = 0 (ii) local minimum

CONCEPT PROBLEMS—D

3. (i) There is no extremum.
(ii) The maximum is f(0) = 0.
(iii) The minimum is f(0) = 0.

4. No extremum at c ; point of inflection at c.

5. Min at t = 1 ; Max at t = – 1

7. (i) local max. at x = 0; local min. at x = ± 1
(ii) local max. at x = –2; local min. at x = 1
(iii) local max. at x = 1; local min. at x = 0, 3/2
(iv) local max. at x = π; local min. at x = 0.
(v) local minima at x = ± 1.

9. local max.

CONCEPT PROBLEMS—E

1. (i) Local maxima at $x = 2$, Local minima at $x = 3$, Global maxima at $x = 2$
(ii) Local minima at $x = -1$, No point of Global minima, no point of local or Global maxima
(iii) Local & Global maxima at $x = 1$, Local & Global minima at $x = 0$.

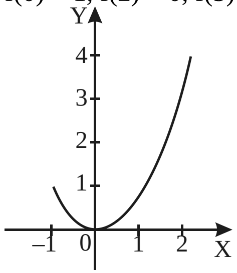
2. (i) Abs. max. at b; local max. at b, e and t; abs. min. at d; local min. at a, d and s.
(ii) Abs. max. at e; local max. at a, c, e and s; abs. min. at t; local min. at b, c, d, r and t.

3. (i) Absolute maximum at $x = c$, absolute minimum at $x = b$.
(ii) Absolute maximum at $x = c$, no absolute minimum.
(iii) No absolute maximum,no absolute minimum
(iv) Absolute minimum at $x = a$, absolute maximum at $x = c$.

4. minimum value of 0 at $x = 0$, maximum value of 1 at $x = -1$

5. (i) Abs. max. $f(4) = 4$; abs. min. $f(7) = 0$; local max. $f(0) = 3$, $f(4) = 4$ and $f(6) = 3$; local min. $f(2) = 1$, $f(5) = 2$ and $f(7) = 0$.
(ii) Abs. max. $f(8) = 5$; abs. min. $f(2) = 0$; local max. $f(1) = 2$, $f(4) = 4$, $f(6) = 3$ and $f(8) = 5$; local min. $f(0) = 1$, $f(2) = 0$, $f(5) = 2$ and $f(7) = 1$.

6. (a)



(b) 4 (c) Yes
(d) No (e) Yes

7. (i) $x = 1, 2$ (ii) $1\ (x = 1)$
(iii) $f(0) = 6$ is the global minimum, $f(1) = 11$ is global maximum

8. max. value $= e^2$, min. value of $= 0$

9. max. at $x = 2\pi$, max. value $= 2\pi$; min. at $x = 0$, min. value $= 0$

11. (i) 8 and 0 (ii) 2 and – 10
(iii) and $-\frac{\pi}{2}$

12. (i) maximum value $3/\sqrt{5}$ at $x = 1$, minimum value $-3/\sqrt{5}$ at $x = -1$;
(ii) maximum value 17 at $x = -5$, minimum value 1 at $x = -3$;
(iii) maximum value $f(-2) = -4$, no minimum.

13. Local minima at $x = 3$, no point of local maximum, global minimum value 2 at $x = 3$, global maximum value is not defined.

14. (i) min 0, no max (ii) max $\frac{2}{3\sqrt{6\pi}}$, no min.

15. (i) true (ii) false

16. No; No

PRACTICE PROBLEMS—D

20. max. at $x = \pi/6$, max. value $= 3/4$; min. at $x = 0$ and 1, min. value $= 1/2$.

21. max. value $= \frac{3}{\sqrt{8}}$, min. value $= 0$

22. (i) the greatest value is $f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{4}\ln 3$, the least value is $f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{4}\ln 3$.
(ii) the greatest value is , $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$, the least value f $\left(\frac{3\pi}{2}\right) = -2$.
(iii) there is no greatest value, the least value is $f(0) = 1$.

23. maximum value 2, minimum value $-\frac{1}{4}$.

25. least value $= -2$; greatest value $= 2$.

26. (i) 9 ; 5 (ii) $\frac{9}{4}$; 0

27. $[0, \infty)$.

28. (i) Greatest value $= \pi/2$; Least value $= 1$
(ii) Greatest value $= 2/\sqrt{3}$; Least value $= 1$

31. the maximum velocity is 60 when $t = 0$.

32. 3, (5, 11)

33. An empty set.

34. Find the minimum of e^x/x.

CONCEPT PROBLEMS—F

1. (a) No (b) Yes
2. False **3.** False
4. True **5.** False
6. (a) False (b) True
7. (a) Yes, it is (b) No, it is not
(c) No, it is not (d) No, it is not
(e) Yes, it is.
8. (a) No, it does not, (b) Yes, it does

PRACTICE PROBLEMS—E

11. (a) $\inf f(x) = 0$ is not attained, $\sup f(x) = 1 = f(1)$
(b) $\inf f(x) = 0 = f(0)$, $\sup f(x) = 100 = f(10)$,
(c) $\inf f(x) = 0$, $\sup\limits_{(-\infty,\infty)} f(x) = \pi/2$ are not attained,
(d) $\inf f(x) = -1 = f(\pi)$, $\sup f(x) = \sqrt{2} = f(\pi/4)$
(e) $\inf f(x) = 0$, $\sup f(x) = 1/2$ are not attained.

12. (a) 4, (b) 2,
(c) 8/(3π), (d) 2/π

CONCEPT PROBLEMS—G

3. No

4. none of the statements is true.

5. (i) Max. f (–5) = e^{48}; min. f (2) = $\frac{1}{e}$

(ii) Max. f(π)= 22 ; Min. $f\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

6. 0, –tan 1.

PRACTICE PROBLEMS—F

7. (i) x = 0,
(ii) x = (2n + 1)π/2, n ∈ I.

8. $\frac{x^2-4x+9}{x^2+4x+9} = 1 - \frac{8}{x+\frac{9}{x}+4}$ .Now consider the max. and min. of right hand side which occurs at ± 3.

9. (i) The minimum is f(–2) = –1, the maximum f(2) = 1.
(ii) The maximum is f(±2) = –1, the minimum f(0) = 7.
(iii) The maximum is f(–3) = $3\sqrt[3]{3}$, the minimum f(2) = $-\sqrt[3]{44}$.

10. (i) min f(– 2) = – 1, max f(2) = 1;
(ii) min f(0) = 7, max f(± 2) = –1 ;
(iii) min f(2) = – $44^{1/3}$, max f(–3) = $3^{4/3}$

11. least value = $\frac{4}{3-\sqrt{2}}$; greatest value = $\frac{4}{8+\sqrt{2}}$
(Put y = sin x + cos x)

12. least value = 3. (Put y = cos x)

13. Represent f(x) in the form f(x) = 2 sin x – 2 $\sin^3$x and, using the substitution t = sin x, reduce the problem to the proof of the validity of the inequality $\min\limits_{t\in[-1,1]} g(t) > -7/9$, where g(t) = 2t – $2t^3$.

14. greatest value = f(3) = $4\sqrt{6}$, least value = f(1) = 0. (Put y = $(x-1)^2$ and use the fact that g(u) = $\sqrt{u}$ is a strictly increasing function)

CONCEPT PROBLEMS—H

1. (i) 4 (ii) 3
(iii) 4 (iv) 1

2. $y_{max} = \frac{1}{2\sqrt{ab}}$, ax + $\frac{b}{x} \geq 2\sqrt{ab}$

3. Max = $\frac{16}{3}$, Min. = 0

4. Max = $\frac{3}{2}$, Min. = 1

5. min. value of y = 75.

6. $2^2 . 3^2 . \left(\frac{4}{5}\right)^5$

PRACTICE PROBLEMS—G

7. $9(20)^4$

8. (i) The greatest difference occurs at x = $\frac{1}{2}$.
(ii) The greatest difference occurs at x = $\frac{1}{\sqrt{3}}$
(iii) The greatest difference occurs at x = $\left(\frac{1}{n}\right)^{1/(n-1)}$

9. f(0) = 1 **11.** [–1/3, 1]

12 c lies between a and b. **13.** min $(|a| + |b|)^2$, no max.

15. $\frac{abc}{3\sqrt{3}}$ **16.** $a_7 = \frac{49}{543}$

17. Maximum : (1, 3), Minimum : (1, 1)

18. If (x,y) is to be the closest point, we must then minimize $(x-11)^2 + (x^3-3x-1)^2$.

19. Maximum at (5, 3); minimum at (–1, –3).

CONCEPT PROBLEMS—I

1. 10

2. The sides of the rectangle which has the largest area is $R\sqrt{2}/2$.

3. $\sqrt{a(a+b)}$ ft

PRACTICE PROBLEMS—H

6. (a) $20\sqrt{3}$ miles/hr; 103.9 rupees
(b) $40\sqrt{2}$ miles/hr; 169.7 rupees
(c) 60 miles/hr; 220 rupees
(d) 60 miles/hr; 270 rupees
(e) 60 miles/hr; 320 rupees

7. V = 48π for 0 ≤ h < 2; V = 4π$(4 + h)^3$/(9h) for h ≥ 2.

8. $\left(\frac{1}{2}, -\frac{1}{4}\right)$ is closest, (–1, –1) is farthest.

9. maximum y = 4 at t = π, 3π; minimum y = 0 at t = 0, 2π.

10. 4000 cubic cm.

11. (i) 8x + y = 16,
(ii) 2x + y = 10 ; min. intercept $5\sqrt{5}$,
(iii) $2\sqrt{2}x + y = 8 + 2\sqrt{2}$]

12. x = 25, y = 10.

13. P is at distance of $\frac{ac}{a+b}$ from c.

14. 1 unit **15.** 4 units

CONCEPT PROBLEMS—J

2. (i) The horizontal asymptote is y = a
(ii) The vertical asymptote is x = b.

3. (i) y = x – 1 (ii) y = 2x – 2

4. Solution 2 is incorrect.

5. (i) x = 0, y = 2x (ii) x = 0; y = –3x
(iii) y = x – 6

PRACTICE PROBLEMS—I

6. $a = \frac{9}{5}, b = \frac{3}{5}$

7. $a = 1, b = 0, c = 1$

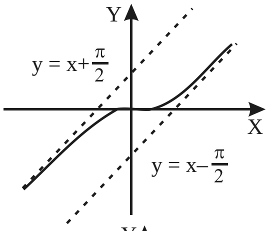
8. (i) $x = 3, y = x - 3$

(ii) $y = \pm\frac{\pi x}{2} - 1$

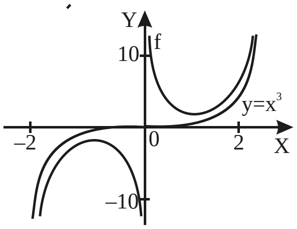
(iii) $y = x$

(iv) $x = \pm 2$ (v) $y = 2x - \frac{\pi}{2}$

9.



11.



CONCEPT PROBLEMS—K

1. (i) increasing in (2, 4), (6, ∞) and decreasing in (0, 2), (4, 6)

(ii) x = 2 (minima), 4 (maxima) and 6 (minima)

(iii) concave up in (1, 3) , (5, 7) , (8, ∞); concave down in (0, 1), (3, 5), (7, 8)

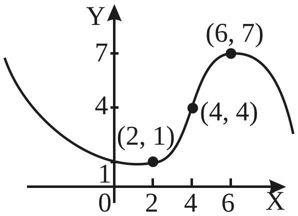
(iv) x = 1, 3, 5, 7, 8

CONCEPT PROBLEMS—L

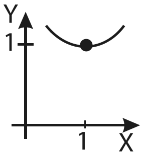
1.

Point	y′	y″
P	–	+
Q	+	+
R	+	–
S	0	–
T	–	–

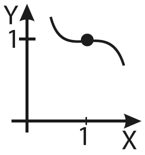
2.



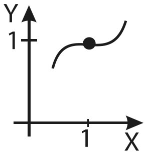
3. (i)



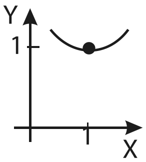
(ii) (a)



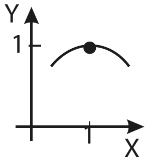
(b)



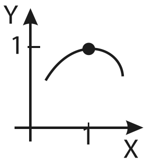
(c)



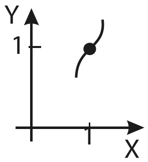
(d)



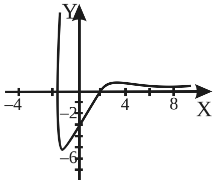
(iii)



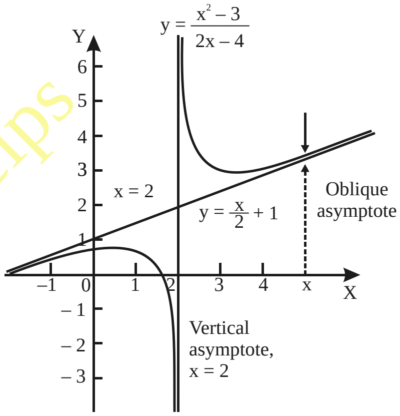
(iv)



4. Relative minimum at (–1, –2e); relative maximum at $(3, 6e^{-3})$



5.



6. (i) c

(ii) d

(iii) b

(iv) f

7. (a)–(D);

(b)–(F);

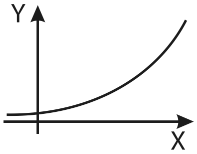
(c)–(B);

(d)–(C);

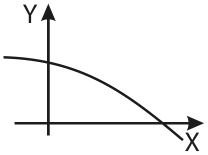
(e)–(A);

(f)–(E)

9. (a)



(b)



(c) No

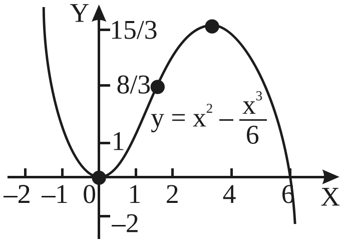
PRACTICE PROBLEMS—J

10. (i) x = 2

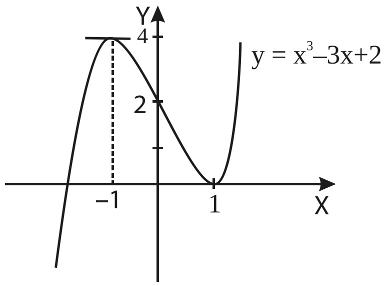
(ii) x = 1, 3, 4

11. A = –3, B = 9 and C = –1

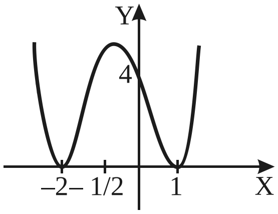
12. (i)



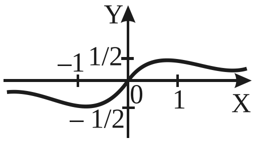
(ii)



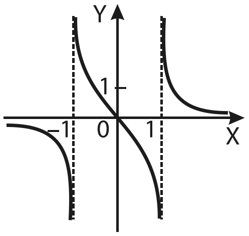
13. (a) On the intervals $(-\infty, -2)$ and $(-1/2, 1)$ the function decreases, on the intervals $(-2, -1/2)$ and $(1, \infty)$ it increases; $x = -2$ and $x = 1$ are points of minimum, $x = -1/2$ is a point of maximum, $y(-2) = y(1) = 0$, $y(-1/2) = 81/16$.



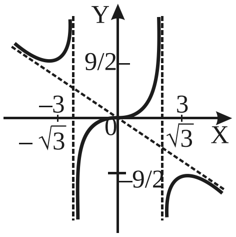
(b) On the intervals $(-\infty, -1)$ and $(1, \infty)$ the function decreases, on the interval $(-1, 1)$ it increases; $x = -1$ is a point of minimum, $x = 1$ is a point of maximum.



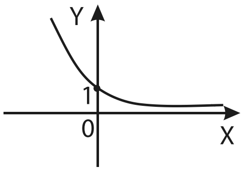
(c) The function decreases on the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$



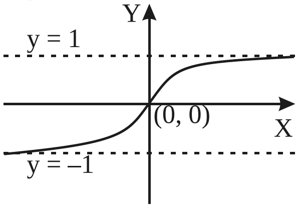
(d) On the intervals $(-\infty, -3)$ and $(3, \infty)$ the function decreases, on the intervals $(-3, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}, 3)$ it increases; $x = -3$ is a point of minimum, $x = 3$ is a point of maximum ; $y(-3) = 9$, $y(3) = -9$.



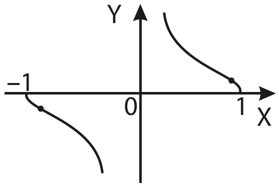
14. (i) y-intercept = 1; horizontal asymptote $y = 0$; decreasing on $(-\infty, \infty)$; concave up on $(-\infty, \infty)$



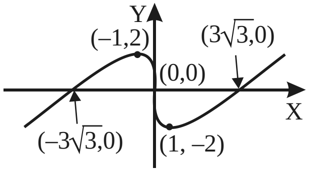
(ii) y-intercept = 0; x-intercept = 0; symmetric about the origin; horizontal asymptote $y = \pm 1$; decreasing on $(-\infty, \infty)$; concave up on $(-\infty, 0)$; concave down on $(0, \infty)$; inflection point $(0, 0)$



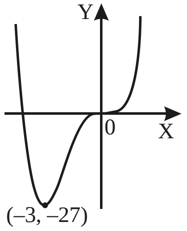
(iii) x-intercept = ±1; symmetric about the origin; vertical asymptote $x = 0$; decreasing on $(-1, 0)$, $(0, 1)$; concave up on $(-1, -\sqrt{2/3})$, $(0, \sqrt{2/3})$; concave down on $(-\sqrt{2/3}, 0)$, $(\sqrt{2/3}, 1)$; inflection points $(\pm\sqrt{2/3}, \pm 1/\sqrt{2})$



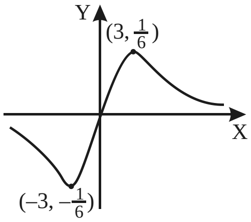
(iv) y-intercept = 0; x-intercept = $\pm 3\sqrt{3}$; symmetric about the origin;
increasing on $(-\infty, -1)$, $(1, \infty)$; decreasing on (–1. 1); local max. $f(-1) = 2$; local min. $f(1) = -2$; concave up on $(0, \infty)$; concave down on $(-\infty, 0)$; inflection point $(0, 0)$



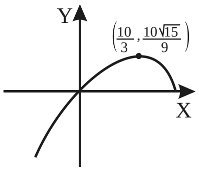
15. (i) y-intercept = 0; x-intercept = –4, 0; increasing on $(-3, \infty)$, decreasing on $(-\infty, -3)$; local min $f(-3) = -27$; concave up on $(-\infty, -2)$ $(0, \infty)$; concave down on $(-2, 0)$; inflection points $(0, 0)$, $(-2, -16)$



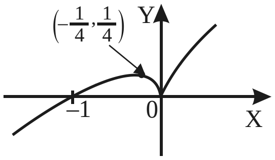
(ii) y-intercept = 0; x-intercept = 0; symmetric about the origin; horizontal asymptote y = 0; increasing on (–3, 3); decreasing on (–∞, –3), (3, ∞); local min f((–3) = $-\frac{1}{6}$; local max. f(3) = $\frac{1}{6}$; concave up on $(-3\sqrt{3},0)$ $(3\sqrt{3},\infty)$; concave down on $(-\infty,-3\sqrt{3})$, $(0,3\sqrt{3})$; inflection points (0, 0), $(\pm3\sqrt{3},\pm\sqrt{3}/12)$



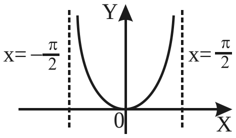
(iii) y-intercept = 0; x-intercept = 0, 5; increasing on $\left(-\infty,\frac{10}{3}\right)$; decreasing on $\left(\frac{10}{3},5\right)$; local max. $f\left(\frac{10}{3}\right)=\frac{10}{9}\sqrt{15}$; concave down on (–∞, 5)



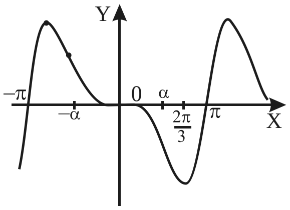
16. (i) y-intercept = 0; x-intercept = –1, 0; increasing on (–∞, $-\frac{1}{4}$), (0, ∞); decreasing on $\left(-\frac{1}{4},0\right)$; local max. $f\left(-\frac{1}{4},0\right)=\frac{1}{4}$; local min. f(0) = 0; concave down on (–∞, 0), (0, ∞)



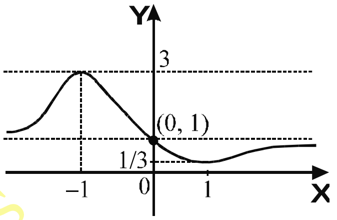
(ii) y-intercept = 0; x-intercept = 0; symmetric about y-axis; vertical asymptote x = ±π/2; increasing on (0, π/2); decreasing on (–π/2, 0); local min. f(0) = 0; concave up on (–π/2, π/2)



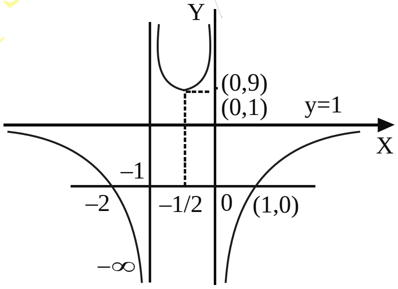
(iii) y-intercept = 0; x-intercept = nπ; symmetric about the origin; period 2π ; On the interval [–π,π], increasing on (–π, –2π/3), (2π/3, π); decreasing on (–2π/3, 2π/3); local max. f(–2π/3) = $\frac{3}{2}\sqrt{3}$; local min. f(2π/3) = $-\frac{3}{2}\sqrt{3}$; concave up on (–α, 0), (α, π), where cosα = $\frac{1}{4}$; concave down on (–π, –α), (0,α); inflection points when x = 0, ±α, ±π



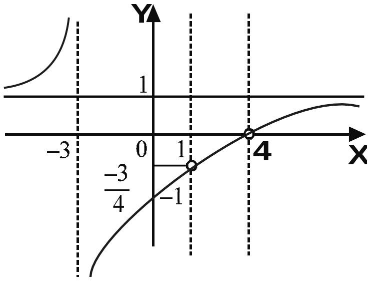
17. (i)



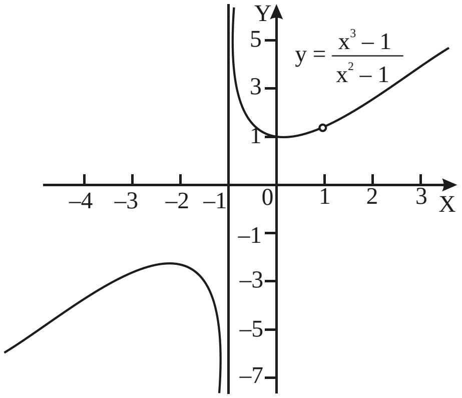
(ii)



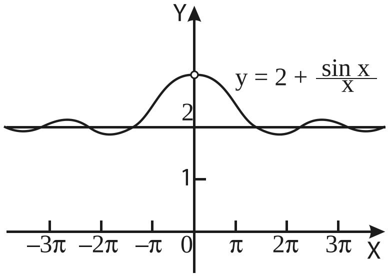
(iii)



(iv)

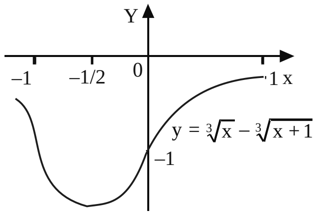


18.

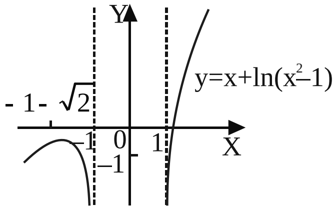


Note: The word asymptote comes from an old Greek word for "never touching." In practice, however, a curve may cross one of its asymptotes a finite number of times or even infinitely often, as does the curvey = $2+\frac{\sin x}{x}$.

19. (i)

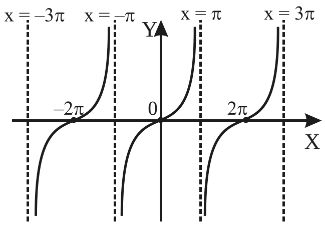


(ii)

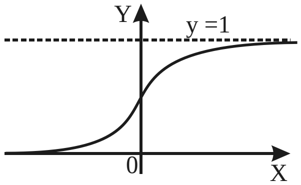


21. (i) $\tan^{-1}6$, $\pi + \tan^{-1}6$; (ii) No

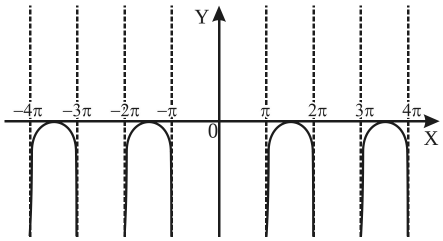
22. (i)



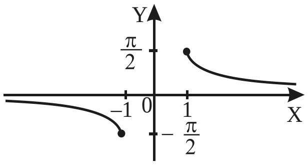
(ii)



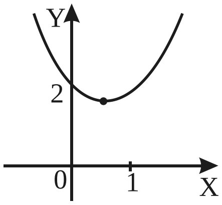
(iii)



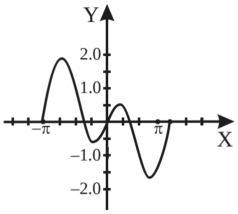
23. (i)



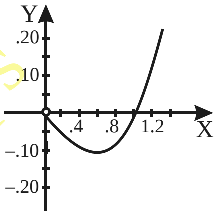
(ii)



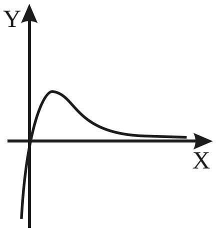
24 .(i)



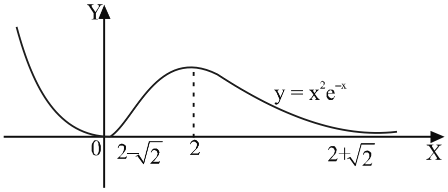
(ii)



(iii)



(iv)



CONCEPT PROBLEMS—M

2. $0.1 < x < 0.2$ **3.** $x = 2$

4. 2

5. (a) 0 ; (b) 1;

(c) $\lim_{x\to\infty} g'(x) = 0$.

6. No

PRACTICE PROBLEMS—K

12. $-0.7 < x_1 < -0.6$ and $0.8 < x_2 < 0.9$.

13. $0.38 < x_1 < 0.39$ and $1.24 < x_2 < 1.25$

14. $1.096 < x < 1.097$

CONCEPT PROBLEMS—N

1. No.
2. (i) – 1, 0, 1 (ii) $\pi/4, 5\pi/4$.
5. not applicable to both functions.
6. (b) No
7. Four, (1, 2), (2, 3), (3, 4)
11. No.
13. No
14. According to the Rolle's Theorem, between the roots of the first derivative there is atleast one root of the second derivative. When passing through one of these roots the second derivative must change sign.
16. Take advantage of the fact that a polynomial of an odd degree (and, hence, also its second derivative) has atleast one real root and changes sign atleast once.
17. No

PRACTICE PROBLEMS—L

18. This can be shown by a graphical construction. If we draw the curves. y = tan x, y = x, their intersections will determine the values of x which make tan x = x.
21. Set $g(x) = f(x)^2 f(1 - x)$. Since $g(0) = g(1) = 0$, g satisfies the hypotheses of Rolle's Theorem. There is a $c \in (0, 1)$ such that $g'(c) = 0 \Rightarrow 2f'(c) f(c) f(1 - c) - f(c)^2 f'(1 - c)$, then assertion.
23. Let $f(x) = x^3 + ax - 1$ and use the intermediate value theorem to show that there is atleast one root. Then assume there are two roots and use Rolle's Theorem to obtain a contradiction.
24. consider g(x) = f(x)/x.

CONCEPT PROBLEMS—O

3. yes; yes.
9. What goes up must come down. Later, consider $e^{kx}f(x)$

PRACTICE PROBLEMS—M

12. Since the original equation vanishes for x = 0, r, by Rolle's Theorem, the derivative must vanish in (0, r).
14. Make sure that the derivative $f'(x) = 4(x^3 - 1)$ has only one real root, x = 1, and apply the Rolle's Theorem.
15. The derivative $f'(x) = nx^{n-1} + p$ has only one real root at an even n and not more than two real roots at an odd n.
16. The derivative is a polynomial of the third degree and has three roots. Take advantage of the fact that between the roots of the polynomial lies the root of its derivative.

CONCEPT PROBLEMS—P

2. (i) c = 3/2 (ii) $c = 3\sqrt{2} - 2$
3. (9/4, 1/2)
5. (i) $\theta = \dfrac{e-2}{e-1}$ (ii) $\theta = \dfrac{1}{2}$

   (iii) $\theta = \dfrac{2}{3}$.

PRACTICE PROBLEMS—N

10. c = 1/2 or $\sqrt{2}$
13. n
14. 0
16. Yes
23. The derivative $f'(x_0)$ is given by $f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ provided this limit exists.

CONCEPT PROBLEMS—Q

1. (a) $2 \sec^2 x \tan x$ (b) 1
2. (a) Corollary 2 (b) Corollary 1
5. Yes
6. No

CONCEPT PROBLEMS—R

1. 16
2. Yes
4. No
6. Let $f(x) = \sqrt{1+x}$
11. 12

PRACTICE PROBLEMS—P

26. Let $f(x) = \sqrt{x}$, a = 3, and b = 4 in the Mean Value Theorem.
29. Let g(x) = Bx – f(x). Then g(b) = g(a) and $g'(x) \geq 0$, so g(x) is a constant.
34. Consider g(x) = ln f(x).
36. By LMVT, (ln y – ln x)/(y – x) = 1/z, where x < z < y. Clearly 1/y < 1/z < 1/x.
38. If $f''(x) \geq 0$ for all x in (a, b), then $f(x) \leq 0$ on (a,b), contrary to hypotheses.

CONCEPT PROBLEMS—S

1. (i) $c = \dfrac{14}{9}$ (ii) $c = \dfrac{\pi}{4}$.
2. No, it is not; $g'(x) \neq 0$ is not satisfied.
3. No, since g(–3) = g(3).

OBJECTIVE EXERCISES

1. A 2. B 3. D
4. B 5. C 6. D
7. A 8. A 9. D
10. A 11. B 12. A
13. B 14. A 15. B
16. C 17. B 18. B

19. C **20.** B **21.** C
22. D **23.** A **24.** B
25. A **26.** A **27.** B
28. C **29.** D **30.** A
31. C **32.** A **33.** D
34. B **35.** C **36.** B
37. A **38.** D **39.** B
40. C **41.** B **42.** D
43. A **44.** D **45.** D
46. D **47.** C **48.** A
49. C **50.** C **51.** ABCD
52. ACD **53.** ACD **54.** ABD
55. ABCD **56.** B **57.** BC
58. AD **59.** ABD **60.** ABCD
61. ACD **62.** AC **63.** AD
64. BCD **65.** BC **66.** AB
67. AC **68.** AC **69.** BD
70. ABCD **71.** B **72.** B
73. A **74.** D **75.** **A**
76. A **77.** D **78.** A
79. A **80.** C **81.** C
82. D **83.** C **84.** B
85. D **86.** A **87.** D
88. C **89.** B **90.** A
91. C **92.** B **93.** A
94. C
95. (A)–(Q) ; (B)–(S) ; (C)–(S) ; (D)–(R)
96. (A)–(S) ; (B)–(Q) ; (C)–(P) ; (D)–(R)
97. (A)–(T) ; (B)–(R) ; (C)–(P) ; (D)–(RST)
98. (A)–(Q) ; (B)–(P) ; (C)–(S) ; (D)–(P)
99. (A)–(R) ; (B)–(P) ; (C)–(Q) ; (D)–(S)

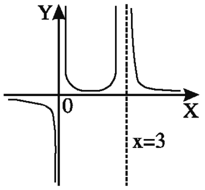
REVIEW EXERCISES for JEE ADVANCED

2. $\pi/8$, $3\pi/8$

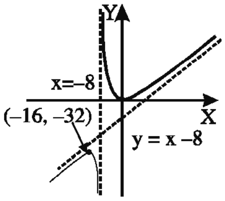
4. Both are minima. **5.** (1, 2]

6. (a) f(1), min; f(–1), min; f((q–p)/(q+p)), max.
(b) f(1), min; f((q–p)/(q+p)), max.
(c) f(1), max; f((q–p)/(q+p)), min.
(d) f((q–p)/(q+p)), max.

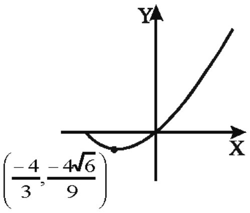
7. (i) horizontal asymptote y = 0; vertical asymptote x = 0, x = 3; decreasing on (–∞, 0), (0,1), (3, ∞); increasing on (1,3); local min. $f(1) = \frac{1}{4}$; concave up on (0, 3), (3, ∞); concave down on (–∞, 0)



(ii) y-intercept = 0, y-intercept = 0 vertical asymptote x = –8; slant asymptote y = x – 8; increasing on (–∞, –16), (0, ∞); decreasing on (–16, –8), (–8, 0); local max. f(–16) = –32; local min. f(0) = 0; concave up on (–8, ∞); concave down on (–∞, –8)



(iii) y-intercept = 0; x-intercept = –2, 0; increasing on $(-\left(\frac{4}{3},\infty\right)$, decreasing on $\left(-2,-\frac{4}{3}\right)$; local min $f\left(-\frac{4}{3}\right) = -4\sqrt{6}/9$; concave up on (–2, ∞)



10. (i) Max. at x = 1, 4; Min. at x = 2
(ii) Max at x = –14/5, 2; Min at x = –2, 1; Neither at 0

11. (i) (a) not applicable (b) not applicable

12. 4, $f(x) = 2x^3 + 3x^2 + x$

13. 2a

17. (a) 0 or 1 (b) 0, 1 or 2
(c) 0, 1, 2, or 3

18. (i) Apply the Lagrange's Theorem to the function ln x on the interval [b, a].
(ii) Apply the Lagrange's Theorem to the function z^p on the interval [y, x].

22. 8/3, 0

23. $\dfrac{a^{\frac{2}{3}}b^{\frac{2}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}$

27. $4\sqrt{3}$ cm from D

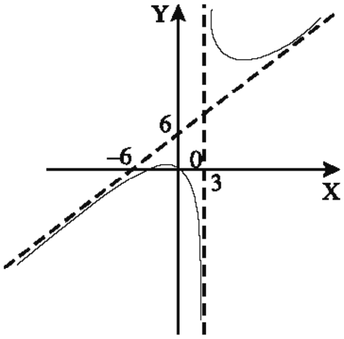
31. (–2, 4), (2, – 4)

TARGET EXERCISES for JEE ADVANCED

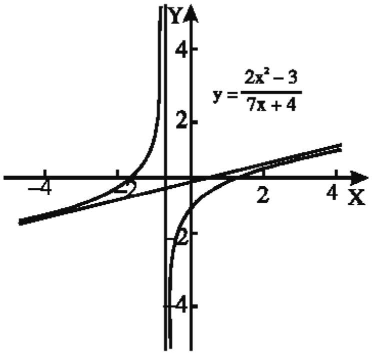
3. If ax + by is a maximum, then a + b(dy/dx) = 0. The relation between x and y gives (2x + y) + (x + 2y)(dy/dx) = 0. Equate the two values of dy/dx.

19. Using the Rolle's Theorem, show that inside the interval $[x_0, x_n]$ there are atleast n roots of the first derivative, n – 1 roots of the second derivative, and so on.

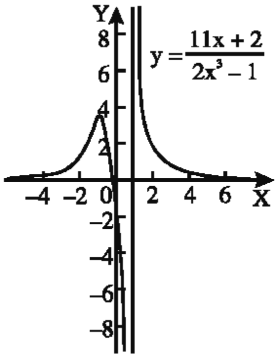
20. (i)



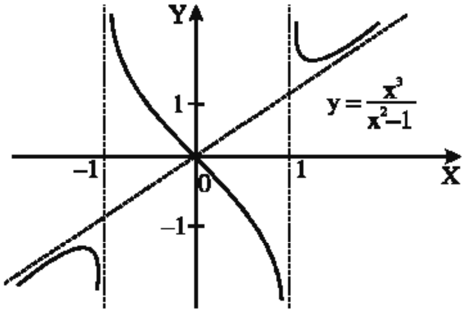
(ii)



(iii)



(iv)



24. This is the condition that y = λ should touch the curve.

26. Consider g(x) = [f(x) – f(a)]/(x – a) for x > a, g(a) = 0. Later consider g(x) = f(x) – f′(a)x.

27. $y = \begin{cases} 16 - 32c + c^2 & \text{if} \quad c \le \frac{1}{2} \\ c^2 & \text{if} \quad c \ge \frac{1}{2} \end{cases}$

28. 256

30. 55

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. (–1, ∞)
2. F
3. A
4. A
5. B
6. C
7. C
8. D
9. B
10. A
11. D
12. A
13. B
14. A
15. D
16. C
17. D
18. A
19. A
20. B
21. BD
22. ABC
23. AB
24. BC
25. ABCD
27. 2
28. e^{π}

30. $\sqrt{c - \frac{1}{4}}$ if $c > \frac{1}{4}$; c if $0 \le c \le \frac{1}{4}$

31. $\frac{ud}{\sqrt{v^2 - u^2}}$

32. (0, 0)

33. $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

35. (0, 2)

36. f is min at $x = \frac{7}{5}$

37. $\frac{10}{3}$ sq. units

39. $\frac{3\sqrt{3r^2}}{4}$

40. 6/(6 + π)

41. $x^3 + x^2 - x + 2$

42. $b \in (-2, -1) \cup [1, \infty)$

43. Max. at $x = \sin^{-1}(1/3)$, $f_{max} = 13/3$; Min at x = π/2 and $f_{min} = 3$

44. 1/3

45. $\left(\frac{4}{3\sqrt{3}}\right)$ sq. units

46. 2 hk

47. min at x = 1/4(b + $\sqrt{b^2-1}$), min at x = 1/4(b – $\sqrt{b^2-1}$)

48. $a = \frac{1}{4}$, $b = \frac{-5}{4}$, c = 2, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

49. $\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$ & $\left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

50. $\left(\pm\frac{a^2}{\sqrt{a^2, b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

51. $\pm\sqrt{3}\,x \pm \sqrt{2}\,y = 5$

55. (2, 1)

57. 6
58. 7
59. 0
60. 1
61. 2
62. 5
63. 9
64. 4

QUESTIONS FROM PREVIOUS YEAR'S (AIEEE/JEE MAIN PAPERS)

1. (c)
2. (d)
3. (c)
4. (c)
5. (d)
6. (d)
7. (a)
8. (b)
9. (d)
10. (b)
11. (d)
12. (c)
13. (c)
14. (a)
15. (a)
16. (c)

HINTS & SOLUTIONS

Objective Exercises Solution

SINGLE CORRECT ANSWER TYPE

1. $f(x)=\begin{cases}2-|(x+2)(x+3)|, & x\neq -2\\ a^2+1, & x=-2\end{cases}$

For maximal at $x=-2 \Rightarrow f(-2)\geq f(-2\pm h)$

$a^2+1\geq 2$

$a^2-1\geq 0$

$\Rightarrow |a|\geq 1$

Hence, A is correct

2. Area of $\Delta=\frac{1}{2}|xy+xy|=|xy|=x(54-2x^2)$

$\frac{dA}{dx}=54-6x^2=0$

$\Rightarrow \quad x^2=9$

$\Rightarrow x=3$

Hence, B is correct

3. $y=1+a^2x-x^3$

$\frac{dy}{dx}=a^2-3x^2=0$

$\Rightarrow x=\pm\frac{a}{\sqrt{3}}$

$\frac{d^2y}{dx^2}=-6x$, For minima, $x=-\frac{a}{\sqrt{3}}(a>0)$

$x=\frac{a}{\sqrt{3}},(a<0)$

Now, $\frac{x^2+x+2}{(x+3)(x+2)}\leq 0 \Rightarrow x\in[-3,-2]$

$\Rightarrow a\in(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$

Hence, D is correct

4. $f(x)=\frac{1+x}{[x]}, \quad x\in[1,3]$

$$f(x)=\begin{cases}1+x, & x\in[1,2)\\ \frac{1+x}{2}, & x\in[2,3)\\ \frac{1+x}{3} & x=3\end{cases}$$

Hence, B is correct

5. $f(x)=\begin{cases}\sqrt{x}, & x\geq 1\\ x^3, & 0\leq x\leq 1\\ \frac{x^3}{3}-4x, & x<0\end{cases}$

$$f'(x)=\begin{cases}\frac{1}{2\sqrt{x}}, & x\geq 1\\ 3x^2, & 0\leq x\leq 1\\ x^2-4, & x<0\end{cases}$$

$f'(x)$ changes its sign when $x=0,-2$

Hence, C is correct

6. $f(x)=\begin{cases}\frac{x^3-2x^2-5x+6}{x-1}, & x\neq 1\\ -6, & x=1\end{cases}$

$=\begin{cases}(x^2-x-6) & ,x\neq 1\\ -6 & ,x=1\end{cases}$

$f(x)$ is continous at $x=1$ & derivable at $x=1$

$f(-2)=0,\quad f(3)=0$

So, Rolle's theorem is applicable

$f'(c)=2c-1=0$

C = 1/2

Hence, D is correct

7. $f(x)=2^{-|x|}(\cos x+\cos\sqrt{3}x)$

$=\begin{cases}2^{-x}(\cos x+\cos\sqrt{3}x), & x\geq 0\\ 2^{x}(\cos x+\cos\sqrt{3}x), & x<0\end{cases}$

As $\cos x+\cos\sqrt{3}x$ can attain maximum only when $\cos x=\cos\sqrt{3}x=1$ & it is possible only when $x=0$

$\Rightarrow$ no of points at which $f(x)$ attain maxima is 1

Hence, A is correct

8. Let $g(x)=x^3+x^2+3x+\sin x$

$g'(x)=3x^2+2x+3+\cos x$

$=3\left(x^2+\frac{2x}{3}+1\right)+\cos x$

$$= 3\left(\left(x+\frac{1}{3}\right)^2 + \frac{8}{9}\right) + \cos x > 0$$

So, $g(x)$ is increasing function $\forall\, x \in R$

Also, $2 < 3 + \sin\left(\frac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$

So, No of points $= 1$

Hence, A is correct

9. Similar to question-6

Hence, D is correct

10. $f(x) = 2x^3 + x^2 - 4x - 2$ is continous and derivable in R

Hence, A is correct

11. $f(\theta) = \dfrac{\left(a^2 - b^2\right)\cos\theta}{a - b\sin\theta}$

$$f'(\theta) = \frac{-(a - b\sin\theta)\left(a^2 - b^2\right)\sin\theta + b\left(a^2 - b^2\right)\cos^2\theta}{(a - b\sin\theta)^2} = 0$$

$-a\sin\theta + b = 0$

$\sin\theta = \dfrac{b}{a}$

Maximum value of $f(\theta) = \dfrac{\left(a^2 - b^2\right)}{\left(a^2 - b^2\right)} a\sqrt{1 - b^2/a^2}$

$$= \sqrt{a^2 - b^2}$$

Hence, B is correct

12. $f(x) = \begin{cases} \sin(x^2 - 3x), & x \le 0 \\ 6x + 5x^2, & x > 0 \end{cases}$

$$f'(x) = \begin{cases} (2x - 3)\cos\left(x^2 - 3x\right), & x \le 0 \\ 6 + 10x, & x > 0 \end{cases}$$

Hence, A is correct

13. $f(x) = \cos(xe^{[x]} + 7x^2 - 3x)$

Greatest value of $\cos\theta$ can occur

When $\theta = 2n\pi, n \in w$

$\Rightarrow$ At $x = 0$, $\quad xe^{[x]} + 7x^2 - 3x = 0$

Hence, B is correct.

14. $f(x) = \left(4\sin^2 x - 1\right)^m \left(e^x - x + 1\right)$

At $x = \dfrac{\pi}{6} + h$, $\quad \sin^2 x = \dfrac{1}{4} + h$

& At $x = \dfrac{\pi}{6} - h$, $\sin^2 x = \dfrac{1}{4} - h$

$$f\left(\frac{\pi}{6} + h\right) = (4h)^n \left(e^{\pi/_6 + h} - \frac{\pi}{6} - h - 1\right)$$

$$f\left(\frac{\pi}{6} - h\right) = (-4h)^n \left(e^{\pi/6 - h} - \frac{\pi}{6} + h - 1\right)$$

for $f\left(\frac{\pi}{6} - h\right) > f\left(\frac{\pi}{6}\right) < f\left(\frac{\pi}{6} + h\right)$

n needs to be even natural number.

$\Rightarrow$ Hence, A is correct

15. $f(x) = \cos\pi x + 10x + 3x^2 + x^3$

$f'(x) = -\pi\sin\pi x + 3x^2 + 6x + 10$

$$= 3((x+1)^2) + 7 - \pi\sin\pi x > 0$$

$f(x)$ is increasing function

Minimum value is at $x = -2$

$f(-2) = 1 - 20 + 12 - 8 = -15$

Hence, B is correct

16. $y = e^{\left(2x^2 - 2x + 1\right)\sin^2 x}$

Now $f(x) = 2x^2 - 2x + 1$ has minima at

$x = \dfrac{1}{2} \quad \& \quad f(1/2) = 1$

So, Minimum value of y occurs when

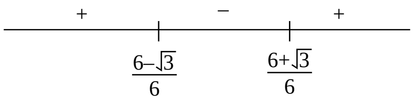
$(2x^2 - 2x + 1)\sin^2 x = 0$

$\Rightarrow y = e^0 = 1$

Hence, C is correct

17. $y = 4x^3 - 12x^2 + 11x - 3$

$\dfrac{dy}{dx} = 12x^2 - 24x + 11 = 0$



$$x = \frac{24 \pm \sqrt{48}}{24} = \frac{6 \pm \sqrt{3}}{6}$$

For $x \in [2, 3]$, y is an increasing function

Maximum value of $f(x)$ is at $x = 3$

$= \log_{10}\left(4(3)^3 - 12(3)^2 + 33 - 3\right)$

$= 1 + \log_{10} 3$

Hence, B is correct

18. $x + \dfrac{a}{x^2} > 2$

use A.M $\geq$ GM

$$\frac{\frac{x}{2} + \frac{x}{2} + \frac{a}{x^2}}{3} \geq \left(\frac{a}{4}\right)^{1/3}$$

$$\Rightarrow 3\left(\frac{a}{4}\right)^{1/3} > 2$$

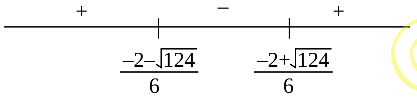
$$\Rightarrow \frac{a}{4} > \frac{8}{27}$$

$$\Rightarrow a > \frac{32}{27}$$

Hence, B is correct

19. $f(x) = x^3 + x^2 - 10x, \qquad -1 \leq x < 0$

$f'(x) = 3x^2 + 2x - 10$



$f(x)$ is decreasing for $-1 \leq x < 0$

So, global maximum is at $x = -1$

Hence, C is correct

20. $y = \dfrac{ab\left(a^2 - b^2\right)\sin 2x}{2\left(a^2\sin^2 x + b^2\cos^2 x\right)} = \dfrac{ab\left(a^2 - b^2\right)\tan x}{a^2\tan^2 x + b^2}$

$y = \dfrac{ab\left(a^2 - b^2\right)}{a^2\tan x + b^2\cot x} \geq \dfrac{ab\left(a^2 - b^2\right)}{2ab}$ (using AM, GM)

$y_{Max} = \dfrac{a^2 - b^2}{2}$

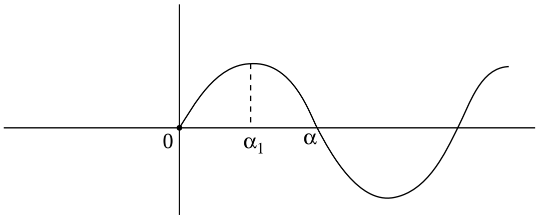
Hence, B is correct

21. $\dfrac{(f(7) - f(2))\left(f^2(7) + f^2(2) + f(2)/f(7)\right.}{3}$

$\Rightarrow \dfrac{f^3(7) - f^3(2)}{3} = \dfrac{5}{3}\dfrac{d}{dc}f^3(c) = 5f^2(c)f'(c)$

Hence, C is correct

22. $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex$



$f'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$

$f''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$

Hence, D is correct

23. Let $f(x) = e^x \sin x - 1$

Applying Rolle's Theorem,

$f'(x) = e^x\cos x + e^x \sin x$

$f'(x) = e^x\cos x + 1 \quad$ for $x \in \{x_1, x_2\}$

Where x_1, x_2 are roots of $e^x \sin x - 1 = 0$

Hence A is correct

24. $y = f(x) + a\ x + b$

$\dfrac{dy}{dx} = f'(x) + a$

As $f'(0) = 0 \Rightarrow a = 0$

$\& \dfrac{d^2y}{dx^2} = f''(x) > 0$

Hence, B is correct

25. As $f''(x)$ is a quadratic polynomial $\& \neq 0$

$\Rightarrow$ $f''(x)$ is always +ve or −ve

$\Rightarrow f'(x)$ will be monotonic and will have only one root.

Hence, A is correct

26. $h(x) = f^2(x) + \left(f'(x)\right)^2$

$h'(x) = 2\left[f(x) + f''(x)\right]\ f'(x)$

$= -2xg(x)\left(f'(x)\right)^2$

As $h'(0) = 0,\ h'\left(0^+\right) < 0,\quad h'\left(0^-\right) > 0$

Hence, A is correct.

27. $y = \cos^2\cos\theta + \sin^2\sin\theta$

$y' = (2\cos\cos\theta)(+\sin\cos\theta)\sin\theta$

$+(2\sin\sin\theta)(\cos\sin\theta)\cos\theta = 0$

$\Rightarrow \theta = \dfrac{\pi}{2}, 0$

As $\cos^2 1 < \sin^2 1$

$\Rightarrow$ Max value of $\cos^2\cos\theta + \sin^2\sin\theta$ will be $1+\sin^2 1$

Hence, B is correct

28. $f(x) = \dfrac{t+3x-x^2}{x-4}$

$$f'(x) = \frac{(x-4)(3-2x)-\left(t+3x-x^2\right)}{(x-4)^2}$$

$$= \frac{-x^2+8x-(12+t)}{(x-4)^2} = 0$$

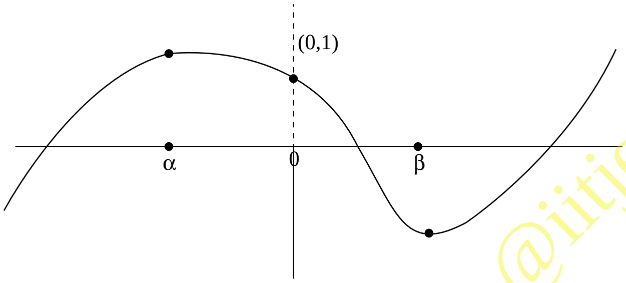
$\Rightarrow 64-4(12+t) > 0$

$\Rightarrow \quad 4-t > 0$

$\Rightarrow t < 4$

Hence C is correct

29. $f(x) = ax^3+bx^2+cx+1$



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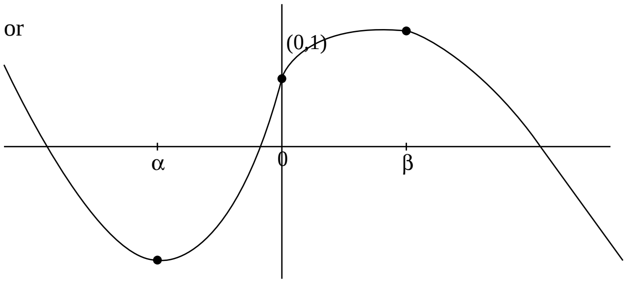


Hence, D is correct

30. Let v be volume

$$v = \frac{1}{3}\pi r^2 h + \pi r^2 y$$

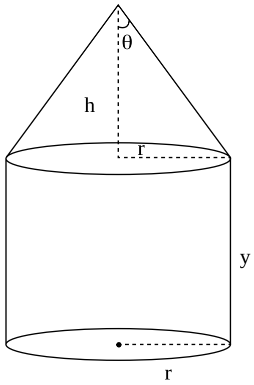
$$v = \pi r^2\left(\frac{n}{3}+y\right) \qquad \text{(I)}$$

$$\& \tan\theta = \frac{r}{h} \qquad \text{(II)}$$

Amount of canvas $k = 2\pi ry + \pi r\left(\sqrt{h^2+r^2}\right)$

$$\Rightarrow k = 2\pi r\left(\left(\frac{v}{\pi r^2}-\frac{h}{3}\right)\right)+\pi r\sqrt{h^2+r^2}$$

$\Rightarrow k = a - bh + c\sqrt{h^2+r^2}$, Where, a, b, c are constant



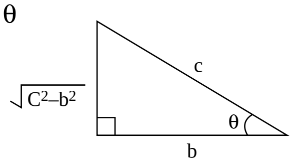
$$\frac{dk}{dh} = -b + \frac{hc}{\sqrt{h^2+r^2}} = 0$$

$$\Rightarrow \quad b^2\left(h^2+r^2\right) = h^2c^2$$

$$\Rightarrow \quad h^2 = \frac{b^2r^2}{c^2-b^2}$$

$$\Rightarrow \quad h = \frac{br}{\sqrt{c^2-b^2}}$$

$$\Rightarrow \quad \frac{r}{h} = \frac{\sqrt{c^2-b^2}}{b} = \tan\theta$$



$$\text{so,} \cos\theta = \frac{b}{c} = \frac{2\pi r}{3\pi r} = \frac{2}{3}$$

Hence, A is correct

31. Material 'm' $= \left(\pi r^2\right)\dfrac{5\omega}{4} + (2\pi r)h \times w$

Where 'w' is width of cylinder

As $v = \pi r^2 h$, where v is volume of cylinder

$$m = \frac{5}{4}\pi r^2 w + 2\pi r w\left(\frac{v}{\pi r^2}\right)$$

$$\frac{dm}{dr} = \omega\left(-\frac{2v}{r^2}+\frac{5\pi r}{2}\right) = 0$$

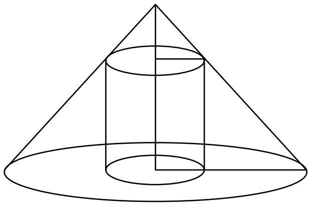
$$\Rightarrow 5\pi r^3 = \ 4v = 4\left(\pi r^2 h\right)$$

$\Rightarrow r : h = 4 : 5$

Hence, C is correct

32. Let r be radius of cone &
y be height

Let r_1 be radius of cylinder
& h be height



Now, $p = \dfrac{1}{3}\pi r^2 y$

As Volume of cylinder $v = \pi r_1^2 h$ .... (1)

& $\dfrac{r}{y} = \dfrac{r_1}{(y-h)}$ .... (11)

Using (1) & (11)

$$v = \pi\left(\frac{r}{y}\right)^2 (y-h)^2 h$$

v will be maximum $h = \dfrac{y}{3}$

$$\Rightarrow v = \pi\frac{r^2}{y^2}\times\frac{y}{3}\times\frac{4y^2}{9} = \frac{4\pi}{27}r^2y = q$$

$$\Rightarrow p : q = 9 : 4$$

Hence, A is correct

33. Let say $x,\ a - x$ are two length

side of square $\rightarrow \dfrac{a-x}{4}$

radius of circle $\Rightarrow \dfrac{x}{2\pi}$

Area 'A' $= \dfrac{(a-x)^2}{16} + \pi\dfrac{x^2}{4\pi^2}$

$$\frac{dA}{dx} = \frac{-1}{8}(a-x) + \frac{x}{2\pi} = 0$$

$$\Rightarrow \quad \frac{a}{8} = \frac{x}{2\pi} + \frac{x}{8}$$

$$\Rightarrow \quad \pi a = 4x + \pi x$$

$$\Rightarrow \quad x = \frac{\pi a}{4+\pi}$$

Now $a - x = \dfrac{4a}{4+\pi}$

$$A = \frac{a^2}{(4+\pi)^2} + \frac{1}{4}\frac{\pi a^2}{(4+\pi)^2}$$

$$= \frac{a^2}{4(4+\pi)}$$

Hence, D is correct

34. $t_n = \dfrac{n^2}{n^4 + 300}$

$$= \frac{1}{n^2 + \dfrac{300}{n^2}}$$

Use AM $\geq$ GM

$$\frac{n^2 + \dfrac{300}{n^2}}{2} \geq \sqrt{300}$$

$$n^2 + \frac{300}{n^2} \geq 20\sqrt{3}$$

Maximum value of n fulfilling the above in equality
$\Rightarrow n = 4$

Hence, B is correct

35. $f(x) = \sin(\ln(\cos x + c))$

$$f'(x) = \frac{\cos(\ln(\cos x + c))}{\cos x + c} \times -\sin x$$

36. Let $f(x) = \cos x$

$f'(x) = -\sin x$

Using LMVT

$$f'(c) = \frac{f(n\alpha) - f(n\beta)}{n\alpha - n\beta}$$

$$-\sin c = \frac{\cos n\alpha - \cos nB}{n(\alpha - \beta)}$$

$$\Rightarrow \left|\frac{\cos n\alpha - \cos n\beta}{n(\alpha-\beta)}\right| < 1$$

Hence, B is correct

37. $y = \sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 2 + \sin^2 x}$

Rationalising the expression

$$y = \frac{2}{\sqrt{\sin^2 x + 2a^2} + \sqrt{2a^2 - 2 + \sin^2 x}}$$

For max value, Denominator should be minimum,
$\Rightarrow \sin x = 0$

$$y = \frac{\sqrt{2}}{a + \sqrt{a^2 - 1}}$$

$\Rightarrow a = 1$ (For y to be maximum)

$$y_{\min} = \sqrt{2}$$

Hence, A is correct

38. i) $f'(C) = \dfrac{\left(Ab^2 + Bb + C\right) - \left(Aa^2 + Ba + C\right)}{b - a}$

$= A(b + a) + B$

$= 2A\dfrac{(b+a)}{2} + B$

So, it is true statement

ii) $f''(c) = \dfrac{3+2}{2} = \dfrac{5}{2}$

$\Rightarrow$ for some $C \in (4, 6)$, there exist

$f''(C) > 0$

So, it is true statement

iii) $f'(x) < 1$

It is a false statement
Hence, D is correct

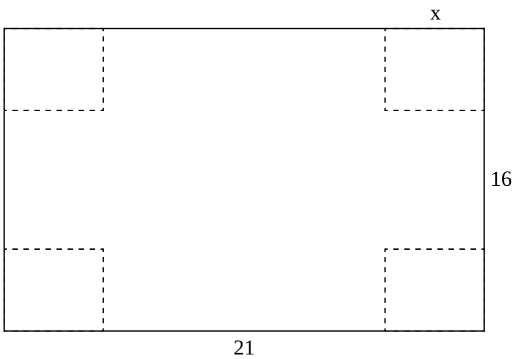
39. Integrating $ax^2 + cx + d$, we get

$$f(x) = \frac{ax^3}{3} + \frac{cx^2}{2} + dx + b$$

$$f(0) = b$$

$$f(1) = \frac{2a + 3c + 6d + 6b}{6}$$

40.



$\text{Volume } = (21-2x)(16-2x)x$

$$\frac{dv}{dx} = -2(16-2x)x - 2(21-2x)x + (21-2x)(16-2x)$$

$$\Rightarrow \quad -32x + 4x^2 - 42x + 4x^2 + 4x^2 - 74x + 336 = 0$$

$$\Rightarrow 12x^2 - 148x + 336 = 0$$

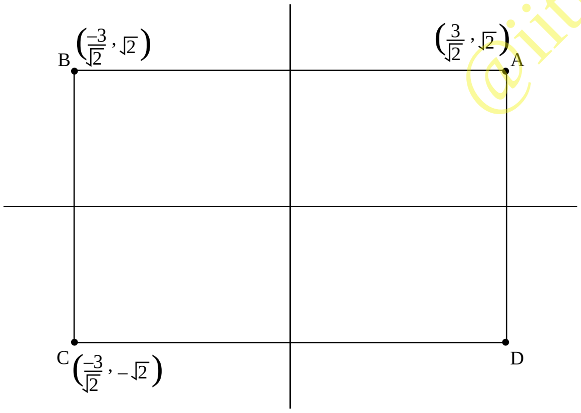
$$\Rightarrow 3x^2 - 37x + 84 = 0$$

$$\Rightarrow (x-3)(3x-28) = 0$$

$$x = 3 \text{ or } \frac{28}{3}$$

Hence, C is correct

41.


B $\left(\frac{-3}{\sqrt{2}}, \sqrt{2}\right)$
$\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ A
C $\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}\right)$
D

Area is maximum if ADC is right angle

$$\Rightarrow \theta = \frac{7\pi}{4}$$

Hence, B is correct

42. As $a^2 + b^2 = k^2$

$$\text{Now} \Rightarrow y = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$$

$$y = a^2 + b^2 + 4 + \frac{a^2 + b^2}{a^2 b^2}$$

$$y = k^2 + 4 + \frac{k^2}{a^2 b^2} \text{ .... } (1)$$

$$\text{As } \frac{a^2 + b^2}{2} \geq ab$$

$$\frac{k^2}{2} \geq ab$$

$$\frac{k^4}{4} \geq a^2 b^2$$

Using in (1)

$$y > k^2 + 4 + \frac{4}{k^2}$$

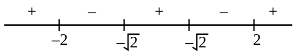
Hence, D is correct

43. Similar to Q. 8

Hence, A is correct

44. $f(x) = \left|\dfrac{x^2 - 2}{x^2 - 4}\right|$

Sign scheme of $\dfrac{x^2 - 2}{x^2 - 4} \Rightarrow$



$$f(x) = \begin{cases} 1 + \dfrac{2}{x^2 - 4} & \dfrac{x^2 - 2}{x^2 - 4} \geq 0 \\ -\left(1 + \dfrac{2}{x^2 - 4}\right), & \dfrac{x^2 - 2}{x^2 - 4} < 0 \end{cases}$$

Hence, D is correct

45. Conceptual as $[f(a+h)] = [f(a-h)] = 2$

Hence, D is correct

46. Between two maxima, there is one minima & otherwise unless there is a pt of inflection

Hence, D is correct

47. $p(x) = 5^{\ln(x+3)}$

As $\ln(x+3)$ is continuous & derivable

in $[0,1]$, LMUT is applicable on $p(x)$

Hence, C is correct

48. $f(x) = x - \log x + \displaystyle\int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz$

$$f'(x) = 1 - \frac{1}{x} + \frac{1}{x} - 2 - 2\cos 4x = 0$$

$$\Rightarrow \cos 4x = \frac{-1}{2}$$

$$4x = \frac{2\pi}{3} \quad \text{or} \quad 2n\pi + \frac{2\pi}{3}$$

$$x = \frac{\pi}{6} \quad \text{or} \quad \frac{n\pi}{2} + \frac{\pi}{6}$$

Hence, A is correct

49. Put $\sin^2\theta / n = x$

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$= 1 - \frac{2}{(x + 1/x) + 1}$$

As $x + \frac{1}{x} \geq 2$ (as $x > 0$)

$$y \in \left[\frac{1}{3}, 1\right)$$

Hence, C is correct.

50. $f(x) = x - 1, \quad 1 < x < 2$

$$g(x) = x - 1 + b\sin\frac{\pi}{2}x, \quad 1 \leq x \leq 2.$$

g(x) is continous & differentiable.

$g(1) = 1,\ g(2) = 1$ for $b = 1$

Hence, C is correct.

51. $$f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$$

$$f'(x) = \begin{cases} 6x + 12, & -1 \leq x \leq 2 \\ -1, & 2 < x \leq 3 \end{cases}$$

Hence, A, B, C, D is correct.

52. A) Use Rolle's Theorem

B) $f''(x) = 0$ is not a necessary condition for point of inflection

C) Conceptual.

D) It is a monotonic function

Hence, A, C, D is correct.

53. $f(x) = (x-1)^4 (x-2)^n$

$f(1) = 0, \quad f(1+h) = h^4 (h-1)^n,\ f(1-h) = h^4(-h-1)^n$

If n is odd,
$f(1-h) < f(1) > f(1+h)$

If n is even,
$f(1-h) > f(1) < f(1+h)$

$f(2) = 0,\ f(2+h) = (1+h)^n (h)^n,\ f(2-h) = (1-h)^4 (-h)^n$

If n is even,
$f(2-h) > f(2) < f(2+h)$

Hence, A, C, D is correct.

54. A) $f(x) = ax^3 + b$

$$f'(x) = 3ax^2 + 1 > 0 \quad \forall x \in (-1, 1)$$

$$\Rightarrow a > 0$$

B) Conceptual

C) If $f'(x) > 0$, f increases. It can be with or without bound.

D) Conceptual

Hence, A, B, D is correct.

55. A) $f(x) = x^5 + 2x - 3$

$f'(x) = 5x^4 + 2 > 0$

$f(0) = -3 \quad \& \quad f(1) = 0$

B) $f(x) = e^{-x} - x + 1$

$f(x) = -e^{-x} - 1 < 0$

$f(1) = \frac{1}{e} > 0 \ \&\ f(2) = \frac{1}{e^2} - 1 < 0$

C) $f(x) = x\ln x - 3$

$f'(x) = 1 + \ln x > 0$

$f(2) = 2\ln 2 - 3 < 0$

$f(4) = 8\ln 2 - 3 > 0$

D) $f(x) = \sin x - 3x + 1$

$f'(x) = \cos x - 3 < 0$

$f(-1) = -\sin 1 + 3 > 0$

$f(1) = \sin 1 - 3 < 0$

Hence, A, B, C, D is correct

56. Let $g(x) = \phi'(x) f(x)$

$g(\alpha) = 0 \quad \& \quad g(\beta) = 0$

Use Rolle's Theorem, $g'(x) = 0$ for atleast one value of $x \in (\alpha, \beta)$

$\&\ g'(x) = \phi'(x) f'(x) + \phi''(x) f(x).$

Hence, B is correct

57. Upon solving $f(x) = 1 - x^4$

$$\Rightarrow f\left(\frac{1}{x}\right) = \frac{x^4 - 1}{x^4}$$

$$\Rightarrow f(x) \cdot f\left(\frac{1}{x}\right) < \frac{-1}{x^4} \leq 0$$

$f'(x) = -4x^3$

As $|f(x)| = k - 2$ as $k \in (2, 3)$

This eqn has 4 distinct roots

Hence B, C is correct.

58. $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c = 0$

Roots of $f'(x) = 0$ are $'\alpha'$ & $'\beta'$.

A) If $f(\alpha)$ & $f(\beta)$ are of opposite signs,

$\Rightarrow$ all roots are real.

B) $f(\alpha)$ & $f(\beta)$ are of same signs.

$\Rightarrow$ all roots may or may not be real.

C) It is not necessary that $\alpha < \beta$.

Sign of 'a' will decide it.

D) If $a > 0$, $f(x)$ is always decreasing where $\beta < \alpha$

Hence, A, D is correct.

59. A) $\tan\theta = 2$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{5}$$

B) $f'(x) \geq 0$ in $(0,3)$

C) It is not a point of entremum

D) No of critical points are two.

Hence, A, B, D is correct.

60. A) $f(x) = |x-2|$ is not differentiable at $x = 2$

& $f'(x) = \pm 1$ at $x = 1$, $f'(x) = 1$ at $x = 4$

B) $f(x) = 1 + |x-1|$ is not differentiable at $x = 1$

& $f'(x) = \pm 1$ at $x = 3$, $f'(x) = -1$ at 0

$f'(c) = \frac{2}{3}$ which is not possible.

C) $f(x) = [x]$ is not derivable at $x = -1, 0, 1$

$f'(c) = \frac{2}{2} = 1$ which is not possible.

D) $f(x) = 3x^{2/3}$

$f'(x) = 2x^{-1/3}$ which is not differentiable at $x = 0$.

$f(1) = 3$, $f(-1) = 3$

$f'(c) = 0$ which is not possible.

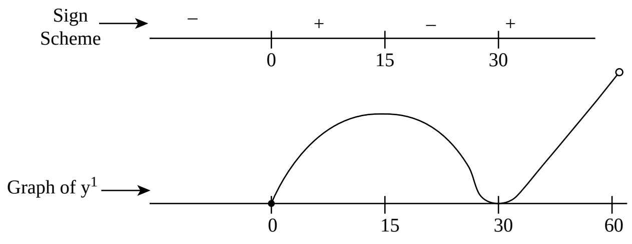
Hence, A, B, C, D is correct

61. $y_1 = x^2(30+y)^2$

$y_1 = x^2(30-x)^2$

$y_1' = 2x(30-x)^2 - 2x^2(30-x)$

$= 2x(30-x)(30-2x)$



Hence, A, C, D is correct.

62. $f(x) = \begin{cases} x^{\alpha}\sin^2\frac{1}{nx}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For continuity, $\alpha > 0$

Now $= f(0) = 0$

$\Rightarrow f(1) = 0$

$\Rightarrow f(1) = \sin^2\frac{1}{n} = 0$

$\Rightarrow$ Greatest value of $n = \frac{1}{\pi}$ & least value of

$n = -\frac{1}{\pi}$

Hence, A, C is correct

63. $f(x) = 3 + 2(a+1)x + \left(a^2+1\right)x^2 - x^3$

$$f'(x) = 2\left(a^2+1\right)x + 2(a+1) - 3x^2$$

$$= -\left(3x^2 - 2\left(a^2+1\right)x - 2(a+1)\right)$$

Now, $g(x) = 3x^2 - 2\left(a^2+1\right)x - 2(a+1)$

& $g(2) < 0$

$12 - 4a^2 - 4 - 2a - 2 < 0$

$4a^2 + 2a - 6 > 0$

$2a^2 + a - 3 > 0$

$(a-1)(2a+3) > 0$

$a \in \left(-\infty, -\frac{3}{2}\right) \cup (1, \infty)$

Hence, A, D is correct

64. $y = \frac{e^x - e^{-x}}{2}$

$2ye^x = \left(e^x\right)^2 - 1$

$\left(e^x\right)^2 - (2y)e^x - 1 = 0$

$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

$\Rightarrow y = g(x) = \ln\left(x + \sqrt{x^2+1}\right)$

Now, $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

$\frac{dy}{dx}$ can never tends to infinity, hence no Vertical tangent

Slope at (0,0)=1

$\Rightarrow$ Eqn of tangent at $(0,0) \Rightarrow y = x$

Hence, B, C, D is correct

65. $f(x) = 4\sin^2 x + 3\cos^2 x - 24\sin\frac{x}{2} - 24\cos\frac{x}{2}$

$= 3 + \sin^2 x - 24\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)$

$f'(x) = \sin 2x - 12\left[\cos\frac{x}{2} - \sin\frac{x}{2}\right] = 0$

$= 2\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\sin x\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - 6\right] = 0$

$\Rightarrow \cos\frac{x}{2} = \sin\frac{x}{2}$

$\Rightarrow x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = 4(1 - 6\sqrt{2})$

$f(0) = -21$

Hence, B, C is correct

66. $f(x) = (x-3)^9 + \left(x - 3^2\right)^9 + \cdots + \left(x - 3^9\right)^9$

$f'(x) = 9\left((x-3)^8 + \left(x - 3^2\right)^8 + \cdots + \left(x - 3^9\right)^8\right) > 0$

$f(x)$ is always increasing

& $f(x) = 0$ has one real & eight imaginary roots.

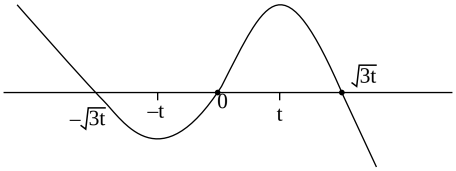
Hence, A, B is correct

67. $f(x) = -\frac{1}{3}x^3 + t^2 x \quad = x\left(t^2 - \frac{x^2}{3}\right)$

$f'(x) = -x^2 + t^2 = 0$

$x = \pm t$

$f''(x) = -2x$



$f(0) = 0, \quad f(1) = t^2 - \frac{1}{3}$

If $t^2 \ge \frac{1}{3}, \quad m(t) = 0$

& If $t^2 < \frac{1}{3}, \quad m(t) = t^2 - \frac{1}{3}$

Hence, A, C is correct

68. As $f(1) > f(-1)$ & $f(x)$ is continous

$\Rightarrow$ there exist atleast an $x \in [-1, 1]$ where

$f(x)$ is increasing $\Rightarrow f'(x) > 0$

As $f'(y) \le 1$

Integrating $\int_{-1}^{1} f'(y)\,dy \le \int_{-1}^{1} 1\,dy$

$= f(1) \le f'(1) + 2$

Hence, A, C is correct

69. At x = 2, $\frac{dy}{dx}$ is changing from negative to positive.

So, there is a local Minima.

As $\frac{dy}{dx}$ is zero thrice in its domain. So, f can have atmost 4 roots.

Hence, B, D is correct

70. $g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)\left(x^2 - 1\right) + \frac{f(1)}{2}x^2(x+1)$

$-f'(0)x(x-1)(x+1)$

$-f^1(0)x(x-1)(x+1)$

Put $x = 0$

$g(0) = f(0)$

Put $x = 1$

$g(1) = f(1)$

Put $x = -1$

$g(-1) = f(-1)$

Now take $h(x) = f(x) - g(x)$

Using Rolles Theorem in (–1, 0)

There exists $x \in (-1, 0)$ such that $f^1(x) = g^1(x)$

71. A: Use Rolles Theorem as $f(1) = f(3) = 0$

R: Conceptual

Hence, B is correct

72. $f(x) = -x^2 + 4x + 1 + \sin^{-1}\frac{x}{2}$

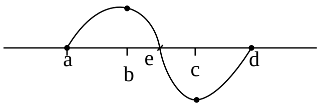
$f'(x) = -2x + 4 + \frac{1}{2} \times \frac{1}{\sqrt{1 - \frac{x^2}{4}}}$

$= -2x + 4 + \frac{1}{\sqrt{4 - x^2}} > 0$

Least value of $f(x) = f(-1) = -4 - \frac{\pi}{6}$

Hence, B is correct

73.



$g(x) = (f'(x))^2 + f(x)f''(x)$

$g(x)$ is derivative of $f(x)f'(x)$.

Now, For $f(x)\ f'(x)$, Rolle's Theorem can be applied in [a, b], [b, e], [e, c], [c, d]

Hence, A is correct.

74. $f(x) = 5 - 4(x-2)^{2/3}$

$f'(x) = -\frac{8}{3} \times \frac{1}{(x-2)^{1/3}}$

$f'(2)$ is not defined

Now, $f(x) = 5 - 4(x-2)^{2/3}$

As $(x-2)^{2/3}$ is always positive

Hence, greatest value of $f(x)=5$

Hence, D is correct

75. $f(x) = \frac{x^3}{3} + \frac{ax^2}{2} + x + 5$

$f'(x) = x^2 + ax + 1$ $\qquad (\because D = a^2 - 4 > 0,\ \text{if } a < -2)$

roots of $x^2 + ax + 1 = 0$

$x = -\frac{a \pm \sqrt{a^2-4}}{2}$

both roots are positive for $a < -2$

$f''(x) = 2x + a < 0$

Hence, A is correct

76. $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$

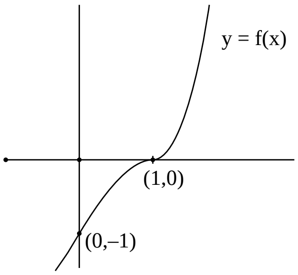
$f'(x) = 12(x^3) - 12x^2 + 12x + a$

$f''(x) = 12(3x^2 - 2x + 1) > 0$

$f'(x)$ is zero at only one point

Hence, A is correct

77. $f(x) = \begin{cases} (x-1)^2, & x \in [1,2] \\ -(x-1)^2, & x \in [0,1) \end{cases}$



$f(x)$ is continous & differentiable in $x \in (0,2)$

Hence D is correct

78. Let $g(x) = ax^4 + bx^3 + cx^2 + dx + e$

$g(0) = e$ & $g(3)=3(27a+9b+3c+d)+e$

$= e$

Using Rolle's Theorem, there is atleast one root of $g'(x) = 0$ in (0, 3)

Hence, A is correct

79. $(x-5)^2 + \left(\sqrt{-3+4x-x^2} - (-4)\right)^2 = y^2$

y is the distance between (5, –4) & a point on

$y^2 = -3 + 4x - x^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = 1$

$\Rightarrow (x-2)^2 + y^2 = 1$

$y_{max} = \sqrt{(5-2)^2 + (4)^2} + 1$

$y_{max} = 6$

Hence, A is correct

80. If Rolle's Theorem is not applicable

& $f(2) = f(-2)$

$\Rightarrow$ $f(x)$ is not derivable in (–2, 2)

$\Rightarrow g(x)$ will have atleast one root in (–2, 2)

Hence, C is correct

Comprehension – 1:

$g(x) = x^3 + g''(1)x^2 + (3g'(1) - g''(1) - 1)x + 3g'(1)$

Put $x = 1$

$g(1) = 6g'(1)$ $\qquad$ (I)

Now,

$g'(x) = 3x^2 + 2g''(1)x + 3g'(1) - g''(1) - 1$

Put $x = 1$

$g''(1) + 2g'(1) + 2 = 0$ $\qquad$ (II)

$g''(x) = 6x + 2g''(1)$

Put $x = 1$

$g''(1) = -6$

Using (II) $\rightarrow g'(1) = 2$

Using (I) $\rightarrow g(1) = 12$

$\Rightarrow g(x) = x^3 - 6x^2 + 11x + 6$

$\Rightarrow f(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$

$= x^2\left(x^2 + \frac{1}{x^2} - 6\left(x + \frac{1}{x}\right) + 11\right)$

$$= x^2\left(\left(x+\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)+9\right)$$

$$= x^2\left(x+\frac{1}{x}-3\right)^2$$

$$\Rightarrow h(x)=x\left(x+\frac{1}{x}-3\right)=x^2-3x+1$$

81. $f(x)=(x^2-3x+1)^2$

$f'(x)=2\left(x^2-3x+1\right)(2x-3)=0$

$$\Rightarrow x=\frac{3}{2},\frac{3\pm\sqrt{5}}{2}$$

At $x=\frac{3}{2}$, sign of $f'(x)$ changes + to – so, Local Maxima

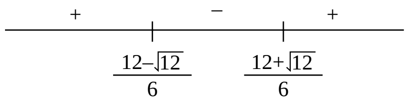
At $x=\frac{3+\sqrt{5}}{2}$, sign of $f'(x)$ changes – to + so, Local Minima

At $x=\frac{3-\sqrt{5}}{2}$, sign of $f'(x)$ changes – to + so, Local Minima.

Hence, C is correct

82. $g(x)=x^3-6x^2+11x+6$

$g'(x)=3x^2-12x+11=0$



$$=2-\frac{1}{\sqrt{3}} \qquad\qquad =2+\frac{1}{\sqrt{3}}$$

Using LMUT,

$g(0)=6,\ g(4)=18$

$\Rightarrow g'(c)=3$

$\Rightarrow 3c^2-12c+11=3$

$\Rightarrow 3c^2-12c+8=0$

$$\Rightarrow \quad C=\frac{12\pm\sqrt{144-96}}{6}$$

$$C=\frac{12\pm\sqrt{48}}{6}$$

Hence, D is correct

83. $h(x)=x^2-3x+1$

$h'(x)=2x-3$

$h''(x)=2$

Hence, C is correct

Comprehension – 2:

$f(x)=4x^2-4ax+a^2-2a+2$

Vertex of parabola is $\left(\frac{9}{2},2-2a\right)$

Case-1 If $0<\frac{9}{2}<2$

$\Rightarrow f(x)$ will attain minimum at $x=\frac{a}{2}$

$\Rightarrow f\left(\frac{9}{2}\right)=3$

or $3=-2a+2$

$\Rightarrow a=\frac{-1}{2}\Rightarrow$ Not possible

Case-2 If $\frac{9}{2}<2$

$f(x)$ attain a global minimum at $x=0$

$\Rightarrow f(0)=3$

$\Rightarrow 3=a^2-2a+2$

$\Rightarrow a=1+\sqrt{3}$

$\Rightarrow a=1+\sqrt{3}$

Case-3 If $\frac{a}{2}\geq 2$

$\Rightarrow f(x)$ attain a minimum at x=2

$\Rightarrow \quad f(2)=3$

$\Rightarrow \quad 3=16-8a+a^2-2a+2$

$\Rightarrow a=5\pm\sqrt{10}$

$\Rightarrow a=5+\sqrt{10}$

84. Hence, B is correct.

85. Hence, D is correct.

86. $f(x)=4x^2-4ax+a^2-2a+2$ is monotonic in $[0,2]$

$f'(x)=8x-4a=0\Rightarrow x=\frac{a}{2}$

If $\frac{a}{2}\in[0,2]$

$\Rightarrow a\in[0,4]$

$\Rightarrow$ For monontonic $a\in(-\infty,0)\cup[4,\infty)$

Hence, A is correct

Comprehension – 3:

Since two point of inflection occur at $x=1$ & $x=0$

$P''(1)=P''(0)=0$

$$\therefore p''(x) = a(x^2 - x)$$

$$p'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + b$$

Also, $\left(\frac{dy}{dx}\right)_{n=0} = \tan^{-1}1 = \left(\frac{dy}{dx}\right)_{x=1}$

$$\Rightarrow p'(0) = 1 \Rightarrow b = 1$$

$$p'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1$$

$$p(x) = a\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + c$$

As $p(-1) = 0$

$$a\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + c = 1$$

$$\Rightarrow \frac{a}{4} + c = 2 \qquad \text{(I)}$$

As $p(1) = 2$

$$a\left(\frac{1}{12} - \frac{1}{6}\right) + 1 + c = 1$$

$$\Rightarrow -\frac{9}{12} + c = 0 \qquad \text{(II)}$$

Solving (I) & (II)

$$a = 6, \quad c = \frac{1}{2}$$

$$\Rightarrow p(x) = 6\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + \frac{1}{2}$$

$$= \frac{x^4}{2} - x^3 + x + \frac{1}{2}$$

87. $\int_{-2}^{2}\left(\frac{x^4}{2} - x^3 + x + \frac{1}{2}\right)dx = 2\int_0^2 \frac{x^4}{2} + \frac{1}{2}dx.$

$$= \int_0^2 \left(x^4 + 1\right)dx$$

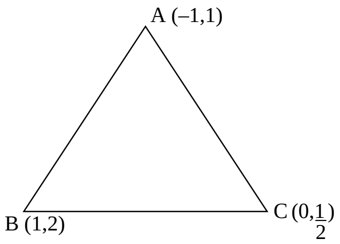
$$= \frac{x^5}{5} + x\Big|_0^2 = \frac{42}{5}$$

so, $k = 42$

88. $P(-1) = 1$

Hence C is correct

89.



$$A_2(\Delta ABC) = \frac{1}{2}\left[\frac{1}{2} - 1\left(-\frac{3}{2}\right)\right] = 1$$

Comprehension – 4:

$$f(x) = x^3 - 3(7-a)x^2 - 3\left(9 - a^2\right)x + 2$$

$$f'(x) = 3x^2 - 6(7-a)x - 3\left(9 - a^2\right)$$

For real root, $D \ge 0$

$$\Rightarrow 49 + a^2 - 14a + 9 - a^2 \ge 0$$

$$\Rightarrow a^2 < \frac{58}{14}$$

90. When point of Minima is negative, point of maxima is also negative.

$\Rightarrow f'(x) = 0$ has both roots negative

Sum of roots $= 2(7-a) < 0$ or $a > 7$

which is not possible as $a \le \frac{58}{14}$

Hence, A is correct

91. When point of Maxima is +ve, point of Minima is also positive

$\Rightarrow f'(x) = 0$ has both roots positive

Sum of roots $= 2(7-a) > 0 \Rightarrow a < 7$

Product of roots $= -(9 - a^2) > 0$

$\Rightarrow a \in (-\infty, -3) \cup (3, \infty)$

$\& \, a^2 < \frac{58}{14}$

Using all results $\Rightarrow a \in \left(\frac{-58}{14}, -3\right) \cup \left(3, \frac{58}{14}\right)$

Hence, D is correct

92. For pt of entrema of opp sign, $f'(x) = 0$ has roots of opp sign.

$a \in \left(-\infty, -\frac{58}{14}\right) \cup (-3, 3)$

Hence, B is correct

Comprehension – 5:

$$f(x) = e^{(p+1)x} - e^x$$

93. $f'(x)=(p+1)e^{(p+1)x}-e^{x}=0$

$$\Rightarrow (p+1)\frac{e^{(p+1)x}}{e^{x}}=1$$

$$\Rightarrow e^{px}=\frac{1}{p+1}$$

$$\Rightarrow px=-\ln(p+1)$$

$$\Rightarrow x=\frac{-\ln(p+1)}{p}$$

Hence, A is correct

94. $g(t)=\int_{t}^{t+1} f(x)e^{t-x}dx$

$$g(t)=e^{t}\int_{t}^{t+1} f(x)e^{-x}dx$$

$$g'(t)=e^{t}\int_{t}^{t+1} f(x)e^{-x}dx+e^{t}f(t+1)e^{-(t+1)}-f(t)$$

$$=e^{t}\int_{t}^{t+1}\left(e^{px}-1\right)dx+e^{t}\left(e^{p(t+1)}-1\right)-e^{(p+1)t}+e^{t}$$

$$=e^{t}\left[\frac{e^{p(t+1)}}{p}-\frac{e^{pt}}{p}\right]-e^{t}+e^{t}\left[e^{p(t+1)}\right]\ -e^{(p+1)t}$$

For $g'(t)=0$

$$e^{p(t+1)}-e^{pt}-p+pe^{p(t+1)}-pe^{(p+1)t}=0$$

$$e-1+pe^{p}-pe^{t}=pe^{-pt}$$

$$\frac{e-1+pe^{p}}{p}=e^{t}+e^{-pt}$$

Match the following

95. A) Put $x=\cos\theta,\quad y=\sin\theta$

$\Rightarrow x+y=\cos\theta+\sin\theta$ whose Minimum value is $-\sqrt{2}$.

B) $y=a\cos x-\frac{1}{3}\cos 3x$

$$\frac{dy}{dx}=\sin 3x-a\sin x$$

$$\frac{d^2y}{dx^2}=3\cos 3x-a\cos x$$

Now Maximum value occurs at $x=\frac{\pi}{6}$

$$\frac{dy}{dx}=1-\frac{a}{2}=0\Rightarrow a=2$$

C) $f(x)=x-2\sin x$

$$f'(x)=1-2\cos x>0$$

$$\Rightarrow \cos x<\frac{1}{2}$$

$$\Rightarrow x\in\left[\frac{\pi}{3},\frac{5\pi}{3}\right]$$

$$\Rightarrow a+b=2$$

D) $y=-e^{-x/2}$

$$\frac{dy}{dx}=\frac{1}{2}e^{-x/2}$$

eqn of tangent $\Rightarrow\quad y-y_1=\frac{1}{2}\cdot\left(e^{-x_1/2}\right)(x-x_1)$

Point of tangency $\rightarrow(0,-1)$

$$\Rightarrow\quad y+1=\frac{1}{2}x$$

$$\Rightarrow \frac{x}{2}-y=1$$

$$\Rightarrow p-q=3$$

Hence (A)–(Q); (B)–(S); (C)–(S); (D)–(R)

96. A) $f(x)=\dfrac{1}{x+\frac{4}{x}+1}$

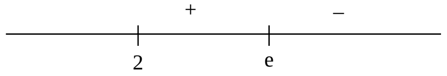
greatest value of $f(x)$ is when $x+\frac{4}{x}$ is min

& Minvalue of $x+\frac{4}{x}=4$

Greatest value of $(x)=\frac{1}{5}$

B) $y=\frac{\ln x}{x}$

$$y'=\frac{1-\ln x}{x}$$



$\Rightarrow$ y is increasing from $(2,e)$ & decreasing from (e,∞)

Max value of $f(x)$ is at $x=e=\frac{1}{e}$

C) $f(x)=\frac{3x}{e^3}+\frac{27e}{x}$

Use AM ≥ GM

$f(x)\ge 2\sqrt{\frac{81}{e^2}}\ \ge\frac{18}{e}$

D) Area of sector 'A' $=\frac{\theta}{360^\circ}\times\pi r^2$

Perimeter of sector $=\frac{\theta}{360^\circ}\times 2\pi r+2r=4e$

$A=\frac{(4e-2r)}{2\pi r}\times\pi r^2$

$A=(2e-r)r$

$A=2er-r^2$

A will be max at $r=e$

Hence → A → (S), B → (Q), C → (P), D → (R).

97. A) eqn of tangent to $y^2=8x$

$\Rightarrow y=mx+\frac{2}{m}$

Solving it with $xy=-1$

$\Rightarrow x\left(mx+\frac{2}{m}\right)=-1$

$\Rightarrow mx^2+\left(\frac{2}{m}\right)x+1=0$

$\Rightarrow$ As $D=0,\quad \frac{4}{m^2}-4m=0$

$\Rightarrow\quad m=1$

eqn of common tangent $\Rightarrow y=x+2$

y intercept $=2$

B) $\frac{\log\left(f(a)+f'(a)+f''(a)\right)-\log\left(f(b)+f'(b)+f''(b)\right)}{a-b}=1$

Let $\quad g(x)=\log\left(f(x)+f'(x)+f''(x)\right)$

using LMUT, there exist a 'c' in $(a,\ b)$

Such that $g'(c)=1$

$\Rightarrow g'(x)=\frac{f'(x)+f''(x)+f'''(x)}{f(x)+f'(x)+f''(x)}$

$\Rightarrow f'''(c)=f(c)\Rightarrow\frac{f'''(c)}{f(c)}=1$

C) $f(x)=\left(x^2-1\right)\left(x^2-4\right)$

$f'(x)=2x\left(x^2-4\right)+2x\left(x^2-1\right)$

$=2x\left(2x^2-5\right)$

Roots of $f'(x)$ are $0,\pm\frac{\sqrt5}{\sqrt2}$

$[\alpha]+[\beta]+[\gamma]=0+1-2=-1$

D) Eqn of normal to $y^2=x$ is

$y=mx-\frac{m}{2}-\frac{m^3}{4}$

As it passes through $(c,\ 0)$

$\Rightarrow mc-\frac{m}{2}-\frac{m^3}{4}=0$

$\Rightarrow m^3+2m-4mc=0$

$\Rightarrow m\left(m^2+2-4c\right)=0$

for three values of m, $4c>2$

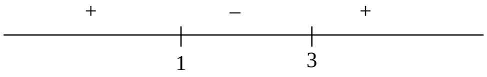
$\Rightarrow c>\frac{1}{2}$

98. A) $h(y)=y^3-6y^2+9y+1$

$h'(y)=3y^2-12y+9$

$=3\left(y^2-4y+3\right)$

$=3(y-3)(y-1)$



In [0,2], Max value of $h(y)$ is at $y=1$

and Min value is at $y=0$ or 2

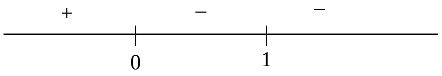
$h(1)=5,\quad h(0)=1,\quad h(2)=3$

$\Rightarrow M+m=6$

B) $f(x)=\tan^{-1}x-\frac{\ln x}{2}$

$f'(x)=\frac{1}{1+x^2}-\frac{1}{2x}$

$=\frac{-(x-1)^2}{2x\left(1+x^2\right)}$



$$M = f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{4}\ln 3$$

$$m = f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{4}\ln 3$$

$$\text{Now}\,[M+m] = \left[\frac{\pi}{2}\right] = 1$$

C) $$h(y) = \begin{cases} 2y^2 + \dfrac{2}{y^2}, & |y| \in [1,2] \\ 1, & |y| < 1 \end{cases}$$

Use AM ≥ GM

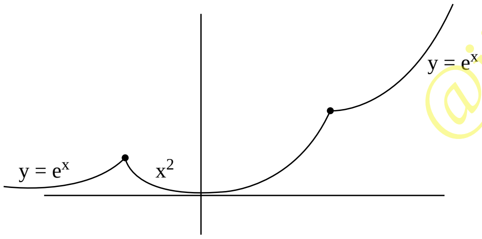
$$2y^2 + \frac{2}{y^2} \geq 4$$

$$M = h(2) = 8 + \frac{1}{2} = \frac{17}{2}$$

m=1

$$M + m = \frac{19}{2}$$

D) Graph of $\min(e^x, x^2)$



From graph n=0 is only value

Hence A → (Q), B → (P), C → (S), D → (P)

99. A) $$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{29}{14^2 \cdot 15^2}$$

$$T_r = \frac{2r+1}{r^2(r+1)^2} = \frac{(r+1)^2 - r^2}{(r^2)(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$s = \sum_{r=1}^{14} T_r = 1 - \frac{1}{(15)^2} = \frac{224}{225}$$

B) $f(x) = x^3 + ax + b$

$$f'(x) = 3x^2 + a$$

$$\left.\frac{dy}{dx}\right|_{x=a} = 3a^2 + a, \quad \left.\frac{dy}{dx}\right|_{x=b} = 3b^2 + a$$

$$\Rightarrow 3a^2 = 3b^2$$

$$\Rightarrow a = -b. \text{ (As } a \neq b)$$

$$\Rightarrow f(x) = x^3 + ax + b = x^3 + ax - a$$

$$f(1) = 1$$

C) $f(x) = 3x^5 - 250x^3 + 735x$

$$f_1'(x) = 15\left(x^4 - 50x^2 + 49\right)$$

$f''(x) = 60\left[x^3 - 25x\right]$ Point of entremum are roots of $x^4 - 50x^2 + 49 = 0$

$$\Rightarrow \left(x^2 - 49\right)\left(x^2 - 1\right) = 0$$

$$\Rightarrow x = \pm 1, \pm 7$$

Point of inflection are roots of $x^3 - 25x = 0$

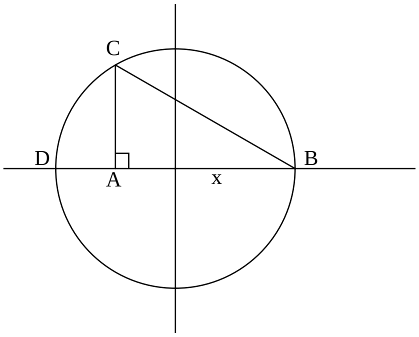
$$x\left(x^2 - 25\right) = 0$$

$$x = 0, \pm 5$$

sum of x coordinate = 0

$\Rightarrow k = 0$

D)



Since, BD = 1

$\Rightarrow AD = 1 - x$

Now, $AC = \sqrt{x(1-x)}$

Area of $\Delta ABC = \frac{1}{2} x \sqrt{n(1-x)}$

Area of ΔABC is maximum when $x^3 - x^4$ is maximum

Let $y = x^3 - x^4$

$$\frac{dy}{dx} = 3x^2 - 4x^3 \Rightarrow x = \frac{3}{4}$$

$$\text{Max Area} = \frac{1}{2} \times \frac{3}{4}\sqrt{\frac{3}{4} \times \frac{1}{4}} = \frac{3\sqrt{3}}{32}$$

Hence, A → (R), B → (P), C → (P), D → (S)

Review Exercises Solution

1. $f(x) = \sin^m x \sin mx + \cos^m x \cos mx$

$f'(x) = m\cos mx \sin^m x + m\sin^{m-1} x \cos x \sin mx$

$-m\sin mx \cos^m x - m\cos^{m-1} x \sin x \cos mx$

For $m = 2$

$f'(x) = 2\left[\cos 2x \sin^2 x - \sin 2x \cos^2 x\right.$

$+\sin x \cos x(\sin 2x - \cos 2x)]$

$= 2[\sin 2x \cos x(\sin x - \cos x) + \cos 2x \sin x$

$(\sin x - \cos x)]$

$= 2[\sin x - \cos x]\sin 3x$

$f'\left(\frac{\pi}{4}\right) = 0, \quad f'\left(\frac{\pi^+}{4}\right) = +ve, \quad f'\left(\frac{\pi^-}{4}\right) = -ve$

$\Rightarrow$ at $x = \frac{\pi}{4}$, there is a minimum

For $m = 4$

$$f'(x) = 4\begin{bmatrix}\cos 4x \sin^4 x + \sin^3 x \cos x \sin 4x \\ -\sin 4x \cos^4 x - \cos^3 x \sin x \cos 4x\end{bmatrix}$$

$$= 4\left[\begin{array}{l}\cos 4x \sin x\left(\sin^3 x - \cos^3 x\right) \\ +\sin 4x \cos x[\sin^3 x - \cos^3 x]\end{array}\right.$$

$= 4\left[\sin^3 x - \cos^3 x\right]\sin 5x$

$f'\left(\frac{\pi}{4}\right) = 0, \quad f'\left(\frac{\pi^+}{4}\right) = +ve, \quad f'\left(\frac{\pi^-}{4}\right) = -ve$

so, $f(x)$ has maximum for $x = \frac{\pi}{4}$

Similarly, for $m = 6$, $f(x)$ has a maximum for $x = \frac{\pi}{4}$.

2. $y = \cos^9 \phi \sec^9 \phi$

3. $y = \frac{\cos x}{\left(1 + \cos^2 x\right)^{3/2}}$

$$y' = \frac{-\left(1+\cos^2 x\right)^{3/2} \sin x + 3\cos^2 x\left(1+\cos^2 x\right)^{1/2} \sin x}{\left(1+\cos^2 x\right)^{3/2}}$$

$$= \frac{3\cos^2 x \sin x - \sin x\left(1+\cos^2 x\right)}{1+\cos^2 x}$$

$$= \frac{\sin x \cos 2x}{1+\cos^2 x} = 0$$

$\Rightarrow$ either $\sin x = 0$ or $\cos 2x = 0$

At $x = \pi$, $y' = 0$

$y'|_{\text{at } x=+} = -ve, \qquad y'|_{\text{at } x=-} = +ve$

at $x = \pi$, y has a maximum value

$y|_{x=\pi} = \frac{-1}{2\sqrt{2}}$

At $x = 3\frac{\pi}{4}, y' = 0$

y' at $x = \frac{3\pi^+}{4} = +ve, \quad y'$ at $x = \frac{3\pi^-}{4} = -ve$

y has a Minimum at $x = \frac{3\pi}{4}$

and its value is

$\frac{-1}{\sqrt{2}} \times \frac{2^{3/2}}{3^{3/2}} = -\frac{2}{3^{3/2}} = -\frac{2\sqrt{3}}{9}$

At $x = \frac{\pi}{4}, y' = 0$

y' changes its sign from $+ve$ to $-ve$,

$\Rightarrow$ y has maximum value $= \frac{2\sqrt{3}}{9}$

At $x = 0$, $y' = 0$

y' changes its sign from $-ve$ to $+ve$.

$\Rightarrow$ y has minimum value $= +\frac{1}{2\sqrt{2}}$

4. $f(x) = x^4\left(x - \frac{\pi}{2}\right)^2 + \sin^4 x$.

$f'(x) = 4x^3\left(x - \frac{\pi}{2}\right)^2 + 2\left(x^4\right)\left(x - \frac{\pi}{2}\right) + 4\sin^3 x \cos x$

$= 2x^3\left(x - \frac{\pi}{2}\right)(3x - \pi) + 4\sin^3 x \cos x$

$\Rightarrow f'(x) = 0$ at $x = 0, \frac{\pi}{2}$

$f'\left(O^+\right) = +ve, \quad f'\left(O^-\right) = -ve$

$\Rightarrow f(x)$ has local Minima at $x = 0$

$$f'\left(\frac{\pi^+}{2}\right) = +\text{ve}, \qquad f'\left(\frac{\pi^-}{2}\right) = -\text{ve}$$

$\Rightarrow f(x)$ has local Minima at $x = \frac{\pi}{2}$

5. $f(x) = \dfrac{x^4 - x^2 - 2(x-3)}{x^4 - x^2 - 2(x-2)}$ Let $x^4 - x^2 - 2x = t$

$$f(x) = \frac{t+6}{t+4} = 1 + \frac{2}{t+4} \qquad \text{(I)}$$

Now $t = x\left(x^3 - x - 2\right)$

$$\frac{dt}{dx} = 4x^3 - 2x - 2 = 2\left(2x^3 - x - 1\right)$$

$$= 2(x-1)\left(2x^2 + 2x + 1\right)$$

$$\frac{d^2t}{dx^2} = 12x^2 - 2$$

$$\left.\frac{d^2t}{dx^2}\right|_{x=1} = \ > 0$$

Min value of $t = -2$

so, $\dfrac{1}{t+4} \in [2, \infty)$

From eq (I)

$\Rightarrow f(x) \in (1, 2]$

6. $f(x) = (x-1)^{p+q}$

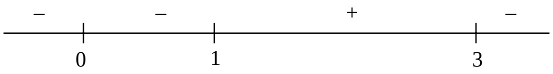
7. i) $y = \dfrac{1}{x(x-3)^2}$, $x \in R \sim \{0, 3\}$

$$y' = -\frac{((x-3)^2 + 2x(x-3))}{\left(x(x-3)^2\right)^2}$$

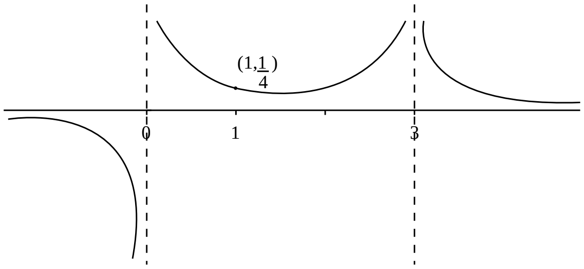
$$= \frac{-3(x-3)(x-1)}{x^2(x-3)^4}$$

$$\lim_{x\to 0^-} f(x) = -\infty, \qquad \lim_{x\to 0^+} f(x) = \infty$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = \infty$$



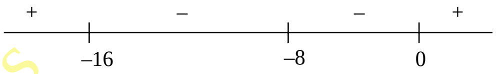
$$f(1) = \frac{1}{4}, \qquad \lim_{x\to\infty} f(x) = 0, \qquad \lim_{x\to-\infty} f(x) = 0$$



ii) $y = \dfrac{x^2}{x+8}$, $x \in R \sim \{-8\}$

$$\frac{dy}{dx} = \frac{2x(x+8) - x^2}{(x+8)^2}$$

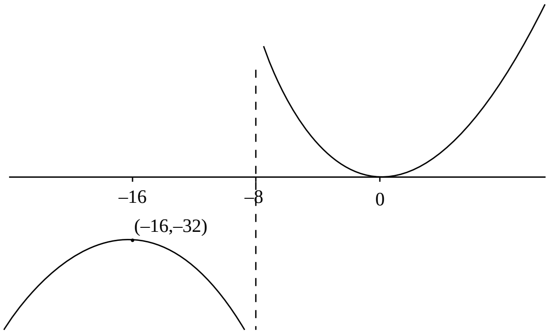
$$= \frac{x^2 + 16x}{(x+8)^2} = \frac{x(x+16)}{(x+8)^2}$$



$$\lim_{x\to-8^+} f(x) = \infty, \quad \lim_{x\to-8^-} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = \infty,$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

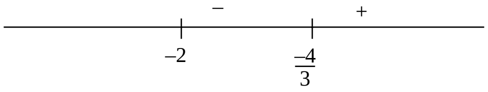
$f(0) = 0, \quad f(-16) = -32$



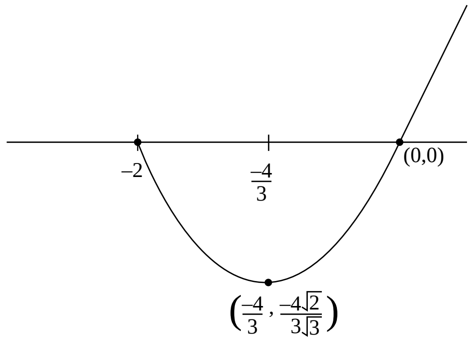
iii) $y = x\sqrt{x+2}$, $x \in [-2, \infty]$

$$\frac{dy}{dx} = \sqrt{x+2} + \frac{x}{2\sqrt{x+2}}$$

$$= \frac{3x+4}{2\sqrt{x+2}}$$



$$\lim_{x\to-2^+} f(x) = 0, \quad f\left(\frac{-4}{3}\right) = \frac{-4\sqrt{2}}{3\sqrt{3}}, \quad \lim_{x\to\infty} f(x) = \infty$$



8. a) If $f'(c)=0$, it doesn't implies that there has to be maxima or minima

Eg. $f(x)=x^3$

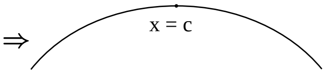
$f'(x)=3x^2=0 \Rightarrow x=0$

Now $f''(x)=6x$

at $x=0,\ f''(x)=0$

b) If f has a maximum at $x=c$ & differentiate at $x=c$

$\Rightarrow$



$\Rightarrow f(c-h)<f(c)>f(c+h)$

Slope of tangent at $x=c$ is always zero

$\Rightarrow \quad f'(c)=0$

9. As $f(x_1)=f(x_2)=f(x_3)=0$

between, x_1 & x_2 there wil be a point 'α'

where $f'(\alpha)=0$

& similarly, between, x_2 & x_3 there wil be

a point 'β' where $f'(\beta)=0$

Now, Let $g(x)=f'(x)$ Where $x\in[\alpha,\beta]$

Apply Rolle's theorem,
There will be atleast one value of 'c'
In between (α,β) where $f''(x)=0$

As $C\in(\alpha,\beta)\Rightarrow C\in(x_1,x_3)$

10. i) $f(x)=\begin{cases}\min\{3t^4-8t^3-6t^2+24t,\ 1\le t\le x\}, & 1\le x<2\\ \max\{3t+\frac{\sin^2\pi t}{4}+2,\ \ 2\le t\le x\}, & 2\le x\le 4\end{cases}$

Let $g(t)=3t^4-8t^3-6t^2+24t$

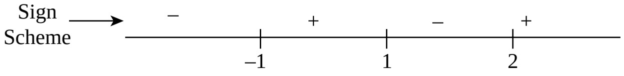
$g'(t)=12t^3-24t^2-12t+24$

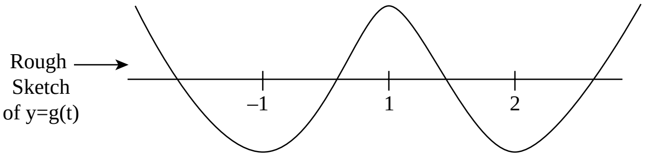
$=12\left[t^3-2t^2-t+2\right]$

$=12[t-1][t+1][t-2]$

$\Rightarrow \min\left\{3t^4-8t^3-6t^2+24t,\ 1\le t\le x\right\}, 1\le x<2$

$=3x^4-8x^3-6x^2+24x$





Let $h(t)=3t+\frac{\sin^2\pi t}{4}+2,$

$h'(t)=3+\frac{\pi\sin 2\pi t}{4}>0$

$\Rightarrow h(t)$ is always increasing

$\Rightarrow \max\left\{3t+\frac{\sin^2\pi t}{4}+2,\ 2\le t\le x\right\}, 2\le x\le 4$

$=3x+\frac{\sin^2\pi x}{4}+2$

$f(x)=\begin{cases}3x^4-8x^3-6x^2+24x, & 1\le x<2\\ 3x+\frac{\sin^2\pi x}{4}+2, & 2\le x\le 4\end{cases}$

∴ Maxmima at $x=1$ & 4 & Minima at $x=2$

ii) $f(x)=\begin{cases}(x+2)^{2/3}(x+4), & x\le 0\\ \frac{1-x}{x^2}, & 0<x\le 1\\ \frac{x-1}{x^2}, & x\ge 1\end{cases}$

$f'(x)=\begin{cases}\frac{(5x+14)}{3(x+2)^{1/3}}, & x\le 0\\ \frac{x(x-2)}{x^4}, & 0<x\le 1\\ -\frac{x(x-2)}{x^4}, & x>1\end{cases}$

f(x) has maximum at $x=\frac{-14}{5}$, 2 & minimum at $x=-2$ and $x=0$, it is neither maxima nor minima.

11. i)

a) $f(x) = x^2 - \sqrt[3]{|x|}, x \in [-1,1]$

$$= \begin{cases} x^2 - x^{1/3}, & 1 > x \geq 0 \\ x^2 - (-x)^{1/3}, & -1 < x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x - \frac{1}{3}x^{-2/3}, 1 > x \geq 0 \\ 2x + \frac{1}{3}(-x)^{-2/3}, 1 < x < 0 \end{cases}$$

$f(x)$ is non derivable at $x = 0$.

So, Rolle's Theorem is not applicable

b) $f(x) = \sec x,\ x \in \left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$

$$f\left(\frac{-\pi}{4}\right) = \sqrt{2}, \quad f\left(\frac{\pi}{3}\right) = 2$$

As $f(a) \neq f(b) \Rightarrow$ Rolle's Theorem is not applicable.

ii) $f(x) = (x-a)(x-b)(x-c),\ a < b < c$

Applying Rolle's Theorem in [a,b]

$f'(x) = 0$ will have atleast one root

& similarly in [b,c], Apply Rolle's Theorem,

$f'(x) = 0$ will have atleast one root.

12. $f(x) = 2x^3 + ax^2 + bx,\ a \in N,\ b \in N$

$f'(x) = 6x^2 + 2ax + b$

For three roots of $f(x)$, $f'(x)$ should have two roots

$\Rightarrow 4a^2 - 24b > 0$

$\Rightarrow a^2 - 6b > 0$

For minimum value of $a + b,\ a = 3,\ b = 1$

$\Rightarrow a + b = 4$

$\Rightarrow f(x) = 2x^3 + 3x^2 + x$

13. $xy(y - x) = 2a^3$ .... (I)

$xy^2 - x^2y = 2a^3$

$$\Rightarrow y^2 + 2xy\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$$

For Min, either $y = 0$ or $y = 2x$.

As $y \neq 0 \Rightarrow y = 2x$

Putting in eq (I)

$2x^3 = 2a^3$

$\Rightarrow x = a\ \&\ y = 2a$

$$\left.\frac{dy}{dx}\right|_{x=a^+} = +ve, \qquad \left.\frac{dy}{dx}\right|_{x=a^-} = -ve$$

$\Rightarrow$ at $x = a$, y has a minimum value & is equal to 2a.

Now, put $x = a$ in eq (I)

$ay^2 - a^2y = 2a^3$

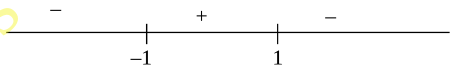
$\Rightarrow y^2 - ay - 2a^2 = 0$

$\Rightarrow (y - 2a)(y + a) = 0$

$y = 2a$ or $y = -a$

14. i) $f(x) = \dfrac{x}{1+x^2}$

$$f'(x) = \frac{1-x^2}{\left(1+x^2\right)^2} = \frac{(1-x)(x+1)}{\left(1+x^2\right)^2}$$



$f(x)$ has Local Minima at $x = -1$

$$f(-1) = \frac{-1}{2}$$

$$\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to\infty} f(x)$$

$f(x)$ has Local Maxima at $x = 1$

$$f(1) = \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}$$

ii) As $\left|\dfrac{x}{1+x^2}\right| \leq \dfrac{1}{2}$

Now $\displaystyle\lim_{b\to a+h}\left|\frac{f(b)-f(a)}{b-a}\right|$ where $h \to 0$

$$= |f'(b)| = \left|\frac{b}{1+b^2}\right| \leq \frac{1}{2}$$

15. $f(x) = \dfrac{x}{(a+x)(b+x)}$

$$f'(x) = \frac{(a+x)(b+x) - x(2x + (a+b)}{((a+x)(b+x))^2}$$

$$= \frac{ab - x^2}{((a+x)(b+x))^2} = 0$$

$$\Rightarrow x = \pm\sqrt{ab}$$

$f'\left(\sqrt{ab}^{+}\right) = -ve,$ $\quad f'(\sqrt{ab}^{-}) = +ve$

$f'\left(-\sqrt{ab}^{+}\right) = +ve,$ $f'(-\sqrt{ab}^{-}) = -ve$

$f(x)$ has a maximum value at $x = \sqrt{ab}$.

& minimum value at $x = -\sqrt{ab}$

16. $y = m_1(x - x_1)^2 + m_2(x - x_2)^2 + \cdots + m_n(x - x_n)^2$

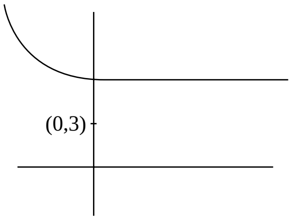
As $y = ax^2 + bx + c$ will have a minimum value at $x = \frac{-b}{2a}$

$$\Rightarrow x = \frac{+(m_1x_1 + m_2x_2 + \cdots + m_nx_n)}{m_1 + m_2 + m_3 + \cdots + m_n}$$

17. a) $f'(x) < 0 \quad \forall x \in R$

$\Rightarrow f(x)$ is decreasing function

$\Rightarrow f(x) = 3$ can have either `0'or 1 solution



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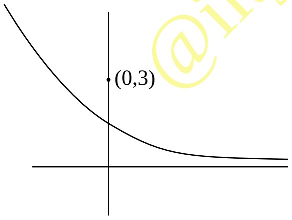
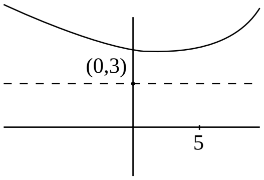


b) $f'(x) < 0 \quad \forall x < 5, f'(x) > 0 \forall x > 5$



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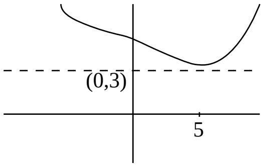

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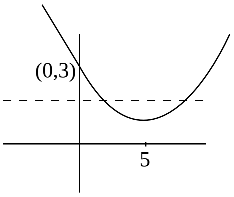
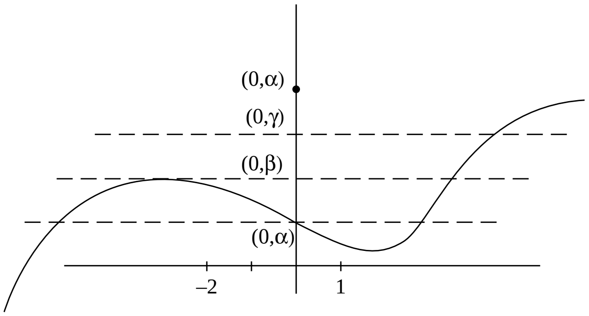


$f(x) = 3$ can be either 0 or 1 or 2 roots

c) $f'(x) > 0 \ \forall \ x < -2, \quad f'(x) < 0 \quad \forall \ x \in (-2,1)$

$\& f'(x) > 0 \ \forall \ x > 1$



$f(x) = 3$ can have 0, 1, 2, 3 roots

Depending upon α, β, γ are 3 respectively,

$f(x) = 3$ can have different roots

18. i) Let $f(x) = \ln x$

& As $m(b-a) \le f(b) - f(a) \le M(b-a)$

Where m, M are minimum & maximum values of

$f'(x) = \frac{1}{x}$ resp in (b, a)

$m = \frac{1}{a}, \ M = \frac{1}{b}$

$$\Rightarrow \frac{-b+a}{a} < -\ln(b) + \ln(a) < \frac{-b+a}{b}$$

$$\Rightarrow \frac{a-b}{a} < \ln\left(\frac{a}{b}\right) < \frac{a-b}{b}$$

ii) Let $f(t) = t^p$

$f'(t) = pt^{p-1}$

$\Rightarrow m = py^{p-1}, \quad M = px^{p-1}$

$py^{p-1}(x-y) \le \ x^p - y^p \le \ px^{p-1}(x-y)$

19. $f(x) = \tan^{-1} x - \frac{3x}{3+x^2}$

$$f'(x) = \frac{1}{1+x^2} - \frac{3\left(3+x^2\right) - 6x^2}{\left(3+x^2\right)^2}$$

$$= \frac{\left(3+x^2\right)^2 - \left(9-3x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)\left(3+x^2\right)^2}$$

$$= \frac{x^4 + 9 + 6x^2 - 9 + 3x^2 - 9x^2 + 3x^4}{\left(1+x^2\right)\left(3+x^2\right)^2}$$

$$=\frac{4x^4}{(1+x^2)(3+x^2)^2}\ge 0$$

$\Rightarrow$ Min value of $f(x)$ in $(0,\infty)=0$

$$\Rightarrow \tan^{-1}x>\frac{x}{1+x^2/3}$$

20. Let $f(x)=\dfrac{\ln x}{x}$

$$f'(x)=\frac{1-\ln x}{x^2}$$

$\Rightarrow f(x)$ is increasing $\forall x\in(0,e)$ & decreasing in (e,∞)
for $n>8$

$$\frac{\ln(\sqrt{n})}{\sqrt{n}}>\frac{\ln(\sqrt{n+1})}{\sqrt{n+1}}$$

$$\Rightarrow \sqrt{n}^{\sqrt{n+1}}>\sqrt{n+1}^{\sqrt{n}}$$

21. $4x^2+6xy+9y^2-8x-24y+4=0 \qquad \text{(I)}$

$$8x+6y+6x\frac{dy}{dx}+18y\frac{dy}{dx}-8\ -24\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{8x+6y-8}{24-6x-18y}=\frac{4x+3y-4}{12-3x-9y}$$

For $\dfrac{dy}{dx}=0$

$\Rightarrow 4x+3y=4 \qquad \text{(II)}$

Solving (I) & (II), we get
$x=1,\ -1$

For $x=1,\quad y=0,\ 2$

For $x=-1,\ y\ =\dfrac{8}{3},\ \dfrac{2}{3}$

Maximum value of $y\ =\dfrac{8}{3}$ & Min value is 0.

22. Let $f(x)=\dfrac{e^{ax}}{1+x^2}$

$$f'(x)=\frac{ae^{ax}(1+x^2)-2xe^{ax}}{(1+x^2)^2}$$

$$=\frac{e^{ax}(ax^2-2x+a)}{(1+x^2)^2}$$

Now, $D=b^2-4ac$

$=4-4a^2$

$=4(1-a^2)$

If $|a|\ge 1$, $D<0\Rightarrow f(x)$ is monotonic
$\Rightarrow$ no turning point
& If $|a|<1$, $D>0\Rightarrow f(x)$ will have
a maximum & minimum value.

23. Let illumination of an Area = f(x)

$$f(x)=\frac{x}{\sqrt{x^2+a^2}}-\frac{x}{\sqrt{x^2+b^2}}$$

$$f'(x)=\frac{\sqrt{x^2+a^2}-\dfrac{x^2}{\sqrt{x^2+a^2}}}{(x^2+a^2)}-\frac{\sqrt{x^2+b^2}-x^2/\sqrt{x^2+b^2}}{x^2+b^2}$$

$$=\frac{a^2}{(x^2+a^2)^{3/2}}-\frac{b^2}{(x^2+b^2)^{3/2}}$$

For $f'(x)=0$

$$\frac{a^2}{b^2}=\left(\frac{x^2+a^2}{x^2+b^2}\right)^{3/2}$$

$$\frac{a^{4/3}}{b^{4/3}}=\frac{x^2+a^2}{x^2+b^2}$$

$$\Rightarrow a^{4/3}x^2+a^{4/3}b^2=b^{4/3}x^2+a^2b^{4/3}$$

$$\Rightarrow \left(a^{4/3}-b^{4/3}\right)x^2=a^{4/3}b^{4/3}\left[a^{4/3}-b^{4/3}\right]$$

$$\Rightarrow\quad x=\pm\sqrt{\frac{a^{4/3}b^{4/3}}{a^{2/3}+b^{2/3}}}=\frac{\pm(ab)^{2/3}}{\left(a^{2/3}+b^{2/3}\right)^{1/2}}$$

For Maximum value of $f(x)$, $f''(x)<0$

$$\Rightarrow x=\frac{(ab)^{2/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

24. $f(x)=xe^{ax}$

$f'(x)=e^{ax}+axe^{ax}$

$\qquad =e^{ax}(1+ax)$

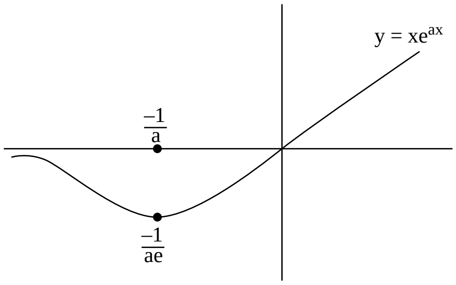
$f'(x)=0\Rightarrow x=\dfrac{-1}{a}$

$f''(x)=ae^{ax}+ae^{ax}+a^2xe^{ax}$

$\qquad =ae^{ax}(2+ax)>0 \quad (\text{at } x=-1/a)$

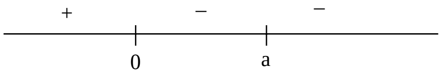
$\Rightarrow$ At $x = \frac{-1}{a}, f(x) = xe^{ax}$ will have a minimum.

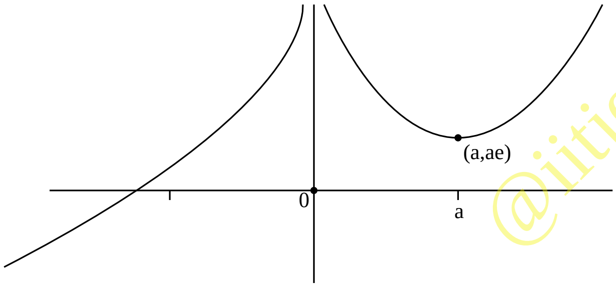
$$f\left(-\frac{1}{a}\right) = \frac{-1}{ae}$$



Now, $f(x) = xe^{a/x}$

$$f'(x) = e^{a/x} - \frac{a}{x}e^{ax} = e^{ax}\frac{(x-a)}{x}$$





Graph of f(x)

So, for $xe^{a/x} = \lambda$ will have two positive roots if $\lambda > ae$

& will have no positive root if $\lambda < ae$.

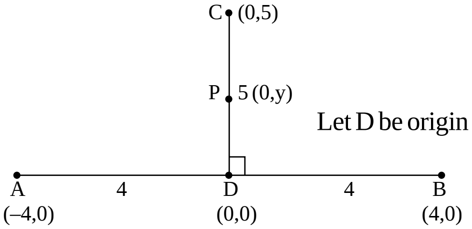
25. $2 \le f'(x) \le 5$

$$\Rightarrow 2 \le \frac{f(4) - f(0)}{4} \le 5$$

$$\Rightarrow 8 + f(0) \le f(4) \le 20 + f(0)$$

$$\Rightarrow 9 \le f(4) \le 21$$

26.



$f(y) = PA + PB + PC$

$$f(y) = 5 - y + \sqrt{16 + y^2} + \sqrt{16 + y^2}$$

$$f'(y) = -1 + \frac{2y}{\sqrt{16 + y^2}}$$

For $f'(y) = 0, \quad \sqrt{16 + y^2} = 2y$

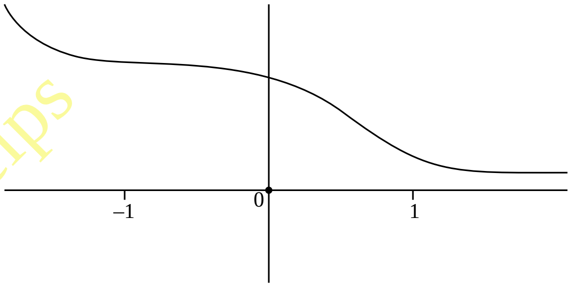
$$16 = 3y^2$$

$$y = \pm\frac{4}{\sqrt{3}}$$

$$f'\left(y^+\right) = +ve,\ f'\left(y^-\right) = -ve \ \text{ at } y = \frac{4}{\sqrt{3}}$$

Hence at $y = \frac{4}{13}$, $f(y)$ will have a minimum value.

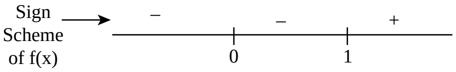
27.



$$\lim_{x \to \pm\infty} [f(x) + x]$$

28. Let $f(x) = \frac{e^x}{x}$

$$f'(x) = \frac{e^x(x-1)}{x^2}$$



Min value of $f(x)$ is at $x = 1 = e$

Min value of $f(x)f(y) = e^2$

$$\Rightarrow \frac{e^{x+y}}{xy} \ge e^2$$

29. Use $AM \ge GM$

$$\frac{x^2 + 4 - x^2}{2} \ge \sqrt{x^2\left(4 - x^2\right)} \quad \forall |x| \le 2, |y| \le 2$$

$$4 \ge x^2\left(4 - x^2\right)$$

Similarly, $4 \ge y^2\left(4 - y^2\right)$

$$\Rightarrow x^2y^2\left(4 - x^2\right)\left(4 - y^2\right) \le 16$$

30. $x^2+xy+y^2=12 \qquad (I)$

$$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-(2x+y)}{(x+2y)}=0$$

$$\Rightarrow y=-2x$$

Solving (I) & (II)

$$x^2-2x^2+4x^2=12$$

$$x^2=4$$

$$x=\pm 2$$

For $x=2 \Rightarrow y^2+2y-8=0$

$$y=\frac{-2+6}{2}=2 \text{ or } -4$$

For $x=-2 \Rightarrow y^2-2y-8=0$

$$y=\frac{2+6}{2}=4 \text{ or } -2$$

Hence

$\Rightarrow (2,-4)$ & $(-2,4)$ are highest & lowest point on the curve.

31. $f'(-1)=\frac{1}{2},\ f'(0)=0,\ f''(x)>0\ \forall x\in R$

If $f''(x)>0$, $f'(x)$ is increasing function

While $f'(-1)=\frac{1}{2}$ & $f'(0)=0$ does not satisfy the condition

So, such a function doesn't exist

Target Exercises Solution

1. $f(x)=\dfrac{\sin^2 x}{\sin(x+a)\sin(x+b)}$

$$f'(x)=\frac{\left(2\sin(x+a)\sin(x+b)\sin x\cos x-\sin^2 x\sin(x+a)\cos(x+b)-\sin^2 x\cos(x+a)\sin(x+b)\right.}{\sin^2(x+a)\sin^2(x+b)}$$

Numerator of $f'(x)$

$$\Rightarrow \sin x\sin(x+a)[\sin(x+b)\cos x-\sin x\cos(x+b)]+\sin x\sin(x+b)[\sin(x+a)\cos x-\sin x\cos(x+a)]$$

$$\Rightarrow \sin x[\sin b\sin(x+a)-\sin a\sin(x+b)]$$

$$\Rightarrow \frac{1}{2}\sin x[\cos(x+a-b)-\cos(x+a+b)-[\cos(x+b-a)-\cos(x+a-b)]$$

$$\Rightarrow \frac{1}{2}\sin x(\cos(x+a-b)-\cos(x+b-a)]=0$$

$$\Rightarrow \sin x\sin x\sin(b-a)\ge 0$$

$$\Rightarrow f'(x)\ge 0$$

At $x=n\pi,\ f'(x)=0$

at $x=n\pi-a$ or $n\pi-b$, $f(x)$ is not defined

At all $x=n\pi$, $f(x)$ has a minimum value = 0

2. $f(x)=\dfrac{\sin mx}{\sin x}$

$$f'(x)=\frac{m\cos mx\sin x-\sin mx\cos x}{\sin^2 x}=0$$

$$\Rightarrow m\cos mx\sin x=\sin mx\cos x$$

$$\Rightarrow m\tan x=\tan mx$$

squaring

$$m^2\tan^2 x=\tan^2 mx$$

$$m^2\cot^2 mx=\cot^2 x$$

$$m^2\left(\operatorname{cosec}^2 mx-1\right)=\operatorname{cosec}^2 x-1$$

$$m^2\operatorname{cosec}^2 mx-\operatorname{cosec}^2 x=m^2-1\ge 0$$

$$\frac{m^2}{\sin^2 mx}\ge\frac{1}{\sin^2 x}$$

$$\Rightarrow m^2\sin^2 x\ge\sin^2 mx$$

3. As $ax+by$ is maximum $\Rightarrow a+b\dfrac{dy}{dx}=0 \qquad (I)$

As $x^2+xy+y^2=3k^2$

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(2x+y)}{x+2y} \qquad (II)$$

Equating (I) & (II)

$$\Rightarrow +\frac{a}{b}=\frac{2x+y}{x+2y}$$

$$\Rightarrow ax+2ay=2bx+by$$

$$\Rightarrow (a-2b)x+(2a-b)y=0 \qquad \text{(III)}$$

Put $x=\dfrac{(b-2a)y}{a-2b}$ in $x^2+yx+y^2=3k^2$

$$\sqrt{\left(\frac{(b-2a)^2}{(a-2b)^2}+\frac{b-2a}{a-2b}+1\right)}=\frac{\sqrt{3}k}{y}$$

$$\Rightarrow \frac{\sqrt{\begin{array}{c}b^2+4a^2-4ab+ab-2a^2-2b^2\\ +4ab+a^2+4b^2-4ab\end{array}}}{a-2b}=\frac{\sqrt{3}k}{y}$$

$$\Rightarrow \frac{\sqrt{3\left(b^2+a^2-ab\right)}}{a-2b}=\frac{\sqrt{3}k}{y}$$

$$\Rightarrow \quad \frac{y}{a-2b}=\frac{k}{\sqrt{b^2+a^2-ab}} \qquad \text{(IV)}$$

$$\text{Now } ax+by=\frac{a(b-2a)k}{\sqrt{b^2+a^2-ab}}+b(a-2b)\,k$$

$$=2k\left(\sqrt{a^2+b^2-ab}\right)$$

4. $y=a\cos\theta+b\cos\phi$

$$y=\frac{a^2}{c-b\sec\phi}+b\cos\phi$$

$$y=\frac{a^2\cos\phi}{c\cos\phi-b}+b\cos\phi$$

$$y=\frac{\left(a^2-b^2\right)\cos\phi+bc\cos^2\phi}{c\cos\phi-b}$$

Now, for $\dfrac{dy}{dx}=0$

$$\Rightarrow (c\cos\phi-b)\left(\left(b^2-a^2\right)\sin\phi-2bc\cos\phi\sin\phi\right)$$

$$+((a^2-b^2)\cos\phi+bc\cos^2\phi)(c\sin\phi)=0$$

$$\Rightarrow c\left(b^2-a^2\right)\cos\phi-b\left(b^2-a^2\right)-2bc^2\cos^2\phi+2b^2c\cos\phi$$

$$+c\left(a^2-b^2\right)\cos\phi+bc^2\cos^2\phi=0$$

$$\Rightarrow -b\left(b^2-a^2\right)+2b^2c\cos\phi-bc^2\cos^2\phi=0$$

$$\Rightarrow -b^3+a^2b+2b^2\frac{(a\cos\phi+b)}{\cos\theta}$$

$$-b\left(\frac{a^2\cos^2\phi}{\cos^2\theta}+b^2+\frac{2ab\cos\phi}{\cos\theta}\right)$$

$$\Rightarrow a^2b+-a^2b\frac{\cos^2\phi}{\cos^2\theta}=0$$

$$\Rightarrow 1-\frac{\cos^2\phi}{\cos^2\theta}=0$$

$$\Rightarrow \cos^2\theta=\cos^2\phi$$

$$\Rightarrow \theta=\phi$$

5. $f(x)=x-\sin x-\tan\alpha(1-\cos x)$

$f'(x)=1-\cos x-\tan\alpha\sin x=0$

$\Rightarrow \cos\alpha=\cos(x-\alpha)$

$\Rightarrow x=2n\pi$ or $x=2n\pi+2\alpha$

Now, $f''(x)=\sin x-\tan\alpha\cos x$

i) For $x=2n\pi$, $\quad f''(x)=-\tan\alpha$

$\forall\alpha\in\left(0,\dfrac{\pi}{2}\right)\Rightarrow f(x)$ is maximum

$\forall\alpha\in\left(\dfrac{\pi}{2},\pi\right)\Rightarrow f(x)$ is minimum

ii) For $x=2n\pi+2\alpha$, $f''(x)=\sin 2\alpha-\tan\alpha\cos 2\alpha$

$=\tan x$

$\forall\alpha\in[0,\pi/2)\Rightarrow f(x)$ is minimum

$\forall\alpha\in\left(x_2,x\right)\Rightarrow f(x)$ is maximum.

Now, $f(x)=0$

$$\Rightarrow x=\sin x+\tan\alpha(1-\cos x)$$

$$\Rightarrow x-\tan\alpha=\frac{\cos\alpha\sin x-\sin\alpha\cos x}{\cos\alpha}$$

$$\Rightarrow x\cos\alpha-\sin\alpha=\sin(x-\alpha)$$

$$\Rightarrow \boxed{x=0}\rightarrow \text{Root of } f(x)=0$$

6. $x^4+4px^3-4qx-1=0$

Let $y=x^4+4px^3-4qx-1$

$$\frac{dy}{dx}=4x^3+12px^2-4q$$

$$\frac{d^2y}{dx^2}=12x^2+24px=12x(x+2p)=0$$

$x=0$ or $x=-2p$

$$(-4q)\left(-32p^3+48p^3+4q\right)<0$$

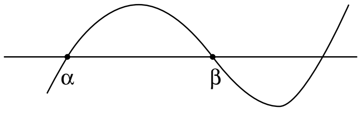
$$(4q)\left(16p^3-4q\right)>0$$

$$q\left(4p^3-q\right)>0$$

7. $g(x)=f'(x)+\lambda f(x)$

$g(\alpha)=f'(\alpha),\ \ g(\beta)=f'(\beta)$

As $f'(\alpha)$ & $f'(\beta)$ are of opposite sign.



$\Rightarrow g(\alpha)\ \&\ g(\beta)$ are of opposite signs

$\Rightarrow$ g(x) will have a root in between pair of root of f(x) = 0

8. Similar of Q.7

9. $f(x)=(a-x)^6-3a(a-x)^5+\frac{5}{2}a^2(a-x)^4-\frac{a^4}{2}(a-x)^2$

Let $a-x=t$

$$f(x)=t^6-3at^5+5a^2t^4-\frac{a^4}{2}2t^2$$

$$=t^2\left(t^4-3at^3+\frac{5/2a^2t^2}{2}-\frac{a^4}{2}\right)$$

Now

$$g(t)=a^4-3at^3+5a^2t^2-\frac{a^4}{2}$$

$$g'(t)=4t^3-9at^2+5a^2t$$

$$=t\left(4t^2-9at+5a^2\right)>0$$

$\Rightarrow g(t)$ is increasing $\forall t\in(0,a)$

$$g(a)=a^4-3a^4+\frac{5}{2}a^4-\frac{a^4}{2}=0$$

$\Rightarrow g(t)<0$

$\Rightarrow f(t)<0$

Hence proved.

10. Let $f(x)=\dfrac{\sin \pi x}{x(1-x)}$

$$f'(x)=\frac{\pi x(1-x)\cos \pi x-\sin \pi x(1-2x)}{x^2(1-x)^2}$$

Let $g(x)\ =\pi\left(x-x^2\right)\cos \pi x-(1-2x)\sin \pi x$

$$g'(x)=\pi(1-2x)\cos \pi x-\pi^2\left(x-x^2\right)\sin \pi x$$

$$-\pi(1-2x)\cos \pi x+2\sin \pi x$$

$$=(\sin \pi x)\left\{2-\pi^2\left(x-x^2\right)\right\}$$

$$=(\sin \pi x)\left(\pi^2x^2-\left(\pi^2\right)x+2\right)$$

as the quadratic ($\pi^2x^2-\pi^2x+2$) has equal values at x = 0 & x = 1

$\Rightarrow$ it is symmetric about $x=\frac{1}{2}$ & the value at $\frac{1}{2}$ is

is negative

Let roots be α , β

$\Rightarrow g(x)$ is increasing in $(0,\alpha)\cup(\beta,1)$

& $g(x)$ is decreasing in (α,β)

$$\alpha=\frac{\pi^4-\sqrt{\pi^4-8\pi^2}}{2\pi^2},\beta=\frac{\pi^4+\sqrt{\pi^4-8\pi^2}}{2\pi^2}$$

11. Let $g(x) = f(x)^n$

Now, $g'(x)=nf'(x)(f(x))^{n-1}$

Using LMVT.

$$g'(c)\ =\frac{2^n-1}{2-1}\ \forall\, c\in(0,1)$$

$$\Rightarrow nf'(c)(f(c))^{n-1}=2^n-1$$

$$\Rightarrow f'(c)(f(c))^{n-1}=\frac{2^n-1}{n}\geq\frac{2^{n-1}}{2}$$

$$\Rightarrow f'(c)(f(c))^{n-1}>2^{\frac{n-1}{2}}=\sqrt{2^{n-1}}$$

Hence proved.

12. As f′(x) is strictly increasing ⇒ f″(x) > 0 in (a, c)

Using LMVT in (a,b) & (b,c)

$$f'(x_1)=\frac{f(b)-f(a)}{b-a},\quad f'(x_2)=\frac{f(c)-f(b)}{c-b}$$

$\Rightarrow$ As $f'(x_1)<f'(x_2)$

$$\Rightarrow\ \frac{f(b)-f(a)}{b-a}<\frac{f(c)-f(b)}{c-b}$$

$$\Rightarrow(b-c)f(a)+(c-a)f(b)+(a-b)f(c)<0$$

Hence proved.

13. $g(x) = f(a) + f'(a)(x-a) + k(x-a)^2$

a) Put $x = a$

$g(a) = f(a)$

b) As $g(b) = f(b)$

& $h(x) = g(x) - f(x)$

Using Rolle's in h(x) over (a,b)

$h'(c_1) = 0$ where $c_1 \in (a,b)$

$\Rightarrow g'(c_1) = f'(c_1)$

c) $g(x) = f(a) + f'(a)(x-a) + k(x-a)^2$

$g'(x) = f'(a) + 2k(x-a)$

Put $x = a,\ g'(a) = f'(a)$

d) Using b, c result for $r(x) = g'(x) - f'(x)\, \forall (a, c_1)$

Apply Rolle's Theorem, we get

$g''(c_2) = f''(c_2)$ where $c_2 \in (a, c_1)$

e) As $g'(x) = f'(a) + 2k(x-a)$

$g''(x) = 2k$

$\Rightarrow g''(c_2) = 2x = f''(c_2)$

& $R = k(x-a)^2$

$\Rightarrow R = f''(c_2)\dfrac{(b-a)^2}{2}$

14. $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_0$

As roots of $P_n(x)$ are real

Then, $P_n(x_1) = P_n(x_2) = \cdots = P_n(x_n) = 0$

Since, $P_n(x)$ is a polynomial function, it is Continuous & differentiable over R.

Apply Rolle's Theorem in $[x_1, x_2]$.

$$P_n'(x_i) = \frac{P_n(x_2) - P_n(x_1)}{x_2 - x_1} = 0$$

$\Rightarrow x_i$ is root of $P_n'(x) = 0$.

Similarly, we can obtain other (n-2) roots of $P_n'(x) = 0$.

As $P_n'(x) = 0$ is also a polynomial function.

Applying Rolle's Theorem, we can find roots of $P_n''(x) = 0$ & so on.

Hence, all those polynomial have real roots.

15. f(x) is continuous & differentiable in $[x_0, x_m]$

By Rolle's Theorem

$$f'(x_i) = \frac{f(x_m) - f(x_0)}{x_m - x_0} = 0$$

Similarly, we can find remaining roots such that

$f'(x_1) = f'(x_2) = f'(x_3) = \cdots$

Repeat the same process we find roots of $f^{n-1}(x) = 0$.

Since $f^{n-1}(x)$ continuous & derivable in $[x_0, x_n]$, Apply Rolle's Theorem

$\Rightarrow f^n(x) = 0.$

16. Let $f(x) = (x^2 - 1)^n$

F(x) is polynomial function in [−1, 1]

& $f(1) = f(-1) = 0$

$\Rightarrow f'(x_1) = 0 \quad \forall x_1 \in (-1, 1)$

17. Let $h(x) = g(x) - f(x)$

$\Rightarrow h(a) = h'(a) = \cdots h^{n-1}(\infty) = 0$

& $h^n(x) \le 0$ for $a < x < b$

For n = 1, Mean value Theorem yields $h(x) \le 0$

In general, $h'(x)$ satisfies given condition for $n - 1$.

Hence, by induction, $h'(x) \le 0$, so $h(x) \le 0$.

18. $f(x) = |x|^m|1 - x|^n$ will have maximum

if $\dfrac{x}{m} = \dfrac{1-x}{n} \Rightarrow x = \dfrac{m}{m+n}$

$\Rightarrow f(x)_{max} = \dfrac{m^m n^n}{(m+n)^{m+n}}.$

19. f(x) is continuous & differentiable in $[x_0, x_m]$

By Rolle's Theorem

$$f'(x_i) = \frac{f(x_m) - f(x_0)}{x_m - x_0} = 0$$

Similarly, we can find remaining roots such that

$f'(x_1) = f'(x_2) = f'(x_3) = \cdots$

Repeat the same process we find roots of $f^{n-1}(x) = 0$.

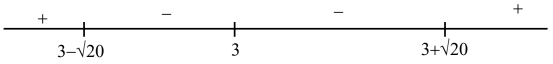
Since $f^{n-1}(x)$ continuous & derivable in $[x_0, x_n]$, Apply Rolle's Theorem

$\Rightarrow f^n(x) = 0.$

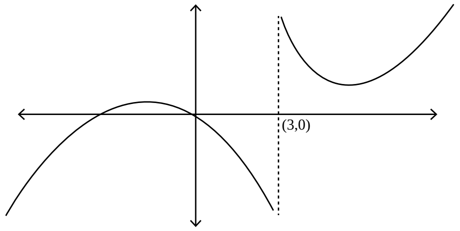
20. i) $y = \dfrac{x^2 + 3x + 2}{x - 3}, \quad x \in R \sim \{3\}$

$$\frac{dy}{dx} = \frac{(x-3)(2x+3) - (x^2 + 3x + 2)}{(x-3)^2} = \frac{x^2 - 6x - 11}{(x-3)^2}$$

Sign scheme for $\dfrac{dy}{dx} \Rightarrow$



$\lim\limits_{x \to 3^+} y = \infty,\ \lim\limits_{x \to 3^-} y = -\infty$

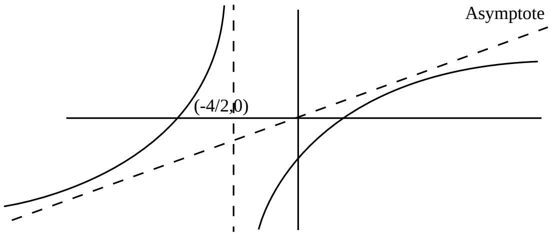


ii) $f(x)=\dfrac{2x^2-3}{7x+4}, \quad x\in R\sim\left\{\dfrac{-4}{7}\right\}$

$$f'(x)=\frac{(7x+4)\,4x-7(2x^2-3)}{(7x+4)^2}=\frac{14x^2+16x+21}{(7x+4)^2}>0$$

$\lim\limits_{x\to\infty} f(x)=\infty,\ \lim\limits_{x\to-\infty} f(x)=-\infty$

$\lim\limits_{x\to\frac{-4^+}{7}} f(x)=-\infty,\ \lim\limits_{x\to\frac{-4^-}{7}} f(x)=\infty$



iii) $f(x)=\dfrac{11x+2}{2x^3-1}, \quad x\in R\sim\{2^{-1/3})$

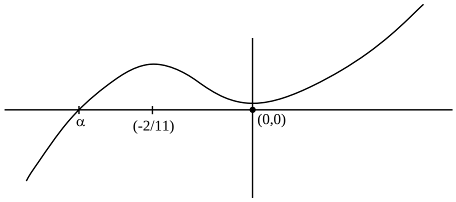
$$f'(x)=\frac{\left(2x^3-1\right)11-6x^2\left(11x+2\right)}{\left(2x^3-1\right)^2}$$

$$=\frac{-44x^3-12x^2-11}{\left(2x^3-1\right)^2}=\frac{-\left(44x^3+12x^2+11\right)}{\left(2x^3-1\right)^2}$$

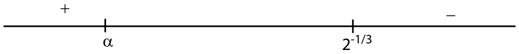
Let $g(x)=44x^3+12x^2+11$

$g'(x)=132x^2+24x=12x(11x+2)$

g(x) ⇒



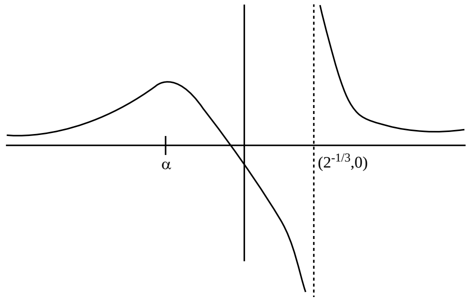
Sign of f′(x)



$\lim\limits_{x\to\infty} f(x)=0,\ \lim\limits_{x\to-\infty} f(x)=0,\ \lim\limits_{x\to 2_+^{-1/3}} f(x)=\infty$

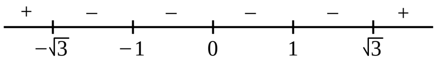
& $\lim\limits_{x\to 2^{\frac{-1}{3}}} f(x)=-\infty$

f(x)→



iv) $f(x)=\dfrac{x^3}{x^2-1}, \quad x\in R\sim\{\pm1\}$

$$f'(x)=\frac{\left(x^2-1\right)3x^2-x^3(2x)}{\left(x^2-1\right)^2}=\frac{x^2\left(x^2-3\right)}{\left(x^2-1\right)^2}$$



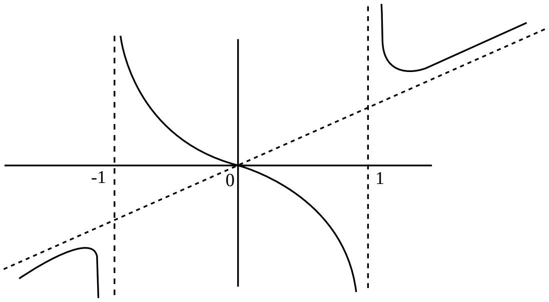
$f(x)=\dfrac{x^3}{x^2-1}=\dfrac{x}{x^2-1}+x.$

⇒ y = x is the asymptote (oblique)

$\lim\limits_{x\to1^+} f(x)=\infty,\ \lim\limits_{x\to1^-} f(x)=-\infty$

$\lim\limits_{x\to-1^+} f(x)=+\infty,\ \lim\limits_{x\to-1^-} f(x)=-\infty$

$f(0)=0.$



21. $y=Ae^{-x^2}+Be^{-(x-c)^2}$

$$\frac{dy}{dx}=-2xAe^{-x^2}-2(x-c)Be^{-(x-c)^2}=0$$

$\Rightarrow xAe^{-x^2}+B(x-c)e^{-(x-c)^2}=0$

Now, $y=Ae^{-x^2}-\dfrac{xAe^{-x^2}}{(x-C)}$

$$y=Ae^{-x^2}\left(1-\frac{x}{x-C}\right)$$

$$y=\frac{ACe^{-x^2}}{C-x}$$

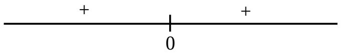
If A, B, C are positive, sign of y is negative for x > C & for x < 0, Sign of y is +ve.

$\lim\limits_{x\to\infty} y = 0 \quad \& \quad \lim\limits_{x\to-\infty} y = 0$

So, there are two roots, one greater than C & other one negative.

22. $y = \frac{1}{x}\ln\left(\frac{e^x - 1}{x}\right)$

Now, $\frac{e^x - 1}{x} > 0$



$\Rightarrow x \in R \sim \{0\}$

As, $\lim\limits_{x\to\infty} \frac{\ln\left(e^x - 1\right) - \ln x}{x}$

$\lim\limits_{x\to\infty} \frac{e^x}{e^x - 1} - \frac{1}{x} = \lim\limits_{x\to\infty} \frac{xe^x - e^x + 1}{x\left(e^x - 1\right)}$

$= \lim\limits_{x\to\infty} \frac{xe^x}{\left(e^x - 1\right) + xe^x} = 1$

$\& \lim\limits_{x\to-\infty} \frac{1}{x}\ln\left(\frac{e^x - 1}{x}\right)$

$\lim\limits_{y\to0} - y\ln\left(e^{-1/y} + 1\right)(y)$

$-\lim\limits_{y\to0}\left(\frac{\ln y}{1/y} + \frac{\ln\left(1 - e^{-1/y}\right)}{1/y}\right)$

$-\left(\lim\limits_{y\to0} \frac{1}{-1/y} + \frac{e^{-1/y}}{\left(1 - e^{-1/y}\right)}\right) = 0$

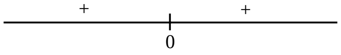
As y ∈ (0,1) when x ∈ (−∞,∞).

$\Rightarrow xy = \ln\left(\frac{e^x - 1}{x}\right)$

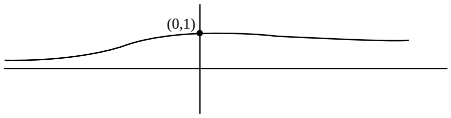
$\Rightarrow xe^{xy} = e^x - 1$

$\Rightarrow y' = \frac{x\left((1 - y)e^x + y\right)}{e^x\left(e^x - 1\right)\left(1 + x^2\right)}$

Sign of y′ →



y →



23. $y = e^{1/x}\sqrt{x^2 + 2x} \quad x \in (-\infty, -2] \cup (0, \infty)$

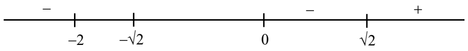
$\lim\limits_{x\to\pm\infty} y = \infty,\ \lim\limits_{x\to0^+} y = \infty, f(-2) = 0$

$y' = \frac{-1}{x^2}e^{1/x}\sqrt{x^2 + 2x} + \frac{e^{1/x}(x + 1)}{\sqrt{x^2 + 2x}}$

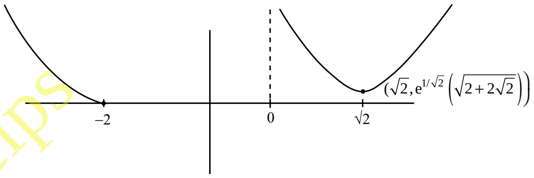
$= \frac{-e^{1/x}\left(x^2 + 2x\right) + e^{1/x}\left(x^3 + x^2\right)}{x^2\sqrt{x^2 + 2x}}$

$= \frac{e^{1/x}}{x^2\sqrt{x^2 + 2x}}\left[x^3 - 2x\right] = \frac{e^{1/x}\left(x^2 - 2\right)}{x\sqrt{x^2 + 2x}}$

Sign of $\frac{dy}{dx}$



Graph of y = f(x) ⇒



It can be seen from graph,

That for $-\infty < \alpha < 0 \Rightarrow e^{1/x}\left(\sqrt{x^2 + 2x}\right) = \infty$ has no roots

0 < α < a ⇒ one negative root

& for α > a, two positive & one negative

Roots are there.

Hence proved.

24. $y = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + c}$

$\Rightarrow (Ay - a)x^2 + (2By - 2b)x + \left(Cy - c_1\right) = 0$

For x to be real,

$(2By - 2b)^2 - 4(Ay - a)\left(cy - c_1\right) \geq 0$

$(By - b)^2 - (Ay - a)\left(cy - c_1\right) \geq 0$

$(b - By)^2 - (a - Ay)\left(c_1 - Cy\right) \geq 0$ .... (1)

As $(a - \lambda A)x^2 + (2b - 2B\lambda)x + \left(C_1 - \lambda C\right) = 0$ is a perfect square

$\Rightarrow (2b - 2B\lambda)^2 - 4(a - \lambda A)\left(c_1 - \lambda c\right) = 0$

$= (b - B\lambda)^2 - (a - \lambda A)\left(c_1 - \lambda c\right) = 0$ .... (11)

From (1) & (11),

Two values of y are the two values of λ.

Hence proved.

25. $f(x) = (x - a)^m (x - b)^n (x - c)^p$

$\ln(x) = m\ln(x-a)+n\ln(x-b)+p\ln(x-c)$

$\frac{1}{f(x)}xf'(x) = \frac{m}{x-a}+\frac{n}{x-b}+\frac{p}{x-c}$

$= \frac{m(x-b)(x-c)+n(x-a)(x-c)+b(x-a)(x-b)}{(x-a)(x-b)(x-c)}$

For f′(x) = 0

$\Rightarrow (m+n+p)x^2-\left(m(b+c)+n(c+a)+p(a+b)\right)x$

$+mbc+ca+pab=0$

Apply Rolle's on f(x) in (a,b) & (b,c)

As f(a) = f(b) = f(c)

⇒f′(x) = 0 has one root in (a,b) & (b,c)

respectively

26. Let $g(x)=\frac{f(x)-f(a)}{x-a}=f'(c)$ where $\frac{x\in(a,b)}{c\in(a,x)}$

$\Rightarrow g(c)=\frac{f(c)-f(a)}{c-a}=f'(c)\forall c\in(a,b)$

Now, let $g(x)=f(x)-f'(a)x$

Using LMVT

$g'(c)=f'(c)-f'(a)=\frac{f(c)-f'(a)c-f(a)-f'(a)a}{c-a}$

$=\frac{f(c)-f(a)}{c-a}-f'(a)$

$g'(c)=\frac{f(c)-f(a)}{c-a}=f'(c)$

27. $y=x^4-8cx^2+c^2$

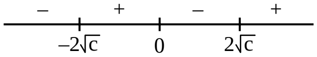
$y'=4x^3-16cx$

$=4x\left(x^2-4c\right)$

$y''=12x^2-16c$

$=4\left(3x^2-4c\right)$

Sign of y' →



y is max at x = 0 or x = 2

y is c² at x = 0 & $c^2-30c+16$

for x = 2

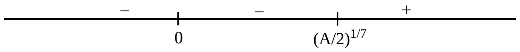
Now if $c\geq\frac{1}{2}, y_{max}=c^2$

if $c<\frac{1}{2}, y_{max}=c^2-3c+16$

28. $f(x)=5x^2+Ax^{-5}$

$f'(x)=10x-5Ax^{-6}$

$=5x\left(2-Ax^{-7}\right)=\frac{5}{x^6}\left(2x^7-A\right)$



f(x) will have maximum value at $x=\left(\frac{A}{2}\right)^{\frac{1}{7}}$

$f\left(\left(\frac{A}{2}\right)^{\frac{1}{7}}\right)=5\left(\frac{A}{2}\right)^{2/7}+A\cdot\frac{A^{-5/7}}{2^{-5/7}}=28$

$\Rightarrow A^{2/7}\left(\frac{5}{2^{2/7}}+2^{5/7}\right)=28$

$\Rightarrow A^{2/7}=4\cdot4^{\frac{1}{7}}$

$\Rightarrow A=4^4=256$

29. $f(x)=\ln(1+x)-x+\frac{x^2}{2}-\frac{x^3}{3}$

$f'(x)=\frac{1}{1+x}-1+x\cdot-x^2$

$=\frac{1+x^2-1-x^2(1+x)}{(1+x)}$

$=\frac{-x^3}{1+x}<0\quad\forall x\in(0,\infty)$

$\Rightarrow f(x)<f(0)$

$\Rightarrow \ln(1+x)<x-\frac{x^2}{2}+\frac{x^3}{3}$

Now, $f(x)=\ln(1+x)-x+\frac{x^2}{2}-\frac{x^3}{3(1+x)}$

$f'(x)=\frac{1}{1+x}-1+x-\frac{(1+x)\left(3x^2\right)-3x^3}{9(1+x)^2}$

$=\frac{3(1+x)+3(x-1)(1+x)^2-x^2}{3(1+x)^2}$

$=\frac{3(x+1)x^2-x^2}{3\left(1+x\right)^2}$

$=\frac{x^2(3x+2)}{3(1+x)^2}>0$

$$\Rightarrow f(x) > f(0)$$

$$\Rightarrow \ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3(1+x)}$$

30. $g(x) = f(x)f'(x)$

$$g'(x) = \left(f'(x)\right)^2 + f(x)f''(x)$$

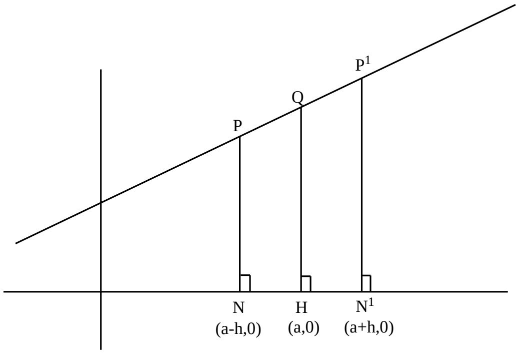
Using LMVT,

$$g'(c) = \frac{f(5)f'(5) - f(0)f'(0)}{5}$$

As $(f(c))^2 f'''(c) < -3f(c)f'(c)f''(c)$

$$\Rightarrow f(c) + 3f'(c) < 0$$

31.



As $\theta H = \frac{1}{2}\left(PN + P'N'\right)$

$$= \frac{1}{2}[\phi(a+h) + \phi(a-h)]$$

$$\Rightarrow Qv = \theta H - vH$$

$$= \frac{1}{2}[\phi(a+h) - 2\phi(a) + \phi(a-h)]$$

$$= \frac{1}{2}[\phi(a+h) - \phi(a) + \phi(a-h) - \phi(a)]$$

$$= \frac{1}{2}h^2\left[\frac{1}{h}\left\{\left[\frac{\phi(a+h)-\phi(a)}{h}\right] - \left(\frac{\phi(a-h)-\phi(a)}{-h}\right)\right\}\right]$$

$$= \frac{1}{2}NH \cdot HN'\phi''(a)$$

Hence proved.

32. Let $y = \dfrac{(n+1)(a_1, a_2, a_3 \cdots a_n)^{1/n+1} x^{1/n+1}}{(a_1 + a_2 + \cdots a_n + x)}$

For

$$\frac{dy}{dx} = 0$$

$$\Rightarrow (a_1 + a_2 + \cdots + x)(a_1 \cdot a_2 \cdots a_n)^{1/n+1} x^{-n/n+1}$$

$$-(n+1)(a_1 \cdot a_2 \cdots a_n)^{1/n+1} x^{1/n+1} = 0$$

$$\Rightarrow (a_1 + a_2 + \cdots + x)x^{-n/n+1} = (n+1)x^{1/n+1}$$

$$\Rightarrow \quad x = \frac{a_1 + a_2 + \cdots + a_n}{n} = A_n$$

Sign of $\frac{dy}{dx}$ will change from +ve to –ve about $x = A_n$.

So, Maximum value of y will be for $x = A_n$
& its value is

$$y = \frac{(n+1)(a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n+1}(a_1 + a_2 + \cdots a_n)^{1/n+1}}{(n)^{1/n+1}\left(\frac{n+1}{n}\right)(a_1 + a_2 + \cdots a_n)}$$

$$= n^{\frac{n}{n+1}} \frac{a_n^{n/n+1}}{(nA_n)^{n/n+1}} = \left(\frac{a_n}{A_n}\right)^{n/n+1}$$

Now, for $x = a_{n+1}$, the function will be decreasing

$$\Rightarrow \frac{\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}}}{\left(\frac{a_1 + \cdots + a_{n+1}}{n}\right)} \le \left(\frac{G_n}{A_n}\right)^{n/n+1}$$

$$\Rightarrow \frac{G_{n+1}}{A_{n+1}} \le \left(\frac{G_n}{A_n}\right)^{n/n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now, from (1)

$$\frac{G_n}{A_n} \le \left(\frac{G_{n-1}}{A_{n-1}}\right)^{\frac{n-1}{n}}$$

Similarly $\frac{G_n}{A_n} \le \left(\frac{G_1}{A_1}\right)^{\frac{1}{2}} = 1$

$\Rightarrow \quad G_n \le A_n$ for n positive numbers.

If all numbers are same,

$$\Rightarrow a_1 = a_2 = a_3 \cdots = a_n$$

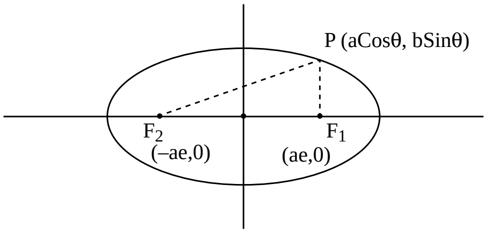
$$\Rightarrow \quad \frac{a_1}{A_1} = \frac{a_2}{A_2} = \frac{a_3}{A_3} = \cdots = 1$$

$$\Rightarrow \frac{a_n}{A_n} = 1 \Rightarrow a_n = A_n$$

Hence proved.

Previous Year's Questions Solutions (JEE Advanced)

1.



$$A=\frac{1}{2}[ae(-b\sin\theta)-aeb\sin\theta]$$

$A = abe\sin\theta$

Area is Maximum for $\sin\theta = 1$

A=abe

2. $f(x)=\dfrac{\log x}{\log a}+\dfrac{\log a}{\log x}$

Use AM $\geq$ GM

$\log_a x+\log_x 9\geq 2$ (only if $a>1\ \&\ a\neq 1$)

Hence, it is False.

3. Let $f(x)=ax^3+bx^2+cx+d$

$f(0)=d,\ f(1)=d$

Using Rolle's Theorem, there exist atleast one value of C where $3ax^2+2bx+c=0$ in $(0,1)$

Hence, A is correct.

4. Property

Hence, A is correct.

5. $y=a\ln|x|+bx^2+x$

For $x=2\Rightarrow \dfrac{dy}{dx}=\dfrac{a}{x}+2bx+1=0$

$\Rightarrow\ a+8b+2=0$ (I)

For $x=-1$

$\dfrac{dy}{dx}=\dfrac{a}{x}+2bx+1=0$

$-a-2b+1=0$ (II)

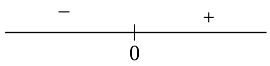
Solving (I) &(II)

$a=+2,\ b=-\dfrac{1}{2}$

Hence, B is correct.

6. $P'(x)=x\left(2a_1+4a_2x^2+\cdots 2na_nx^{2n-2}\right)$

For $p'(x)=0\Rightarrow x=0$



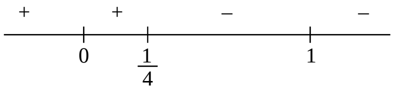
$\Rightarrow$ P(x) has minima at x = 0

Hence, C is correct.

7. $f(x)=x^{25}(1-x)^{75}$

$f'(x)=25x^{24}(1-x)^{75}-75(1-x)^{74}x^{25}$

$=25x^{24}(1-x)^{74}(1-4x)$



Hence, B is correct.

8. $f(x)=\dfrac{x^2-1}{x^2+1}=1-\dfrac{2}{x^2+1}$

For $x=0$, $f(x)$ is minimum $=-1$

Hence, D is correct.

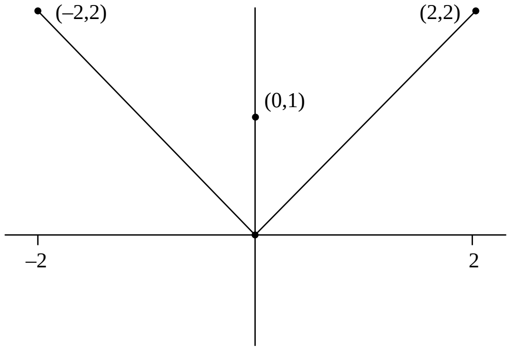
9. $f(x)=\cos x+\cos\sqrt{2}x$

For maximum, both cos x & $\cos\sqrt{2}x=1$

Which is possible at x = 0 only

Hence, B is correct.

10.



Hence, A is correct.

11. $f(x)=\left(1+b^2\right)x^2+2bx+1$

Min value of $f(x)=-\dfrac{\left(4b^2-4-4b^2\right)}{4\left(1+b^2\right)}$

$=\dfrac{1}{1+b^2}=m(b)$

range of m(b) = (0,1]

Hence, D is correct.

12. $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$

As f(x) is not derivable at $x = \frac{1}{2}$

LMVT is not applicable

Hence, A is correct.

13. eqn of tangent $\rightarrow \frac{x\cos\theta}{\sqrt{27}} + y\sin\theta = 1$

Sum of intercept 's' = $\operatorname{cosec}\theta + \sqrt{27}\sec\theta$

$\frac{ds}{d\theta} = -\operatorname{cosec}\theta\cot\theta + \sqrt{27}\sec\theta\tan\theta = 0$

$\Rightarrow \tan^3\theta = \left(\frac{1}{\sqrt{3}}\right)^3$

$\Rightarrow \theta = \pi/6$

Hence, B is correct.

14. $f(x) = x^3 + bx^2 + cx + d$

$f'(x) = 3x^2 + 2bx + c$

$D = 4b^2 - 12c = 4\left(b^2 - 3c\right) < 0$

$\Rightarrow f'(x) > 0 \quad \forall\ x \in R$,

Hence, A is correct.

15. $f(x) = x^\alpha \log x \quad \& \quad f(0) = 0$

$f(1) = 1^\alpha (0) = 0$

For Rolle's Theorem, f(x) needs to be continuous & derivable

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\log x}{x^{-\alpha}} \lim_{x\to 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}} = 0$

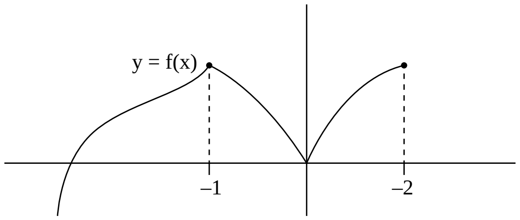
$\Rightarrow \alpha > 0$

Hence, D is correct.

16. $f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 \leq x < 2 \end{cases}$

$f'(x) = \begin{cases} 3(2+x)^2, & 3 < x \leq -1 \\ \frac{2}{3}x^{-1/3}, & -1 < x < 2 \end{cases}$

Point of entremum are x = –1 and x = 0



Hence, C is correct.

17. $f(x) = e^{x^2} + \frac{1}{e^{x^2}}$

as $1 \geq x \geq x^2 \quad \forall\ x \in [0,1]$

$e^{x^2} > xe^{x^2} \geq x^2 e^{x^2} \quad \forall x \in [0,1]$

$\Rightarrow e^{-x^2} + e^{x^2} \geq e^{-x^2} + xe^{x^2} \geq e^{-x^2} + x^2 e^{x^2}$

equality holds at $x = 1$

$\Rightarrow f(x) \Rightarrow g(x) \geq h(x) \Rightarrow a = b = c$

Hence, D is correct.

18. $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1} = 1 - \frac{2ax}{x^2 + ax + 1}$

$f'(x) = \frac{-2a\left(x^2 + ax + 1 - x(2x+a)\right)}{\left(x^2 + ax + 1\right)^2} = \frac{+2a\left(x^2 - 1\right)}{\left(x^2 + ax + 1\right)^2}$

$4ax\left(x^2 + ax + 1\right)^2$

$f''(x) = \frac{-4ax\left(x^2 - 1\right)(2x+a)\left(x^2 + ax + 1\right)}{(x^2 + ax + 1)^4}$

$f''(1) = \frac{4a}{(2+a)^2}, \quad f''(-1) = \frac{-4a}{(2-a)^2}$

$\Rightarrow (2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$

Hence, A is correct.

19. As $f'(x) = \frac{2a(x-1)(x+1)}{\left(x^2 + ax + 1\right)^2}$

$\Rightarrow f(x)$ is decreasing in $(-1,1)$ & minima at $x = 1$

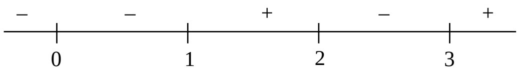
Hence, A is correct.

20. $g'(x) = \frac{f'\left(e^x\right)e^x}{1 + e^{2x}}$

Hence $g'(x) > 0$ for $(0, \infty)$ & negative for $(-\infty, 0)$

Hence, B is correct.

21. $f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$



So, f(x) has local Minimum x = 1 & 3

Hence, B, D is correct.

22. Let $f(x)=ax^3+bx^2+cx+d$

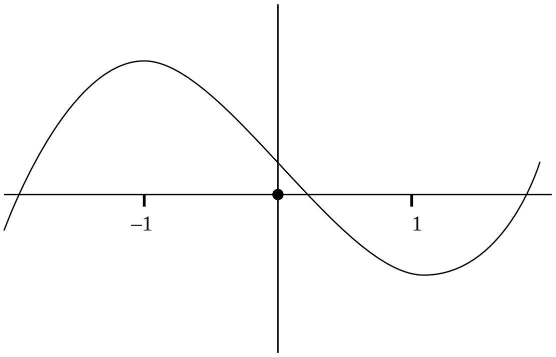
Using $f(3)=18,\quad f(-1)=2\quad f'(-1)=0,\quad f''(0)=0$

$$f(x)=\frac{\left(19x^3-57x+34\right)}{4}$$

f(x) has a local Maximum at

$x=-1$ & local minimum at $x=1$

f(x) is increasing for $x\in[1,2\sqrt{5}]$



(a, f(a)) is (1, –1)

Distance between $(1,-1)$ & $(-1,2)=\sqrt{13}$

Hence, B C is correct.

23. $$g'(x)=f(x)=\begin{cases} e^x, & 0\le x<1 \\ 2-e^{x-1}, & 1<x\le 2 \\ x-e, & 2<x\le 3 \end{cases}$$

$g'(x)=0$ when $x=1+\ln 2$ & $x=e$

$$g''(x)=\begin{cases} -e^{x-1}, & 1<x\le 2 \\ 1, & 2<x\le 3 \end{cases}$$

$g''(1+\ln 2)<0$, so, local maximum at $x=1+\ln 2$

$g''(e)>0$, so, local Minimum at $x=e$

& $f(x)$ is discontinous at $x=1$, we get

a local Maxima at $x=1$ & Local minima at x = 2.

Hence, A, B is correct.

24. $f(x)=\ln x+\int_0^x\sqrt{1+\sin t}\,dt$

$$f'(x)=\frac{1}{x}+\sqrt{1+\sin x},\quad x>0$$

Clearly, $f'(x)$ exists $\forall x\in(0,\infty)$

& $f'(x)$ is continous $\forall x\in(0,\infty)$

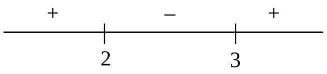
but not differentiable on $(0,\infty)$

More over, $f(x)>f'(x)$. $\forall x\in(1,\infty)$

$$\Rightarrow \ln x+\int_0^{x_0}\sqrt{1+\sin t}\,dt>\frac{1}{x}+\sqrt{1+\sin x}$$

Hence, B, C is correct.

25. $f'(x)=e^{x^2}(x-2)(x-3)$



$f(x)$ has local Maxima at x = 2 & Local Minima at x = 3

$f(x)$ is decreasing in (2,3) as $f'(x)<0$

Hence, A, B, C, D is correct.

26. $$y=\frac{(a+x)(b+x)}{c+x}$$

Let x + c = t

$$y=\frac{((a-c)+t)(b-c+t))}{t}$$

$$=t+\frac{(a-c)(b-c)}{t}+(a-c)+(b-c)$$

$$y=\left(\sqrt{t}-\sqrt{\frac{(a-c)(b-c)}{t}}\right)^2+\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$$

Minimum value of y = $\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$

27. Min value of $x+y=x+\frac{1}{x}=2$

as x,y both are positive.

28. Apply LMUT Theorem

$$\frac{f(1)-f(0)}{1-0}=f'(c),\ C\in(0,1)$$

$$\Rightarrow f'(c)=0$$

Again $\frac{f'(x)-f'(c)}{x-c'}=f''(d),\ x\in(0,1)$

As $|f''(d)|\le 1$

$$\Rightarrow\left|\frac{f'(x)-f'(c)}{x-c}\right|\le 1$$

$$\Rightarrow\left|\frac{f'(x)}{x-c}\right|\le 1$$

$$\Rightarrow|f'(x)|\le|x-c|<1$$

29. Let $h(x) = f(x) - 2\,g(x)$

$h(0) = 2, \quad h(1) = 2$

As $h(x)$ is continous & derivable

Apply Rolle's Theorem

$\Rightarrow h'(c) = 0$

$\Rightarrow f'(c) - 2g'(c) = 0 \quad \forall\ C \in (0,1)$

30. $y = x^2,\ 0 \le c \le 5$

Any pt on parabola is $\left(x, x^2\right)$

Distance between $\left(x, x^2\right)$ & $(0, c)$ is

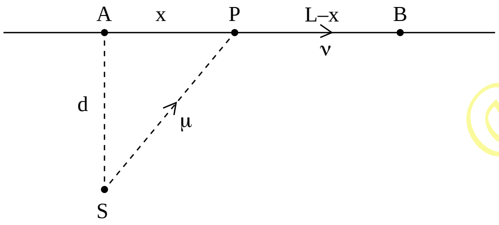
$$D = \sqrt{x^2 + \left(x^2 - c\right)^2}$$

$$\text{or } D^2 = x^4 - (2c-1)x^2 + c^2$$

$$= \left(x^2 - \frac{(2c-1)}{2}\right)^2 + c - \frac{1}{4}$$

is Minimum when, $x^2 = \dfrac{2c-1}{2}$ & $D_{\min} = \sqrt{c - \dfrac{1}{4}}$

31.



Given that AB = LKm

$PB = (L - x)\,\text{km}$

$t =$ Total time from S to B

$= (\text{Time taken from S to P}) + (\text{Time taken from P to B})$

$$t = \frac{SP}{\mu} + \frac{PB}{v}$$

$$t = \frac{\sqrt{d^2 + x^2}}{\mu} + \frac{L - x}{v^2}$$

$$\frac{dt}{dx} = \frac{x}{\mu\sqrt{d^2 + x^2}} - \frac{1}{v}$$

$$\& \quad \frac{d^2t}{dx^2} = \frac{d^2}{\mu\left(d^2 + x^2\right)^{3/2}} > 0$$

If $\dfrac{dt}{dx} = 0 \Rightarrow v^2x^2 = u^2\left(d^2 + x^2\right)$

$$x = \frac{ud}{\sqrt{v^2 - \mu^2}}$$

Therefore, t is minimum for this value of x

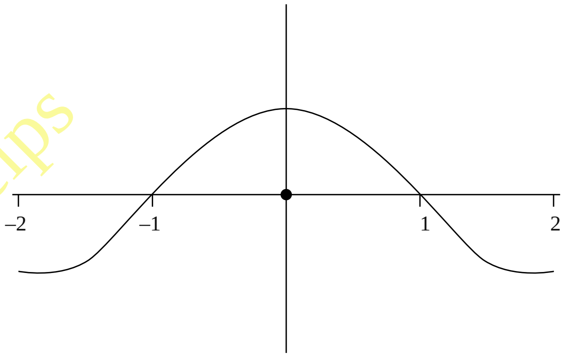
$$\& \quad L - x = L - \frac{\mu d}{\sqrt{v^2 - \mu^2}}$$

32. $y = \dfrac{x}{1 + x^2}$ is an odd function

Also $x > 0 \Rightarrow y > 0$ & $x < 0 \Rightarrow y < 0$

when $x \to \pm\infty,\ y \to 0$

$$\text{As } \frac{dy}{dx} = \frac{1 - x^2}{\left(1 + x^2\right)^2}$$



Which has greatest value at $x = 0$

33. $f(x) = \sin^3 x + \lambda \sin^2 x$

$f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x$

For $f'(x) = 0$

$\sin x = 0,\ \cos x = 0, \sin x = -2\lambda/3$

Since, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2},\quad \cos x \ne 0$

$\Rightarrow \sin x = 0$ or $x = 0$

& $\sin x = -\dfrac{2d}{3}$ or $x = \sin^{-1}\left(-\dfrac{2\lambda}{3}\right)$

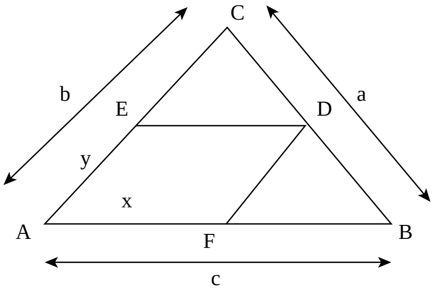
As $-1 < -\dfrac{2d}{3} \le 1$

$$\Rightarrow -\frac{3}{2} < \lambda < \frac{3}{2}$$

If $\lambda = 0$, $\sin x = 0$ has only one soln.

$\Rightarrow \lambda \in \left(-\dfrac{3}{2}, 0\right) \cup \left(0, \dfrac{3}{2}\right)$ there are two distinct solutions.

34.



As $\Delta ABC \,\&\, \Delta ABD$ are similar

$$\Rightarrow \frac{c-x}{c}=\frac{y}{b}$$

$$\Rightarrow y=\frac{b}{c}(c-x)$$

$$\therefore z = \text{Area of AFDE} = xy\sin A=\frac{b\sin A}{c}\left(Cx-x^2\right) \quad \text{(I)}$$

$$\frac{dz}{dx}=\frac{b\sin A}{c}(C-2x)=0$$

$$\rightarrow x=\frac{C}{2}$$

$$\frac{d^2z}{dx^2}=\frac{-2b\sin A}{c}<0$$

$$\Rightarrow z \text{ has maxima at } \frac{c}{2}$$

$$Z_{\max}=\frac{1}{2}\times\frac{1}{2}bc\sin A=\frac{1}{2}\times\frac{1}{2}\begin{vmatrix} p^2 & -p & 1\\ q^2 & q & 1\\ r^2 & -r & 1\end{vmatrix}$$

$$=\frac{1}{4}(p+q)(q+r)(p-r)$$

35. Given curve is $4x^2+a^2y^2=4a^2$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{4}=1 \qquad \text{(I)}$$

Let $P=\left(a\cos\phi, 2\sin\phi\right)$ & $Q(0,-2)$

Let $\mu=(PQ)^2=(a\cos\phi)^2+(2\sin\phi+2)^2$

For $\frac{du}{dp}=0$

$$\Rightarrow \cos\phi\left(\left(8-2a^2\right)\sin\phi+8\right)=0$$

$$\Rightarrow \phi=\frac{\pi}{2} \quad \text{or} \quad \sin\phi=\frac{4}{a^2-4}$$

Since $4<a^2<8 \Rightarrow 0<a^2-4<4$

$$\Rightarrow \frac{a^2-4}{4}<1 \quad \text{or} \quad \frac{y}{a^2-4}>1$$

So $\phi=\frac{\pi}{2}$

Also, $\frac{d^2u}{d\phi^2}=\left(8-2a^2\right)\cos^2\phi+\left(2a^2-8\right)\sin^2\phi-8\sin\phi$

$\frac{d^2u}{d\phi^2}$ at $\phi=\frac{\pi}{2}$ is negative

$\Rightarrow \sqrt{PQ}$ is minimum at $\phi=\frac{\pi}{2}$

$\Rightarrow$ Coordinates of P are $(0, 2)$

36. $f(x)=\int_1^x\left(2(t-1)(t-2)^3+3(t-1)^2(t-2)^2\right)dt$

$$f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$$

$$=(x-1)(x-2)(5x-7)$$

Critical points of $x=1, 2, 7/5$

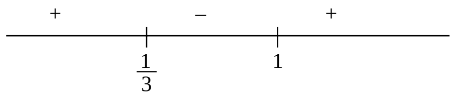
At x=1 is point of maxima

$x=\frac{7}{5}$ is point of minima

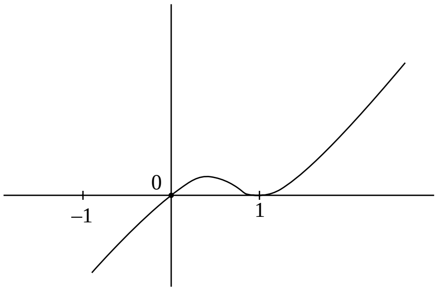
$x=2$ is point of inflection as $\frac{d^2y}{dx^2}$ doesn't change its sign.

37. As $y=x(x-1)^2, \quad 0\le x\le 2$

$$\frac{dy}{dx}=(x-1)(3x-1)$$



Clearly, at x = 1 is the point of minima & $x=\frac{1}{3}$ is the point of maxima.



38. $f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

$= \sec^2 x(1-\cos x)^2(1+2\cos x)$

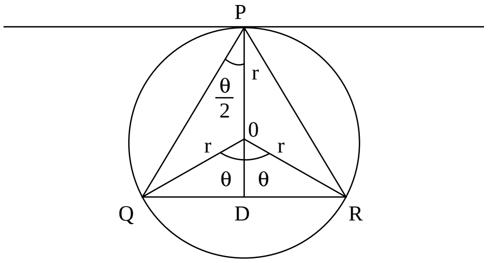
$\Rightarrow f'(x) \geq 0 \ \forall \ x \in (0, \pi/2)$

$\Rightarrow f(x)$ is increasing function

$\Rightarrow f(x) > f(0)$

$\Rightarrow 2\sin x + \tan x \geq 3x \ \forall x \in (0, \pi/2)$

39.



Let O be the centre & 'r' be the radius of the circle

Let QR be the chord parallel to the tangent at point P on the circle

Area of $\Delta PQR = \frac{1}{2}(QR)(PD)$

$A = r\sin\theta(r + r\cos\theta)$

$\frac{dA}{d\theta} = r^2(\cos\theta + \cos 2\theta)$

& $\frac{d^2A}{dQ^2} = r^2(\sin\theta + 2\sin 2\theta)$

For $\frac{dA}{d\theta} = 0$

$\Rightarrow \cos 2\theta + \cos\theta = 0$

$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$

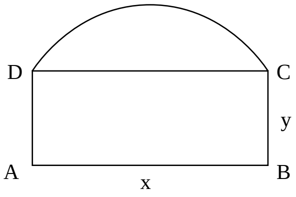
$\Rightarrow \cos\theta = \frac{1}{2}$ or -1

$\Rightarrow \theta = \frac{\pi}{3}$

At $\theta = \frac{\pi}{3}, \frac{d^2A}{d\theta^2} = \frac{-3\sqrt{3}}{2} < 0$

Max Area $= 3\sqrt{3}r^2$

40.



Perimeter of window $= 2x + 2y + \frac{\pi x}{2} = c$

$\Rightarrow y = \frac{1}{2}\left(c - 2x - \frac{\pi x}{2}\right)$ (I)

Let K be the light per square meter, by colored glass,

$\Rightarrow 3k$ is light transmitted by clear glass

Total light $(A) = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 k + xy3x$

$A = \frac{1}{8}k\left(-5\pi x^2 - 24x^2 + 12cx\right)$

$\frac{dA}{dx} = +\frac{k}{8}(-10\pi x - 48x + 12c)$

$\frac{d^2A}{dx^2} = -\frac{1}{4}(5\pi + 24)k < 0$

For $\frac{dA}{dx} = 0 \Rightarrow x = \frac{6c}{5\pi + 24}$

From (1), $y = \frac{(\pi + 6)C}{5\pi + 24}$

ratio of $\frac{x}{y} = 6(\pi + 6)$

41. $f(x) = ax^3 + bx^2 + cx + d$

As $f(-2) = 0$

$\Rightarrow -8a + 4b - 2c + d = 0$ (1)

$f'(x) = 3ax^2 + 2bx + c$

As $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$ (2)

Also, $f(x)$ has relative max/min at $x = -1$ & $\frac{1}{3}$

$f'\left(\frac{1}{3}\right) = 0 \Rightarrow a + 2b + 3c = 0$ (3)

Also, $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$

$\Rightarrow \left.\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right|_{-1}^{1} = \frac{14}{3}$

$\Rightarrow \frac{b}{3} + d = \frac{7}{3}$ (4)

Solving (1), (2), (3), (4),

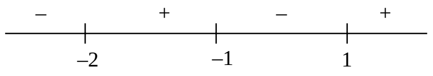
$a = 1, \quad b = 1, \quad c = -1, \quad d = 2$

Required cubic is $x^3 + x^2 - x + 2$

42. For point of minima at x = 1

$$\Rightarrow \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq 0$$

$$\Rightarrow \frac{(b^2+1)(b-1)}{(b+2)(b+1)} \geq 0$$



$$\Rightarrow b \in (-2, -1) \cup (1, \infty)$$

43. $f(x) = 1 + 2\sin x + 3\cos^2 x$

$f'(x) = 2\cos x - 6\cos x \sin x$

$= 2\cos x(1 - 3\sin x)$

For $f'(x) = 0 \Rightarrow \cos x = 0 \quad \text{or} \quad \theta^2 \sin x = \frac{1}{3}$

$$\Rightarrow x = \frac{\pi}{2} \text{ or } x = \sin^{-1}\frac{1}{3}$$

Now, $f'\left(\frac{\pi^+}{2}\right) > 0 \quad \& \quad f'\left(\frac{\pi^-}{2}\right) < 0$

$\Rightarrow$ Minima at $x = \frac{\pi}{2}$

$f'\left(\sin^{-1}\frac{1}{3}\right) > 0 \quad \& \quad f'\left(\sin^{-1}\frac{1}{3}\right) < 0$

$\Rightarrow$ Maxima at $x = \sin^{-1} 1/3$

$f\left(\frac{x}{2}\right) = 3 \quad \& \quad f\left(\sin^{-1}\frac{1}{3}\right) = \frac{13}{3}$

44. As $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

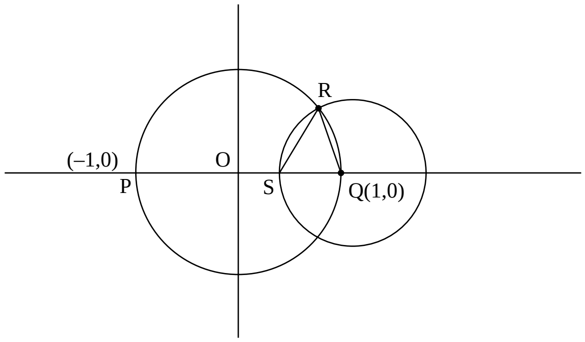
$\Rightarrow \sqrt{3} - \sqrt{3}\tan A \tan B = \tan A + \tan B$

$\Rightarrow \sqrt{3}\tan A \tan B = \sqrt{3} - (\tan A + \tan B)$

$\geq \sqrt{3} - \left(\frac{2}{\sqrt{3}}\right)$ (using AM ≥ GM)

$\tan A \tan B \geq \frac{1}{3}$

45.



Since circle $x^2 + y^2 = 1$ cuts x-axis P and Q

$P \equiv (-1, 0) \;\&\; Q \equiv (1, 0)$

Equation of circle with centre at $Q(1,0)$ & having `r' radius is

$(x-1)^2 + y^2 = x^2$

Solving two curves,

$$x = 1 - \frac{r^2}{2}, \quad y = \pm\sqrt{1 - x^2} = \sqrt{r^2 - \frac{r^4}{4}}$$

$$\rightarrow A(\text{Area of } \Delta QSR) = \frac{1}{2} r \sqrt{\frac{r^2 - r^y}{4}}$$

$$A^2 = \frac{r^4}{4} - \frac{r^6}{16} = 2 \quad \text{(Say)}$$

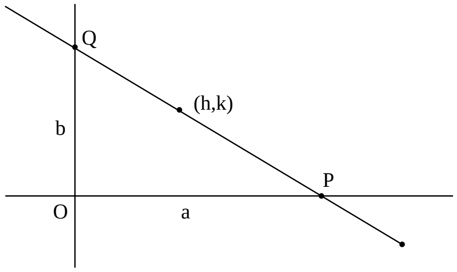
$$\frac{dz}{dr} = r^3 - \frac{6r^5}{16} = 0$$

$$\Rightarrow r = \sqrt{\frac{8}{3}} \;\&\; \frac{d^2z}{dr^2} = -\frac{16}{3} < 0 \text{ at } z = \sqrt{\frac{8}{3}}$$

$$\Rightarrow \text{A is maximum at } r = \sqrt{\frac{8}{3}}$$

$$\text{Area of } \Delta QRS = \frac{1}{2}\sqrt{\frac{8}{3}}\sqrt{\frac{8}{3} - \frac{64}{36}} = \frac{4}{3\sqrt{3}} \text{ sq units}$$

46.



eqn of line $\Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$

It passes theough (h, k)

$$\Rightarrow \frac{h}{a}+\frac{k}{b}=1$$

$$\Rightarrow \quad b=\frac{ak}{a-h}$$

$$\Delta=\frac{1}{2}ab=\frac{1}{2}\frac{a^2k}{a-h}$$

Δ is min when $y=\frac{a-h}{a^2}=\frac{1}{a}-\frac{h}{a^2}$ is max.

$$\frac{dy}{da}=-\frac{1}{a^2}+\frac{2h}{a^3}\Rightarrow 0\Rightarrow a=2h$$

$$\frac{d^2y}{da^2}=\frac{2}{a^3}-\frac{6h}{a^4}$$

$$\frac{d^2y}{dx^2} \text{ at a=2 h}=-\frac{1}{a^3}<0$$

$$\Rightarrow \Delta_{Min}=\frac{1}{2}4h^2*\frac{k}{h}=2hk.$$

47. $f(x)=\frac{1}{8}\log x-bx+x^2$

$$f'(x)=\frac{1}{8x}-b+2x=\frac{16x^2-8bx+1}{8x}$$

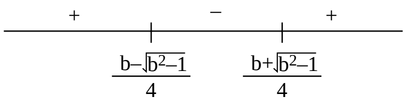
For $f'(x)=0$

$$\Rightarrow 16x^2-8bx+1=0$$

$$x=\frac{b\pm\sqrt{b^2-1}}{4}$$

$\Rightarrow b>1$ for x to be real.

sign scheme for $f'(x)$



$$\frac{b-\sqrt{b^2-1}}{4} \quad \frac{b+\sqrt{b^2-1}}{4}$$

$\rightarrow f(x)$ has maxima at $x=\frac{b-\sqrt{b^2-1}}{4}$ &

minima at $x=\frac{b+\sqrt{b^2-1}}{4}$

& If b = 1, $f'(x)=\frac{(4x-1)^2}{x}>0$

$\Rightarrow$ neither maxima nor minima if b = 1

48. Applying $R_3\rightarrow R_3-R_1-2R_2$

$$f'(x)=\begin{vmatrix}2ax & 2ax-a & 2ax+b+1\\ b & b+1 & -1\\ 0 & 0 & 1\end{vmatrix}$$

=2ax + b

Integrating, $f(x)=ax^2+bx+c.$

Since f (x) has maxima at x = 5/2

$$f'(5/2)=0\Rightarrow 5a+b=0$$

$$\&\ f(0)=2\Rightarrow c=2$$

$$\&\ f(1)=1\Rightarrow \quad a+b+c=1$$

Solving, we get $a=\frac{1}{4}, b=-\frac{5}{4}$

$$\Rightarrow f(x)=\frac{x^2}{4}-\frac{5x}{4}+2$$

49. As $ax^2+2bxy+ay^2=c$

$$2ax+2by+2bx\frac{dy}{dx}+2ay\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(ax+by)}{bx+ay}$$

Let point be P(h, k)

Now, $\left.\frac{dy}{dx}\right|_{at\,P}=-\frac{(ah+bk)}{bh+ak}$

$\Rightarrow$ As distance from origin is minimum,

$$\frac{k}{h}\times\frac{(ah+bk)}{(bh+ak)}=1$$

$$\Rightarrow ahk+bk^2=bh^2+ahk$$

$$\Rightarrow k^2=h^2$$

$$\Rightarrow h=\pm k.$$

As (h, k) lies on the curve $ax^2+2bxy+ay^2=c$

$$\Rightarrow ah^2+2bhk+ak^2-c=0$$

Put h = k

$$h=\pm\sqrt{\frac{c}{2(a+b)}}$$

$\Rightarrow$ Point P can be

$$\left(\sqrt{\frac{c}{2(a+b)}},\sqrt{\frac{c}{2(a+b)}}\right)$$

or $$\left(-\sqrt{\frac{c}{2(a+b)}},-\sqrt{\frac{c}{2(a+b)}}\right)$$

50. Let $P(a\cos\theta, b\sin\theta)$

Tangent at $p \Rightarrow \dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

Foot of $\perp$ from (0, 0) to tangent is N.

$$\Rightarrow \text{N is}\left(\frac{ab^2\cos\theta}{b^2\cos^2\theta+a^2\sin^2\theta},\frac{a^2b\sin\theta}{b^2\cos^2\theta+a^2\sin^2\theta}\right)$$

Area of

$$\Delta PON=\frac{1}{2}\left[a\cos\theta\left(\frac{-a^2b\sin\theta}{b^2\cos^2\theta+a^2\sin^2\theta}\right.\left.+\frac{ab^2\cos\theta}{b^2\cos^2\theta+a^2\sin^2\theta}(b\sin\theta)\right)\right]$$

$$A=\frac{1}{2}\frac{(ab\cos\theta\sin\theta)\left(b^2-a^2\right)}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$=\frac{ab\left(b^2-a^2\right)\tan\theta}{2\left(b^2+a^2\tan^2\theta\right)}=\frac{ab\left(b^2-a^2\right)}{2\left(b^2+a^2\tan^2\theta\right)}$$

A_{max} will be when,

$\Rightarrow\ b^2\cot\theta=a^2\tan\theta$ (Using AM $\ge$ GM)

$$\Rightarrow \tan^2\theta=\frac{b^2}{a^2}$$

$$\Rightarrow \tan\theta=\pm\frac{b}{a}$$

$$\Rightarrow \text{ Point P is }\left(\frac{\pm a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

51. Eqn of normal to ellipse is $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$

$\Rightarrow 3x\sec\theta-2y\operatorname{cosec}\theta=5$

Distance from $(0,0)=\dfrac{5}{\sqrt{9\sec^2\theta+4\operatorname{cosec}^2\theta}}$

Minimize $f(\theta)=9\sec^2\theta+4\operatorname{cosec}^2\theta$

$f'(\theta)=18\sec^2\theta\tan\theta-8\operatorname{cosec}^2\theta\cot\theta=0$

$$\Rightarrow \tan^4\theta=\frac{4}{9}\Rightarrow \tan^2\theta=\frac{2}{3}$$

$$\Rightarrow \tan\theta=\pm\frac{\sqrt{2}}{\sqrt{3}}$$

$\Rightarrow$ eqn of normal $\Rightarrow \pm\sqrt{3}x\pm\sqrt{2}y=5$

52. Let $x=\cos\theta$

$\Rightarrow 4x^3-3x-p=0 \quad \Rightarrow 4\cos^3\theta-3\cos\theta-p=0$

$\Rightarrow \cos 3\theta=p$

Since, $x=\cos\theta\in\left[\dfrac{1}{2},1\right],\theta\in[0,\pi/3]$

$\Rightarrow p\in[-1,1]$

Also, $\cos 3\theta=p$

$$\Rightarrow \theta=\frac{1}{3}\cos^{-1}p$$

$$\Rightarrow x=\cos\theta=\cos\left(\frac{1}{3}\cos^{-1}p\right)$$

53. $y-2=-m(x-8)$

$P\rightarrow\left(8+\dfrac{2}{m},0\right)$ & $Q\rightarrow(0,2+8m)$

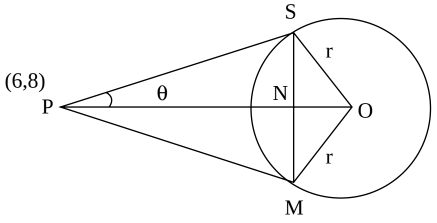
$$OP+OQ=l=\sqrt{\left(8+\frac{2}{m}\right)^2+(2+8m)^2}$$

$$\Rightarrow l=\sqrt{68+\frac{4}{m^2}+64m^2+\frac{32}{m}+32m}$$

$l_{min}=\sqrt{68+32+64}$ (Using AM $\ge$ GM)

$=\sqrt{164}$

54.



As $\tan\theta = \frac{r}{PS}$

$= \frac{r}{\sqrt{100-r^2}}$

Area of $\Delta PSM = \frac{1}{2}\times SM \times PN = SN \times PN$

$A = \sqrt{100-r^2}\,\sin\theta\cos\theta$

$A = \frac{r\left(100-r^2\right)^{3/2}}{100}$

$\frac{dA}{dr} = \frac{3}{200}(100-r^2)^{1/2}\ \ (-2r)r\ + \frac{(100-r^2)^{3/2}}{100} = 0$

$\Rightarrow r = 5$

55. Let $P = (\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$

Slope of tangent at P = –1

Now, $x^2 + 2y^2 = 6$

$2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} = \frac{-1}{\sqrt{2}}\cot\theta = -1$

$\Rightarrow \cot\theta = \sqrt{2}$

$\Rightarrow P \Rightarrow \left(\sqrt{6}\cdot\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{3}.\frac{1}{\sqrt{3}}\right) = (2,1)$

56. Let $g(x) = \int p(x)dx = \frac{51x^{102}}{102} - \frac{2323x^{101}}{101}$

$-\frac{45x^2}{2} + 1035x + c$

$g\left(45^{1/100}\right) = c \ \ \& \, g(46) = c$

$\Rightarrow g'(x) = \phi(x)$ will have atleast one root in $\left(45^{1/100}, 46\right)$

57. Let $P(x) = ax^3 + bx^2 + cx + d$

As $(P(-1) = 10 \Rightarrow -a + b - c + d = 10$

$P(1) = -6 \Rightarrow a + b + c + d = -6$

$p'(-1) = 3a - 2b + c = 0$

$p''(1) = 6a + 2b = 0$

Solving, we get $P(x) = x^3 - 3x^2 - 9x + 5$

$P'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$

x = –1 is point of maximum & x = 3 is point of minimum.

Hence distance between $(-1,10)\,\&\,(3,-2)$ is $4\sqrt{65}$ units.

58. $g(x) = \frac{d}{dx}\left(f(x)f'(x)\right)$

Let $h(x) = f(x)\,f'(x)$

Now, $f(x) = 0$ has min 4 soln & $f'(x) = 0$ has min 3 soln

$\Rightarrow f(x) = 0$ has min 7 soln

$\Rightarrow f'(x) = g(x) = 0$ has min 6 solutions

59. $f(x) = 2x^3 - 15x^2 + 36x - 48$

$f'(x) = 6(x-2)(x-3)$

$\because A = \left\{x \mid x^2 + 20 - 9x \le 0\right\}$

$4 \le x < 5$

$\Rightarrow f(x)$ is increasing for $x \in [3, \infty)$

Maxima of $f(x) = f(5) = 7$

60. Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

As $\lim\limits_{x\to 0}\frac{p(x)}{x^2} = 1$

$\Rightarrow \lim\limits_{x\to 0} ax^2 + bx + c + \frac{d}{x} + \frac{e}{x^2} = 1$

$\Rightarrow d = e = 0 \quad \& \quad c = 1$

$\Rightarrow P(x) = ax^4 + bx^3 + x^2$

$P'(x) = 4ax^3 + 3bx^2 + 2x$

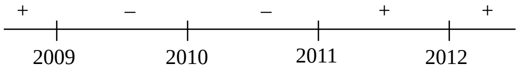
$P'(1) = P'(2) = 0$ So

4a + 3b + 2 = 0 & 32a + 12b + 4 = 0

Solving, $a = \frac{1}{4}, b = -1$

$\Rightarrow P(2) = 16a + 8b + 4c = 0$

61. Sign scheme of $f'(x)$



$\Rightarrow$ Local Maxima of f(x) occurs at x = 2009

$\Rightarrow$ Local Maxima of g(x) occurs at x = 2009

No of point of local Maxima = 1

62. $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24$

Hence, $f''(x) > 0 \quad \forall x \in R$

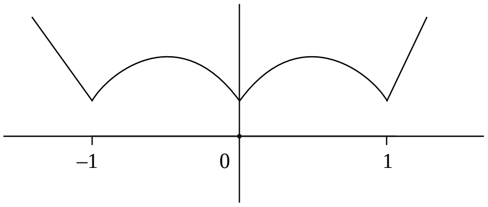
$f'(x)$ is increasing function,

So, f(x) will have only one pt of extrema,
$f(0) = -1$

Hence, f (x) = 0 has two distinct real roots.

63. $f(x) = |x| + |x^2 - 1|$

$$f(x) = \begin{cases} x^2 - x - 1, & x < -1 \\ -x^2 - x + 1, & -1 \le x \le 0 \\ -x^2 + x + 1, & 0 < x < 1 \\ x^2 + x - 1, & x \ge 1 \end{cases}$$



Hence, there are 5 points at which f (x) attains local maxima or minima.

64. $P'(x) = \lambda(x-1)(x-3) = \lambda\left(x^2 - 4x + 3\right)$

$P(x) = \lambda\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$

As $P(1) = 6$

$6 = \lambda\left(\frac{1}{3} - 2 + 3\right) + c$

$\Rightarrow 18 = 4d + 3c$ (I)

$P(3) = 2$

$2 = \lambda\left(\frac{27}{3} - 2 \times 9 + 9\right) + C$

$\Rightarrow c = 2$

$\Rightarrow \lambda = 3$

$P'(x) = 3(x-1)(x-3)$

$P'(0) = 9$

Previous Year's Questions (AIEEE/JEE MAINS)

1. As $\cos t\,(c) = av\frac{b}{v}$

Now, $75 = 30a + \frac{b}{30}$ .... (1)

$65 = 40a + \frac{b}{40}$ .... (2)

Solving (1) and (11), we get $a = \frac{1}{2}, b = 1800$

$\Rightarrow c = \frac{v}{2} + \frac{1800}{v}$

Using AM ≥ GM

$c \ge 2\sqrt{\frac{v}{2} + \frac{1800}{v}} = 6$

Hence, C is correct

2. Area $(A) = \frac{1}{2} B \times H$

$= \frac{1}{2} B \times \sqrt{h^2 - B^2}$

$= \frac{1}{2}\sqrt{B^2(h^2 - B^2)}$

Using AM ≥ GM

$\frac{B^2 + h^2 - B^2}{2} \ge \sqrt{B^2(h^2 - B^2)}$

$\frac{h^2}{2} \ge \sqrt{B^2(h^2 - B^2)}$

Max Area $= \frac{h^2}{4}$

Hence, D is correct

3. $f(x) = \alpha \log|x| + \beta x^2 + x$

$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

$f'(-1) = 0 \Rightarrow \frac{-\alpha}{1} - 2\beta^{+1} = 0 \Rightarrow -\alpha = 2\beta - 1$ .... (1)

$f'(2) = 0 \Rightarrow \frac{\alpha}{2} + 4\beta^{+1} = 0$ .... (11)

Upon solving (1) & (11)

$\alpha = 2, \beta = -\frac{1}{2}$

Hence, C is correct.

4. As $\lim\limits_{x\to 0} \frac{f(x)}{x^2} = 2$

$\Rightarrow f(0) = 0$ for limit to exist

$\lim\limits_{x\to 0} \frac{f'(x)}{2x} = 2$

$\Rightarrow f'(0) = 0$ for limit to exist

$$\lim_{x\to 0}\frac{f''(x)}{2} = 2 \Rightarrow f''(0) = 4$$

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$e = d = 0 \quad$ as $f(0) = f'(0) = 0$

$\Rightarrow f(x) = ax^4 + bx^3 + cx^2$

$f'(x) = 4ax^3 + 3bx^2 + 2cx$

$f''(x) = 12ax^2 + 6bx + 2c$

$c = 2$ as $f''(0) = 4$

$f'(x) = x\left(4ax^2 + 3bx + 4\right)$

Now $f'(1) = f'(2) = 0$

$\Rightarrow 4a + 3b + 4 = 0$

& $16a + 6b + 4 = 0$

solving together, $a = +\frac{1}{2}, b = -2$

$f(x) = +\frac{x^4}{2} - 2x^3 + 2x^2$

$f(2) = +8 - 16 + 8 = 0$

Hence, C is correct.

5. $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$

$$f'(x) = \frac{0.6(1+x)^{-0.4}\left(1+x^{0.6}\right) - 0.6x^{-0.4}(1+x)^{0.6}}{\left(1+x^{0.6}\right)^2}$$

$$= \frac{0.6\left(1+x^{0.6}\right) - x^{-0.4}(1+x)}{\left(1+x^{0.6}\right)^2(1+x)^{0.4}}$$

$$= \frac{0.6\left(x^{0.4}-1\right)}{\left(1+x^{0.6}\right)^2(1+x)^{0.4}x^{0.4}} < 0$$

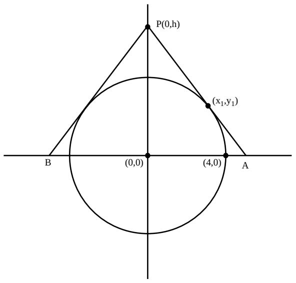
$\Rightarrow f(x)$ is decreasing function

$\Rightarrow \quad k = f(0) = 1 \Rightarrow$ max

$k = f(1) = \frac{2^{0.6}}{2} = 2^{-0.4} =$ Min

Hence, D is correct.

6.



As $x^2 + y^2 = 16$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x_1}{y_1}$

eqn of tangent $\Rightarrow xx_1 + yy_1 = 16$

$\Rightarrow hy_1 = 16$

$y_1 = \frac{16}{h} \qquad -(1)$

$$x_1^2 + y_1^2 = 16 \Rightarrow x_1 = \sqrt{16 - \frac{16^2}{h^2}} = \frac{4}{h}\sqrt{h^2 - 16}$$

eqn of tangent$= \frac{4x}{h}\sqrt{h^2 - 16} + y\frac{16}{h} = 16$

Put $y = 0, \quad x = \frac{4h}{\sqrt{h^2 - 16}}$

As $\Delta PAB = 2 \times \frac{1}{2} \times h \times \frac{4h}{\sqrt{h^2 - 16}}$

$$A = \frac{4h^2}{\sqrt{h^2 - 16}}$$

$$\frac{dA}{dh} = \frac{\left(\sqrt{h^2 - 16}\right)(8h) - 4h^2 \times \frac{h}{\sqrt{h^2 - 16}}}{h^2 - 16} = 0$$

$\Rightarrow h = 0$ or $h = \sqrt{32}$

$\Rightarrow h = 4\sqrt{2}$

Hence, D is correct.

7. $f(x) = 4 + \frac{1}{2}\sin^2 2x - 2\cos^4 x$

$= 4 + \frac{1}{2}\sin^2 2x - \frac{1}{2}(1 + \cos 2x)^2$

$= \frac{7}{2} - \frac{1}{2}\cos 4x - \cos 2x$

$= \frac{7}{2} - \frac{1}{2}\left(2\cos^2 2x - 1\right) - \cos 2x$

$= -\cos^2 2x - \cos 2x + 4$

$= -\left(\cos 2x + \frac{1}{2}\right)^2 + \frac{17}{4}$

$M = \frac{17}{4}$

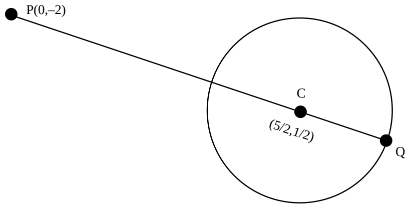
$$M = \frac{17}{4}$$

$$m = \frac{17}{4} - \frac{9}{4} = 2$$

$$M - m = \frac{9}{4}$$

Hence, A is correct.

8.



$x^2 + y^2 - 5x - y + 5 = 0$

$$PQ_{max} = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2} + \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 5}$$

$$= \frac{5}{\sqrt{2}} + \frac{\sqrt{6}}{2} = \frac{5+\sqrt{3}}{\sqrt{2}}$$

$$(PQ)^2 = \left(\frac{5+\sqrt{3}}{\sqrt{2}}\right)^2 = \frac{1}{2}\left[25+3+10\sqrt{3}\right]$$

$$= 14 + 5\sqrt{3}$$

Hence, B is correct.

9. Length of arc $= \theta r$

Area of sector $= \frac{\theta}{2} r^2$

$\Rightarrow 2r + \theta r = 20$

$$\Rightarrow \theta = \frac{20-2r}{r}$$

$$A = \left(\frac{10-x}{r}\right) r^2 = (10-r)r$$

A is max when $10 - r = r \Rightarrow r = 5$

So, Area = 25

Hence, D is correct.

10. $$\lim_{x\to 0} \frac{f(x)}{x^2} = 2$$

$\Rightarrow f'(0) = 0, f(0) = 0, f''(0) = 4$

$\quad \& f'(1) = f'(2) = 0$

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

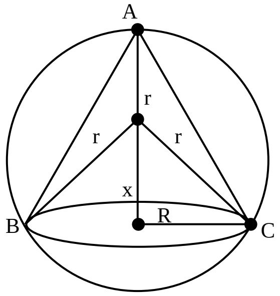
Upon solving,

$$f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$$

$$f(-1) = \frac{1}{2} + 2 + 2 = \frac{9}{2}$$

Hence, B is correct.

11. $x^2 + R^2 = r^2 = 9$



$x^2 = 9 - R^2 \qquad -(1)$

Curved Area $= \pi R l$

$= \pi R\sqrt{h^2 + R^2}$

$= \pi R\sqrt{(3+x)^2 + R^2}$

$= \pi\sqrt{9-x^2}\sqrt{(3+x)^2 + 9 - x^2}$

$= \pi\sqrt{9-x^2}\sqrt{18+6x}$

$= \pi\sqrt{6}(x+3)\sqrt{3-x}$

Max Area is possible when

$$\frac{x+3}{1} = \frac{3-x}{1/2}$$

$\Rightarrow x + 3 = 6 - 2x$

$\Rightarrow x = 1$

Area $= 8\sqrt{3}\pi$

Hence, D is correct

12. $$h(x) = \frac{x^2 + 1/x^2}{x - \frac{1}{x}} = \frac{(x-1/x)^2 + 2}{x - 1/x}$$

$$= x - 1/x + \frac{2}{x - 1/x}$$

Let $x - \frac{1}{x} = t$

$$r(t) = t + \frac{2}{t}$$

$$r'(t) = 1 - \frac{2}{t^2} = 0 \Rightarrow t = \pm\sqrt{2}$$

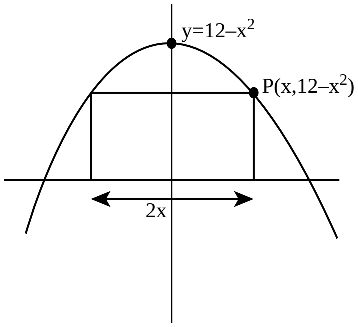
$$r''(t) = \frac{4}{t^3}$$

$\Rightarrow r(t)$ will have minima at $\sqrt{2}$

$$h(x) = \sqrt{2} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$$

Hence, C is correct.

13. Area(A) $= 2x\left(12 - x^2\right)$



$\Rightarrow A = 24x - 2x^3$

$$\frac{dA}{dx} = 24 - 6x^2$$

For

$$\frac{dA}{dx} = 0 \Rightarrow x = \pm 2$$

$\Rightarrow x = 2$

As $$\frac{d^2A}{dx^2} = -12x < 0$$

$\Rightarrow$ Maximum Area = 32

Hence, C is correct.

14. $y = 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$

$= 3\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta - \frac{5}{2}\cos\theta$

$y = \frac{1}{2}[-\cos\theta + 5\sqrt{3}\sin\theta]$

$y_{max} = \frac{1}{2}\sqrt{1+75} = \frac{\sqrt{76}}{2} = \sqrt{19}$

Hence, Ais correct.

15. $S = \left\{x \in R, x^2 + 30 \le 11x\right\}$

$\Rightarrow x^2 - 11x + 30 \le 0$

$\Rightarrow (x-6)(x-5) \le 0$

$x \in [5, 6]$

As $f(x) = 3x^3 - 18x^2 + 27x - 40$

$f'(x) = 9x^2 - 36x + 27$

$= 9\left(x^2 - 4x + 3\right)$

$= 9(x-3)(x-1)$

$\Rightarrow$ in $x \in [5, 6]$, $f'(x) > 0$

$f(x)_{max} = 3(6)^3 - 18 \cdot 6^2 + 27 \cdot 6 - 40$

$= 648 - 648 + 162 - 40$

$= 122$

Hence, A is correct.

16. Volume $(V) = \frac{1}{3}\pi r^2 h$

As $h = l\cos\theta = 3\cos\theta$

$r = l\sin\theta = 3\sin\theta$

$\Rightarrow v = 9\pi\sin^2\theta\cos\theta$

$\frac{dv}{d\theta} = 9\pi\left(-\sin^3\theta + 2\sin\theta\cos^2\theta\right)$

$= 9\pi\sin\theta\left(-\sin^2\theta + 2\cos^2\theta\right)$

For $\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}, \sin\theta = \sqrt{\frac{2}{3}}$

$v = 9\pi \times \frac{2}{3} \times \frac{1}{\sqrt{3}} = 2\pi\sqrt{3}m^3$

Hence, C is correct.